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Global Existence and Blow-Up Results for a Classical Semilinear Parabolic Equation*

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Abstract The author studies the boundary value problem of the classical semilinear parabolic equations

$$u_t - \Delta u = |u|^{p-1}u \quad \text{in } \Omega \times (0,T),$$

and $u = 0$ on the boundary $\partial\Omega \times [0,T)$ and $u = \phi$ at $t = 0$, where $\Omega \subset \mathbb{R}^n$ is a compact C^1 domain, $1 < p \le p_S$ is a fixed constant, and $\phi \in C_0^1(\Omega)$ is a given smooth function. Introducing a new idea, it is shown that there are two sets $\widetilde{W}$ and $\widetilde{Z}$, such that for $\phi \in \widetilde{W}$, there is a global positive solution $u(t) \in \widetilde{W}$ with H^1 omega limit 0 and for $\phi \in \widetilde{Z}$, the solution blows up at finite time.

Keywords Positive solution, Global existence, Blow-up, Omega limit
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1 Introduction

In this paper, we study the Dirichlet boundary value problem of the classical semilinear parabolic equation

$$u_t - \Delta u = |u|^{p-1}u, \quad \text{in } \Omega \times (0,T) \tag{1.1}$$

with $u = 0$ on the boundary $\partial\Omega \times [0,T)$ and $u = \phi \in C_0^1(\Omega)$ at $t = 0$, where $T > 0$, $\Omega \subset \mathbb{R}^n$ is a compact C^1 domain, $p > 1$ is a fixed constant, and $\phi \in C_0^1(\Omega)$ is a given smooth function. This kind of problems arose from the models from the reaction-diffusion phenomenon (see [1–2]). Assume that $p \le p_S = \frac{n+2}{n-2}$ for $n \ge 3$ and $p < \infty$ for $n = 1, 2$. By the standard theory, we know that there is a local time positive solution to (1.1) provided $\phi > 0$. With the help of Nehari functional, one may find the threshold of the initial data, such that the solution either exists globally or blows up in finite time. The stationary solutions to (1.1) share the similar variational structure as the solitary waves studied in [3]. Other related results about (1.1) can be found in the recent work (see [4–5]). Since equation (1.1) is a model problem, it deserves to have more understanding. Introducing a new idea, we show in this paper that there are two new sets $\widetilde{W}$ and $\widetilde{Z}$, such that for $\phi \in \widetilde{W}$, there is a global positive solution in $\widetilde{W}$ with the H^1 omega limit 0 and for $\phi \in \widetilde{Z}$, the solution blows up at finite time. We may extend the method used in this paper to treat Neumann boundary value problem of the semilinear parabolic equation with critical power or with negative power (see [6–7]). To define the invariant set $\widetilde{Z}$, we shall use the fact that the cones

$$C_+ = \{u \in C_0^1(\Omega);\ u \ge 0,\ u \ne 0\},$$
$$C_- = \{u \in C_0^1(\Omega);\ u \le 0,\ u \ne 0\}$$

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are invariant sets of (1.1). This fact can be proved by applying the maximum principle.

We now recall the standard way to construct the invariant sets for (1.1). Formally, (1.1) has a Lyapunov functional, namely,

$$J(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - \frac{1}{1+p}u^{p+1}.$$

Here and hereafter, we use $\int_\Omega \cdot$ to denote the integration $\int_\Omega \cdot \mathrm{d}x$. In fact, we may consider (1.1) as the negative L^2-gradient flow of the functional $J(\,\cdot\,)$, that is, abstractly, (1.1) can be written as

$$u_t = -J'(u).$$

Hence, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}J(u(t)) = \langle J'(u), u_t\rangle = -|u_t|_2^2 = -|J'(u)|_{L^2}^2.$$

Let $f(u) = u^p$ and its primitive

$$F(u) = \frac{u^{p+1}}{p+1}.$$

Introduce the working space

$$\Sigma = \Big\{u \in H_0^1;\ u \neq 0, \int_\Omega F(u) < \infty\Big\}.$$

The condition $\int_\Omega F(u) < \infty$ is always true by using the Sobolev inequality.

Define on Σ, the functional

$$M(u) = \frac{1}{2}\int_\Omega |u|^2$$

and the Nehari functional

$$I(u) = \int_\Omega |\nabla u|^2 - uf(u) = \int_\Omega |\nabla u|^2 - |u|^{p+1}.$$

Note that these two functionals are well-defined on Σ.

Along the flow (1.1), we can see that

$$\frac{\mathrm{d}}{\mathrm{d}t}M(u) = \int_\Omega uu_t = -I(u). \tag{1.2}$$

Let

$$d = \inf\{J(u); u \in \Sigma, I(u) = 0\}.$$

Define

$$W = \{u \in \Sigma; J(u) < d, I(u) > 0\} \cup \{0\}$$

and

$$Z = \{u \in \Sigma; J(u) < d, I(u) < 0\}.$$

The classical result says that W and Z are invariant sets of (1.1). Furthermore, for $1 < p < p_S$ and for any initial data $\phi \in W$, the solution exists globally; for $1 < p \leq p_S$ and for any initial data $\phi \in Z$, the solution blows up at finite time. One may see [2] for more results and references.

We now introduce new functionals. For $\lambda \in \mathbb{R}_+$, define

$$E_\lambda(u) = J(u) + \lambda M(u).$$

Then along the flow (1.1), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E_\lambda(u) = -|J'(u)|_2^2 - \lambda I(u). \tag{1.3}$$

From this, it is clear that for $\lambda \geq 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E_\lambda(u) < 0$$

for $I(u) \geq 0$ and $J'(u) \neq 0$.

Introduce

$$d_\lambda = \inf\{E_\lambda(u); u \in \Sigma, I(u) = 0\}.$$

As in the case for the quantity d, we can give it the mountain-pass characterization.

Assume that it is finite at this moment. Define

$$W_\lambda = \{u \in \Sigma; E_\lambda(u) < d_\lambda, I(u) > 0\} \cup \{0\}.$$

For convenience, we set $W_0 = W$. Arguing as in W, one can see that W_λ with $\lambda > 0$ is non-empty.

Then by (1.3) and the standard argument, we know that for $\lambda \geq 0$, W_λ is an invariant set of the flow (1.1).

One of our main results for (1.1) is to show the following conclusion.

Theorem 1.1 *Fixing any power* $1 < p < p_S$*, we have for* $\lambda > 0$ *that*

(1) d_λ *is finite, and* $d_\lambda > d$ *for* $\lambda > 0$.

(2) *For* $\phi \in W_\lambda$ *with* $\lambda \geq 0$*, the flow exists globally and its omega limit is* 0*. Hence,*

$$\widetilde{W} := \bigcup_{\lambda \geq 0} W_\lambda$$

is an invariant set of (1.1).

We remark that since $d_\lambda > d$, we know that the set W_λ is different from the set W.

For $0 \leq \lambda < \lambda_1(\Omega)$, where $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$, we may let

$$E^\lambda(u) = E(u) - \lambda M(u), \quad I^\lambda(u) = I(u) - 2\lambda M(u)$$

and

$$\overset{\vee}{d}_\lambda = \inf\{E^\lambda(u); u \in \Sigma; I^\lambda(u) = 0\}.$$

Set

$$\overset{\vee}{W}_\lambda = \{u \in \Sigma; E^\lambda(u) < \overset{\vee}{d}_\lambda, I^\lambda(u) < 0\} \cup \{0\}$$

and

$$\overset{\vee}{W} = \bigcup_{0 \leq \lambda \leq \lambda_1(\Omega)} \overset{\vee}{W}_\lambda.$$

Then we can also show the result as follows.

Proposition 1.1 $\overset{\vee}{W} \cap C_+$ *is a set of initial data with the global flow of* (1.1).

Since the proof of Proposition 1.1 is similar to that of Theorem 1.1, we omit the full proof.

To find the set for finite-time blow-up solutions to (1.1), we need to use the comparison argument. We shall restrict the initial data to being positive. Let $\delta \geq 0$. Consider the boundary value problem of the following semilinear parabolic equation:

$$v_t - \Delta v + \delta v = v^p, \quad u > 0, \quad \text{in } \Omega \times (0, T) \tag{1.4}$$

with $u = 0$ on the boundary $\partial\Omega \times [0, T)$ and $u = \phi$ at $t = 0$, where $T := T_{\max}(\phi) > 0$ is the maximal existence time of the solution $v(t)$. The key point in our construction is that the positive solution to (1.4) is a sub-solution to (1.1) with the same initial and boundary data. Define on $\Sigma_+ = \Sigma \cap C_+$,

$$J_\delta(v) = J(v) + \delta M(v),$$
$$I_\delta(v) = I(v) + 2\delta M(v),$$

and on the set where $\{I_\delta(v) = 0\}$,

$$E^\delta(v) = J_\delta(v) = \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\int_\Omega |u|^{p+1}.$$

Define

$$d_\delta = \inf\{E^\delta(v); v \in \Sigma_+, I_\delta(u) = 0\}.$$

For $\epsilon > 0$, we have

$$d_{\delta,\epsilon} = \inf\{J_\delta(u); u \in \Sigma_+; I_\delta(u) = \epsilon\},$$
$$Z_\delta = \{u \in \Sigma_+; J_\delta(u) < d_\delta, I_\delta(u) < 0\}.$$

Clearly, Z_δ is non-empty and it is an invariant set of the flow (1.4). We remark that one may make similar construction on $\Sigma_- = \Sigma \cap C_-$.

Theorem 1.2 *Fix $1 < p \le p_S$.*

(1) *For $\phi \in Z_\delta$, the flow $v(t)$ to* (1.4) *blows up in finite time.*

(2) *Let $u(t)$ be the flow to* (1.1) *with the initial data ϕ as* (1) *above. Then $u(t) \ge v(t)$ and $u(t)$ blows up at some $t < \infty$.*

As a consequence of Theorem 1.2, we have the corollary below.

Corollary 1.1 *Set $\widetilde{Z} = \bigcup_{\delta \ge 0} Z_\delta$. Then for any $\phi \in \widetilde{Z}$, the solution to* (1.1) *blows up at finite time.*

The results above will be proved in the next section.

2 Global Solution and Finite Time Blow-Up Solution

Proof of Theorem 1.1 (1) The finiteness of d_λ can be obtained in the similar way as in [2]. Since $1 < p < p_S$, we know that d_λ can also be achieved by some function u_λ (see [8–10]). By this, we know that d_λ is different from d for $\lambda > 0$. Hence, we have $d_\lambda > d$ for $\lambda > 0$.

(2) Since $I(\phi) > 0$, we have $I(u(t)) > 0$ for all $t \in [0, T)$. Otherwise, for some $t > 0$, $I(u(t)) = 0$. Using the definition of d_λ, we have $E_\lambda(u(t)) \ge d_\lambda$. This is a contradiction to the fact that

$$\frac{\mathrm{d}}{\mathrm{d}t} E_\lambda(u(t)) < 0 \quad \text{and} \quad E_\lambda(u(t)) < E_\lambda(\phi) < d_\lambda.$$

Using (1.2), we know that $M(u(t)) < M(\phi)$. With the help of the condition $E_\lambda(u(t)) < d$ and $1 < p < p_S$, we know that $u(t) \in H^1$ is uniformly bounded and the bounding constant depends only on d, p, $|\Omega|$ and $M(\phi)$.

The H^1 omega limit at $t = \infty$ can be determined below. It is a classical fact that the H^1 omega limit set $\omega(\phi)$ consists of classical equilibria (see [2]). If $v \in \omega(\phi)$, we have $I(v) = 0$. If v is nontrivial, we have

$$E_\lambda(v) \ge d_\lambda.$$

It is impossible. Hence, $v = 0$, that is, $\omega(\phi) = \{0\}$.

This completes the proof of Theorem 1.1.

Proof of Theorem 1.2 Introduce

$$A = \inf\Big\{\frac{|\nabla u|_2^2 + \delta |u|_2^2}{|u|_{p+1}^2};\ u \in H_0^1(\Omega),\ u \ne 0\Big\}.$$

Then it is easy to see that $d_\delta = \frac{p-1}{2(p+1)} A^{\frac{p+1}{p-1}}$ (see [8–10]). Assume that $0 \neq v \in H_0^1(\Omega)$, such that $I_\delta(v) = -\epsilon$. Then

$$E^\delta(v) = \frac{p-1}{2(p+1)} \int_\Omega (|\nabla v|^2 + \delta v^2) - \frac{\epsilon}{p+1}. \tag{2.1}$$

Using the definition of A, we have

$$\int_\Omega (|\nabla v|^2 + \delta v^2) \le \int_\Omega |v|^{p+1} \le A^{-\frac{p+1}{2}} \Big(\int_\Omega (|\nabla v|^2 + \delta v^2) \Big)^{\frac{p+1}{2}}.$$

Hence,

$$\int_\Omega (|\nabla v|^2 + \delta v^2) \ge A^{\frac{p+1}{p-1}}.$$

Combining this with (2.1), we have

$$d_{\delta,\epsilon} \ge d_\delta - \frac{\epsilon}{p+1}. \tag{2.2}$$

We now prove (1) in the statement of Theorem 1.2.

(1) Take $\epsilon > 0$, such that

$$\epsilon < \min(-I_\delta(\phi), d_\delta - J_\delta(\phi)).$$

Then by using

$$\frac{\mathrm{d}}{\mathrm{d}t} J_\delta(v(t)) = -|J'_\delta(v(t))|^2 \le 0$$

and (2.2), we know that

$$J_\delta(v(t)) \le J_\delta(\phi) < d_{\delta,\epsilon}$$

for $t \in [0, T)$. Since $I_\delta(\phi) < -\epsilon$, by using the definition of $d_{\delta,\epsilon}$ and the continuity, we know that

$$I_\delta(v(t)) < -\epsilon.$$

Note that

$$I_\delta(v) = 2J_\delta(v) - \Big(1 - \frac{2}{p+1}\Big) \int_\Omega |v|^{p+1}.$$

Assume that $T = T_{\max} > 0$ is the maximal time of the flow $v(t)$. Assume that $T = \infty$. On one hand, applying the similar formula to (1.2), we have

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega v^2 = -I_\delta(v) \ge \epsilon > 0,$$

and then

$$\int_\Omega v^2 \ge \int_\Omega \phi^2 + 2\epsilon t \to \infty,$$

that is, $|v(t)|_{L^2} \to \infty$ as $t \to \infty$.

On the other hand,

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega v^2 = -I_\delta(v) \ge -2d_\epsilon + \Big(1 - \frac{2}{p+1}\Big) \int_\Omega |v|^{p+1}.$$

Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t} M(v(t)) \ge -2d_\epsilon + C(p, |\Omega|) M(v(t))^{\frac{p+1}{2}}$$

for some uniform constant $C(p, |\Omega|) > 0$. Then, using $M(v(t)) \to \infty$, we know that there exists a $T_1 > 0$, such that for any $t > T_1$,

$$\frac{\mathrm{d}}{\mathrm{d}t} M(v(t)) \geq \frac{1}{2} C(p, |\Omega|) M(v(t))^{\frac{p+1}{2}}.$$

However, this implies that $T < \infty$, which is a contradiction. Hence, $T < \infty$ and $|v(t)|_\infty \to \infty$ as $t \to T$.

We shall prove (2) in the statement of Theorem 1.2 by using the comparison lemma.

(2) Let $T_{\max} < \infty$ be the blow-up time of the flow $v(t)$. Recall that $v(t) > 0$ for $t \in (0, T_{\max})$. Let $w(t) = u(t) - v(t)$, $t < T_{\max}$. Then $w(t)$ is bounded in any finite time before the blowing up time of the solution $u(t)$. Note that

$$w_t - \Delta w = p\xi^{p+1} w + \delta v. \tag{2.3}$$

Here ξ is some smooth function between u and v. Recall that $w(0) = 0$ and $w(t)|_{\partial\Omega} = 0$. Let $w_-(t)$ be the negative part of $w(t)$. Multiplying both sides of (2.3) by $w_-(t)$ and integrating over Ω by $w_-(t)$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |w_-(t)|^2 = -\int_\Omega |\nabla w_-(t)|^2 + p \int_\Omega \xi^{p+1} |w_-(t)|^2 + \delta \int_\Omega v w_-(t).$$

We remark that the last term is non-positive. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |w_-(t)|^2 \leq C \int_\Omega |w_-(t)|^2.$$

By the Gronwall inequality, we know that $\int_\Omega |w_-(t)|^2 = 0$ for any $t > 0$. Hence, we have $u(t) \geq v(t)$, and then

$$|u(t)|_\infty \geq |v(t)|_\infty \to \infty,$$

as $t \to T_{\max} < \infty$.

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