Chin. Ann. Math.
34B(4), 2013, 575–586
DOI: 10.1007/s11401-013-0779-7

Chinese Annals of Mathematics, Series B

Holomorphic Maps from Sasakian Manifolds into Kähler Manifolds*

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Abstract The authors consider $\pm(\Phi, J)$-holomorphic maps from Sasakian manifolds into Kähler manifolds, which can be seen as counterparts of holomorphic maps in Kähler geometry. It is proved that those maps must be harmonic and basic. Then a Schwarz lemma for those maps is obtained. On the other hand, an invariant in its basic homotopic class is obtained. Moreover, the invariant is just held in the class of basic maps.

Keywords Sasakian manifold, Harmonic, Schwarz lemma, Homotopic invariant
2000 MR Subject Classification 53C21, 55P91

1 Introduction

An odd dimensional Riemannian manifold (M, g) is said to be a Sasakian manifold if the cone manifold $(C(M), \widetilde{g}) = (M \times R^+,\ r^2 g + \mathrm{d}r^2)$ is Kähler. Sasakian geometry was introduced by Sasaki [11] and is often described as an odd dimensional counterpart of Kähler geometry. The following equivalent conditions provide three alternative characterizations of the Sasakian property, and the proof can be found in [1]. Let (M, g) be a $(2n+1)$-dimensional Riemannian manifold. Then the following conditions are equivalent:

(1) There exists a Killing vector field ξ of unit length on M, so that the $(1,1)$ type tensor field Φ, defined by $\Phi(X) = \nabla_X \xi$, satisfies the condition

$$(\nabla_X \Phi)(Y) = \langle \xi, Y\rangle_g X - \langle X, Y\rangle_g \xi \tag{1.1}$$

for any pair of vector fields X and Y on M.

(2) There exists a Killing vector field ξ of unit length on M, so that the Riemann curvature satisfies the condition

$$R(X, \xi)Y = \langle \xi, Y\rangle_g X - \langle X, Y\rangle_g \xi \tag{1.2}$$

for any pair of vector fields X and Y on M.

(3) The metric cone $(M \times R^+, r^2 g + \mathrm{d}r^2)$ is Kähler.

Set $\eta(X) = \langle X, \xi\rangle_g$ for any vector field X on M. In view of the above equivalent conditions, (ξ, η, Φ) is called a contact structure on M. The Killing vector field ξ is called the characteristic

Manuscript received December 27, 2011. Revised July 20, 2012.
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*Project supported by the National Natural Science Foundation of China (Nos. 10771188, 10831008, 11071212, 11171297) and the Doctoral Program Foundation of the Ministry of Education of China (No. 20060335133).

or Reeb vector field, η is called the contact 1-form, and Φ is a $(1,1)$ tensor field which defines a complex structure on the contact sub-bundle $D = \ker \eta$ which annihilates ξ. Sasakian geometry is a special kind of contact metric geometry, such that the structure transverse to the Reeb vector field ξ is Kähler and invariant under the flow of ξ. Recently, influenced by the recently found relevance of Sasakian manifolds in string theory (see [14]), Sasakian geometry was extensively studied. Various differential geometric aspects of Sasakian manifolds were studied by Boyer, Galicki and their collaborators (see [1–5]). A Sasakian manifold (M, g) is said to be Sasakian-Einstein if the Ricci tensor of the metric g satisfies the Einstein condition, i.e., $\mathrm{Ric}_g = \lambda g$. The existence of Sasakian-Einstein metrics is of great interest in the physics of the CFT/Ads duality conjecture (see [6–10, 12, 14, 16–18, 20–21]).

On the other hand, harmonic map is a very useful tool in the study of differential geometry. A lot of results about harmonic map were discovered, combining both global and local aspects. A map between Riemannian manifolds is called harmonic if it is a critical point of the energy functional. It is well-known that every holomorphic map between Kähler manifolds must be harmonic. In this paper, we consider some special maps from Sasakian manifolds into Kähler manifolds. Let (M, ξ, η, Φ, g) be a Sasakian manifold, and N be a smooth manifold. A map $f : M \to N$ is called basic if it satisfies $\mathrm{d}f(\xi) \equiv 0$.

Definition 1.1 *A smooth map $f : M \to N$ from a Sasakian manifold (M, ξ, η, Φ, g) to an almost complex manifold (N, J) is called $(\pm)$ (Φ, J)-holomorphic if it satisfies*

$$\mathrm{d}f \circ \Phi = \pm J \circ \mathrm{d}f. \tag{1.3}$$

In the following, $\pm(\Phi, J)$-holomorphic will also be called holomorphic (anti-holomorphic) for simplicity. It is easy to check that the Hopf map $S^{2n+1} \to CP^n$ must be holomorphic, where S^{2n+1} and CP^n have their natural Sasakian structure and Kähler structure. So, the Hopf map is a non-trivial example of the holomorphic map from a Sasakian manifold into a Kähler manifold. In this paper, we show that every holomorphic map from a Sasakian manifold into Kähler must be basic and harmonic. We also get a Schwarz lemma about these holomorphic maps.

Theorem 1.1 *Let (M, ξ, η, Φ, g) be a complete Sasakian manifold with Ricci curvature bounded from below by $-K_1$, and (N, H, J) be a Hermitian manifold with holomorphic bisectional curvature bounded from above by $-K_2$, where K_1, K_2 are constants, and $K_1 \geq 2, K_2 > 0$. Then for any $\pm(\Phi, J)$-holomorphic mapping f from M to N, we have*

$$f^*\mathrm{d}S_N^2 \leq \frac{K_1 - 2}{K_2}\mathrm{d}S_M^2. \tag{1.4}$$

If $K_1 \leq 2$ and $K_2 > 0$, then any $\pm(\Phi, J)$-holomorphic map f from M to N must be trivial.

As an application of the Bochner's inequality obtained in the proof of Schwarz lemma, we use it to get the following theorem.

Theorem 1.2 *Let (M, g, ξ, η, Φ) be a complete noncompact Sasakian manifold without a boundary of dimension $2n + 1$. Let $R(x)$ denote the pointwise lower bound of the transverse Ricci curvature of M, and $R_-(x)$ be the negative part of $R(x)$. Assume that $R_-(x)$ satisfies*

$$\int_M R_- \mathrm{d}V < \infty, \qquad \int_{B_r(y)} R_-(x) \mathrm{d}V = o(r^{\beta(p-1)})$$

for some $p > n$, and some $\beta < \frac{2}{n}$, where $B_r(y)$ denotes the ball centered at y with radius r. Suppose that f is a non-constant (Φ, J)-holomorphic map from M to a Hermitian manifold (N, h, J), which has holomorphic bisectional curvature bounded from the above by $K(z)$ for all $z \in N$. Suppose that the curvature of the image of M under f satisfies $K(f(x)) \leq -B$ for some constant $B > 0$ and for all $x \in M$. Then, it must satisfy the inequality

$$\int_M R\mathrm{d}V \leq \int_M K(f(x))|\mathrm{d}f|^2\mathrm{d}V.$$

In particular, if $\int_M R\mathrm{d}V \geq 0$, then f has to be identically constant.

Furthermore, we discuss the basic map from Sasakian manifolds to Kähler manifolds. Firstly, we get a lemma about $*\mathrm{d}\eta$. Then, using this identity, we get the following theorem.

Theorem 1.3 *Let (M, g, ξ, η, Φ) be a compact Sasakian manifold, and (N, h, J) be a Kähler manifold. Suppose that f is a (Φ, J)-holomorphic map from M to N. Let $E'(f)$, $E''(f)$ be the integrals of $e'(f) = |\mathrm{d}f - J \circ \mathrm{d}f \circ \Phi|^2$ and $e''(f) = |\mathrm{d}f + J \circ \mathrm{d}f \circ \Phi|^2$, respectively. Then $E'(f) - E''(f)$ is an invariant in its basic homotopic class. Moreover, the invariant is just held in the class of basic maps.*

2 Preliminary Results in Sasakian Geometry

Let (M, g, ξ, η, Φ) be a $(2n+1)$-dimensional smooth Sasakian manifold. Denote by ∇ the Levi-Civita connection of g, and $R(X, Y)$ the Riemann curvature tensor of ∇. We list the following elementary properties of Sasakian structures, and the proof can be found in [1]:

$$\eta(\xi) = 1, \quad \mathrm{d}\eta(\xi, X) = 0, \tag{2.1}$$

$$\Phi(\xi) = 0, \quad \eta(\Phi(Y)) = 0, \tag{2.2}$$

$$\langle \Phi(X), Y\rangle_g = -\langle X, \Phi(Y)\rangle_g, \quad \Phi^2 = -\mathrm{Id} + \eta \otimes \xi, \tag{2.3}$$

$$\mathrm{d}\eta(X, Y) = 2\langle \Phi(X), Y\rangle_g, \tag{2.4}$$

$$\langle \Phi(X), \Phi(Y)\rangle_g = \langle X, Y\rangle_g - \eta(X)\eta(Y). \tag{2.5}$$

The contact 1-form η defines a $2n$-dimensional vector bundle D over M, where at each point $p \in M$ the fiber D_p of D is given by $D_p = \ker \eta_p$. There is a decomposition of the tangent bundle TM

$$TM = D \otimes L\xi, \tag{2.6}$$

where $L\xi$ is the trivial bundle generated by the Reeb vector field ξ. On the sub-bundle D, it is naturally endowed with both a complex structure $\Phi|_D$ and a symplectic structure $\mathrm{d}\eta$. $(D, \Phi|_D, \mathrm{d}\eta)$ gives M a transverse Kähler structure with a Kähler form $\mathrm{d}\eta$. The transverse metric g^{T} defined by

$$g^{\mathrm{T}}(X, Y) = \frac{1}{2}\mathrm{d}\eta(X, \Phi(Y)) \tag{2.7}$$

for any $X, Y \in D$. The transverse metric g^{T} is related to the Sasakian metric g by

$$g = g^{\mathrm{T}} + \eta \otimes \eta. \tag{2.8}$$

From the transverse metric g^{T}, one can define the transverse Levi-Civita connection on D by

$$\nabla^{\mathrm{T}}_X Y = \begin{cases} (\nabla_X Y)^p, & X \in D, \\ [\xi, Y]^p, & X = \xi, \end{cases} \tag{2.9}$$

where Y is a section of D, and X^p is the projection of X onto D. It is easy to check that the connection satisfies

$$45\nabla^{\mathrm{T}}_X Y - \nabla^{\mathrm{T}}_Y X - [X, Y]^p = 0 \tag{2.10}$$

and

$$X g^{\mathrm{T}}(Z, W) = g^{\mathrm{T}}(\nabla^{\mathrm{T}}_X Z, W) + g^{\mathrm{T}}(Z, \nabla^{\mathrm{T}}_X W) \tag{2.11}$$

for any $X, Y \in TM$ and $Z, W \in D$. This means that the transverse Levi-Civita connection is torsion-free and metric compatible. The transverse curvature relating with the above transverse connection is defined by

$$R^{\mathrm{T}}(V, W)Z = \nabla^{\mathrm{T}}_V \nabla^{\mathrm{T}}_W Z - \nabla^{\mathrm{T}}_W \nabla^{\mathrm{T}}_V Z - \nabla^{\mathrm{T}}_{[V,W]} Z, \tag{2.12}$$

where $V, W \in TM$ and $Z \in D$. From the above transverse curvature operator, we define the transverse Ricci curvature by

$$\mathrm{Ric}^{\mathrm{T}}(X, Y) = \langle R^{\mathrm{T}}(X, e_i)e_i, Y\rangle_g, \tag{2.13}$$

where e_i is an orthonormal basis of D and $X, Y \in D$. On the other hand, one can check easily that

$$\mathrm{Ric}^{\mathrm{T}}(X, Y) = \mathrm{Ric}(X, Y) + 2g^{\mathrm{T}}(X, Y) \tag{2.14}$$

for any $X, Y \in D$.

A p-form θ on the Sasakian manifold (M, g, ξ, η, ϕ) is called basic if

$$i_\xi \theta = 0, \quad L_\xi \theta = 0, \tag{2.15}$$

where i_ξ is the contraction with the Killing vector field ξ, and L_ξ is the Lie derivative with respect to ξ. It is easy to see that the exterior differential preserves basic forms. Namely, if θ is a basic form, so is $\mathrm{d}\theta$. Let $\wedge^p_B(M)$ be the sheaf of germs of basic p-forms and $\Omega^p_B(M) = \Gamma(M, \wedge^p_B(M))$ the set of all sections of $\wedge^p_B(M)$. The basic cohomology can be defined in a usual way (see [13]).

Let $T^C M$ be the complexification of the tangent bundle of (M, g), and D^C be the complexification of the sub-bundle D. The contact structure (ξ, η, Φ) on (M, g) defines the decomposition

$$T^C M = C \otimes \xi \oplus D^{1,0} \oplus D^{0,1} \tag{2.16}$$

and

$$D^C = D^{1,0} \oplus D^{0,1}, \tag{2.17}$$

where $C \otimes \xi$, $D^{1,0}$ and $D^{0,1}$ are eigenspaces of Φ with eigenvalues 0, $\sqrt{-1}$ and $-\sqrt{-1}$, respectively. For $p, q \geq 0$, we define

$$D^{p,q} = (\wedge^p D^{1,0}) \otimes (\wedge^q D^{0,1}). \tag{2.18}$$

Therefore, we have a decomposition

$$\wedge^s D^C = \bigoplus_{i=0}^{s} D^{i,s-i} \tag{2.19}$$

for all $s > 0$.

Let θ be a basic complex p-form. Since $i_\xi \theta = 0$, the evaluation of θ on $\wedge^d T^C M$ is determined by the evaluation of θ on the sub-bundle $\wedge D^C$. We will say that a basic form θ is of type $(i, p-i)$, if the evaluation of θ on $D^{j,p-j}$ vanishes for all $j \neq i$. Let $\wedge_B^{i,j}(M)$ denote the sheaf of germs of basic (i, j)-type forms. Similarly, we have the following decomposition:

$$\wedge_B^p(M) \otimes C = \bigoplus_{i+j=p} \wedge_B^{i,j}(M). \tag{2.20}$$

In the end of this section, we will show the harmonicity of $\pm(\Phi, J)$ holomorphic maps. Let $f : M \to N$ be a smooth map from a Sasakian manifold (M, ξ, η, Φ, g) to an almost Hermitian manifold (N, J, H), and suppose that it is $\pm(\Phi, J)$-holomorphic. By the definition, it is easy to check that

$$\mathrm{d}f(\xi) = \mp J \circ \mathrm{d}f \circ \Phi(\xi) = 0 \tag{2.21}$$

and

$$J(\nabla \mathrm{d}f(X, Y)) + (\nabla^N_{\mathrm{d}f(X)} J)(\mathrm{d}f(Y)) = \pm\{\mathrm{d}f((\nabla^M_X \Phi)Y) + \nabla \mathrm{d}f(X, \Phi Y)\} \tag{2.22}$$

for any (Φ, J)-holomorphic (anti-holomorphic) map $f : M \to N$.

Choosing an orthonormal basis $\{e_i\}_{i=1}^{2m+1}$ on M, such that $e_{2m+1} = \xi$, we have

$$\begin{aligned} \xi|\mathrm{d}f|^2 &= 2\langle \nabla_\xi \mathrm{d}f, \mathrm{d}f\rangle \\ &= 2\sum_i \langle (\nabla_\xi \mathrm{d}f)(e_i), \mathrm{d}f(e_i)\rangle_H \\ &= -2\sum_i \langle \mathrm{d}f(\nabla^M_\xi e_i), \mathrm{d}f(e_i)\rangle_H \\ &= -2\sum_{i,k} \langle \mathrm{d}f(e_k), \mathrm{d}f(e_i)\rangle_H \langle \nabla^M_\xi e_i, e_k\rangle_g \\ &= 2\sum_{i,k} \langle \mathrm{d}f(e_k), \mathrm{d}f(e_i)\rangle_H \langle \nabla^M_\xi e_k, e_i\rangle_g \\ &= 0. \end{aligned} \tag{2.23}$$

Theorem 2.1 *Any $\pm(\varphi, J)$-holomorphic map f from a Sasakian manifold to a Kähler manifold must be basic and harmonic.*

Proof Observing (2.21), we only need to prove that f is harmonic. Letting $X, Y \in \ker \eta$, and by (2.22), we have

$$\begin{aligned} J(\nabla \mathrm{d}f(X, Y)) &= \pm\{\mathrm{d}f((\nabla_X \Phi)Y) + \nabla \mathrm{d}f(X, \Phi Y)\} \\ &= \pm\{\mathrm{d}f(\langle Y, \xi\rangle X - \langle X, Y\rangle \xi) + \nabla \mathrm{d}f(X, \Phi Y)\} \\ &= \pm \nabla \mathrm{d}f(X, \Phi Y), \end{aligned} \tag{2.24}$$

where we have used (1.1) and $\nabla^N J = 0$. Then, Considering the symmetry of $\nabla \mathrm{d}f(\,\cdot\,,\,\cdot\,)$, we have

$$\nabla \mathrm{d}f(\Phi X, \Phi Y) = \pm J(\nabla \mathrm{d}f(\Phi X, Y)) = -\nabla \mathrm{d}f(X, Y). \tag{2.25}$$

Choosing an orthonormal basis $\{e_i\}_{i=1}^{2m+1}$, such that $e_{2m+1} = \xi$, we have

$$\begin{aligned}\tau(f) &= \sum_{i=1}^{2m} \nabla \mathrm{d}f(e_i, e_i) + \nabla \mathrm{d}f(\xi, \xi)\\ &= \frac{1}{2}\Big\{\sum_{i=1}^{2m} \nabla \mathrm{d}f(e_i, e_i) + \sum_{i=1}^{2m} \nabla \mathrm{d}f(\Phi e_i, \Phi e_i)\Big\}\\ &= \frac{1}{2}\sum_{i=1}^{2m}\{\nabla \mathrm{d}f(e_i, e_i) + \nabla \mathrm{d}f(\Phi e_i, \Phi e_i)\} = 0.\end{aligned} \tag{2.26}$$

3 A Schwarz Lemma for (Φ, J)-Holomorphic Maps

We first review local coordinates on a Sasakian manifold. In [15], it was proved that every Sasakian manifold can be locally generated by a free real function of $2n$ variables. This function is a Sasakian analogue of the Kähler potential for the Kähler geometry. More precisely, for any point P in M, one can choose local coordinates $(x, z^1, z^2, \cdots, z^n) \in R \times C^n$ on a small neighborhood U around P, such that

$$\begin{cases}\xi &= \dfrac{\partial}{\partial x},\\ \eta &= \mathrm{d}x - \sqrt{-1}(h_j \mathrm{d}z^j - h_{\overline{j}} \mathrm{d}\overline{z}^j),\\ \Phi &= \sqrt{-1}\{X_j \otimes \mathrm{d}z^j - \overline{X}_j \otimes \mathrm{d}\overline{z}^j\},\\ g &= \eta \otimes \eta + 2h_{i\overline{j}} \mathrm{d}z^i \mathrm{d}\overline{z}^l,\end{cases} \tag{3.1}$$

where $h : U \to R$ is a local basic function, i.e., $\frac{\partial h}{\partial x} = 0$, $h_i = \frac{\partial h}{\partial z^i}$, $h_{i\overline{j}} = \frac{\partial^2 h}{\partial z^i \partial \overline{z}^j}$, $X_j = \frac{\partial}{\partial z^j} + \sqrt{-1} h_j \frac{\partial}{\partial x}$ and $\overline{X}_j = \frac{\partial}{\partial \overline{z}^j} - \sqrt{-1} h_{\overline{j}} \frac{\partial}{\partial x}$. In the above, we set $2\mathrm{d}z^i \mathrm{d}\overline{z}^j = \mathrm{d}z^i \otimes \mathrm{d}\overline{z}^j + \mathrm{d}\overline{z}^j \otimes \mathrm{d}z^i$. In such local coordinates, $D \otimes C$ is spanned by X_i and $\overline{X}_i$. It is clear that

$$\begin{cases}\Phi X_i = \sqrt{-1} X_i, \quad \Phi \overline{X}_i = -\sqrt{-1}\,\overline{X}_i,\\ [X_i, X_j] = [\overline{X}_i, \overline{X}_j] = [\xi, X_i] = [\xi, \overline{X}_i] = 0,\\ [X_i, \overline{X}_j] = -2\sqrt{-1} h_{i\overline{j}} \xi.\end{cases} \tag{3.2}$$

Obviously, $\{\eta, \mathrm{d}z^i, \mathrm{d}\overline{z}^j\}$ is the dual basis of $\{\frac{\partial}{\partial x}, X_i, \overline{X}_j\}$, and

$$\mathrm{d}\eta = 2\sqrt{-1} h_{i\overline{j}} \mathrm{d}z^i \wedge \mathrm{d}\overline{z}^j. \tag{3.3}$$

Then, the transverse metric is as follows:

$$g^{\mathrm{T}} = 2g^{\mathrm{T}}_{i\overline{j}} \mathrm{d}z^i \mathrm{d}\overline{z}^j = 2h_{i\overline{j}} \mathrm{d}z^i \mathrm{d}\overline{z}^j, \tag{3.4}$$

where $g^{\mathrm{T}}_{i\overline{j}} = g^{\mathrm{T}}(X_i, \overline{X}_j) = h_{i\overline{j}}$. By (2.9), we know that $\nabla^{\mathrm{T}}_{\frac{\partial}{\partial x}} X_i = \nabla^{\mathrm{T}}_{\frac{\partial}{\partial x}} \overline{X}_j = 0$. Define Γ^A_{BC} by

$$\nabla^{\mathrm{T}}_{X_B} X_C = \Gamma^A_{BC} X_C \tag{3.5}$$

for $A, B, C = 1, 2, \cdots, n, \overline{1}, \overline{2}, \cdots, \overline{n}$, where $X_{\overline{j}} = \overline{X}_j$. Since ∇^{T} is torsion free, metric compatible, and $\nabla^{\mathrm{T}} J = 0$, by (2.9), it is easy to check that only Γ^i_{jk} and $\Gamma^{\overline{i}}_{\overline{j}\,\overline{k}}$ may not vanish as in the Kähler case, where $i, j, k = 1, 2, \cdots, n$. Moreover,

$$\Gamma^i_{jk} = \Gamma^i_{kj} = h^{i\overline{l}} \frac{\partial h_{j\overline{l}}}{\partial z^k}, \tag{3.6}$$

where $\sum\limits_{l=1}^{n} h^{i\overline{l}} h_{j\overline{l}} = \delta^i_j$. One can check that the transverse Ricci curvature can be expressed by

$$R^{\mathrm{T}}_{i\overline{j}} = -\frac{\partial^2}{\partial z^i \partial \overline{z}^j} \log \det(h_{s\overline{t}}). \tag{3.7}$$

Remark 3.1 For a fixed point $P \in M$, one can always choose the above local coordinates $(x, z^1, \cdots, z^n)$ centered at P satisfying additionally that $\{\frac{\partial}{\partial z^i}|_P\} \in D^c$ or equivalently $h_i(P) = 0$ for all j. Indeed, one can only change local coordinates by $(y, u^1, \cdots, u^n)$, where $y = x - \sqrt{-1} h_i(P) z^i + \sqrt{-1} h_{\overline{j}}(P) \overline{z}^j$ and $u^k = z^k$ for all $k = 1, \cdots, n$, and change the potential function by $h^* = h - h_i(P) u^i - h_{\overline{j}}(P) \overline{u}^j$. Furthermore, in the same way as that in Kähler's case, one can choose a normal coordinate system $(x, z^1, \cdots, z^n)$, such that $h_i(P) = 0$, $h_{i\overline{j}}(P) = \delta^i_j$ and $\mathrm{d}(h_{i\overline{j}})|_P = 0$, i.e., $\Gamma^i_{jk}|_P = 0$ for all i, j, k. This local coordinate system also be called a normal coordinate system on the Sasakian manifold.

Proof of Theorem 1.1 Let $\widetilde{\nabla}^N$ be the Chern connection on the Hermitian manifold (N, H, J). We know that the Chern connection must be compatible with a metric structure and a complex structure, i.e.,

$$\widetilde{\nabla}^N H = 0, \quad \widetilde{\nabla}^N J = 0, \tag{3.8}$$

where J is the almost complex structure of N. Usually, the Chern connection is not torsion free, but the $(1,1)$ part of its torsion tensor vanishes.

Let us choose a normal coordinate $(x, z^1, \cdots, z^m)$ as explained in the above remark on the considered point. Since f is $\pm(\Phi, J)$-holomorphic, we have

$$J(\widetilde{\nabla}^N_{\mathrm{d}f(X_i)} \mathrm{d}f(\overline{X}_j)) = \pm \widetilde{\nabla}^N_{\mathrm{d}f(X_i)} \mathrm{d}f(\Phi \overline{X}_j) = \mp \sqrt{-1} \widetilde{\nabla}^N_{\mathrm{d}f(X_i)} \mathrm{d}f(\overline{X}_j) \tag{3.9}$$

and

$$J(\widetilde{\nabla}^N_{\mathrm{d}f(\overline{X}_j)} \mathrm{d}f(X_i)) = \pm \widetilde{\nabla}^N_{\mathrm{d}f(\overline{X}_j)} \mathrm{d}f(\Phi X_i) = \pm \sqrt{-1} \widetilde{\nabla}^N_{\mathrm{d}f(\overline{X}_j)} \mathrm{d}f(X_i). \tag{3.10}$$

On the other hand, since the $(1,1)$ part of torsion tensor vanished, we get

$$\begin{aligned} \widetilde{\nabla}^N_{\mathrm{d}f(X_i)} \mathrm{d}f(\overline{X}_j) &= \widetilde{\nabla}^N_{\mathrm{d}f(\overline{X}_j)} \mathrm{d}f(X_i) + [\mathrm{d}f(X_i), \mathrm{d}f(\overline{X}_j)] \\ &= \widetilde{\nabla}^N_{\mathrm{d}f(\overline{X}_j)} \mathrm{d}f(X_i) + \mathrm{d}f(-2\sqrt{-1} h_{i\overline{j}} \xi) \\ &= \widetilde{\nabla}^N_{\mathrm{d}f(\overline{X}_j)} \mathrm{d}f(X_i). \end{aligned} \tag{3.11}$$

So we have

$$\widetilde{\nabla}^N_{\mathrm{d}f(X_i)} \mathrm{d}f(\overline{X}_j) = \widetilde{\nabla}^N_{\mathrm{d}f(\overline{X}_j)} \mathrm{d}f(X_i) = 0 \tag{3.12}$$

for all $i, j = 1, \cdots, m$. By (1.1) and (3.2), we have

$$\begin{aligned} -\sqrt{-1} \nabla^M_{X_i} \overline{X}_j &= \nabla^M_{X_i} (\Phi \overline{X}_j) \\ &= (\nabla^M_{X_i} \Phi) \overline{X}_j + \Phi(\nabla^M_{X_i} \overline{X}_j) \\ &= -\langle X_i, \overline{X}_j \rangle_g \xi + \Phi(\nabla^M_{X_i} \overline{X}_j), \end{aligned} \tag{3.13}$$

and similarly,

$$\sqrt{-1}\nabla^M_{\overline{X}_j} X_i = -\langle X_i, \overline{X}_j\rangle_g \xi + \Phi(\nabla^M_{\overline{X}_j} X_i). \tag{3.14}$$

From (3.2), we have

$$\nabla^M_{\overline{X}_j} X_i - \nabla^M_{X_i} \overline{X}_j = [\overline{X}_j, X_i] = 2\sqrt{-1}h_{i\bar{j}}\xi. \tag{3.15}$$

So, we have

$$\nabla^M_{\overline{X}_j} X_i + \nabla^M_{X_i} \overline{X}_j = -\sqrt{-1}\Phi(\nabla^M_{\overline{X}_j} X_i - \nabla^M_{X_i} \overline{X}_j) = 0, \tag{3.16}$$

and then

$$\nabla^M_{\overline{X}_j} X_i = -\nabla^M_{X_i} \overline{X}_j = \sqrt{-1}h_{i\bar{j}}\xi. \tag{3.17}$$

By direct calculation and (2.23), we have

$$\begin{aligned}
\triangle|\mathrm{d}f|^2 &= \nabla\mathrm{d}(|\mathrm{d}f|^2)(\xi,\xi) + 2h^{i\bar{j}}\nabla^M\mathrm{d}(|\mathrm{d}f|^2)(X_i,\overline{X}_j)\\
&= 2h^{i\bar{j}}\{X_i\overline{X}_j|\mathrm{d}f|^2\} + 2h^{i\bar{j}}\mathrm{d}(|\mathrm{d}f|^2)(\nabla^M_{X_i}\overline{X}_j)\\
&= 4h^{i\bar{j}}\{X_i\overline{X}_j(h^{k\bar{l}}\langle \mathrm{d}f(X_k),\mathrm{d}f(\overline{X}_l)\rangle_H)\}\\
&= 4h^{i\bar{j}}\{X_i\overline{X}_j(h^{k\bar{l}})\langle \mathrm{d}f(X_k),\mathrm{d}f(\overline{X}_l)\rangle_H + h^{k\bar{l}}X_i\overline{X}_j(\langle \mathrm{d}f(X_k),\mathrm{d}f(\overline{X}_l)\rangle_H)\}\\
&= 4h^{i\bar{j}}\{X_i\overline{X}_j(h^{k\bar{l}})\}\langle \mathrm{d}f(X_k),\mathrm{d}f(\overline{X}_l)\rangle_H\\
&\quad + 4h^{i\bar{j}}h^{k\bar{l}}\{\langle\widetilde{\nabla}^N_{\mathrm{d}f(X_i)}\mathrm{d}f(X_k),\widetilde{\nabla}^N_{\mathrm{d}f(\overline{X}_j)}\mathrm{d}f(\overline{X}_l)\rangle_H\\
&\quad + \langle \mathrm{d}f(X_k),\widetilde{\nabla}^N_{\mathrm{d}f(X_i)}\widetilde{\nabla}^N_{\mathrm{d}f(\overline{X}_j)}\mathrm{d}f(\overline{X}_l)\rangle_H\}\\
&= 4h^{i\bar{j}}\{X_i\overline{X}_j(h^{k\bar{l}})\}\langle \mathrm{d}f(X_k),\mathrm{d}f(\overline{X}_l)\rangle_H\\
&\quad + 4h^{i\bar{j}}h^{k\bar{l}}\{\langle\widetilde{\nabla}^N_{\mathrm{d}f(X_i)}\mathrm{d}f(X_k),\widetilde{\nabla}^N_{\mathrm{d}f(\overline{X}_j)}\mathrm{d}f(\overline{X}_l)\rangle_H\\
&\quad + \langle \mathrm{d}f(X_k),\widetilde{R}^N(\mathrm{d}f(X_i),\mathrm{d}f(\overline{X}_j))\mathrm{d}f(\overline{X}_l)\rangle_H\}\\
&= 4h^{i\bar{j}}h^{k\bar{l}}\langle\widetilde{\nabla}^N_{\mathrm{d}f(X_i)}\mathrm{d}f(X_k),\widetilde{\nabla}^N_{\mathrm{d}f(\overline{X}_j)}\mathrm{d}f(\overline{X}_l)\rangle_H\\
&\quad - 4h^{k\bar{n}}h^{m\bar{l}}\frac{\partial^2}{\partial z^m\partial\bar{z}^n}(\log\det(h_{s\bar{t}}))\langle \mathrm{d}f(X_k),\mathrm{d}f(\overline{X}_l)\rangle_H\\
&\quad - 4h^{i\bar{j}}h^{k\bar{l}}\langle\widetilde{R}^N(\mathrm{d}f(X_i),\mathrm{d}f(\overline{X}_j))\mathrm{d}f(X_k),\mathrm{d}f(\overline{X}_l)\rangle_H.
\end{aligned}$$

By the assumption and the above identity, we obtain the following Bochner type inequality for the $\pm(\Phi,J)$-holomorphic map f:

$$\triangle|\mathrm{d}f|^2 \geq -2(K_1-2)|\mathrm{d}f|^2 + K_2|\mathrm{d}f|^4, \tag{3.18}$$

where we have used (2.14) and (3.7). In the following, we follow Yau's discuss in [20]. Set $\rho = (|\mathrm{d}f|^2 + C)^{-\frac{1}{2}}$, where C is a positive constant. Since ρ is bounded, using Yau's maximum principle on complete Riemannian manifolds, we have that, for all $\epsilon > 0$, there exists a point $p \in M$, such that at p,

$$|\nabla\rho| < \epsilon, \quad \triangle\rho > -\epsilon, \quad \rho(p) < \inf\rho + \epsilon. \tag{3.19}$$

On the other hand, direct computation shows that

$$\rho\triangle\rho = -\frac{1}{2}\rho^{-2}\triangle|\mathrm{d}f|^2 + 3|\nabla\rho|^2. \tag{3.20}$$

Applying (3.18)–(3.20), at point $p \in M$, we have

$$-\epsilon(\inf\rho + \epsilon) - 3\epsilon^2 \leq \frac{1}{2}\rho^4(2(K_1 - 2)|\mathrm{d}f|^2 - K_2|\mathrm{d}f|^4). \tag{3.21}$$

When $\epsilon \to 0$, ρ goes to its infimum, and $|\mathrm{d}f|^2$ goes to its supremum. If $K_1 \leq 2$ and $K_2 > 0$, then $|\mathrm{d}f|^2 \equiv 0$. If $K_1 > 2$ and $K_2 > 0$, we have

$$|\mathrm{d}f|^2 \leq \frac{2(K_1 - 2)}{K_2}. \tag{3.22}$$

So we get the above Schwarz lemma, i.e., we complete the proof of Theorem 1.1.

Now we turn to prove Theorem 1.2.

Proof of Theorem 1.2 From the process of deducing (3.18), by noticing

$$\begin{aligned}&\langle \widetilde{\nabla}^N_{\mathrm{d}f(X_i)}\mathrm{d}f(X_k), \widetilde{\nabla}^N_{\mathrm{d}f(\overline{X}_j)}\mathrm{d}f(\overline{X}_l)\rangle_H \\ &\geqslant h^{m\overline{n}}\langle \widetilde{\nabla}^N_{\mathrm{d}f(X_i)}\mathrm{d}f(X_k), \mathrm{d}f(X_m)\rangle_H \langle \widetilde{\nabla}^N_{\mathrm{d}f(\overline{X}_j)}\mathrm{d}f(\overline{X}_l), \mathrm{d}f(\overline{X}_n)\rangle_H,\end{aligned}$$

the assumption implies the Bochner's inequality

$$\triangle|\mathrm{d}f|^2 - \frac{|\nabla|\mathrm{d}f|^2|^2}{|\mathrm{d}f|^2} \geq 2R(x)|\mathrm{d}f|^2 - K(f(x))|\mathrm{d}f|^4. \tag{3.23}$$

By the Hölder inequality and the assumption,

$$\int_{B_r(y)} R_-^{p'}\mathrm{d}V \leq \Big(\int_{B_r(y)} R_-\mathrm{d}V\Big)^{\frac{p-p'}{p-1}}\Big(\int_{B_r(y)} R_-^p\mathrm{d}V\Big)^{\frac{p'-1}{p-1}} = o(r^{\beta(p'-1)}),$$

where $\beta < \frac{2}{n}$ and $p' = \frac{2}{\beta} + 1 > n + 1$.

There is some $p' > n+1$, such that $\int_{B_r(y)} R_-^{p'}\mathrm{d}V = o(r^2)$. By [11, Corollary 1.2], $V_r(y) = o(r^{2(p+1)})$ as $r \to \infty$. Applying those and (3.23) to [11, Theorem 2.3], we know

$$\int_{B_r(y)} (|\mathrm{d}f|)^{2p'}\mathrm{d}V = o(r^2). \tag{3.24}$$

Using the notations above and considering (3.24) with [11, Theorem 2.1], it is easy to see

$$0 \geq \int_M -K|\mathrm{d}f|^2\mathrm{d}V + \int_{M+} 2R\mathrm{d}V.$$

If we further assume $\int_M R\mathrm{d}V \geq 0$, we get

$$0 \leq \int_{M+} 2R\mathrm{d}V \leq \int_M K(f(x))|\mathrm{d}f|^2\mathrm{d}V \leq -B\int_M |\mathrm{d}f|^2\mathrm{d}V < 0.$$

We get a contradiction, because the set of $|\mathrm{d}f| \neq 0$ is a null set, where f is a non-constant map.

Remark 3.2 Recently, Tosatti [19] proved the Schwarz lemma in an almost Hermitian case. By the concept of almost Hermitian, our theorems can be generalized to the target manifold to as the almost Hermitian one. It is just because there exists a unique almost Hermitian connection ∇ on (M, J, H) whose torsion has an everywhere vanishing (1,1) part.

4 A Homotopic Invariant for Basic Maps

As in Kähler geometry, we can find some homotopic invariants to study the properties of the basic map in the Sasakian case. We begin from a decomposition of the energy density.

Let M be a Sasakian manifold with dimension $m = 2n + 1$, and N be a Kähler manifold with an even dimension. Let f be a smooth map defined as above. The energy density can be expressed as

$$\begin{aligned}|\mathrm{d}f|^2 = \frac{1}{4}|\mathrm{d}f - J\circ \mathrm{d}f\circ\Phi|^2 + \frac{1}{4}|\mathrm{d}f + J\circ \mathrm{d}f\circ\Phi|^2 \\ + \frac{1}{2}\langle \mathrm{d}f - J\circ \mathrm{d}f\circ\Phi, \mathrm{d}f + J\circ \mathrm{d}f\circ\Phi\rangle.\end{aligned}$$

Use $e'(f)$ and $e''(f)$ to denote $|\mathrm{d}f - J\circ \mathrm{d}f\circ\Phi|^2$ and $|\mathrm{d}f + J\circ \mathrm{d}f\circ\Phi|^2$, respectively. Then

$$e'(f) - e''(f) = -4\langle \mathrm{d}f, J\circ \mathrm{d}f\circ\Phi\rangle = -4\langle f^*\omega_N, \mathrm{d}\eta\rangle,$$

where ω_N is the Kähler form on N.

Define $E'(f)$, $E''(f)$ to be the integrals of $e'(f)$ and $e''(f)$, respectively. Then $E'(f)-E''(f)$ is $-4\int\langle f^*\omega_N, \mathrm{d}\eta\rangle$. To prove Theorem 4.1, we need the following lemma. It shows $*\mathrm{d}\eta = |\mathrm{d}\eta|^2\eta\wedge(\mathrm{d}\eta)^{n-1}$ by direct computation.

Lemma 4.1 *Let (M, ξ, η, Φ, g) be a Sasakian manifold. Then $*(\mathrm{d}\eta) = n^2\eta\wedge(\mathrm{d}\eta)^{n-1}$.*

Proof We can express η and $\mathrm{d}\eta$ as below in local coordinates in a Sasakian manifold of dimension $m = 2n + 1$ (see [15]),

$$\begin{aligned}\eta &= \mathrm{d}x + \sqrt{-1}\sum_{j=1}^m h_j\mathrm{d}z^j - \sqrt{-1}\sum_{j=1}^m h_{\overline{j}}\mathrm{d}\overline{z}^j,\\ \mathrm{d}\eta &= \sqrt{-1}\sum_{i,j=1}^m h_{j\overline{i}}\mathrm{d}\overline{z}^i\wedge \mathrm{d}z^j - \sqrt{-1}\sum_{i,j=1}^m h_{\overline{j}i}\mathrm{d}z^i\wedge\mathrm{d}\overline{z}^j.\end{aligned}$$

So

$$\begin{aligned}*(\mathrm{d}\eta) = &\frac{\sqrt{-1}\sqrt{G}}{2!(m-2)!}\delta^{1,2,\cdots,m}_{i,j,1,i_4,\cdots,i_m}g^{\overline{i}k}g^{j\overline{l}}h_{k\overline{l}}\mathrm{d}x\wedge\mathrm{d}z^{i_4}\wedge\cdots\wedge\mathrm{d}\overline{z}^j\\ &- \frac{\sqrt{-1}\sqrt{G}}{2!(m-2)!}\delta^{1,2,\cdots,m}_{i,j,1,i_4,\cdots,i_m}g^{i\overline{k}}g^{\overline{j}l}h_{\overline{k}l}\mathrm{d}x\wedge\mathrm{d}z^{i_4}\wedge\cdots\wedge\mathrm{d}\overline{z}^j.\end{aligned}$$

Since $\eta\wedge(\mathrm{d}\eta)^n = \sqrt{G}\mathrm{d}x\wedge\mathrm{d}z^1\wedge\cdots\wedge\mathrm{d}z^n\wedge\mathrm{d}\overline{z}^1\wedge\cdots\wedge\mathrm{d}\overline{z}^n$, we derive the equation below:

$$\begin{aligned}&\sqrt{-1}\eta\wedge(\mathrm{d}\eta)^{n-1}h_{j\overline{i}}\\ &= \sqrt{G}\delta^{1,2,\cdots,m}_{1,i_4,\cdots,i_m,j,i}\mathrm{d}x\wedge\mathrm{d}z^1\wedge\cdots\wedge\widehat{dz^j}\wedge\cdots\wedge\mathrm{d}z^n\wedge\mathrm{d}\overline{z}^1\wedge\cdots\wedge\widehat{\mathrm{d}\overline{z}^i}\wedge\cdots\wedge\mathrm{d}\overline{z}^n,\\ &\sqrt{-1}\eta\wedge(\mathrm{d}\eta)^{n-1}h_{\overline{j}i}\\ &= \sqrt{G}\delta^{1,2,\cdots,m}_{1,i_4,\cdots,i_m,i,j}\mathrm{d}x\wedge\mathrm{d}z^1\wedge\cdots\wedge\widehat{\mathrm{d}z^i}\wedge\cdots\wedge\mathrm{d}z^n\wedge\mathrm{d}\overline{z}^1\wedge\cdots\wedge\widehat{\mathrm{d}\overline{z}^j}\wedge\cdots\wedge\mathrm{d}\overline{z}^n,\end{aligned}$$

where $\widehat{x}$ means taking away the x term in the wedge product. Then,

$$*(\mathrm{d}\eta) = (\sqrt{-1})^2\eta\wedge(\mathrm{d}\eta)^{n-1}(-2|h|^2) = |\mathrm{d}\eta|^2\eta\wedge(\mathrm{d}\eta)^{n-1}.$$

We have $|\mathrm{d}\eta|^2 = n^2$, since $|\mathrm{d}\eta| = \sum\limits_i \mathrm{d}\eta(e_i, \overline{e}_i) = \sum\limits_i \langle \Phi e_i, \overline{e}_i \rangle = n$.

With this lemma, we can prove Theorem 1.3 as follows.

Proof of Theorem 1.3 In a general Riemannian manifold, we have $\frac{\partial}{\partial t} f_t^* \omega_N = \mathrm{d}\alpha_t$, where $\alpha = f_t^* i(\frac{\partial f_t}{\partial t})\omega$ (see [7]).

With the above lemma, the quantum is

$$\begin{aligned}\frac{\partial}{\partial t}(E'(f_t) - E''(f_t)) &= -4\frac{\partial}{\partial t}\int_M |\mathrm{d}\eta|^2 f_t^* \omega_N \wedge \eta \wedge (\mathrm{d}\eta)^{n-1} \\ &= -4n^2 \int_M \mathrm{d}\alpha_t \wedge \eta \wedge (\mathrm{d}\eta)^{n-1} \\ &= -4n^2 \int_M \{\mathrm{d}[\alpha_t \wedge \eta \wedge (\mathrm{d}\eta)^{n-1}] + \alpha_t \wedge (\mathrm{d}\eta)^n\}.\end{aligned}$$

$\alpha_t \wedge (\mathrm{d}\eta)^n = 0$ is equal to $\alpha_t(\xi) = 0$, i.e.,

$$\Big[f_t^* i\Big(\frac{\partial f_t}{\partial t}\Big)\omega\Big](\xi) = i\Big(\frac{\partial f_t}{\partial t}\Big)\omega(f_{t*}\xi) = \omega\Big(\frac{\partial f_t}{\partial t}, f_{t*}\xi\Big) = 0,$$

which implies

$$f_{t*}\xi = 0.$$

Suppose that f is not a basic map. There is a variation of f in its homotopic class with $f_0 = f$, such that $f_{t*}\xi \neq 0$. Let

$$\frac{\partial f_t}{\partial t}\Big|_{t=0} = f_{0*}\xi.$$

Then

$$\omega\Big(\frac{\partial f_t}{\partial t}, f_{t*}\xi\Big)\Big|_{t=0} \neq 0.$$

$E'(f) - E''(f)$ is not a constant in a small neighborhood of $f_0 = f$, by the continuous property.

Definition 4.1 *We call two functions f_1, f_2 (or forms ω_1, ω_2) basic homotopy to each other, if f_1, f_2 (or ω_1, ω_2) are homotopy to each other and f_t (or ω_t) are all basic.*

The basic homotopy is stronger than homotopy. Since the 1-form η is defined on M unitarily, the basic forms on the Sasakian manifold have the following property.

Proposition 4.1 *Let α_1, α_2 be basic 1-forms on a Sasakian manifold (M, g, ξ, η, Φ). If they are homotopy to each other, they are basic homotopy.*

Corollary 4.1 *Let f_1, f_2 be holomorphic and anti-holomorphic maps from a Sasakian manifold (M, g, ξ, η, Φ) to a Kähler manifold (N, h, J), respectively. Then f_1, f_2 are impossible to be basic homotopy to each other, only if they are constant maps.*

Proof Let f_1, f_2 be basic holomorphic and basic anti-holomorphic maps, respectively. If they are basic homotopy, then

$$\begin{aligned}E(f_1) &= E'(f_1) + E''(f_1) = E'(f_1) - E''(f_1) \\ &= E'(f_2) - E''(f_2) = -E'(f_2) - E''(f_2) \\ &= -E(f_2).\end{aligned}$$

It implies $E(f_1) = E(f_2) = 0$.

Similarly, we have the following corollary.

Corollary 4.2 *The holomorphic or anti-holomorphic map from a Sasakian manifold to a Kähler manifold is the energy minimal map in its basic homotopy class.*

Acknowledgement The authors would like to thank Dr. B. Chen for the useful discussion with respect to the last part.

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