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Iterative Algorithm with Mixed Errors for Solving a New System of Generalized Nonlinear Variational-Like Inclusions and Fixed Point Problems in Banach Spaces

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Abstract A new system of generalized nonlinear variational-like inclusions involving A-maximal m-relaxed η-accretive (so-called, (A,η)-accretive in [36]) mappings in q-uniformly smooth Banach spaces is introduced, and then, by using the resolvent operator technique associated with A-maximal m-relaxed η-accretive mappings due to Lan et al., the existence and uniqueness of a solution to the aforementioned system is established. Applying two nearly uniformly Lipschitzian mappings $\mathcal{S}_1$ and $\mathcal{S}_2$ and using the resolvent operator technique associated with A-maximal m-relaxed η-accretive mappings, we shall construct a new perturbed N-step iterative algorithm with mixed errors for finding an element of the set of the fixed points of the nearly uniformly Lipschitzian mapping $\mathcal{Q}=(\mathcal{S}_1,\mathcal{S}_2)$ which is the unique solution of the aforesaid system. We also prove the convergence and stability of the iterative sequence generated by the suggested perturbed iterative algorithm under some suitable conditions. The results presented in this paper extend and improve some known results in the literature.

Keywords A-Maximal m-relaxed η-accretive mapping, System of generalized nonlinear variational-like inclusion, Resolvent operator technique, Convergence and stability, Variational convergence
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1 Introduction

The theory of variational inequalities, which was initially introduced by Stampacchia [55] in 1964, is a branch of the mathematical sciences dealing with general equilibrium problems. It has a wide range of applications in economics, operations research, industry, physics, and engineering sciences. Many research papers have been written lately, both on the theory and applications of this field. Important connections with main areas of pure and applied sciences have been made, see for example [7, 19, 21] and the references cited therein. The development of the variational inequality theory can be viewed as the simultaneous pursuit of two different lines of research. On the one hand, it reveals the fundamental facts on the qualitative aspects of the solution to important classes of problems; on the other hand, it also enables us to develop highly efficient and powerful new numerical methods to solve, for example, obstacle, unilateral, free, moving and the complex equilibrium problems. These activities have motivated researchers to

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generalize and extend the variational inequalities and related optimization problems in several directions using new and novel techniques.

Variational inclusions introduced and studied by Hassouni and Moudafi [22], as the generalization of variational inequalities, have been widely studied in recent years. Many efficient ways have been studied to find solutions of variational inclusions. These methods include the projection method and its various forms, linear approximation, descent and Newton's method, and the method based on the auxiliary principle technique, etc. The method based on the resolvent operator technique is a generalization of the projection method and has been widely used to solve variational inclusions.

Some new and interesting problems, which are called systems of variational inequality problems were introduced and studied. Pang [48], Cohen and Chaplais [13], Bianchi [8] and Ansari and Yao [6] considered a system of scalar variational inequalities and Pang showed that the traffic equilibrium problem, the spatial equilibrium problem, the Nash equilibrium problem and the general equilibrium programming problem can be modeled as a variational inequality. He decomposed the original variational inequality into a system of variational inequalities which are easy to solve, and studied the convergence of such methods. Ansari et al. [5] introduced and studied a system of vector variational inequalities by a fixed point theorem. Allevi et al. [4] considered a system of generalized vector variational inequalities and established some existence results under relative pseudo monotonicity. Kassay and Kolumban [29] introduced a system of variational inequalities and proved an existence theorem by the Ky Fan lemma. Kassay et al. [30] studied Minty and Stampacchia variational inequality systems with the help of the Kakutani-Fan-Glicksberg fixed point theorem. Peng [49–51] introduced a system of quasi-variational inequality problems and proved its existence theorem by the maximal element theorems. Verma [57–59, 61–62] introduced and studied some systems of variational inequalities and developed some iterative algorithms for approximating the solution to this system of generalized nonlinear quasi-variational inequalities in Hilbert spaces. Kim and Kim [33] introduced a new system of generalized nonlinear quasi-variational inequalities and obtained some existence and uniqueness results of the solution to this system of generalized nonlinear quasi-variational inequalities in Hilbert spaces. Cho et al. [10] introduced a new system of nonlinear variational inequalities and proved some existence and uniqueness theorems of the solution to this system of nonlinear variational inequalities in Hilbert spaces. As generalizations of the system of variational inequalities, Agarwal et al. [2] introduced a system of generalized nonlinear mixed quasi-variational inclusions and investigated the sensitivity of solutions to this system of generalized nonlinear mixed quasi-variational inclusions in Hilbert spaces. Kazmi and Bhat [32] introduced a system of nonlinear variational-like inclusions and gave an iterative algorithm for finding its approximate solution. It is known that accretivity of the underlying operator plays an indispensable role in the theory of variational inequality and its generalizations. In 2001, Huang and Fang [26] were the first to introduce a generalized m-accretive mapping and gave the definition of the resolvent operator for generalized m-accretive mappings in Banach spaces. They also proved some properties of the resolvent operator for generalized m-accretive mappings in Banach spaces. Subsequently, Fang and Huang [16], Yan et al. [65], Fang et al. [18], Lan et al. [37,39], Fang and Huang [17], Peng et al. [52] introduced and investigated many new systems of variational inclusions involving H-monotone operators, (H, η)-monotone operators in Hilbert spaces, generalized m-accretive mappings, H-accretive mappings and (H, η)-accretive mappings in Banach spaces, respectively. In [56, 63], Verma introduced new notions of A-monotone and (A, η)-monotone operators. Further, Cho et al. [12], Lan et al. [38] and Verma [56, 63] studied some properties of A-monotone and (A, η)-monotone operators in Hilbert spaces. In [36], Lan et al. first introduced a new concept of (A, η)-accretive mappings, which generalizes the existing monotone or accretive operators, studied some properties of (A, η)-accretive mappings and defined resolvent operators associated with (A, η)-accretive mappings (also, see [11]). They also investigated a class of variational inclusions using the resolvent operator associated with (A, η)-accretive mappings. Subsequently, Lan [35] by using the concept of (A, η)-accretive mappings and the new resolvent operator technique associated with (A, η)-accretive mappings, introduced and studied a system of general mixed quasivariational inclusions involving (A, η)-accretive mappings in Banach spaces and constructed a perturbed iterative algorithm with mixed errors to this system of nonlinear (A, η)-accretive variational inclusions in q-uniformly smooth Banach spaces.

Recently, Liu et al. [45] introduced and studied a new system of nonlinear variational-like inclusions involving s-(G, η)-maximal monotone operators in Hilbert spaces and showed the existence and uniqueness of a solution to the system of nonlinear variational-like inclusions.

Very recently, Liu et al. [44] introduced and investigated a new system of generalized nonlinear variational-like inclusions involving s-(G, η)-maximal monotone operators in Hilbert spaces. They also suggested a perturbed Mann iterative method with errors for approximating the solution of the aforesaid system and discussed the convergence and stability of the iterative sequence generated by their proposed algorithm.

On the other hand, related to the variational inequalities, we have the problem of finding the fixed points of the nonexpansive mappings, which is the subject of current interest in functional analysis. It is natural to consider a unified approach to these two different problems. Motivated and inspired by the research going in this direction, Noor and Huang [47] considered the problem of finding the common element of the set of the solutions of variational inequalities and the set of the fixed points of the nonexpansive mappings. It is well known that every nonexpansive mapping is a Lipschitzian mapping. Lipschitzian mappings have been generalized by various authors. Sahu [53] introduced and investigated nearly uniformly Lipschitzian mappings as the generalization of Lipschitzian mappings.

Inspired and motivated by the above achievements, in this paper, we shall consider and study a new system of generalized nonlinear variational-like inclusions with A-maximal m-relaxed η-accretive (so-called (A, η)-accretive) mappings in Banach spaces. By using the resolvent operator technique associated with A-maximal m-relaxed η-accretive mappings due to Lan et al., we prove a few existence and uniqueness theorems of the solution to the system of generalized nonlinear variational-like inclusions in q-uniformly smooth Banach spaces. Applying two nearly uniformly Lipschitzian mappings $\mathcal{S}_1$ and $\mathcal{S}_2$ and using the resolvent operator technique associated with A-maximal m-relaxed η-accretive mappings, we shall construct a new perturbed N-step iterative algorithm with mixed errors for finding an element of the set of the fixed points of the nearly uniformly Lipschitzian mapping $\mathcal{Q} = (\mathcal{S}_1, \mathcal{S}_2)$ which is the unique solution of the aforesaid system. We also establish the convergence and stability of the iterative sequence generated by the suggested iterative algorithm under some suitable conditions. The results presented in this paper improve and extend the corresponding results of [18, 23–24, 32, 43–45, 54] and many other recent works.

2 Preliminaries and Basic Results

Let X be a real Banach space with a dual space X^*, $\langle \cdot, \cdot \rangle$ be the dual pair between X and X^* and $CB(X)$ denotes the family of all nonempty closed bounded subsets of X. The generalized duality mapping $J_q : X \multimap X^*$ is defined by

$$J_q(x) = \{f^* \in X^* : \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1}\}, \quad \forall x \in X,$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is known that, in general, $J_q(x) = \|x\|^{q-2} J_2(x)$ for all $x \neq 0$ and J_q is single-valued if X^* is strictly convex. In the sequel, we always assume that X is a real Banach space such that J_q is single-valued. If X is a Hilbert space, then J_2 becomes the identity mapping on X.

The modulus of the smoothness of X is the function $\rho_X : [0, \infty) \to [0, \infty)$ defined by

$$\rho_X(t) = \sup\left\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| \leq 1, \|y\| \leq t\right\}.$$

A Banach space X is called uniformly smooth if

$$\lim_{t \to 0} \frac{\rho_X(t)}{t} = 0.$$

X is called q-uniformly smooth if there exists a constant $c > 0$ such that

$$\rho_X(t) \leq ct^q, \quad q > 1.$$

Note that J_q is single-valued if X is uniformly smooth.

We recall that a nonlinear mapping $T : X \to X$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \text{for all } x, y \in X.$$

In the next definitions, several generalizations of the nonexpansive mappings which have been introduced by various authors in recent years are stated.

Definition 2.1 *A nonlinear mapping $T : X \to X$ is called*

(a) *L-Lipschitzian if there exists a constant $L > 0$ such that*

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in X;$$

(b) *generalized Lipschitzian if there exists a constant $L > 0$ such that*

$$\|Tx - Ty\| \leq L(\|x - y\| + 1), \quad \forall x, y \in X;$$

(c) *generalized (L, M)-Lipschitzian (see [53]) if there exist two constants $L, M > 0$ such that*

$$\|Tx - Ty\| \leq L(\|x - y\| + M), \quad \forall x, y \in X;$$

(d) *asymptotically nonexpansive (see* [20]*) if there exists a sequence* $\{k_n\} \subseteq [1,\infty)$ *with* $\lim\limits_{n\to\infty} k_n = 1$ *such that for each* $n \in \mathbb{N}$,

$$\|T^n x - T^n y\| \le k_n \|x-y\|, \quad \forall x, y \in X;$$

(e) *pointwise asymptotically nonexpansive* [34] *if, for each integer* $n \ge 1$,

$$\|T^n x - T^n y\| \le \alpha_n(x)\|x-y\|, \quad x, y \in X,$$

where $\alpha_n \to 1$ *pointwise on* X*;*

(f) *uniformly* L*-Lipschitzian if there exists a constant* $L > 0$ *such that for each* $n \in \mathbb{N}$,

$$\|T^n x - T^n y\| \le L\|x-y\|, \quad \forall x, y \in X.$$

Definition 2.2 (see [53]) *A nonlinear mapping* $T : X \to X$ *is said to be*

(a) *nearly Lipschitzian with respect to the sequence* $\{a_n\}$ *if for each* $n \in \mathbb{N}$*, there exists a constant* $k_n > 0$ *such that*

$$\|T^n x - T^n y\| \le k_n(\|x-y\| + a_n), \quad \forall x, y \in X, \tag{2.1}$$

where $\{a_n\}$ *is a fix sequence in* $[0,\infty)$ *with* $a_n \to 0$ *as* $n \to \infty$.

For an arbitrary, but fixed $n \in \mathbb{N}$*, the infimum of constants* k_n *in* (2.1) *is called a nearly Lipschitz constant and is denoted by* $\eta(T^n)$*. Notice that*

$$\eta(T^n) = \sup\Big\{\frac{\|T^n x - T^n y\|}{\|x-y\| + a_n} : x, y \in X, x \ne y\Big\}.$$

A nearly Lipschitzian mapping T *with the sequence* $\{(a_n, \eta(T^n))\}$ *is said to be*

(b) *nearly nonexpansive if* $\eta(T^n) = 1$ *for all* $n \in \mathbb{N}$*, that is,*

$$\|T^n x - T^n y\| \le \|x-y\| + a_n, \quad \forall x, y \in X;$$

(c) *nearly asymptotically nonexpansive if* $\eta(T^n) \ge 1$ *for all* $n \in \mathbb{N}$ *and* $\lim\limits_{n\to\infty} \eta(T^n) = 1$*, in other words,* $k_n \ge 1$ *for all* $n \in \mathbb{N}$ *with* $\lim\limits_{n\to\infty} k_n = 1$*;*

(d) *nearly uniformly* L*-Lipschitzian if* $\eta(T^n) \le L$ *for all* $n \in \mathbb{N}$*, in other words,* $k_n = L$ *for all* $n \in \mathbb{N}$.

Remark 2.1 It should be pointed that

(a) Every nonexpansive mapping is an asymptotically nonexpansive mapping and every asymptotically nonexpansive mapping is a pointwise asymptotically nonexpansive mapping. Also, the class of Lipschitzian mappings properly includes the class of pointwise asymptotically nonexpansive mappings.

(b) It is obvious that every Lipschitzian mapping is a generalized Lipschitzian mapping. Furthermore, every mapping with a bounded range is a generalized Lipschitzian mapping. It is easy to see that the class of generalized (L, M)-Lipschitzian mappings is more general than the class of generalized Lipschitzian mappings.

(c) Clearly, the class of nearly uniformly L-Lipschitzian mappings properly includes the class of generalized (L, M)-Lipschitzian mappings and that of uniformly L-Lipschitzian mappings. Note that every nearly asymptotically nonexpansive mapping is nearly uniformly L-Lipschitzian.

Now, we present some new examples to investigate the relations between these mappings.

Example 2.1 Let $X = \mathbb{R}$ and define $T : X \to X$ as follows:

$$T(x) = \begin{cases} \dfrac{1}{\gamma}, & x \in [0, \gamma], \\ 0, & x \in (-\infty, 0) \cup (\gamma, \infty), \end{cases}$$

where $\gamma > 1$ is a real constant. Evidently, the mapping T is discontinuous at the points $x = 0, \gamma$. Since every Lipschitzian mapping is continuous, it follows that T is not Lipschitzian. For each $n \in \mathbb{N}$, take $a_n = \frac{1}{\gamma^n}$. Then

$$|Tx - Ty| \leq |x - y| + \frac{1}{\gamma} = |x - y| + a_1, \quad \forall x, y \in \mathbb{R}.$$

Since $T^n z = \frac{1}{\gamma}$, for all $z \in \mathbb{R}$ and $n \geq 2$, it follows that for all $x, y \in \mathbb{R}$ and $n \geq 2$,

$$|T^n x - T^n y| \leq |x - y| + \frac{1}{\gamma^n} = |x - y| + a_n.$$

Hence, T is a nearly nonexpansive mapping with respect to the sequence $\{a_n\} = \{\frac{1}{\gamma^n}\}$.

The following example shows that the nearly uniformly L-Lipschitzian mappings are not necessarily continuous.

Example 2.2 Let $X = [0, b]$, where $b \in (0, 1]$ is an arbitrary real constant, and let the self-mapping T of X be defined as below:

$$T(x) = \begin{cases} \gamma x, & x \in [0, b), \\ 0, & x = b, \end{cases}$$

where $\gamma \in (0, 1)$ is also an arbitrary real constant. It is plain that the mapping T is discontinuous at the point b. Hence, T is not a Lipschitzian mapping. For each $n \in \mathbb{N}$, take $a_n = \gamma^{n-1}$. Then for all $n \in \mathbb{N}$ and $x, y \in [0, b)$, we have

$$\begin{aligned} |T^n x - T^n y| &= |\gamma^n x - \gamma^n y| = \gamma^n |x - y| \leq \gamma^n |x - y| + \gamma^n \\ &\leq \gamma |x - y| + \gamma^n = \gamma(|x - y| + a_n). \end{aligned}$$

If $x \in [0, b)$ and $y = b$, then for each $n \in \mathbb{N}$, we have $T^n x = \gamma^n x$ and $T^n y = 0$. Since $0 < |x - y| \leq b \leq 1$, it follows that for all $n \in \mathbb{N}$,

$$\begin{aligned} |T^n x - T^n y| &= |\gamma^n x - 0| = \gamma^n x \leq \gamma^n b \leq \gamma^n < \gamma^n |x - y| + \gamma^n \\ &\leq \gamma |x - y| + \gamma^n = \gamma(|x - y| + a_n). \end{aligned}$$

Hence, T is a nearly uniformly γ-Lipschitzian mapping with respect to the sequence $\{a_n\} = \{\gamma^{n-1}\}$.

Obviously, every nearly nonexpansive mapping is a nearly uniformly Lipschitzian mapping. In the following example, we show that the class of nearly uniformly Lipschitzian mappings properly includes the class of nearly nonexpansive mappings.

Example 2.3 Let $X = \mathbb{R}$ and let the self-mapping T of X be defined as follows:

$$T(x) = \begin{cases} \dfrac{1}{2}, & x \in [0,1) \cup \{2\}, \\ 2, & x = 1, \\ 0, & x \in (-\infty, 0) \cup (1,2) \cup (2, +\infty). \end{cases}$$

Evidently, the mapping T is discontinuous at the points $x = 0, 1, 2$. Hence, T is not a Lipschitzian mapping. For each $n \in \mathbb{N}$, take $a_n = \frac{1}{2^n}$. Then T is not a nearly nonexpansive mapping with respect to the sequence $\{\frac{1}{2^n}\}$, because taking $x = 1$ and $y = \frac{1}{2}$, we have $Tx = 2$, $Ty = \frac{1}{2}$ and

$$|Tx - Ty| > |x - y| + \frac{1}{2} = |x - y| + a_1.$$

However,

$$|Tx - Ty| \leq 4\Big(|x - y| + \frac{1}{2}\Big) = 4(|x - y| + a_1), \quad \forall x, y \in \mathbb{R}$$

and for all $n \geq 2$,

$$|T^n x - T^n y| \leq 4\Big(|x - y| + \frac{1}{2^n}\Big) = 4(|x - y| + a_n), \quad \forall x, y \in \mathbb{R},$$

since $T^n z = \frac{1}{2}$, for all $z \in \mathbb{R}$ and $n \geq 2$. Hence, for each $L \geq 4$, T is a nearly uniformly L-Lipschitzian mapping with respect to the sequence $\{\frac{1}{2^n}\}$.

It is clear that every uniformly L-Lipschitzian mapping is a nearly uniformly L-Lipschitzian mapping. In the next example, we show that the class of nearly uniformly L-Lipschitzian mappings properly includes the class of uniformly L-Lipschitzian mappings.

Example 2.4 Let $X = \mathbb{R}$ and let the self-mapping T of X be defined in the same way as in Example 2.3. Then T is not a uniformly 4-Lipschitzian mapping. Since if $x = 1$ and $y \in (1, \frac{3}{2})$, then we have $|Tx - Ty| > 4|x - y|$, because $0 < |x - y| < \frac{1}{2}$. But, in view of Example 2.3, T is a nearly uniformly 4-Lipschitzian mapping.

The following example shows that the class of generalized Lipschitzian mappings properly includes the class of Lipschitzian mappings and that of mappings with a bounded range.

Example 2.5 (see [9]) Let $X = \mathbb{R}$ and let $T : X \to X$ be defined by

$$T(x) = \begin{cases} x - 1, & x \in (-\infty, -1), \\ x - \sqrt{1 - (x+1)^2}, & x \in [-1, 0), \\ x + \sqrt{1 - (x-1)^2}, & x \in [0, 1], \\ x + 1, & x \in (1, \infty). \end{cases}$$

Then T is a generalized Lipschitzian mapping which is not Lipschitzian and whose range is not bounded.

Concerned with the characteristic inequalities in q-uniformly smooth Banach spaces, Xu [64] proved the following result.

Lemma 2.1 *The real Banach space X is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $x, y \in X$,*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q.$$

Lemma 2.2 (see [40]) *For any two nonnegative real numbers r and s, we have*

$$(r + s)^q \leq 2^q(r^q + s^q).$$

Definition 2.3 *A set-valued mapping $T : X \multimap X$ is called ξ-$\widehat{H}$-Lipschitz continuous, if there exists a constant $\xi > 0$ such that*

$$\widehat{H}(T(x), T(y)) \leq \xi\|x - y\|, \quad \forall x, y \in X,$$

where $\widehat{H}$ is the Hausdorff pseudo-metric, that is, for any two nonempty subsets A, B of X,

$$\widehat{H}(A, B) = \max\Big\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\Big\},$$

where $d(u, K) = \inf\limits_{v\in K} \|u - v\|$.

It should be pointed that if the domain of $\widehat{H}$ is restricted to closed bounded subsets $CB(X)$, then $\widehat{H}$ is the Hausdorff metric.

Definition 2.4 *Let X be a q-uniformly smooth Banach space, and $T, A : X \to X$, $\eta : X \times X \to X$ be single-valued mappings.*

(a) *T is said to be accretive if*

$$\langle T(x) - T(y), J_q(x - y)\rangle \geq 0, \quad \forall x, y \in X;$$

(b) *T is called strictly accretive if T is accretive and*

$$\langle T(x) - T(y), J_q(x - y)\rangle = 0,$$

if and only if $x = y$;

(c) *T is said to be r-strongly accretive if there exists a constant $r > 0$ such that*

$$\langle T(x) - T(y), J_q(x - y)\rangle \geq r\|x - y\|^q, \quad \forall x, y \in X;$$

(d) *T is called m-relaxed accretive if there exists a constant $m > 0$ such that*

$$\langle T(x) - T(y), J_q(x - y)\rangle \geq -m\|x - y\|^q, \quad \forall x, y \in X;$$

(e) *T is said to be (ζ, ς)-relaxed cocoercive if there exist constants $\zeta, \varsigma > 0$ such that*

$$\langle T(x) - T(y), J_q(x - y)\rangle \geq -\zeta\|T(x) - T(y)\|^q + \varsigma\|x - y\|^q, \quad \forall x, y \in X;$$

(f) *T is said to be ϱ-Lipschitz continuous if there exists a constant $\varrho > 0$ such that*

$$\|T(x) - T(y)\| \leq \varrho\|x - y\|, \quad \forall x, y \in X;$$

(g) *η is said to be τ-Lipschitz continuous if there exists a constant $\tau > 0$ such that*

$$\|\eta(x, y)\| \leq \tau\|x - y\|, \quad \forall x, y \in X;$$

(h) *η is said to be ϵ-Lipschitz continuous in the first argument if there exists a constant $\epsilon > 0$ such that*

$$\|\eta(x, u) - \eta(y, u)\| \leq \epsilon\|x - y\|, \quad \forall x, y, u \in X;$$

(i) *η is said to be (ρ, ξ)-relaxed cocoercive with respect to A in the first argument if there exist constants $\rho, \xi > 0$ such that*

$$\langle \eta(x, u) - \eta(y, u), J_q(A(x) - A(y))\rangle \geq -\rho\|\eta(x, u) - \eta(y, u)\|^q + \xi\|x - y\|^q, \quad \forall x, y, u \in X.$$

In a similar way to (h) and (i), we can define the Lipschitz continuity of the mapping η in the second argument and the relaxed cocoercivity of η with respect to A in the second argument.

Definition 2.5 *Let X be a q-uniformly smooth Banach space and $T : X \times X \to X$ be a single-valued mapping. Then T is said to be (α, β)-Lipschitz continuous if there exist constants $\alpha, \beta > 0$ such that*

$$\|T(x, y) - T(x', y')\| \leq \alpha\|x - x'\| + \beta\|y - y'\|, \quad \forall x, x', y, y' \in X.$$

Definition 2.6 *Let X be a q-uniformly smooth Banach space, $\eta : X \times X \to X$ and $H, A : X \to X$ be three single-valued mappings. A set-valued mapping $M : X \multimap X$ is said to be*

(a) *accretive if*

$$\langle u - v, J_q(x - y)\rangle \geq 0, \quad \forall x, y \in X,\ u \in Mx,\ v \in My;$$

(b) *η-accretive if*

$$\langle u - v, J_q(\eta(x, y))\rangle \geq 0, \quad \forall x, y \in X,\ u \in Mx,\ v \in My;$$

(c) *strictly η-accretive if M is η-accretive and the equality holds if and only if $x = y$;*

(d) *r-strongly η-accretive if there exists a constant $r > 0$ such that*

$$\langle u - v, J_q(\eta(x, y))\rangle \geq r\|x - y\|^q, \quad \forall x, y \in X,\ u \in Mx,\ v \in My;$$

(e) *α-relaxed η-accretive if there exists a constant $\alpha > 0$ such that*

$$\langle u - v, J_q(\eta(x, y))\rangle \geq -\alpha\|x - y\|^q, \quad \forall x, y \in X,\ u \in Mx,\ v \in My;$$

(f) *m-accretive if M is accretive and $(I + \lambda M)(X) = X$ for all $\lambda > 0$, where I denotes the identity operator on X;*

(g) *generalized m-accretive if M is η-accretive and $(I + \lambda M)(X) = X$ for all $\lambda > 0$;*

(h) *H-accretive if M is accretive and $(H + \lambda M)(X) = X$ for all $\lambda > 0$;*

(i) *(H, η)-accretive if M is η-accretive and $(H + \lambda M)(X) = X$ for all $\lambda > 0$.*

Remark 2.2 It should be noticed that

(1) The class of generalized m-accretive operators was first introduced by Huang and Fang [26] and includes that of m-accretive operators as a special case. The class of H-accretive operators was first introduced and studied by Fang and Huang [14] and also includes that of m-accretive operators as a special case.

(2) When $X = \mathcal{H}$, (a)–(i) of Definition 2.6 reduce to the definitions of monotone operators, η-monotone operators, strictly η-monotone operators, strongly η-monotone operators, relaxed η-monotone operators, maximal monotone operators, maximal η-monotone operators, H-monotone operators, and (H, η)-monotone operators, respectively.

Definition 2.7 *Let $A : X \to X$, $\eta : X \times X \to X$ be two single-valued mappings and $M : X \multimap X$ be a set-valued mapping. Then M is called A-maximal m-relaxed η-accretive (so-called (A, η)-accretive in* [36]*) if M is m-relaxed η-accretive and $(A + \lambda M)(X) = X$ for every $\lambda > 0$.*

Remark 2.3 For appropriate and suitable choices of m, A, η and the space X, it is easy to see that Definition 2.7 includes a number of definitions of monotone operators and accretive operators (see [36]).

In [36], Lan et al. showed that $(A + \rho M)^{-1}$ is a single-valued operator if $M : X \multimap X$ is an A-maximal m-relaxed η-accretive mapping and $A : X \to X$ is a r-strongly η-accretive mapping. Based on this fact, we can define the resolvent operator $R^{\eta,M}_{\rho,A}$ associated with an A-maximal m-relaxed η-accretive mapping M as follows.

Definition 2.8 *Let $A : X \to X$ be a strictly η-accretive mapping and $M : X \multimap X$ be an A-maximal m-relaxed η-accretive mapping. The resolvent operator $R^{\eta,M}_{\rho,A} : X \to X$ associated with A and M is defined by*

$$R^{\eta,M}_{\rho,A}(x) = (A + \rho M)^{-1}(x), \quad \forall x \in X.$$

Proposition 2.1 (see [36]) *Let X be a q-uniformly smooth Banach space, $\eta : X \times X \to X$ be τ-Lipschitz continuous, $A : X \to X$ be an r-strongly η-accretive mapping and $M : X \multimap X$ be an A-maximal m-relaxed η-accretive mapping. Then the resolvent operator $R^{\eta,M}_{\rho,A} : X \to X$ is $\frac{\tau^{q-1}}{r-\rho m}$-Lipschitz continuous, i.e.,*

$$\|R^{\eta,M}_{\rho,A}(x) - R^{\eta,M}_{\rho,A}(y)\| \le \frac{\tau^{q-1}}{r - \rho m}\|x - y\|, \quad \forall x, y \in X,$$

where $\rho \in (0, \frac{r}{m})$ is a constant.

3 A New System of Generalized Nonlinear Variational-Like Inclusions

In this section, we introduce a new system of generalized nonlinear variational-like inclusions in q-uniformly smooth Banach spaces and prove the existence and uniqueness theorems of the solution to the aforesaid system.

For $i = 1,2$, let X_i be a q_i-uniformly smooth Banach space with $q_i > 1$, and $S_1, Q_2 : X_1 \to X_2$, $T_1, P_2 : X_2 \to X_1$, $P_1, T_2 : X_1 \to X_1$, $Q_1, S_2 : X_2 \to X_2$, $A_i, f_i, g_i, \theta_i : X_i \to X_i$, $\eta_i : X_i \times X_i \to X_i$ and $N_i : X_2 \times X_1 \times X_1 \times X_2 \to X_i$, be single-valued mappings. Further, suppose that for $i \in \{1,2\}$ and $j \in \{1,2\} \setminus \{i\}$, $h_i : X_j \times X_i \to X_i$ and $M_i : X_i \times X_j \multimap X_i$ are any nonlinear mappings such that for all $x_j \in X_j$, $M_i(\cdot, x_j) : X_i \multimap X_i$ is an A_i-maximal m_i-relaxed η_i-accretive mapping with $\mathrm{Range}(f_i - g_i) \cap \mathrm{dom}\, M_i(\cdot, x_j) \neq \varnothing$. For any given $a \in X_1$, $b \in X_2$, $\lambda_1, \lambda_2 > 0$, our problem is finding $(x, y) \in X_1 \times X_2$ such that

$$\begin{cases} a \in N_1(S_1(x), T_1(y), P_1(x), Q_1(y)) - (h_1(y, x) + \theta_1(x) - \lambda_1 M_1((f_1 - g_1)(x), y)), \\ b \in N_2(S_2(y), T_2(x), P_2(y), Q_2(x)) - (h_2(x, y) + \theta_2(y) - \lambda_2 M_2((f_2 - g_2)(y), x)), \end{cases} \tag{3.1}$$

where $(f_i - g_i)(x) = f_i(x) - g_i(x)$ for all $x \in X_i$ and $i \in \{1,2\}$.

The problem (3.1) is called a system of generalized nonlinear variational-like inclusions involving A-maximal m-relaxed η-accretive mappings in uniformly smooth Banach spaces. Next, we denote by SGNVLI the set of the solutions of the system (3.1).

Remark 3.1 For appropriate and suitable choices of X_i, S_i, T_i, P_i, Q_i, M_i, N_i, η_i, λ_i, A_i, f_i, g_i, h_i, θ_i $(i = 1,2)$, a and b in the above system, one can obtain different problems considered and studied in [24, 28, 42, 44–45, 54, 60] and the references therein.

Now, we prove the existence and uniquness of solution of the system (3.1). For this purpose, we need the following lemma which offers a good approach to solve the system (3.1).

Lemma 3.1 *For $i = 1,2$, let X_i, S_i, T_i, P_i, Q_i, M_i, N_i, η_i, λ_i, A_i, f_i, g_i, h_i, θ_i, a and b be the same as in the system* (3.1)*. Then an element $(x^*, y^*) \in X_1 \times X_2$ is a solution of the system* (3.1)*, if and only if it satisfies*

$$\begin{cases} f_1(x^*) = g_1(x^*) + R^{\eta_1, M_1(\cdot, y^*)}_{\rho_1 \lambda_1, A_1}[A_1(f_1 - g_1)(x^*) \\ \qquad - \rho_1(N_1(S_1(x^*), T_1(y^*), P_1(x^*), Q_1(y^*)) - h_1(y^*, x^*) - \theta_1(x^*) - a)], \\ f_2(y^*) = g_2(y^*) + R^{\eta_2, M_2(\cdot, x^*)}_{\rho_2 \lambda_2, A_2}[A_2(f_2 - g_2)(y^*) \\ \qquad - \rho_2(N_2(S_2(y^*), T_2(x^*), P_2(y^*), Q_2(x^*)) - h_2(x^*, y^*) - \theta_2(y^*) - b)], \end{cases} \tag{3.2}$$

where $\rho_1, \rho_2 > 0$ are two constants.

Proof The conclusion follows directly from Definition 2.8 and some simple arguments.

Theorem 3.1 *For $i = 1,2$, let X_i, S_i, T_i, P_i, Q_i, M_i, N_i, η_i, λ_i, A_i, f_i, g_i, h_i, θ_i, a and b be the same as in the system* (3.1)*. Further, suppose that for $i = 1,2$,*

(a) *S_i, T_i, P_i and Q_i are ξ_i-Lipschitz continuous, ζ_i-Lipschitz continuous, ς_i-Lipschitz continuous and δ_i-Lipschitz continuous, respectively;*

(b) *f_i and g_i are ω_i-Lipschitz continuous and π_i-Lipschitz continuous, respectively;*

(c) *$f_i - g_i$ is (κ_i, ϱ_i)-relaxed cocoercive and h_i is (ε_i, o_i)-Lipschitz continuous;*

(d) *η_i and θ_i are τ_i-Lipschitz continuous and ι_i-Lipschitz continuous, respectively;*

(e) *A_i is r_i-strongly η_i-accretive and γ_i-Lipschitz continuous;*

(f) *N_i is ϵ_i-Lipschitz continuous in the first argument, υ_i-Lipschitz continuous in the second argument, ι'_i-Lipschitz continuous in the third argument and ν_i-Lipschitz continuous in the fourth argument;*

(g) *$N_i(S_i(.), u, v, w)$ is (σ_i, ϖ_i)-relaxed cocoercive with respect to f'_i, for all $u, v \in X_1$ and $w \in X_2$, where $f'_i : X_i \to X_i$ is defined by*

$$f'_i(x) = A_i \circ (f_i - g_i)(x) = A_i(f_i - g_i)(x),$$

for all $x \in X_i$;

(h) *there exist constants $\rho_i \in (0, \frac{r_i}{\lambda_i m_i})$ and $\mu_i > 0$ such that for $i \in \{1,2\}$ and $j \in \{1,2\} \setminus \{i\}$,*

$$\|R^{\eta_i, M_i(\cdot,x)}_{\rho_i\lambda_i, A_i}(z) - R^{\eta_i, M_i(\cdot,y)}_{\rho_i\lambda_i, A_i}(z)\| \le \mu_i \|x - y\|, \quad \forall z \in X_i,\ x, y \in X_j \tag{3.3}$$

and

$$\begin{cases} \mu_2 + \sqrt[q_1]{1 - q_1\varrho_1 + 2^{q_1}(c_{q_1} + q_1\kappa_1)(\omega_1^{q_1} + \pi_1^{q_1})} < 1, \\ \mu_1 + \sqrt[q_2]{1 - q_2\varrho_2 + 2^{q_2}(c_{q_2} + q_2\kappa_2)(\omega_2^{q_2} + \pi_2^{q_2})} < 1, \\ \sqrt[q_i]{2^{q_i}\gamma_i^{q_i}(\omega_i^{q_i} + \pi_i^{q_i}) - q_i\rho_i(-\sigma_i\epsilon_i^{q_i}\xi_i^{q_i} + \varpi_i) + c_{q_i}(\rho_i\epsilon_i\xi_i)^{q_i}} \\ < \tau_i^{1-q_i}(r_i - \rho_i\lambda_i m_i)\chi_i - \rho_i(\iota'_i\varsigma_i + o_i + \iota_i), \end{cases} \tag{3.4}$$

where

$$\chi_1 = 1 - (\mu_2 + \sqrt[q_1]{1 - q_1\varrho_1 + 2^{q_1}(c_{q_1} + q_1\kappa_1)(\omega_1^{q_1} + \pi_1^{q_1})}) - \frac{\rho_2\tau_2^{q_2-1}(\nu_2\delta_2 + \upsilon_2\zeta_2 + \varepsilon_2)}{r_2 - \rho_2\lambda_2 m_2},$$
$$\chi_2 = 1 - (\mu_1 + \sqrt[q_2]{1 - q_2\varrho_2 + 2^{q_2}(c_{q_2} + q_2\kappa_2)(\omega_2^{q_2} + \pi_2^{q_2})}) - \frac{\rho_1\tau_1^{q_1-1}(\nu_1\delta_1 + \upsilon_1\zeta_1 + \varepsilon_1)}{r_1 - \rho_1\lambda_1 m_1}$$

and c_{q_1} and c_{q_2} are two constants guaranteed by Lemma 2.1. *Then the system* (3.1) *admits a unique solution in $X_1 \times X_2$.*

Proof For any given $\rho_1 > 0$ and $\rho_2 > 0$, define

$$\Psi_{\rho_1} : X_1 \times X_2 \to X_1 \quad \text{and} \quad \Phi_{\rho_2} : X_1 \times X_2 \to X_2$$

by

$$\begin{cases} \Psi_{\rho_1}(x,y) = x - (f_1 - g_1)(x) + R^{\eta_1, M_1(\cdot,y)}_{\rho_1\lambda_1, A_1}[A_1(f_1 - g_1)(x) \\ \qquad\qquad - \rho_1(N_1(S_1(x), T_1(y), P_1(x), Q_1(y)) - h_1(y,x) - \theta_1(x) - a)], \\ \Phi_{\rho_2}(x,y) = y - (f_2 - g_2)(y) + R^{\eta_2, M_2(\cdot,x)}_{\rho_2\lambda_2, A_2}[A_2(f_2 - g_2)(y) \\ \qquad\qquad - \rho_2(N_2(S_2(y), T_2(x), P_2(y), Q_2(x)) - h_2(x,y) - \theta_2(y) - b)] \end{cases} \tag{3.5}$$

for all $(x,y) \in X_1 \times X_2$, where $a \in X_1$ and $b \in X_2$ are the same as in the system (3.1). Also, define $F_{\rho_1,\rho_2} : X_1 \times X_2 \to X_1 \times X_2$ as follows:

$$F_{\rho_1,\rho_2}(x,y) = (\Psi_{\rho_1}(x,y), \Phi_{\rho_2}(x,y)), \quad (x,y) \in X_1 \times X_2. \tag{3.6}$$

Consider a function $\|\cdot\|_*$ on $X_1 \times X_2$ which is defined by

$$\|(x,y)\|_* = \|x\| + \|y\|, \quad \forall (x,y) \in X_1 \times X_2.$$

Obviously, $(X_1 \times X_2, \|\cdot\|_*)$ is a Banach space. Now, we prove that F is a contraction mapping on $(X_1 \times X_2, \|\cdot\|_*)$. Let $(x,y),(x',y') \in X_1 \times X_2$ be given. It follows from (3.3) and Proposition 2.1, that

$$\begin{aligned}
&\|\Psi_{\rho_1}(x,y) - \Psi_{\rho_1}(x',y')\| \\
&= \|x - (f_1 - g_1)(x) + R^{\eta_1, M_1(\cdot,y)}_{\rho_1\lambda_1, A_1}[A_1(f_1 - g_1)(x) - \rho_1(N_1(S_1(x), T_1(y), P_1(x), Q_1(y)) \\
&\quad - h_1(y,x) - \theta_1(x) - a)] - (x' - (f_1 - g_1)(x') + R^{\eta_1, M_1(\cdot,y')}_{\rho_1\lambda_1, A_1}[A_1(f_1 - g_1)(x') \\
&\quad - \rho_1(N_1(S_1(x'), T_1(y'), P_1(x'), Q_1(y')) - h_1(y',x') - \theta_1(x') - a)])\| \\
&\le \|x - x' - [(f_1 - g_1)(x) - (f_1 - g_1)(x')]\| \\
&\quad + \|R^{\eta_1, M_1(\cdot,y)}_{\rho_1\lambda_1, A_1}[A_1(f_1 - g_1)(x) - \rho_1(N_1(S_1(x), T_1(y), P_1(x), Q_1(y)) \\
&\quad - h_1(y,x) - \theta_1(x) - a)] \\
&\quad - R^{\eta_1, M_1(\cdot,y')}_{\rho_1\lambda_1, A_1}[A_1(f_1 - g_1)(x') - \rho_1(N_1(S_1(x'), T_1(y'), P_1(x'), Q_1(y')) \\
&\quad - h_1(y',x') - \theta_1(x') - a)]\| \\
&\le \|x - x' - [(f_1 - g_1)(x) - (f_1 - g_1)(x')]\| \\
&\quad + \|R^{\eta_1, M_1(\cdot,y)}_{\rho_1\lambda_1, A_1}[A_1(f_1 - g_1)(x) - \rho_1(N_1(S_1(x), T_1(y), P_1(x), Q_1(y)) \\
&\quad - h_1(y,x) - \theta_1(x) - a)] \\
&\quad - R^{\eta_1, M_1(\cdot,y')}_{\rho_1\lambda_1, A_1}[A_1(f_1 - g_1)(x) - \rho_1(N_1(S_1(x), T_1(y), P_1(x), Q_1(y)) \\
&\quad - h_1(y,x) - \theta_1(x) - a)]\| \\
&\quad + \|R^{\eta_1, M_1(\cdot,y')}_{\rho_1\lambda_1, A_1}[A_1(f_1 - g_1)(x) - \rho_1(N_1(S_1(x), T_1(y), P_1(x), Q_1(y)) \\
&\quad - h_1(y,x) - \theta_1(x) - a)] \\
&\quad - R^{\eta_1, M_1(\cdot,y')}_{\rho_1\lambda_1, A_1}[A_1(f_1 - g_1)(x') - \rho_1(N_1(S_1(x'), T_1(y'), P_1(x'), Q_1(y')) \\
&\quad - h_1(y',x') - \theta_1(x') - a)]\| \\
&\le \|x - x' - [(f_1 - g_1)(x) - (f_1 - g_1)(x')]\| + \mu_1\|y - y'\| \\
&\quad + \frac{\tau_1^{q_1-1}}{r_1 - \rho_1\lambda_1 m_1}\{\rho_1(\|N_1(S_1(x), T_1(y), P_1(x), Q_1(y)) - N_1(S_1(x), T_1(y), P_1(x), Q_1(y'))\| \\
&\quad + \|N_1(S_1(x), T_1(y), P_1(x), Q_1(y')) - N_1(S_1(x), T_1(y), P_1(x'), Q_1(y'))\| \\
&\quad + \|N_1(S_1(x), T_1(y), P_1(x'), Q_1(y')) - N_1(S_1(x), T_1(y'), P_1(x'), Q_1(y'))\| \\
&\quad + \|h_1(y,x) - h_1(y',x')\| + \|\theta_1(x) - \theta_1(x')\|) \\
&\quad + \|A_1(f_1 - g_1)(x) - A_1(f_1 - g_1)(x') - \rho_1(N_1(S_1(x), T_1(y'), P_1(x'), Q_1(y')) \\
&\quad - N_1(S_1(x'), T_1(y'), P_1(x'), Q_1(y')))\|\}.
\end{aligned} \tag{3.7}$$

By Lemma 2.1, there exists $c_{q_1} > 0$ such that

$$\begin{aligned}
&\|x - x' - [(f_1 - g_1)(x) - (f_1 - g_1)(x')]\|^{q_1} \\
&\le \|x - x'\|^{q_1} - q_1\langle (f_1 - g_1)(x) - (f_1 - g_1)(x'), J_{q_1}(x - x')\rangle \\
&\quad + c_{q_1}\|(f_1 - g_1)(x) - (f_1 - g_1)(x')\|^{q_1}.
\end{aligned}$$

Since $f_1 - g_1$ is (κ_1, ϱ_1)-relaxed cocoercive and f_1, g_1 are ω_1-lipschitz continuous and π_1-

Lipschitz continuous, respectively, by Lemma 2.2, we obtain

$$\begin{aligned}
&\|x-x'-[(f_1-g_1)(x)-(f_1-g_1)(x')]\|^{q_1}\\
&\le \|x-x'\|^{q_1}+(c_{q_1}+q_1\kappa_1)\|(f_1-g_1)(x)-(f_1-g_1)(x')\|^{q_1}-q_1\varrho_1\|x-x'\|^{q_1}\\
&\le 2^{q_1}(c_{q_1}+q_1\kappa_1)(\|f_1(x)-f_1(x')\|^{q_1}+\|g_1(x)-g_1(x')\|^{q_1})+(1-q_1\varrho_1)\|x-x'\|^{q_1}\\
&=(1-q_1\varrho_1+2^{q_1}(c_{q_1}+q_1\kappa_1)(\omega_1^{q_1}+\pi_1^{q_1}))\|x-x'\|^{q_1}.
\end{aligned}\tag{3.8}$$

In view of that N_1 is υ_1-Lipschitz continuous in the second argument, ι_1'-Lipschitz is continuous in the third argument, ν_1-Lipschitz is continuous in the fourth argument and T_1, P_1, Q_1 are ζ_1-Lipschitz continuous, and ς_1-Lipschitz continuous, δ_1-Lipschitz continuous, respectively, we conclude that

$$\begin{cases}
\quad\|N_1(S_1(x),T_1(y),P_1(x),Q_1(y))-N_1(S_1(x),T_1(y),P_1(x),Q_1(y'))\|\\
\le \nu_1\|Q_1(y)-Q_1(y')\|\le \nu_1\delta_1\|y-y'\|,\\
\quad\|N_1(S_1(x),T_1(y),P_1(x),Q_1(y'))-N_1(S_1(x),T_1(y),P_1(x'),Q_1(y'))\|\\
\le \iota_1'\|P_1(x)-P_1(x')\|\le \iota_1'\varsigma_1\|x-x'\|,\\
\quad\|N_1(S_1(x),T_1(y),P_1(x'),Q_1(y'))-N_1(S_1(x),T_1(y'),P_1(x'),Q_1(y'))\|\\
\le \upsilon_1\|T_1(y)-T_1(y')\|\le \upsilon_1\zeta_1\|y-y'\|.
\end{cases}\tag{3.9}$$

Since A_1, f_1, g_1 are γ_1-Lipschitz continuous, ω_1-Lipschitz continuious, and π_1-Lipschitz continuous, $N_1(S_1(\cdot),u,v,w)$ is (σ_1,ϖ_1)-relaxed cocoercive with respect to $f'=A_1\circ(f_1-g_1)$ in the first argument, for all $u,v\in X_1$ and $w\in X_2$, N_1 is ϵ_1-Lipschitz continuous in the first argument and S_1 is ξ_1-Lipschitz continuous, by Lemmas 2.1–2.2, we get

$$\begin{aligned}
&\|A_1(f_1-g_1)(x)-A_1(f_1-g_1)(x')-\rho_1(N_1(S_1(x),T_1(y'),P_1(x'),Q_1(y'))\\
&\quad-N_1(S_1(x'),T_1(y'),P_1(x'),Q_1(y')))\|^{q_1}\\
&\le \|A_1(f_1-g_1)(x)-A_1(f_1-g_1)(x')\|^{q_1}+c_{q_1}\rho_1^{q_1}\|N_1(S_1(x),T_1(y'),P_1(x'),Q_1(y'))\\
&\quad-N_1(S_1(x'),T_1(y'),P_1(x'),Q_1(y')))\|^{q_1}-q_1\rho_1\langle N_1(S_1(x),T_1(y'),P_1(x'),Q_1(y'))\\
&\quad-N_1(S_1(x'),T_1(y'),P_1(x'),Q_1(y')),J_{q_1}(A_1(f_1-g_1)(x)-A_1(f_1-g_1)(x'))\rangle\\
&\le \gamma_1^{q_1}(\|f_1(x)-f_1(x')\|+\|g_1(x)-g_1(x')\|)^{q_1}+c_{q_1}\rho_1^{q_1}\epsilon_1^{q_1}\|S_1(x)-S_1(x')\|^{q_1}\\
&\quad-q_1\rho_1(-\sigma_1\|N_1(S_1(x),T_1(y'),P_1(x'),Q_1(y'))\\
&\quad-N_1(S_1(x'),T_1(y'),P_1(x'),Q_1(y'))\|^{q_1}+\varpi_1\|x-x'\|^{q_1})\\
&\le 2^{q_1}\gamma_1^{q_1}(\|f_1(x)-f_1(x')\|^{q_1}+\|g_1(x)-g_1(x')\|^{q_1})+c_{q_1}(\rho_1\epsilon_1\xi_1)^{q_1}\|x-x'\|^{q_1}\\
&\quad-q_1\rho_1(-\sigma_1\epsilon_1^{q_1}\|S_1(x)-S_1(x')\|^{q_1}+\varpi_1\|x-x'\|^{q_1})\\
&\le [2^{q_1}\gamma_1^{q_1}(\omega_1^{q_1}+\pi_1^{q_1})-q_1\rho_1(-\sigma_1\epsilon_1^{q_1}\xi_1^{q_1}+\varpi_1)+c_{q_1}(\rho_1\epsilon_1\xi_1)^{q_1}]\|x-x'\|^{q_1},
\end{aligned}\tag{3.10}$$

where c_{q_1} is the constant as in Lemma 2.1.

It follows from (ε_1,o_1)-Lipschitz continuity of h_1 and ι_1-Lipschitz continuity of θ_1, that

$$\|h_1(y,x)-h_1(y',x')\|\le\varepsilon_1\|y-y'\|+o_1\|x-x'\|\tag{3.11}$$

and

$$\|\theta_1(x)-\theta_1(x')\|\le\iota_1\|x-x'\|.\tag{3.12}$$

Combining (3.8)–(3.12) with (3.7), we get

$$\|\Psi_{\rho_1}(x,y)-\Psi_{\rho_1}(x',y')\| \leq \varphi_1\|x-x'\|+\phi_1\|y-y'\|, \tag{3.13}$$

where

$$\begin{aligned}
\varphi_1 &= \sqrt[q_1]{1-q_1\varrho_1+2^{q_1}(c_{q_1}+q_1\kappa_1)(\omega_1^{q_1}+\pi_1^{q_1})}+\frac{\tau_1^{q_1-1}(\rho_1(\iota_1'\varsigma_1+o_1+\iota_1)+\Gamma_1)}{r_1-\rho_1\lambda_1 m_1},\\
\Gamma_1 &= \sqrt[q_1]{2^{q_1}\gamma_1^{q_1}(\omega_1^{q_1}+\pi_1^{q_1})-q_1\rho_1(-\sigma_1\epsilon_1^{q_1}\xi_1^{q_1}+\varpi_1)+c_{q_1}(\rho_1\epsilon_1\xi_1)^{q_1}},\\
\phi_1 &= \mu_1+\frac{\rho_1\tau_1^{q_1-1}(\nu_1\delta_1+\upsilon_1\zeta_1+\varepsilon_1)}{r_1-\rho_1\lambda_1 m_1}.
\end{aligned}$$

Similarly, one can easily get

$$\|\Phi_{\rho_2}(x,y)-\Phi_{\rho_2}(x',y')\| \leq \varphi_2\|x-x'\|+\phi_2\|y-y'\|, \tag{3.14}$$

where

$$\begin{aligned}
\phi_2 &= \sqrt[q_2]{1-q_2\varrho_2+2^{q_2}(c_{q_2}+q_2\kappa_2)(\omega_2^{q_2}+\pi_2^{q_2})}+\frac{\tau_2^{q_2-1}(\rho_2(\iota_2'\varsigma_2+o_2+\iota_2)+\Gamma_2)}{r_2-\rho_2\lambda_2 m_2},\\
\Gamma_2 &= \sqrt[q_2]{2^{q_2}\gamma_2^{q_2}(\omega_2^{q_2}+\pi_2^{q_2})-q_2\rho_2(-\sigma_2\epsilon_2^{q_2}\xi_2^{q_2}+\varpi_2)+c_{q_2}(\rho_2\epsilon_2\xi_2)^{q_2}},\\
\varphi_2 &= \mu_2+\frac{\rho_2\tau_2^{q_2-1}(\nu_2\delta_2+\upsilon_2\zeta_2+\varepsilon_2)}{r_2-\rho_2\lambda_2 m_2}
\end{aligned}$$

and c_{q_2} is the constant as in Lemma 2.1. The inequalities (3.13)–(3.14) imply that

$$\|\Psi_{\rho_1}(x,y)-\Psi_{\rho_1}(x',y')\|+\|\Phi_{\rho_2}(x,y)-\Phi_{\rho_2}(x',y')\| \leq \vartheta(\|x-x'\|+\|y-y'\|), \tag{3.15}$$

where $\vartheta=\max\{\varphi_1+\varphi_2,\phi_1+\phi_2\}$. By using (3.6) and (3.15), we obtain

$$\|F_{\rho_1,\rho_2}(x,y)-F_{\rho_1,\rho_2}(x',y')\|_* \leq \vartheta\|(x,y)-(x',y')\|_*. \tag{3.16}$$

From (3.4), we know that $0\leq\vartheta<1$ and it follows from (3.16) that $F_{\rho_1,\rho_2}: X_1\times X_2\to X_1\times X_2$ is a contraction mapping. According to the Banach fixed point theorem, there exists a unique point $(x^*,y^*)\in X_1\times X_2$ such that $F_{\rho_1,\rho_2}(x^*,y^*)=(x^*,y^*)$. By (3.5)–(3.6), we conclude that

$$\begin{aligned}
f_1(x^*) &= g_1(x^*)+R^{\eta_1,M_1(\cdot,y^*)}_{\rho_1\lambda_1,A_1}[A_1(f_1-g_1)(x^*)-\rho_1(N_1(S_1(x^*),T_1(y^*),P_1(x^*),Q_1(y^*))\\
&\qquad -h_1(y^*,x^*)-\theta_1(x^*)-a)],\\
f_2(y^*) &= g_2(y^*)+R^{\eta_2,M_2(\cdot,x^*)}_{\rho_2\lambda_2,A_2}[A_2(f_2-g_2)(y^*)-\rho_2(N_2(S_2(y^*),T_2(x^*),P_2(y^*),Q_2(x^*))\\
&\qquad -h_2(x^*,y^*)-\theta_2(y^*)-b)].
\end{aligned}$$

Now, in view of Lemma 3.1, (x^*,y^*) is a unique solution of the system (3.1) and this is the desired result.

4 Variational Convergence and the Perturbed Iterative Algorithm

In this section, by using two nearly uniformly Lipschitzian mappings $\mathcal{S}_1$ and $\mathcal{S}_2$, the resolvent operator technique associated with A-maximal m-relaxed η-accretive mappings and the equivalent formulation (3.2), we construct a new perturbed N-step iterative algorithm with mixed errors for finding an element of the set of the fixed points $\mathcal{Q}$ which is the unique solution of the system (3.1).

Definition 4.1 *For each $n \geq 0$, let $M_n, M : X \multimap X$ be set-valued mappings. We say that the sequence $\{M_n\}$ is graph-convergent to M (denoted by $M_n \xrightarrow{G} M$), if for every $(x,u) \in \mathrm{Gph}(M)$, there exists $(x_n, u_n) \in \mathrm{Gph}(M_n)$ such that $x_n \to x$ and $u_n \to u$ as $n \to \infty$, where $\mathrm{Gph}(M)$ is defined as follows:*

$$\mathrm{Gph}(M) = \{(x,u) \in X \times X : u \in M(x)\}.$$

Theorem 4.1 *Suppose that, for each $n \geq 0$, $\eta_n, \eta : X \times X \to X$ are τ_n-Lipschitz continuous and τ-Lipschitz continuous, respectively, $A_n : X \to X$ is r_n-strongly η_n-accretive and α_n-Lipschitz continuous and $A : X \to X$ is an r-strongly η-accretive mapping. For each $n \geq 0$, let $M_n, M : X \multimap X$ be A_n-maximal m_n-relaxed η_n-accretive and A-maximal m-relaxed η-accretive mappings, respectively. Further, assume that for any given constant $\rho > 0$, the sequences $\left(\frac{\tau_n^{q-1}}{r_n - \rho m_n}\right)_{n=0}^{\infty}$ and $\left(\frac{\alpha_n \tau_n^{q-1}}{r_n - \rho m_n}\right)_{n=0}^{\infty}$ are bounded and $\lim\limits_{n\to\infty} A_n(x) = A(x)$, for any $x \in X$. Then for any given constant $\rho > 0$, the sequence $\{M_n\}$ is graph-convergent to M, if and only if $R^{\eta_n, M_n}_{\rho, A_n}(z) \to R^{\eta, M}_{\rho, A}(z)$ for all $z \in X$.*

Proof Suppose that $\{M_n\}$ is graph-convergent to M and $z \in X$, $\rho > 0$ are arbitrary. Since $(A + \rho M)(X) = X$, there exists $(x,u) \in \mathrm{Gph}(M)$ for which $z = A(x) + \rho u$ and thus by Definition 4.1, there exists a sequence $\{(x_n, u_n)\} \subseteq \mathrm{Gph}(M_n)$ such that $x_n \to x$ and $u_n \to u$ as $n \to \infty$. Clearly, $(x,u) \in \mathrm{Gph}(M)$ and $\{(x_n, u_n)\} \subseteq \mathrm{Gph}(M_n)$ imply that

$$x = R^{\eta, M}_{\rho, A}[A(x) + \rho u], \quad x_n = R^{\eta_n, M_n}_{\rho, A_n}[A_n(x_n) + \rho u_n]. \tag{4.1}$$

Put $z_n = A_n(x_n) + \rho u_n$. By using Proposition 2.1, the relation (4.1) and the assumptions, we obtain

$$\begin{aligned}
&\|R^{\eta_n, M_n}_{\rho, A_n}(z) - R^{\eta, M}_{\rho, A}(z)\| \\
&\leq \|R^{\eta_n, M_n}_{\rho, A_n}(z_n) - R^{\eta, M}_{\rho, A}(z)\| + \|R^{\eta_n, M_n}_{\rho, A_n}(z_n) - R^{\eta_n, M_n}_{\rho, A_n}(z)\| \\
&\leq \|R^{\eta_n, M_n}_{\rho, A_n}(A_n(x_n) + \rho u_n) - R^{\eta, M}_{\rho, A}(A(x) + \rho u)\| + \frac{\tau_n^{q-1}}{r_n - \rho m_n}\|z_n - z\| \\
&\leq \|x_n - x\| + \frac{\tau_n^{q-1}}{r_n - \rho m_n}(\|A_n(x_n) - A(x)\| + \rho\|u_n - u\|) \\
&\leq \Big(1 + \frac{\alpha_n \tau_n^{q-1}}{r_n - \rho m_n}\Big)\|x_n - x\| + \frac{\tau_n^{q-1}}{r_n - \rho m_n}(\|A_n(x) - A(x)\| + \rho\|u_n - u\|).
\end{aligned}$$

In view of the assumptions, the right side of the above inequality approaches zero as $n \to \infty$ hence we conclude that $R^{\eta_n, M_n}_{\rho, A_n}(z) \to R^{\eta, M}_{\rho, A}(z)$ as $n \to \infty$.

Conversely, assume that for any given constant $\rho > 0$, $R^{\eta_n, M_n}_{\rho, A_n}(z) \to R^{\eta, M}_{\rho, A}(z)$ as $n \to \infty$ for all $z \in X$. Then for any given $u \in M(x)$, we have $x = R^{\eta, M}_{\rho, A}(A(x) + \rho u)$ and hence $R^{\eta_n, M_n}_{\rho, A_n}(A(x) + \rho u) \to x$. Now, taking $x_n = R^{\eta_n, M_n}_{\rho, A_n}(A(x) + \rho u)$, we deduce that $A(x) + \rho u \in (A_n + \rho M_n)(x_n)$ and thus there exists $u_n \in M_n(x_n)$ such that $A(x) + \rho u = A_n(x_n) + \rho u_n$. Then one has

$$\begin{aligned}\rho\|u_n - u\| &= \|A_n(x_n) - A(x)\| \le \|A_n(x_n) - A_n(x)\| + \|A_n(x) - A(x)\| \\ &\le \beta_n\|x_n - x\| + \|A_n(x) - A(x)\|.\end{aligned}$$

Now, $x_n \to x$ and $A_n(x) \to A(x)$ as $n \to \infty$, guarantee that $u_n \to u$ as $n \to \infty$. This completes the proof.

Let $\mathcal{S}_1 : X_1 \to X_1$ be a nearly uniformly L_1-Lipschitzian mapping with the sequence $\{a_n\}_{n=1}^{\infty}$ and $\mathcal{S}_2 : X_2 \to X_2$ be a nearly uniformly L_2-Lipschitzian mapping with the sequence $\{b_n\}_{n=1}^{\infty}$. We define the self-mapping $\mathcal{Q}$ of $X_1 \times X_2$ as follows:

$$\mathcal{Q}(x,y) = (\mathcal{S}_1 x, \mathcal{S}_2 y), \quad \forall (x,y) \in X_1 \times X_2. \tag{4.2}$$

Then $\mathcal{Q} = (\mathcal{S}_1, \mathcal{S}_2) : X_1 \times X_2 \to X_1 \times X_2$ is a nearly uniformly $\max\{L_1, L_2\}$-Lipschitzian mapping with the sequence $\{a_n + b_n\}_{n=1}^{\infty}$ with respect to the norm $\|\cdot\|_*$ in $X_1 \times X_2$. Because, for any $(x,y), (x',y') \in X_1 \times X_2$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}&\|\mathcal{Q}^n(x,y) - \mathcal{Q}^n(x',y')\|_* \\ &= \|(\mathcal{S}_1^n x, \mathcal{S}_2^n y) - (\mathcal{S}_1^n x', \mathcal{S}_2^n y')\|_* \\ &= \|(\mathcal{S}_1^n x - \mathcal{S}_1^n x', \mathcal{S}_2^n y - \mathcal{S}_2^n y')\|_* \\ &= \|\mathcal{S}_1^n x - \mathcal{S}_1^n x'\| + \|\mathcal{S}_2^n y - \mathcal{S}_2^n y'\| \\ &\le L_1(\|x - x'\| + a_n) + L_2(\|y - y'\| + b_n) \\ &\le \max\{L_1, L_2\}(\|x - x'\| + \|y - y'\| + a_n + b_n) \\ &= \max\{L_1, L_2\}(\|(x,y) - (x',y')\|_* + a_n + b_n).\end{aligned}$$

We denote the sets of all the fixed points of $\mathcal{S}_i$ $(i = 1,2)$ and $\mathcal{Q}$ by $\mathrm{Fix}(\mathcal{S}_i)$ and $\mathrm{Fix}(\mathcal{Q})$, respectively. Evidently, for any $(x,y) \in X_1 \times X_2$, $(x,y) \in \mathrm{Fix}(\mathcal{Q})$ if and only if $x \in \mathrm{Fix}(\mathcal{S}_1)$ and $y \in \mathrm{Fix}(\mathcal{S}_2)$, that is, $\mathrm{Fix}(\mathcal{Q}) = \mathrm{Fix}(\mathcal{S}_1, \mathcal{S}_2) = \mathrm{Fix}(\mathcal{S}_1) \times \mathrm{Fix}(\mathcal{S}_2)$. We now characterize the problem. If $(x^*, y^*) \in \mathrm{Fix}(\mathcal{Q}) \cap \mathrm{SGNVLI}$, then $x^* \in \mathrm{Fix}(\mathcal{S}_1)$, $y^* \in \mathrm{Fix}(\mathcal{S}_2)$ and $(x^*, y^*) \in$ SGNVLI. Accordingly, it follows from Lemma 3.1 that for all $n \in \mathbb{N}$,

$$\begin{cases} x^* = \mathcal{S}_1^n x^* = x^* - (f_1 - g_1)(x^*) + R^{\eta_1, M_1(\cdot, y^*)}_{\rho_1\lambda_1, A_1}(\Omega(x^*, y^*)) \\ \quad\; = \mathcal{S}_1^n[x^* - (f_1 - g_1)(x^*) + R^{\eta_1, M_1(\cdot, y^*)}_{\rho_1\lambda_1, A_1}(\Omega(x^*, y^*))], \\ y^* = \mathcal{S}_2^n y^* = y^* - (f_2 - g_2)(y^*) + R^{\eta_2, M_2(\cdot, x^*)}_{\rho_2\lambda_2, A_2}(\Theta(x^*, y^*)) \\ \quad\; = \mathcal{S}_2^n[y^* - (f_2 - g_2)(y^*) + R^{\eta_2, M_2(\cdot, x^*)}_{\rho_2\lambda_2, A_2}(\Theta(x^*, y^*))], \end{cases} \tag{4.3}$$

where

$$\begin{cases} \Omega(x^*, y^*) = A_1(f_1 - g_1)(x^*) - \rho_1(N_1(S_1(x^*), T_1(y^*), P_1(x^*), Q_1(y^*)) \\ \qquad\qquad\quad - h_1(y^*, x^*) - \theta_1(x^*) - a), \\ \Theta(x^*, y^*) = A_2(f_2 - g_2)(y^*) - \rho_2(N_2(S_2(y^*), T_2(x^*), P_2(y^*), Q_2(x^*)) \\ \qquad\qquad\quad - h_2(x^*, y^*) - \theta_2(y^*) - b). \end{cases} \tag{4.4}$$

The fixed point formulation (4.3) enables us to suggest the following perturbed N-step iterative algorithm with mixed errors for finding an element of the set of the fixed points of the nearly uniformly Lipschitzian mapping $\mathcal{Q} = (\mathcal{S}_1, \mathcal{S}_2)$ which is the unique solution of the system (3.1).

Algorithm 4.1 For $i = 1, 2$, let X_i, S_i, T_i, P_i, Q_i, M_i, N_i, η_i, λ_i, A_i, f_i, g_i, h_i, θ_i, a and b be the same as in the system (3.1), and for $i = 1, 2$ and for all $n \in \mathbb{N}$, let $\eta_{n,i} : X_i \times X_i \to X_i$, $A_{n,i} : X_i \to X_i$ be single-valued mappings. Assume that for $i \in \{1, 2\}$, $j \in \{1, 2\} \setminus \{i\}$ and for all $n \in \mathbb{N}$, $M_{n,i} : X_i \times X_j \multimap X_i$ are any nonlinear operators such that for all $x_j \in X_j$ and $n \in \mathbb{N}$, $M_{n,i}(\cdot, x_j) : X_i \multimap X_i$ is an $A_{n,i}$-maximal $m_{n,i}$-relaxed $\eta_{n,i}$-accretive mapping with $\mathrm{Range}(f_i - g_i) \cap \mathrm{dom}\, M_{n,i}(\cdot, x_j) \neq \varnothing$. Further, for $i = 1, 2$, let $\mathcal{S}_i : X_i \to X_i$ be a nearly uniformly Lipschitzian mapping. For any given $(x_1, y_1) \in X_1 \times X_2$, define the iterative sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ in the following way:

$$\begin{cases} x_{n+1} = (1 - \alpha_{n,1} - \beta_{n,1})x_n + \alpha_{n,1}\mathcal{S}_1^n[z_{n,1} - (f_1 - g_1)(z_{n,1}) \\ \qquad + R_{\rho_1\lambda_1, A_{n,1}}^{\eta_{n,1}, M_{n,1}(\cdot, t_{n,1})}(\Omega(z_{n,1}, t_{n,1}))] + \alpha_{n,1}e_{n,1} + \beta_{n,1}j_{n,1} + r_{n,1}, \\ y_{n+1} = (1 - \alpha_{n,1} - \beta_{n,1})y_n + \alpha_{n,1}\mathcal{S}_2^n[t_{n,1} - (f_2 - g_2)(t_{n,1}) \\ \qquad + R_{\rho_2\lambda_2, A_{n,2}}^{\eta_{n,2}, M_{n,2}(\cdot, z_{n,1})}(\Theta(z_{n,1}, t_{n,1}))] + \alpha_{n,1}p_{n,1} + \beta_{n,1}s_{n,1} + k_{n,1}, \\ z_{n,i} = (1 - \alpha_{n,i+1} - \beta_{n,i+1})x_n + \alpha_{n,i+1}\mathcal{S}_1^n[z_{n,i+1} - (f_1 - g_1)(z_{n,i+1}) \\ \qquad + R_{\rho_1\lambda_1, A_{n,1}}^{\eta_{n,1}, M_{n,1}(\cdot, t_{n,i+1})}(\Omega(z_{n,i+1}, t_{n,i+1}))] \\ \qquad + \alpha_{n,i+1}e_{n,i+1} + \beta_{n,i+1}j_{n,i+1} + r_{n,i+1}, \\ t_{n,i} = (1 - \alpha_{n,i+1} - \beta_{n,i+1})y_n + \alpha_{n,i+1}\mathcal{S}_2^n[t_{n,i+1} - (f_2 - g_2)(t_{n,i+1}) \\ \qquad + R_{\rho_2\lambda_2, A_{n,2}}^{\eta_{n,2}, M_{n,2}(\cdot, z_{n,i+1})}(\Theta(z_{n,i+1}, t_{n,i+1}))] \\ \qquad + \alpha_{n,i+1}p_{n,i+1} + \beta_{n,i+1}s_{n,i+1} + k_{n,i+1}, \\ \qquad\qquad \vdots \\ z_{n,N-1} = (1 - \alpha_{n,N} - \beta_{n,N})x_n + \alpha_{n,N}\mathcal{S}_1^n[x_n - (f_1 - g_1)(x_n) \\ \qquad + R_{\rho_1\lambda_1, A_{n,1}}^{\eta_{n,1}, M_{n,1}(\cdot, y_n)}(\Omega(x_n, y_n))] + \alpha_{n,N}e_{n,N} + \beta_{n,N}j_{n,N} + r_{n,N}, \\ t_{n,N-1} = (1 - \alpha_{n,N} - \beta_{n,N})y_n + \alpha_{n,N}\mathcal{S}_2^n[y_n - (f_2 - g_2)(y_n) \\ \qquad + R_{\rho_2\lambda_2, A_{n,2}}^{\eta_{n,2}, M_{n,2}(\cdot, x_n)}(\Theta(x_n, y_n))] + \alpha_{n,N}p_{n,N} + \beta_{n,N}s_{n,N} + k_{n,N}, \\ i = 1, 2, \cdots, N - 2, \end{cases} \tag{4.5}$$

where, for all $n \in \mathbb{N}$ and $i = 1, 2, \cdots, N - 1$,

$$\begin{aligned} \Omega(z_{n,i}, t_{n,i}) &= A_1(f_1 - g_1)(z_{n,i}) - \rho_1(N_1(S_1(z_{n,i}), T_1(t_{n,i}), P_1(z_{n,i}), Q_1(t_{n,i})) \\ &\quad - h_1(t_{n,i}, z_{n,i}) - \theta_1(z_{n,i}) - a), \\ \Theta(z_{n,i}, t_{n,i}) &= A_2(f_2 - g_2)(t_{n,i}) - \rho_2(N_2(S_2(t_{n,i}), T_2(z_{n,i}), P_2(t_{n,i}), Q_2(z_{n,i})) \\ &\quad - h_2(z_{n,i}, t_{n,i}) - \theta_2(t_{n,i}) - b), \\ \Omega(x_n, y_n) &= A_1(f_1 - g_1)(x_n) - \rho_1(N_1(S_1(x_n), T_1(y_n), P_1(x_n), Q_1(y_n)) \\ &\quad - h_1(y_n, x_n) - \theta_1(x_n) - a), \\ \Theta(x_n, y_n) &= A_2(f_2 - g_2)(y_n) - \rho_2(N_2(S_2(y_n), T_2(x_n), P_2(y_n), Q_2(x_n)) \\ &\quad - h_2(x_n, y_n) - \theta_2(y_n) - b), \end{aligned}$$

$\rho_1, \rho_2 > 0$ are two constants, $\{\alpha_{n,i}\}_{n=1}^{\infty}$, $\{\beta_{n,i}\}_{n=1}^{\infty}$ $(i = 1, 2, \cdots, N)$, are $2N$ sequences in $[0, 1]$ such that for all $n \in \mathbb{N}$ and $i = 1, 2, \cdots, N$, $\sum\limits_{n=1}^{\infty}\prod\limits_{i=1}^{N}\alpha_{n,i} = \infty$, $\sum\limits_{n=1}^{\infty}\beta_{n,i} < \infty$, $\alpha_{n,i} + \beta_{n,i} \leq 1$, and $\{e_{n,i}\}_{n=1}^{\infty}$, $\{p_{n,i}\}_{n=1}^{\infty}$, $\{j_{n,i}\}_{n=1}^{\infty}$, $\{s_{n,i}\}_{n=1}^{\infty}$, $\{r_{n,i}\}_{n=1}^{\infty}$, $\{k_{n,i}\}_{n=1}^{\infty}$ $(i = 1, 2, \cdots, N)$ are $6N$ sequences to take into account a possible inexact computation of the resolvent operator point satisfying the following conditions: For $i = 1, 2, \cdots, N$, $\{j_{n,i}\}_{n=1}^{\infty}$ are N bounded sequences in X_1, $\{s_{n,i}\}_{n=1}^{\infty}$ are N bounded sequences in X_2 and $\{(e_{n,i}, p_{n,i})\}_{n=1}^{\infty}$, $\{(r_{n,i}, k_{n,i})\}_{n=1}^{\infty}$ are $2N$

sequences in $X_1 \times X_2$ such that for all $n \in \mathbb{N}$ and $i = 1, 2, \cdots, N$,

$$\begin{cases} e_{n,i} = e'_{n,i} + e''_{n,i}, \quad p_{n,i} = p'_{n,i} + p''_{n,i}, \\ \lim_{n\to\infty} \|(e'_{n,i}, p'_{n,i})\|_* = 0, \\ \sum_{n=1}^{\infty} \|(e''_{n,i}, p''_{n,i})\|_* < \infty, \quad \sum_{n=1}^{\infty} \|(r_{n,i}, k_{n,i})\|_* < \infty. \end{cases} \tag{4.6}$$

Let $\{(u_n, v_n)\}_{n=1}^{\infty}$ be any sequence in $X_1 \times X_2$ and define $\{\epsilon_n\}_{n=1}^{\infty}$ by

$$\begin{cases} \epsilon_n = \|(u_{n+1}, v_{n+1}) - (E_n, D_n)\|_*, \\ E_n = (1-\alpha_{n,1}-\beta_{n,1})u_n + \alpha_{n,1}\mathcal{S}_1^n[\nu_{n,1} - (f_1-g_1)(\nu_{n,1}) \\ \qquad + R_{\rho_1\lambda_1,A_{n,1}}^{\eta_{n,1},M_{n,1}(\cdot,w_{n,1})}(\Omega(\nu_{n,1},w_{n,1}))] + \alpha_{n,1}e_{n,1} + \beta_{n,1}j_{n,1} + r_{n,1}, \\ D_n = (1-\alpha_{n,1}-\beta_{n,1})v_n + \alpha_{n,1}\mathcal{S}_2^n[w_{n,1} - (f_2-g_2)(w_{n,1}) \\ \qquad + R_{\rho_2\lambda_2,A_{n,2}}^{\eta_{n,2},M_{n,2}(\cdot,\nu_{n,1})}(\Theta(\nu_{n,1},w_{n,1}))] + \alpha_{n,1}p_{n,1} + \beta_{n,1}s_{n,1} + k_{n,1}, \\ \nu_{n,1} = (1-\alpha_{n,2}-\beta_{n,2})u_n + \alpha_{n,2}\mathcal{S}_1^n[\nu_{n,2} - (f_1-g_1)(\nu_{n,2}) \\ \qquad + R_{\rho_1\lambda_1,A_{n,1}}^{\eta_{n,1},M_{n,1}(\cdot,w_{n,2})}(\Omega(\nu_{n,2},w_{n,2}))] + \alpha_{n,2}e_{n,2} + \beta_{n,2}j_{n,2} + r_{n,2}, \\ w_{n,1} = (1-\alpha_{n,2}-\beta_{n,2})v_n + \alpha_{n,2}\mathcal{S}_2^n[w_{n,2} - (f_2-g_2)(w_{n,2}) \\ \qquad + R_{\rho_2\lambda_2,A_{n,2}}^{\eta_{n,2},M_{n,2}(\cdot,\nu_{n,2})}(\Theta(\nu_{n,2},w_{n,2}))] + \alpha_{n,2}p_{n,2} + \beta_{n,2}s_{n,2} + k_{n,2}, \\ \qquad \vdots \\ \nu_{n,N-2} = (1-\alpha_{n,N-1}-\beta_{n,N-1})u_n + \alpha_{n,N-1}\mathcal{S}_1^n[\nu_{n,N-1} - (f_1-g_1)(\nu_{n,N-1}) \\ \qquad + R_{\rho_1\lambda_1,A_{n,1}}^{\eta_{n,1},M_{n,1}(\cdot,w_{n,N-1})}(\Omega(\nu_{n,N-1},w_{n,N-1}))] \\ \qquad + \alpha_{n,N-1}e_{n,N-1} + \beta_{n,N-1}j_{n,N-1} + r_{n,N-1}, \\ w_{n,N-2} = (1-\alpha_{n,N-1}-\beta_{n,N-1})v_n + \alpha_{n,N-1}\mathcal{S}_2^n[w_{n,N-1} - (f_2-g_2)(w_{n,N-1}) \\ \qquad + R_{\rho_2\lambda_2,A_{n,2}}^{\eta_{n,2},M_{n,2}(\cdot,\nu_{n,N-1})}(\Theta(\nu_{n,N-1},w_{n,N-1}))] \\ \qquad + \alpha_{n,N-1}p_{n,N-1} + \beta_{n,N-1}s_{n,N-1} + k_{n,N-1}, \\ \nu_{n,N-1} = (1-\alpha_{n,N}-\beta_{n,N})u_n + \alpha_{n,N}\mathcal{S}_1^n[u_n - (f_1-g_1)(u_n) \\ \qquad + R_{\rho_1\lambda_1,A_{n,1}}^{\eta_{n,1},M_{n,1}(\cdot,v_n)}(\Omega(u_n,v_n))] + \alpha_{n,N}e_{n,N} + \beta_{n,N}j_{n,N} + r_{n,N}, \\ w_{n,N-1} = (1-\alpha_{n,N}-\beta_{n,N})v_n + \alpha_{n,N}\mathcal{S}_2^n[v_n - (f_2-g_2)(v_n) \\ \qquad + R_{\rho_2\lambda_2,A_{n,2}}^{\eta_{n,2},M_{n,2}(\cdot,u_n)}(\Theta(u_n,v_n))] + \alpha_{n,N}p_{n,N} + \beta_{n,N}s_{n,N} + k_{n,N}, \end{cases} \tag{4.7}$$

where for all $n \in \mathbb{N}$ and $i = 1, 2, \cdots, N-1$,

$$\begin{aligned} \Omega(\nu_{n,i}, w_{n,i}) &= A_1(f_1-g_1)(\nu_{n,i}) - \rho_1(N_1(S_1(\nu_{n,i}), T_1(w_{n,i}), P_1(\nu_{n,i}), Q_1(w_{n,i})) \\ &\quad - h_1(w_{n,i}, \nu_{n,i}) - \theta_1(\nu_{n,i}) - a), \\ \Theta(\nu_{n,i}, w_{n,i}) &= A_2(f_2-g_2)(w_{n,i}) - \rho_2(N_2(S_2(w_{n,i}), T_2(\nu_{n,i}), P_2(w_{n,i}), Q_2(\nu_{n,i})) \\ &\quad - h_2(\nu_{n,i}, w_{n,i}) - \theta_2(w_{n,i}) - b), \\ \Omega(u_n, v_n) &= A_1(f_1-g_1)(u_n) - \rho_1(N_1(S_1(u_n), T_1(v_n), P_1(u_n), Q_1(v_n)) \\ &\quad - h_1(v_n, u_n) - \theta_1(u_n) - a), \\ \Theta(u_n, v_n) &= A_2(f_2-g_2)(v_n) - \rho_2(N_2(S_2(v_n), T_2(u_n), P_2(v_n), Q_2(u_n)) \\ &\quad - h_2(u_n, v_n) - \theta_2(v_n) - b). \end{aligned}$$

Definition 4.2 *For $i = 1, 2$, let X_i be a real Banach space and T be a self-mapping of $X_1 \times X_2$. Suppose that $(x_1, y_1) \in X_1 \times X_2$ and $(x_{n+1}, y_{n+1}) = f(T, x_n, y_n)$ defines an iterative* *procedure which yields a sequence of points* $\{(x_n, y_n)\}_{n=1}^{\infty}$ *in* $X_1 \times X_2$. *Assume that*

$$\mathrm{Fix}(T) = \{(x, y) \in X_1 \times X_2 : (x, y) = T(x, y)\} \neq \varnothing$$

and $\{(x_n, y_n)\}_{n=1}^{\infty}$ *converges to* $(x^*, y^*) \in \mathrm{Fix}(T)$. *Further, let* $\{(z_n, w_n)\}_{n=1}^{\infty}$ *be an arbitrary sequence in* $X_1 \times X_2$ *and* $\epsilon_n = \|(z_{n+1}, w_{n+1}) - f(T, z_n, w_n)\|$ *for each* $n \in \mathbb{N}$. *If* $\lim\limits_{n\to\infty} \epsilon_n = 0$ *implies that* $\lim\limits_{n\to\infty} (z_n, w_n) = (x^*, y^*)$, *then the iteration procedure defined by* $(x_{n+1}, y_{n+1}) = f(T, x_n, y_n)$ *is said to be* T*-stable or stable with respect to* T.

Remark 4.1 Some stability results of the iteration procedures for variational inequalities (inclusions) have been established by various authors, see for example [1, 3, 25, 27, 31, 46].

5 Convergence and Stability of the Resolvent Iterative Algorithm

In this section, we establish the convergence and stability of the iterative sequence generated by the suggested perturbed N-step iterative algorithm under some suitable conditions. For this purpose, we need the following lemma.

Lemma 5.1 *Let* $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *be three nonnegative real sequences satisfying the following condition: there exists a natural number* n_0 *such that*

$$a_{n+1} \leq (1 - t_n)a_n + b_n t_n + c_n, \quad \forall n \geq n_0,$$

where $t_n \in [0, 1]$, $\sum\limits_{n=1}^{\infty} t_n = \infty$, $\lim\limits_{n\to\infty} b_n = 0$, $\sum\limits_{n=1}^{\infty} c_n < \infty$. *Then* $\lim\limits_{n\to 0} a_n = 0$.

Proof The proof directly follows from Lemma 2 in [41].

Theorem 5.1 *For* $i = 1, 2$ *and for all* $n \in \mathbb{N}$, *suppose that* X_i, S_i, T_i, P_i, Q_i, M_i, N_i, η_i, λ_i, A_i, f_i, g_i, h_i, θ_i, $A_{n,i}$, $\eta_{n,i}$, $M_{n,i}$, a *and* b *are the same as in* Algorithm 4.1 *and let all the conditions of* Theorem 3.1 *hold. Assume that* $\mathcal{S}_1 : X_1 \to X_1$ *is a nearly uniformly* L_1*-Lipschitzian mapping with the sequence* $\{\widetilde{c}_n\}_{n=1}^{\infty}$, $\mathcal{S}_2 : X_2 \to X_2$ *is a nearly uniformly* L_2*-Lipschitzian mapping with the sequence* $\{\widetilde{d}_n\}_{n=1}^{\infty}$, *and* $\mathcal{Q}$ *is a self-mapping of* $X_1 \times X_2$ *defined by* (4.2) *such that* $\mathrm{Fix}(\mathcal{Q}) \cap \mathrm{SGNVLI} \neq \varnothing$. *Further, for all* $n \in \mathbb{N}$ *and* $i = 1, 2$, *let*

(a) $\eta_{n,i} : X_i \times X_i \to X_i$ *be* $\tau_{n,i}$*-Lipschitz continuous;*

(b) $A_{n,i} : X_i \to X_i$ *be* $r_{n,i}$*-strongly* $\eta_{n,i}$*-accretive and* $\gamma_{n,i}$*-Lipschitz continuous;*

(c) $\lim\limits_{n\to\infty} A_{n,i}(x) = A_i(x)$ *and* $M_{n,i}(\cdot, x_j) \xrightarrow{G} M_i(\cdot, x_j)$, *for* $i \in \{1, 2\}$, $j \in \{1, 2\} \setminus \{i\}$ *and for any* $x_j \in X_j$;

(d) *there exist constants* ρ_i *such that for all* $n \in \mathbb{N}$ *and* $i = 1, 2$, $\rho_i \in (0, \frac{r_{n,i}}{\lambda_i m_{n,i}})$ *and*

$$\|R^{\eta_{n,i}, M_{n,i}(\cdot,x)}_{\rho_i\lambda_i, A_{n,i}}(z) - R^{\eta_{n,i}, M_{n,i}(\cdot,y)}_{\rho_i\lambda_i, A_{n,i}}(z)\| \leq \mu_{n,i}\|x - y\|, \quad \forall z \in X_i,\ x, y \in X_j;$$

(e) *the sequences* $\left(\frac{\tau_{n,i}^{q_i-1}}{r_{n,i}-\rho_i\lambda_i m_{n,i}}\right)_{n=1}^{\infty}$ *and* $\left(\frac{\gamma_{n,i}\tau_{n,i}^{q-1}}{r_{n,i}-\rho_i\lambda_i m_{n,i}}\right)_{n=1}^{\infty}$ *be bounded;*

(f) $\tau_{n,i} \to \tau_i$, $r_{n,i} \to r_i$, $\gamma_{n,i} \to \gamma_i$, $m_{n,i} \to m_i$, $\mu_{n,i} \to \mu_i$, *as* $n \to \infty$;

(g) *there exist a constant* $\alpha > 0$ *such that* $\prod\limits_{i=1}^{N} \alpha_{n,i} \geq \alpha$ *for all* $n \in \mathbb{N}$;

(h) $L_i\widetilde{\theta} < 1$ *for any* $\widetilde{\theta} \in (\vartheta, 1)$ *and* $i = 1, 2$, *where* ϑ *is the same as in* (3.15).

Then

(1) *The iterative sequence* $\{(x_n, y_n)\}_{n=1}^{\infty}$ *generated by Algorithm* 4.1, *converges strongly to the only element* (x^*, y^*) *of* $\mathrm{Fix}(\mathcal{Q}) \cap \mathrm{SGNVLI}$.

(2) *Moreover,* $\lim_{n\to\infty}(u_n, v_n) = (x^*, y^*)$ *if and only if* $\lim_{n\to\infty} \epsilon_n = 0$, *where* $\{(u_n, v_n)\}_{n=1}^{\infty}$ *is any sequence in* $X_1 \times X_2$ *defined by* (4.7).

Proof According to Theorem 3.1, the system (3.1) has a unique solution (x^*, y^*) in $X_1 \times X_2$. Since SGNVLI is a singleton set and $\mathrm{Fix}(\mathcal{Q}) \cap \mathrm{SGNVLI} \neq \varnothing$, it follows that $(x^*, y^*) \in \mathrm{Fix}(\mathcal{Q})$ and so $x^* \in \mathrm{Fix}(\mathcal{S}_1)$ and $y^* \in \mathrm{Fix}(\mathcal{S}_2)$. Hence, in view of Lemma 3.1, for all $n \in \mathbb{N}$, we can write

$$\begin{cases} x^* = \mathcal{S}_1^n x^* = \mathcal{S}_1^n[x^* - (f_1 - g_1)(x^*) + R_{\rho_1\lambda_1,A_1}^{\eta_1,M_1(\cdot,y^*)}(\Omega(x^*, y^*))], \\ y^* = \mathcal{S}_2^n y^* = \mathcal{S}_2^n[y^* - (f_2 - g_2)(y^*) + R_{\rho_2\lambda_2,A_2}^{\eta_2,M_2(\cdot,x^*)}(\Theta(x^*, y^*))], \end{cases} \tag{5.1}$$

where $\Omega(x^*, y^*)$ and $\Theta(x^*, y^*)$ are the same as in (4.4). Letting

$$K = \max\Big\{\sup_{n\geq 1}\|j_{n,i} - x^*\|,\ \sup_{n\geq 1}\|s_{n,i} - y^*\|,\ i = 1, 2, \cdots, N\Big\}$$

and applying (4.5), (5.1), Proposition 2.1 and the assumptions, it follows that

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
\leq{}& (1 - \alpha_{n,1} - \beta_{n,1})\|x_n - x^*\| + \alpha_{n,1}\|\mathcal{S}_1^n[z_{n,1} - (f_1 - g_1)(z_{n,1}) \\
&+ R_{\rho_1\lambda_1,A_{n,1}}^{\eta_{n,1},M_{n,1}(\cdot,t_{n,1})}(\Omega(z_{n,1}, t_{n,1}))] - \mathcal{S}_1^n[x^* - (f_1 - g_1)(x^*) \\
&+ R_{\rho_1\lambda_1,A_1}^{\eta_1,M_1(\cdot,y^*)}(\Omega(x^*, y^*))]\| + \beta_{n,1}\|j_{n,1} - x^*\| + \alpha_{n,1}\|e_{n,1}\| + \|r_{n,1}\| \\
\leq{}& (1 - \alpha_{n,1} - \beta_{n,1})\|x_n - x^*\| + \alpha_{n,1}L_1(\|z_{n,1} - x^* - [(f_1 - g_1)(z_{n,1}) - (f_1 - g_1)(x^*)]\| \\
&+ \|R_{\rho_1\lambda_1,A_{n,1}}^{\eta_{n,1},M_{n,1}(\cdot,t_{n,1})}[A_1(f_1 - g_1)(z_{n,1}) - \rho_1(N_1(S_1(z_{n,1}), T_1(t_{n,1}), P_1(z_{n,1}), Q_1(t_{n,1})) \\
&- h_1(t_{n,1}, z_{n,1}) - \theta_1(z_{n,1}) - a)] - R_{\rho_1\lambda_1,A_1}^{\eta_1,M_1(\cdot,y^*)}[A_1(f_1 - g_1)(x^*) \\
&- \rho_1(N_1(S_1(x^*), T_1(y^*), P_1(x^*), Q_1(y^*)) - h_1(y^*, x^*) - \theta_1(x^*) - a)]\| + \widetilde{c}_n) \\
&+ \alpha_{n,1}(\|e'_{n,1}\| + \|e''_{n,1}\|) + \|r_{n,1}\| + \beta_{n,1}K \\
\leq{}& (1 - \alpha_{n,1} - \beta_{n,1})\|x_n - x^*\| + \alpha_{n,1}L_1(\|z_{n,1} - x^* - [(f_1 - g_1)(z_{n,1}) - (f_1 - g_1)(x^*)]\| \\
&+ \|R_{\rho_1\lambda_1,A_{n,1}}^{\eta_{n,1},M_{n,1}(\cdot,t_{n,1})}[A_1(f_1 - g_1)(z_{n,1}) - \rho_1(N_1(S_1(z_{n,1}), T_1(t_{n,1}), P_1(z_{n,1}), Q_1(t_{n,1})) \\
&- h_1(t_{n,1}, z_{n,1}) - \theta_1(z_{n,1}) - a)] - R_{\rho_1\lambda_1,A_{n,1}}^{\eta_{n,1},M_{n,1}(\cdot,t_{n,1})}[A_1(f_1 - g_1)(x^*) \\
&- \rho_1(N_1(S_1(x^*), T_1(y^*), P_1(x^*), Q_1(y^*)) - h_1(y^*, x^*) - \theta_1(x^*) - a)]\| \\
&+ \|R_{\rho_1\lambda_1,A_{n,1}}^{\eta_{n,1},M_{n,1}(\cdot,t_{n,1})}[A_1(f_1 - g_1)(x^*) - \rho_1(N_1(S_1(x^*), T_1(y^*), P_1(x^*), Q_1(y^*)) \\
&- h_1(y^*, x^*) - \theta_1(x^*) - a)] - R_{\rho_1\lambda_1,A_{n,1}}^{\eta_{n,1},M_{n,1}(\cdot,y^*)}[A_1(f_1 - g_1)(x^*) \\
&- \rho_1(N_1(S_1(x^*), T_1(y^*), P_1(x^*), Q_1(y^*)) - h_1(y^*, x^*) - \theta_1(x^*) - a)]\| \\
&+ \|R_{\rho_1\lambda_1,A_{n,1}}^{\eta_{n,1},M_{n,1}(\cdot,y^*)}[A_1(f_1 - g_1)(x^*) - \rho_1(N_1(S_1(x^*), T_1(y^*), P_1(x^*), Q_1(y^*)) \\
&- h_1(y^*, x^*) - \theta_1(x^*) - a)] - R_{\rho_1\lambda_1,A_1}^{\eta_1,M_1(\cdot,y^*)}[A_1(f_1 - g_1)(x^*) \\
&- \rho_1(N_1(S_1(x^*), T_1(y^*), P_1(x^*), Q_1(y^*)) - h_1(y^*, x^*) - \theta_1(x^*) - a)]\| + \widetilde{c}_n) \\
&+ \alpha_{n,1}\|e'_{n,1}\| + \|e''_{n,1}\| + \|r_{n,1}\| + \beta_{n,1}K
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\alpha_{n,1}-\beta_{n,1})\|x_n-x^*\| + \alpha_{n,1}L_1(\|z_{n,1}-x^*-[(f_1-g_1)(z_{n,1})-(f_1-g_1)(x^*)]\| \\
&\quad + \mu_{n,1}\|t_{n,1}-y^*\| + \frac{\tau_{n,1}^{q_1-1}}{r_{n,1}-\rho_1\lambda_1 m_{n,1}}\{\rho_1[\|N_1(S_1(z_{n,1}),T_1(t_{n,1}),P_1(z_{n,1}),Q_1(t_{n,1})) \\
&\quad - N_1(S_1(z_{n,1}),T_1(t_{n,1}),P_1(z_{n,1}),Q_1(y^*))\| + \|N_1(S_1(z_{n,1}),T_1(t_{n,1}),P_1(z_{n,1}),Q_1(y^*)) \\
&\quad - N_1(S_1(z_{n,1}),T_1(t_{n,1}),P_1(x^*),Q_1(y^*))\| + \|N_1(S_1(z_{n,1}),T_1(t_{n,1}),P_1(x^*),Q_1(y^*)) \\
&\quad - N_1(S_1(z_{n,1}),T_1(y^*),P_1(x^*),Q_1(y^*))\| + \|h_1(t_{n,1},z_{n,1})-h_1(y^*,x^*)\| \\
&\quad + \|\theta_1(z_{n,1})-\theta_1(x^*)\|] + \|A_1(f_1-g_1)(z_{n,1})-A_1(f_1-g_1)(x^*) \\
&\quad - \rho_1(N_1(S_1(z_{n,1}),T_1(y^*),P_1(x^*),Q_1(y^*))-N_1(S_1(x^*),T_1(y^*),P_1(x^*),Q_1(y^*)))\|\} \\
&\quad + \|\Upsilon(n)\| + \widetilde{c}_n) + \alpha_{n,1}\|e'_{n,1}\| + \|e''_{n,1}\| + \|r_{n,1}\| + \beta_{n,1}K \\
&\leq (1-\alpha_{n,1}-\beta_{n,1})\|x_n-x^*\| + \alpha_{n,1}L_1(\varphi_1(n)\|z_{n,1}-x^*\| + \phi_1(n)\|t_{n,1}-y^*\|) \\
&\quad + \alpha_{n,1}(L_1(\|\Upsilon(n)\|+\widetilde{c}_n) + \|e'_{n,1}\|) + \|e''_{n,1}\| + \|r_{n,1}\| + \beta_{n,1}K, \qquad (5.2)
\end{aligned}
$$

where

$$
\begin{aligned}
\varphi_1(n) &= \sqrt[q_1]{1-q_1\varrho_1+2^{q_1}(c_{q_1}+q_1\kappa_1)(\omega_1^{q_1}+\pi_1^{q_1})} + \frac{\tau_{n,1}^{q_1-1}(\rho_1(\iota'_1\varsigma_1+o_1+\iota_1)+\Gamma_1(n))}{r_{n,1}-\rho_1\lambda_1 m_{n,1}}, \\
\Gamma_1(n) &= \sqrt[q_1]{2^{q_1}\gamma_{n,1}^{q_1}(\omega_1^{q_1}+\pi_1^{q_1})-q_1\rho_1(-\sigma_1\epsilon_1^{q_1}\xi_1^{q_1}+\varpi_1)+c_{q_1}(\rho_1\epsilon_1\xi_1)^{q_1}}, \\
\phi_1(n) &= \mu_{n,1} + \frac{\rho_1\tau_{n,1}^{q_1-1}(\nu_1\delta_1+\upsilon_1\zeta_1+\varepsilon_1)}{r_{n,1}-\rho_1\lambda_1 m_{n,1}}, \\
\Upsilon(n) &= R_{\rho_1\lambda_1,A_{n,1}}^{\eta_{n,1},M_{n,1}(\cdot,y^*)}(\Omega(x^*,y^*)) - R_{\rho_1\lambda_1,A_1}^{\eta_1,M_1(\cdot,y^*)}(\Omega(x^*,y^*)).
\end{aligned}
$$

Similarly, by (4.5), Proposition 2.1 and the assumptions, we get

$$
\begin{aligned}
\|y_{n+1}-y^*\| \leq (1-\alpha_{n,1}-\beta_{n,1})\|y_n-y^*\| &+ \alpha_{n,1}L_2(\varphi_2(n)\|z_{n,1}-x^*\| \\
&+ \phi_2(n)\|t_{n,1}-y^*\|) + \alpha_{n,1}(L_2(\|\Delta(n)\|+\widetilde{d}_n)+\|p'_{n,1}\|) \\
&+ \|p''_{n,1}\| + \|k_{n,1}\| + \beta_{n,1}K, \qquad (5.3)
\end{aligned}
$$

where

$$
\begin{aligned}
\phi_2(n) &= \sqrt[q_2]{1-q_2\varrho_2+2^{q_2}(c_{q_2}+q_2\kappa_2)(\omega_2^{q_2}+\pi_2^{q_2})} + \frac{\tau_{n,2}^{q_2-1}(\rho_2(\iota'_2\varsigma_2+o_2+\iota_2)+\Gamma_2(n))}{r_{n,2}-\rho_2\lambda_2 m_{n,2}}, \\
\Gamma_2(n) &= \sqrt[q_2]{2^{q_2}\gamma_{n,2}^{q_2}(\omega_2^{q_2}+\pi_2^{q_2})-q_2\rho_2(-\sigma_2\epsilon_2^{q_2}\xi_2^{q_2}+\varpi_2)+c_{q_2}(\rho_2\epsilon_2\xi_2)^{q_2}}, \\
\varphi_2(n) &= \mu_{n,2} + \frac{\rho_2\tau_{n,2}^{q_2-1}(\nu_2\delta_2+\upsilon_2\zeta_2+\varepsilon_2)}{r_{n,2}-\rho_2\lambda_2 m_{n,2}}, \\
\Delta(n) &= R_{\rho_2\lambda_2,A_{n,2}}^{\eta_{n,2},M_{n,2}(\cdot,x^*)}(\Theta(x^*,y^*)) - R_{\rho_2\lambda_2,A_2}^{\eta_2,M_2(\cdot,x^*)}(\Theta(x^*,y^*)).
\end{aligned}
$$

Letting $L=\max\{L_1,L_2\}$, and using (5.2)–(5.3), we get

$$
\begin{aligned}
&\|(x_{n+1},y_{n+1})-(x^*,y^*)\|_* \\
&\leq (1-\alpha_{n,1}-\beta_{n,1})\|(x_n,y_n)-(x^*,y^*)\|_* + \alpha_{n,1}L\vartheta(n)\|(z_{n,1},t_{n,1})-(x^*,y^*)\|_* \\
&\quad + \alpha_{n,1}(L(\|(\Upsilon(n),\Delta(n))\|_* + \widetilde{c}_n + \widetilde{d}_n) + \|(e'_{n,1},p'_{n,1})\|_*)
\end{aligned}
$$

$$+\|(e''_{n,1},p''_{n,1})\|_* + \|(r_{n,1},k_{n,1})\|_* + 2\beta_{n,1}K, \tag{5.4}$$

where

$$\vartheta(n) = \max\{\varphi_1(n)+\varphi_2(n), \phi_1(n)+\phi_2(n)\}.$$

In a similar way to the proof of the inequalities (5.2)–(5.4), for $i = 1, 2, \cdots, N-2$, we can prove that

$$\begin{aligned}
&\|(z_{n,i},t_{n,i}) - (x^*,y^*)\|_* \\
\le\ & (1-\alpha_{n,i+1}-\beta_{n,i+1})\|(x_n,y_n)-(x^*,y^*)\|_* + \alpha_{n,i+1}L\vartheta(n)\|(z_{n,i+1},t_{n,i+1})-(x^*,y^*)\|_* \\
&+ \alpha_{n,i+1}(L(\|(\Upsilon(n),\Delta(n))\|_* + \widetilde{c}_n + \widetilde{d}_n) + \|(e'_{n,i+1},p'_{n,i+1})\|_*) \\
&+ \|(e''_{n,i+1},p''_{n,i+1})\|_* + \|(r_{n,i+1},k_{n,i+1})\|_* + 2\beta_{n,i+1}K
\end{aligned} \tag{5.5}$$

and

$$\begin{aligned}
&\|(z_{n,N-1},t_{n,N-1}) - (x^*,y^*)\|_* \\
\le\ & (1-\alpha_{n,N}-\beta_{n,N})\|(x_n,y_n)-(x^*,y^*)\|_* + \alpha_{n,N}L\vartheta(n)\|(z_{n,N},t_{n,N})-(x^*,y^*)\|_* \\
&+ \alpha_{n,N}(L(\|(\Upsilon(n),\Delta(n))\|_* + \widetilde{c}_n + \widetilde{d}_n) + \|(e'_{n,N},p'_{n,N})\|_*) \\
&+ \|(e''_{n,N},p''_{n,N})\|_* + \|(r_{n,N},k_{n,N})\|_* + 2\beta_{n,N}K.
\end{aligned} \tag{5.6}$$

Clearly, $\vartheta(n) \to \vartheta = \max\{\varphi_1+\varphi_2, \phi_1+\phi_2\}$ as $n \to \infty$, where φ_1, φ_2, ϕ_1, ϕ_2 are the same as in (3.13)–(3.14). Then for $\widehat{\vartheta} = \frac{1}{2}(\vartheta+1) \in (\vartheta, 1)$, there exists $n_0 \ge 1$ such that $\vartheta(n) < \widehat{\vartheta}$ for all $n \ge n_0$. Accordingly, it follows from (5.5)–(5.6), that for all $n \ge n_0$,

$$\begin{aligned}
&\|(z_{n,1},t_{n,1}) - (x^*,y^*)\|_* \\
\le\ & (1-\alpha_{n,2}-\beta_{n,2})\|(x_n,y_n)-(x^*,y^*)\|_* + \alpha_{n,2}L\widehat{\vartheta}\|(z_{n,2},t_{n,2})-(x^*,y^*)\|_* \\
&+ \alpha_{n,2}(L(\|(\Upsilon(n),\Delta(n))\|_* + \widetilde{c}_n + \widetilde{d}_n) + \|(e'_{n,2},p'_{n,2})\|_*) \\
&+ \|(e''_{n,2},p''_{n,2})\|_* + \|(r_{n,2},k_{n,2})\|_* + 2\beta_{n,2}K \\
\le\ & (1-\alpha_{n,2}-\beta_{n,2})\|(x_n,y_n)-(x^*,y^*)\|_* \\
&+ \alpha_{n,2}L\widehat{\vartheta}[(1-\alpha_{n,3}-\beta_{n,3})\|(x_n,y_n)-(x^*,y^*)\|_* \\
&+ \alpha_{n,3}L\widehat{\vartheta}\|(z_{n,3},t_{n,3})-(x^*,y^*)\|_* + \alpha_{n,3}(L(\|(\Upsilon(n),\Delta(n))\|_* + \widetilde{c}_n + \widetilde{d}_n) \\
&+ \|(e'_{n,3},p'_{n,3})\|_*) + \|(e''_{n,3},p''_{n,3})\|_* + \|(r_{n,3},k_{n,3})\|_* + 2\beta_{n,3}K] \\
&+ \alpha_{n,2}(L(\|(\Upsilon(n),\Delta(n))\|_* + \widetilde{c}_n + \widetilde{d}_n) + \|(e'_{n,2},p'_{n,2})\|_*) \\
&+ \|(e''_{n,2},p''_{n,2})\|_* + \|(r_{n,2},k_{n,2})\|_* + 2\beta_{n,2}K \\
=\ & (1-\alpha_{n,2}-\beta_{n,2}+\alpha_{n,2}(1-\alpha_{n,3}-\beta_{n,3})L\widehat{\vartheta})\|(x_n,y_n)-(x^*,y^*)\|_* \\
&+ \alpha_{n,2}\alpha_{n,3}L^2\widehat{\vartheta}^2\|(z_{n,3},t_{n,3})-(x^*,y^*)\|_* + (\alpha_{n,2}L+\alpha_{n,2}\alpha_{n,3}L^2\widehat{\vartheta})\|(\Upsilon(n),\Delta(n))\|_* \\
&+ \alpha_{n,2}\|(e'_{n,2},p'_{n,2})\|_* + \alpha_{n,2}\alpha_{n,3}L\widehat{\vartheta}\|(e'_{n,3},p'_{n,3})\|_* + \|(e''_{n,2},p''_{n,2})\|_* \\
&+ \alpha_{n,2}L\widehat{\vartheta}\|(e''_{n,3},p''_{n,3})\|_* + \|(r_{n,2},k_{n,2})\|_* + \alpha_{n,2}L\widehat{\vartheta}\|(r_{n,3},k_{n,3})\|_* \\
&+ (\alpha_{n,2}L+\alpha_{n,2}\alpha_{n,3}L^2\widehat{\vartheta})(\widetilde{c}_n+\widetilde{d}_n) + 2(\beta_{n,2}+\alpha_{n,2}\beta_{n,3}L\widehat{\vartheta})K \\
\le\ & \cdots \\
\le\ & [1-\alpha_{n,2}-\beta_{n,2}+\alpha_{n,2}(1-\alpha_{n,3}-\beta_{n,3})L\widehat{\vartheta}+\alpha_{n,2}\alpha_{n,3}(1-\alpha_{n,4}-\beta_{n,4})L^2\widehat{\vartheta}^2
\end{aligned}$$

$$
\begin{aligned}
&+\cdots+\prod_{i=2}^{N-1}\alpha_{n,i}(1-\alpha_{n,N}-\beta_{n,N})L^{N-2}\widehat{\vartheta}^{N-2}]\|(x_n,y_n)-(x^*,y^*)\|_*\\
&+\Big(\alpha_{n,2}L+\alpha_{n,2}\alpha_{n,3}L^2\widehat{\vartheta}+\cdots+\prod_{i=2}^{N}\alpha_{n,i}L^{N-1}\widehat{\vartheta}^{N-2}\Big)\|(\Upsilon(n),\Delta(n))\|_*\\
&+\alpha_{n,2}\|(e'_{n,2},p'_{n,2})\|_*+\alpha_{n,2}\alpha_{n,3}L\widehat{\vartheta}\|(e'_{n,3},p'_{n,3})\|_*+\cdots+\prod_{i=2}^{N}\alpha_{n,i}L^{N-2}\widehat{\vartheta}^{N-2}\|(e'_{n,N},p'_{n,N})\|_*\\
&+\|(e''_{n,2},p''_{n,2})\|_*+\alpha_{n,2}L\widehat{\vartheta}\|(e''_{n,3},p''_{n,3})\|_*+\cdots+\prod_{i=2}^{N-1}\alpha_{n,i}L^{N-2}\widehat{\vartheta}^{N-2}\|(e''_{n,N},p''_{n,N})\|_*\\
&+\|(r_{n,2},k_{n,2})\|_*+\alpha_{n,2}L\widehat{\vartheta}\|(r_{n,3},k_{n,3})\|_*+\cdots+\prod_{i=2}^{N-1}\alpha_{n,i}L^{N-2}\widehat{\vartheta}^{N-2}\|(r_{n,N},k_{n,N})\|_*\\
&+\Big(\alpha_{n,2}L+\alpha_{n,2}\alpha_{n,3}L^2\widehat{\vartheta}+\alpha_{n,2}\alpha_{n,3}\alpha_{n,4}L^3\widehat{\vartheta}^2+\cdots+\prod_{i=2}^{N}\alpha_{n,i}L^{N-1}\widehat{\vartheta}^{N-2}\Big)(\widetilde{c}_n+\widetilde{d}_n)\\
&+2\Big(\beta_{n,2}+\alpha_{n,2}\beta_{n,3}L\widehat{\vartheta}+\alpha_{n,2}\alpha_{n,3}\beta_{n,4}L^2\widehat{\vartheta}^2+\cdots+\prod_{i=2}^{N-1}\alpha_{n,i}\beta_{n,N}L^{N-2}\widehat{\vartheta}^{N-2}\Big)K. \qquad (5.7)
\end{aligned}
$$

Applying (5.4), (5.7) and the fact that $0<\alpha\le\prod_{i=1}^{N}\alpha_{n,i}$, for all $n\in\mathbb{N}$, we get

$$
\begin{aligned}
&\|(x_{n+1},y_{n+1})-(x^*,y^*)\|_*\\
\le&\Big[1-\alpha_{n,1}-\beta_{n,1}+\alpha_{n,1}(1-\alpha_{n,2}-\beta_{n,2})L\widehat{\vartheta}\\
&+\alpha_{n,1}\alpha_{n,2}(1-\alpha_{n,3}-\beta_{n,3})L^2\widehat{\vartheta}^2+\alpha_{n,1}\alpha_{n,2}\alpha_{n,3}(1-\alpha_{n,4}-\beta_{n,4})L^3\widehat{\vartheta}^3\\
&+\cdots+\prod_{i=1}^{N-2}\alpha_{n,i}(1-\alpha_{n,N-1}-\beta_{n,N-1})L^{N-2}\widehat{\vartheta}^{N-2}\\
&+\prod_{i=1}^{N-1}\alpha_{n,i}(1-\alpha_{n,N}-\beta_{n,N})L^{N-1}\widehat{\vartheta}^{N-1}\Big]\|(x_n,y_n)-(x^*,y^*)\|_*\\
&+\Big(\alpha_{n,1}L+\alpha_{n,1}\alpha_{n,2}L^2\widehat{\vartheta}+\cdots+\prod_{i=1}^{N}\alpha_{n,i}L^{N}\widehat{\vartheta}^{N-1}\Big)\|(\Upsilon(n),\Delta(n))\|_*\\
&+\alpha_{n,1}\|(e'_{n,1},p'_{n,1})\|_*+\alpha_{n,1}\alpha_{n,2}L\widehat{\vartheta}\|(e'_{n,2},p'_{n,2})\|_*\\
&+\cdots+\prod_{i=1}^{N}\alpha_{n,i}L^{N-1}\widehat{\vartheta}^{N-1}\|(e'_{n,N},p'_{n,N})\|_*\\
&+\|(e''_{n,1},p''_{n,1})\|_*+\alpha_{n,1}L\widehat{\vartheta}\|(e''_{n,2},p''_{n,2})\|_*+\cdots+\prod_{i=1}^{N-1}\alpha_{n,i}L^{N-1}\widehat{\vartheta}^{N-1}\|(e''_{n,N},p''_{n,N})\|_*\\
&+\|(r_{n,1},k_{n,1})\|_*+\alpha_{n,1}L\widehat{\vartheta}\|(r_{n,2},k_{n,2})\|_*+\cdots+\prod_{i=1}^{N-1}\alpha_{n,i}L^{N-1}\widehat{\vartheta}^{N-1}\|(r_{n,N},k_{n,N})\|_*\\
&+\Big(\alpha_{n,1}L+\alpha_{n,1}\alpha_{n,2}L^2\widehat{\vartheta}+\cdots+\prod_{i=1}^{N}\alpha_{n,i}L^{N}\widehat{\vartheta}^{N-1}\Big)(\widetilde{c}_n+\widetilde{d}_n)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + 2\Big(\beta_{n,1} + \alpha_{n,1}\beta_{n,2}L\widehat{\vartheta} + \cdots + \prod_{i=1}^{N-1}\alpha_{n,i}\beta_{n,N}L^{N-1}\widehat{\vartheta}^{N-1}\Big)K \\
&\le \Big(1 - \prod_{i=1}^{N}\alpha_{n,i}L^{N-1}\widehat{\vartheta}^{N-1}\Big)\|(x_n,y_n)-(x^*,y^*)\|_* + \sum_{i=1}^{N}\prod_{j=1}^{i}\alpha_{n,j}L^{i}\widehat{\vartheta}^{i-1}\|(\Upsilon(n),\Delta(n))\|_* \\
&\quad + \sum_{i=1}^{N}\prod_{j=1}^{i}\alpha_{n,j}L^{i-1}\widehat{\vartheta}^{i-1}\|(e'_{n,i},p'_{n,i})\|_* + \sum_{i=2}^{N}\prod_{j=1}^{i-1}\alpha_{n,j}L^{i-1}\widehat{\vartheta}^{i-1}\|(e''_{n,i},p''_{n,i})\|_* \\
&\quad + \|(r_{n,1},k_{n,1})\|_* + \|(e''_{n,1},p''_{n,1})\|_* + \sum_{i=2}^{N}\prod_{j=1}^{i-1}\alpha_{n,j}L^{i-1}\widehat{\vartheta}^{i-1}\|(r_{n,i},k_{n,i})\|_* \\
&\quad + \sum_{i=1}^{N}\prod_{j=1}^{i}\alpha_{n,j}L^{i}\widehat{\vartheta}^{i-1}(\widetilde{c}_n+\widetilde{d}_n) + 2\Big(\beta_{n,1} + \sum_{i=2}^{N}\prod_{j=1}^{i-1}\alpha_{n,j}\beta_{n,i}L^{i-1}\widehat{\vartheta}^{i-1}\Big)K \\
&\le \Big(1 - \prod_{i=1}^{N}\alpha_{n,i}L^{N-1}\widehat{\vartheta}^{N-1}\Big)\|(x_n,y_n)-(x^*,y^*)\|_* \\
&\quad + \prod_{i=1}^{N}\alpha_{n,i}L^{N-1}\widehat{\vartheta}^{N-1}\Big(\frac{\sum_{i=1}^{N}\prod_{j=1}^{i}\alpha_{n,j}L^{i}\widehat{\vartheta}^{i-1}\|(\Upsilon(n),\Delta(n))\|_*}{\alpha L^{N-1}\widehat{\vartheta}^{N-1}} \\
&\quad + \frac{\sum_{i=1}^{N}\prod_{j=1}^{i}\alpha_{n,j}L^{i-1}\widehat{\vartheta}^{i-1}\|(e'_{n,i},p'_{n,i})\|_* + \sum_{i=1}^{N}\prod_{j=1}^{i}\alpha_{n,j}L^{i}\widehat{\vartheta}^{i-1}(\widetilde{c}_n+\widetilde{d}_n)}{\alpha L^{N-1}\widehat{\vartheta}^{N-1}}\Big) \\
&\quad + \|(e''_{n,1},p''_{n,1})\|_* + \sum_{i=2}^{N}\prod_{j=1}^{i-1}\alpha_{n,j}L^{i-1}\widehat{\vartheta}^{i-1}\|(e''_{n,i},p''_{n,i})\|_* + \|(r_{n,1},k_{n,1})\|_* \\
&\quad + \sum_{i=2}^{N}\prod_{j=1}^{i-1}\alpha_{n,j}L^{i-1}\widehat{\vartheta}^{i-1}\|(r_{n,i},k_{n,i})\|_* + 2\Big(\beta_{n,1} + \sum_{i=2}^{N}\prod_{j=1}^{i-1}\alpha_{n,j}\beta_{n,i}L^{i-1}\widehat{\vartheta}^{i-1}\Big)K. \qquad (5.8)
\end{aligned}
$$

It follows from Theorem 4.1 that $\|(\Upsilon(n),\Delta(n))\|_* \to 0$ as $n\to\infty$. The condition (h) implies that $L\widetilde{\vartheta} < 1$. Since $\lim_{n\to\infty}\widetilde{c}_n = \lim_{n\to\infty}\widetilde{d}_n = 0$ and for each $i\in\{1,2,\cdots,N\}$, $\sum_{n=1}^{\infty}\beta_{n,i} < \infty$, in view of (4.6), it is clear that the conditions of Lemma 5.1 are satisfied. Now, Lemma 5.1 and (5.8) guarantee that $(x_{n+1},y_{n+1}) \to (x^*,y^*)$ as $n\to\infty$. So $\{(x_n,y_n)\}_{n=1}^{\infty}$ converges strongly to the only element (x^*,y^*) of $\mathrm{Fix}(\mathcal{Q})\cap\mathrm{SGNVLI}$.

Now, we establish the conclusion (2). By (4.7), we obtain

$$
\begin{aligned}
\|(u_{n+1},v_{n+1})-(x^*,y^*)\|_* &\le \|(u_{n+1},v_{n+1})-(E_n,D_n)\|_* + \|(E_n,D_n)-(x^*,y^*)\|_* \\
&= \epsilon_n + \|E_n - x^*\| + \|D_n - y^*\|. \qquad (5.9)
\end{aligned}
$$

In a similar way to the proof of inequalities (5.2)–(5.3), we can prove that

$$
\begin{aligned}
\|E_n - x^*\| \le{}& (1-\alpha_{n,1}-\beta_{n,1})\|u_n - x^*\| + \alpha_{n,1}L_1(\varphi_1(n)\|\nu_{n,1}-x^*\| \\
&+ \phi_1(n)\|w_{n,1}-y^*\|) + \alpha_{n,1}(L_1(\|\Upsilon(n)\|+\widetilde{c}_n) + \|e'_{n,1}\|) \\
&+ \|e''_{n,1}\| + \|r_{n,1}\| + \beta_{n,1}K \qquad (5.10)
\end{aligned}
$$

and

$$\begin{aligned}\|D_n-y^*\| &\leq (1-\alpha_{n,1}-\beta_{n,1})\|v_n-y^*\|+\alpha_{n,1}L_2(\varphi_2(n)\|\nu_{n,1}-x^*\| \\ &\quad+\phi_2(n)\|w_{n,1}-y^*\|)+\alpha_{n,1}(L_2(\|\Delta(n)\|+\widetilde{d}_n)+\|p'_{n,1}\|) \\ &\quad+\|p''_{n,1}\|+\|k_{n,1}\|+\beta_{n,1}K, \end{aligned}\tag{5.11}$$

where $\varphi_1(n)$, $\phi_1(n)$, $\Upsilon(n)$ are the same as in (5.2) and $\varphi_2(n)$, $\phi_2(n)$, $\Delta(n)$ are the same as in (5.3).

Since $0<\alpha\leq\prod\limits_{i=1}^{N}\alpha_{n,i}$ for all $n\in\mathbb{N}$, by using (5.9)–(5.11), as the proof of inequality (5.8), we obtain that

$$\begin{aligned}&\|(u_{n+1},v_{n+1})-(x^*,y^*)\|_* \\ &\leq\Big(1-\prod_{i=1}^{N}\alpha_{n,i}L^{N-1}\widehat{\vartheta}^{N-1}\Big)\|(u_n,v_n)-(x^*,y^*)\|_* \\ &\quad+\prod_{i=1}^{N}\alpha_{n,i}L^{N-1}\widehat{\vartheta}^{N-1}\Big(\frac{\sum\limits_{i=1}^{N}\prod\limits_{j=1}^{i}\alpha_{n,j}L^{i}\widehat{\vartheta}^{i-1}\|(\Upsilon(n),\Delta(n))\|_*}{\alpha L^{N-1}\widehat{\vartheta}^{N-1}} \\ &\quad+\frac{\sum\limits_{i=1}^{N}\prod\limits_{j=1}^{i}\alpha_{n,j}L^{i-1}\widehat{\vartheta}^{i-1}\|(e'_{n,i},p'_{n,i})\|_*+\sum\limits_{i=1}^{N}\prod\limits_{j=1}^{i}\alpha_{n,j}L^{i}\widehat{\vartheta}^{i-1}(\widetilde{c}_n+\widetilde{d}_n)+\epsilon_n}{\alpha L^{N-1}\widehat{\vartheta}^{N-1}}\Big) \\ &\quad+\|(e''_{n,1},p''_{n,1})\|_*+\sum_{i=2}^{N}\prod_{j=1}^{i-1}\alpha_{n,j}L^{i-1}\widehat{\vartheta}^{i-1}\|(e''_{n,i},p''_{n,i})\|_*+\|(r_{n,1},k_{n,1})\|_* \\ &\quad+\sum_{i=2}^{N}\prod_{j=1}^{i-1}\alpha_{n,j}L^{i-1}\widehat{\vartheta}^{i-1}\|(r_{n,i},k_{n,i})\|_*+2\Big(\beta_{n,1}+\sum_{i=2}^{N}\prod_{j=1}^{i-1}\alpha_{n,j}\beta_{n,i}L^{i-1}\widehat{\vartheta}^{i-1}\Big)K.\end{aligned}\tag{5.12}$$

Suppose that $\lim\limits_{n\to\infty}\epsilon_n=0$. Then it follows from

$$\lim_{n\to\infty}\|(\Upsilon(n),\Delta(n))\|_*=0,\quad \lim_{n\to\infty}\widetilde{c}_n=\lim_{n\to\infty}\widetilde{d}_n=0,\quad \sum_{n=1}^{\infty}\beta_{n,i}<\infty$$

for each $i\in\{1,2,\cdots,N\}$, (4.6), (5.12) and Lemma 5.1 that $\lim\limits_{n\to\infty}(u_n,v_n)=(x^*,y^*)$.

Conversely, if $\lim\limits_{n\to\infty}(u_n,v_n)=(x^*,y^*)$, then we get

$$\begin{aligned}\epsilon_n&=\|(u_{n+1},v_{n+1})-(E_n,D_n)\|_* \\ &\leq\|(u_{n+1},v_{n+1})-(x^*,y^*)\|+\|(E_n,D_n)-(x^*,y^*)\| \\ &\leq\|(u_{n+1},v_{n+1})-(x^*,y^*)\|+\Big(1-\prod_{i=1}^{N}\alpha_{n,i}L^{N-1}\widehat{\vartheta}^{N-1}\Big)\|(u_n,v_n)-(x^*,y^*)\|_* \\ &\quad+\prod_{i=1}^{N}\alpha_{n,i}L^{N-1}\widehat{\vartheta}^{N-1}\Big(\frac{\sum\limits_{i=1}^{N}\prod\limits_{j=1}^{i}\alpha_{n,j}L^{i}\widehat{\vartheta}^{i-1}\|(\Upsilon(n),\Delta(n))\|_*}{\alpha L^{N-1}\widehat{\vartheta}^{N-1}} \\ &\quad+\frac{\sum\limits_{i=1}^{N}\prod\limits_{j=1}^{i}\alpha_{n,j}L^{i-1}\widehat{\vartheta}^{i-1}\|(e'_{n,i},p'_{n,i})\|_*+\sum\limits_{i=1}^{N}\prod\limits_{j=1}^{i}\alpha_{n,j}L^{i}\widehat{\vartheta}^{i-1}(\widetilde{c}_n+\widetilde{d}_n)}{\alpha L^{N-1}\widehat{\vartheta}^{N-1}}\Big)\end{aligned}$$

$$
\begin{aligned}
&+\|(e''_{n,1},p''_{n,1})\|_*+\sum_{i=2}^{N}\prod_{j=1}^{i-1}\alpha_{n,j}L^{i-1}\widehat{\vartheta}^{i-1}\|(e''_{n,i},p''_{n,i})\|_*\\
&+\|(r_{n,1},k_{n,1})\|_*+\sum_{i=2}^{N}\prod_{j=1}^{i-1}\alpha_{n,j}L^{i-1}\widehat{\vartheta}^{i-1}\|(r_{n,i},k_{n,i})\|_*\\
&+2\Big(\beta_{n,1}+\sum_{i=2}^{N}\prod_{j=1}^{i-1}\alpha_{n,j}\beta_{n,i}L^{i-1}\widehat{\vartheta}^{i-1}\Big)K. \qquad (5.13)
\end{aligned}
$$

From (4.6) and $\sum\limits_{n=1}^{\infty}\beta_{n,i}<\infty$ for each $i\in\{1,2,\cdots,N\}$, we deduce that $\lim\limits_{n\to\infty}(e''_{n,i},p''_{n,i})=\lim\limits_{n\to\infty}(r_{n,i},k_{n,i})=0$ and $\lim\limits_{n\to\infty}\beta_{n,i}=0$ for each $i\in\{1,2,\cdots,N\}$. Now, it follows from $\lim\limits_{n\to\infty}\|(\Upsilon(n),\Delta(n))\|_*=0$, $\lim\limits_{n\to\infty}\|(e'_{n,i},p'_{n,i})\|_*=0$ and $\lim\limits_{n\to\infty}\widetilde{c}_n=\lim\limits_{n\to\infty}\widetilde{d}_n=0$ that the right side of the inequality (5.13) tends to zero as $n\to\infty$. This completes the proof.

Remark 5.1 It should be pointed that

(1) Theorem 3.1 is an extension of Theorem 3.1 in [44] in Hilbert spaces. Moreover, Theorem 3.1 improves and extends Theorems 3.2–3.4 in [45].

(2) Theorem 5.1 generalizes and improves Theorem 4.1 in [44] and Theorems 3.5–3.8 in [43].

Remark 5.2 If $M_{n,i}$ and M_i, $n\in\mathbb{N}$, $i=1,2$, are A-accretive mappings, A-monotone operators, (H,η)-accretive mappings, (H,η)-monotone operators or H-monotone operators, respectively, then from Theorems 3.1 and 5.1, we can obtain the existence and convergence results of the solutions to the system (3.1). In brief, for a suitable and appropriate choices of the mappings S_i, T_i, P_i, Q_i, M_i, N_i, η_i, A_i, f_i, g_i, h_i, θ_i, $A_{n,i}$, $\eta_{n,i}$, $M_{n,i}$, $\mathcal{S}_i$, $\mathcal{Q}$ $(n\in\mathbb{N})$, $i=1,2$, the sequences $\{\alpha_{n,i}\}_{n=1}^{\infty}$, $\{\beta_{n,i}\}_{n=1}^{\infty}$, $\{e_{n,i}\}_{n=1}^{\infty}$, $\{p_{n,i}\}_{n=1}^{\infty}$, $\{j_{n,i}\}_{n=1}^{\infty}$, $\{s_{n,i}\}_{n=1}^{\infty}$, $\{r_{n,i}\}_{n=1}^{\infty}$, $\{k_{n,i}\}_{n=1}^{\infty}$, $n\in\mathbb{N}$, $i=1,2,\cdots,N$, the constants a, b, λ_i $i=1,2$, and the spaces X_1, X_2, Theorems 3.1 and 5.1 include many known results of variational (variational-like) inclusions as special cases, see [15, 18, 23–24, 32, 43–45, 54] and the references therein.

6 Conclusion

In this paper, we have introduced and considered a new system of generalized nonlinear variational-like inclusions (SGNVLI) with A-maximal m-relaxed η-accretive (so-called (A,η)-accretive) mappings in Banach spaces. By using the resolvent operator technique associated with A-maximal m-relaxed η-accretive mappings due to Lan et al., we have established the equivalence between SGNVLI and the fixed point problem, and then, by this equivalent formulation, we have discussed the existence and uniqueness of solution of SGNVLI. This equivalence and two nearly uniformly Lipschitzian mappings $\mathcal{S}_i$ $(i=1,2)$ are used to suggest and analyze a new perturbed N-step iterative algorithm with mixed errors for finding an element of the set of the fixed points of the nearly uniformly Lipschitzian mapping $\mathcal{Q}=(\mathcal{S}_1,\mathcal{S}_2)$ which is the unique solution of SGNVLI. It is expected that the results proved in this paper may stimulate further research regarding the numerical methods and their applications in various fields of pure and applied sciences.

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