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Operator Equations and Duality Mappings in Sobolev Spaces with Variable Exponents

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Abstract After studying in a previous work the smoothness of the space

$$U_{\Gamma_0} = \{u \in W^{1,p(\cdot)}(\Omega); u = 0 \text{ on } \Gamma_0 \subset \Gamma = \partial\Omega\},$$

where $\mathrm{d}\Gamma - \mathrm{meas}\Gamma_0 > 0$, with $p(\cdot) \in \mathcal{C}(\overline{\Omega})$ and $p(x) > 1$ for all $x \in \overline{\Omega}$, the authors study in this paper the strict and uniform convexity as well as some special properties of duality mappings defined on the same space. The results obtained in this direction are used for proving existence results for operator equations having the form $J_\varphi u = N_f u$, where J_φ is a duality mapping on U_{Γ_0} corresponding to the gauge function φ, and N_f is the Nemytskij operator generated by a Carathéodory function f satisfying an appropriate growth condition ensuring that N_f may be viewed as acting from U_{Γ_0} into its dual.

Keywords Monotone operators, Smoothness, Strict convexity, Uniform convexity, Duality mappings, Sobolev spaces with a variable exponent, Nemytskij operators
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1 Notations

All vector and function spaces considered in this paper are real.

The Euclidean norm in $\mathbb{R}^N$ is denoted by $|\cdot|$, and $z \cdot y = \sum_{i=1}^{N} z_i y_i$ for $z = (z_1, \cdots, z_N)$ and $y = (y_1, \cdots, y_N) \in \mathbb{R}^N$.

The Lebesgue measure in $\mathbb{R}^N$ is denoted by $\mathrm{d}x$. *Throughout this paper*, Ω designates a domain in $\mathbb{R}^N$ $(N \geq 2)$, i.e., a bounded and connected open subset of $\mathbb{R}^N$ whose boundary Γ is Lipschitz continuous, with the set Ω being locally on the same side of Γ. A measure, denoted by $\mathrm{d}\Gamma$, can then be defined on Γ. For details, see, e.g., [1] or [21]. No distinction will be made between $\mathrm{d}x$-measurable (resp. $\mathrm{d}\Gamma$-measurable), functions and their equivalence classes modulo the relation of $\mathrm{d}x$-almost everywhere (resp. $\mathrm{d}\Gamma$-almost everywhere) equality.

Unless a specific notation is used, $\|\cdot\|_V$ denotes the norm in a normed vector space V, and $\overline{A}^{\|\cdot\|_V}$ designates the closure in V of a subset A of V with respect to the norm $\|\cdot\|_V$.

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The notation V^* designates the dual space of the normed vector space V. The duality pairing between V^* and V is denoted by $\langle \cdot, \cdot \rangle_{V^*,V}$. We shall often omit to indicate the spaces in such a duality, i.e., we shall simply write $\langle \cdot, \cdot \rangle$.

Given two normed vector spaces V and W, the notation $V \hookrightarrow W$ (resp. $V \Subset W$) means that $V \subset W$ and the canonical injection from V into W is continuous (resp. compact).

Strong and weak convergence are respectively denoted by $\to$ and $\rightharpoonup$.

The notation $\mathcal{D}(\Omega)$ denotes the space of functions that are infinitely differentiable in Ω and whose support is a compact subset of Ω.

Given a real number $p \geq 1$, the notations $L^p(\Omega)$, $W^{1,p}(\Omega)$, and

$$W_0^{1,p}(\Omega) := \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{W^{1,p}(\Omega)}} = \{v \in W^{1,p}(\Omega); \operatorname{tr} v = 0 \text{ on } \Gamma\}$$

designate the usual Lebesgue and Sobolev spaces, where "usual" means that the exponent $p \geq 1$ is a constant.

2 Some Abstract Existence Results

The main result of this section is given by the following theorem.

Theorem 2.1 *Let X be a real reflexive Banach space. Given a demicontinuous operator $T : X \to X^*$, i.e., such that*

$$u_n \to u \textit{ implies } Tu_n \rightharpoonup Tu \quad \textit{as } n \to \infty,$$

with the following properties:

(a) *For any $f \in X^*$, $T^{-1}(f) = \{v \in X; Tv = f\}$ is a nonempty, closed and convex subset in X;*

(b) *T satisfies the condition* $(\mathrm{S})_+$, *i.e., as $n \to \infty$, the following holds:*

$$u_n \rightharpoonup u \textit{ and } \limsup_{n\to\infty} \langle Tu_n, u_n - u \rangle \leq 0 \textit{ imply } u_n \to u.$$

In addition, let there be given a nonempty, closed and convex set $C \subset X$ and a demicontinuous operator $K : C \to X^*$ with the following properties:

(c) *$(v_n) \subset C$ and (Tv_n) bounded imply that (v_n) is bounded;*

(d) *$T^{-1}(Ku) = \{v \in X; Tv = Ku\} \subset C$ for all $u \in C$;*

(e) *$K(C)$ is relatively compact.*

Then the equation

$$Tu = Ku$$

has a solution in C.

Proof Define the set-valued mapping

$$S : C \to 2^C \text{ by } Su = T^{-1}(Ku) \quad \text{for all } u \in C.$$

Then, by virtue of (d), S is unambiguously defined.

We now prove that S has the following properties:

(P_1) For any $u \in C$, Su is a nonempty, closed and convex subset of C.

This property follows by (d) and (a).

(P_2) S is closed.

Let $(u_n) \subset C$ satisfy that $u_n \to u$ as $n \to \infty$, and let $v_n \in Su_n$ be such that $v_n \to v$ as $n \to \infty$. Since T and K are demicontinuous, one has $Tv_n \rightharpoonup Tv$ and $Ku_n \rightharpoonup Ku$ as $n \to \infty$. Since $Tv_n = Ku_n$, we conclude that $Tv = Ku$, i.e., $v \in Su$.

(P_3) $S(C) = \bigcup_{u \in C} Su$ is relatively compact.

Let (v_n) be a sequence in $S(C)$. We will show that (v_n) contains a strongly convergent subsequence. Let $u_n \in C$ be such that $v_n \in Su_n$, or equivalently $Tv_n = Ku_n$ for all n. Since $K(C)$ is relatively compact (see (e)), we may assume, passing if necessary to a subsequence, that $Ku_n \to f$ as $n \to \infty$. Thus, $Tv_n = Ku_n \to f$ as $n \to \infty$. In particular, (Tv_n) is bounded. In view of (c), we conclude that (v_n) is bounded. Again passing if necessary to a subsequence, we may assume that $v_n \rightharpoonup v$ as $n \to \infty$. Since $Tv_n \to f$ and $v_n \rightharpoonup v$, it follows that $\langle Tv_n, v_n - v\rangle \to 0$ as $n \to \infty$. Since T satisfies the condition $(S)_+$, we conclude that $v_n \to v$.

Properties (P_1)–(P_3) of S allow us to apply the *Bohnenblust-Karlin fixed point theorem* (see, e.g., [24, Theorem 9.2.3]) and to conclude that there exists $u \in C$ such that $u \in Su$. Equivalently, there exists $u \in C$ such that $Tu = Ku$.

Remark 2.1 It is easily seen that the assumptions that $K : C \to X^*$ is demicontinuous and that $K(C)$ is relatively compact imply that, in fact, K is continuous.

Corollary 2.1 *Let X be a real reflexive Banach space. Let there be given a hemicontinuous operator $T : X \to X^*$, i.e., such that*

$$\langle T(u + \lambda v), w\rangle \to \langle Tu, w\rangle \quad \text{as } \lambda \to 0 \text{ for all } u, v, w \in X,$$

with the following properties:

(a) *T is monotone:*

$$\langle Tu - Tv, u - v\rangle \geq 0 \quad \text{for all } u, v \in X,$$

(b) *T is coercive:*

$$\frac{\langle Tu, u\rangle}{\|u\|} \to \infty \quad \text{as } \|u\|_X \to \infty,$$

(c) *T satisfies the condition* $(S)_+$ *of Theorem* 2.1.

In addition, let there be given a nonempty, closed convex set $C \subset X$ and a demicontinuous operator $K : C \to X^*$ with the following properties:

(d) *$T^{-1}(Ku) = \{v \in X; Tv = Ku\} \subset C$ for all $u \in C$;*

(e) *$K(C)$ is relatively compact.*

Then the equation

$$Tu = Ku$$

has a solution in C.

Proof It is sufficient to show that T is demicontinuous and that assumptions (a) and (c) of Theorem 2.1 are fulfilled. Indeed, a classical result in the theory of monotone operators asserts that, if X is a real reflexive Banach space and $T : X \to X^*$ is monotone and hemicontinuous, then T is demicontinuous (see [2, 3]). Moreover, due to a well-known surjectivity result of

Browder (see [2, 3]), if X is reflexive and $T : X \to X^*$ is monotone, hemicontinuous, and coercive then, for any $f \in X^*$, the inverse image $T^{-1}(f)$ is nonempty, bounded, closed and convex (see, e.g., [11] for more details). Finally, the coerciveness of T implies that, if $T(A)$ is bounded in X^*, then A is bounded in X. Thus Theorem 2.1 applies and the result follows.

Remark 2.2 Corollary 2.1 is nothing else but [11, Theorem 1].

The operator T appearing in Corollary 2.1 will now be specialized as being a duality mapping on a real reflexive and smooth Banach space. To this end, we will first recall some definitions and basic results related to duality mapping on such a space. A result (which is new to the best of our knowledge) characterizing the strict convexity of a real reflexive and smooth Banach space in terms of the injectivity of any duality mapping will be given (see Theorem 3.1 below). This result will play a crucial role in proving the strict convexity of the space $(U_{\Gamma_0}, \|\cdot\|_{0,p(\cdot),\nabla})$ (see Corollary 4.3 below).

3 Duality Mappings on Smooth Banach Spaces

A real Banach space X is said to be *smooth* if, given any nonzero element $x \in X$, there exists a unique *support functional*, i.e., there exists a unique element $x^*(x) \in X^*$ having the properties that $\langle x^*(x), x\rangle = \|x\|$ and $\|x^*(x)\| = 1$.

A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a *gauge function* if it is continuous, strictly increasing, $\varphi(0) = 0$ and $\varphi(r) \to \infty$ as $r \to \infty$.

If X is a smooth real Banach space and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a gauge function, then, by definition, the *duality mapping on* X, *subordinated to* φ, is the mapping $J_\varphi : X \to X^*$ defined by

$$J_\varphi 0_X := 0_{X^*}, \tag{3.1}$$

$$J_\varphi x := \varphi(\|x\|)x^*(x) \quad \text{for all } x \in X, x \neq 0_X, \tag{3.2}$$

where $x^*(x)$ is the unique support functional at x.

It easily follows from (3.1)–(3.2) that

$$\|J_\varphi x\| = \varphi(\|x\|), \tag{3.3}$$

$$\langle J_\varphi x, x\rangle = \varphi(\|x\|)\|x\| = \|J_\varphi x\|\|x\| \quad \text{for all } x \in X. \tag{3.4}$$

The duality mapping subordinated to the identity gauge function defined by $\varphi(t) = t$, $t \geq 0$, will be called the *normalized duality mapping* and will be denoted by J. Thus $J : X \to X^*$ is defined by

$$J0_X := 0_{X^*}, \tag{3.5}$$

$$Jx := \|x\|\ x^*(x) \quad \text{for all } x \in X\backslash\{0_X\}. \tag{3.6}$$

Clearly J_φ and J are related by

$$J_\varphi x = \frac{\varphi(\|x\|)}{\|x\|} Jx$$

for all nonzero $x \in X$.

Thanks to this relation, many properties of J (which are usually easier to prove) may be converted into corresponding properties of J_φ. Here is a useful example.

Proposition 3.1 *Let X be a smooth real Banach space. Then $J_\varphi : X \to X^*$ is injective if and only if $J : X \to X^*$ is injective.*

Proof Assume that J_φ is injective. We have to show that J is injective, i.e., that $Jx = Jy$ implies $x = y$. If $Jx = Jy = 0_{X^*}$, then $x = y = 0_X$. If $Jx = Jy \neq 0_{X^*}$, then $x \neq 0_X$, $y \neq 0_X$ and $\|x\| = \|y\|$. Consequently,

$$J_\varphi x = \frac{\varphi(\|x\|)}{\|x\|} Jx = \frac{\varphi(\|y\|)}{\|y\|} Jy = J_\varphi y,$$

and, since J_φ is assumed to be injective, it follows that $x = y$.

In a similar manner we can see that, conversely, the injectivity of J implies that of J_φ.

According to a classical result (see, e.g., [9, Theorem 1 in Chapter 2]), a *real Banach space X is smooth if and only if the norm of X is Gâteaux-differentiable at any nonzero $x \in X$*, i.e., if and only if there exists a mapping

$$(\mathrm{grad}\, \|\cdot\|) : X\backslash\{0_X\} \to X^*\backslash\{0_{X^*}\},$$

such that, at any nonzero element $x \in X$,

$$\langle (\mathrm{grad}\, \|\cdot\|)(x), h\rangle = \lim_{t\to 0} \frac{\|x + th\| - \|x\|}{t} \quad \text{for all } h \in X.$$

Since, for all nonzero $x \in X$, the gradient of the norm satisfies (see [25, Lemma 2.5])

$$\langle (\mathrm{grad}\, \|\cdot\|)(x), x\rangle = \|x\|, \tag{3.7}$$

$$\|(\mathrm{grad}\, \|\cdot\|)(x)\| = 1, \tag{3.8}$$

we conclude that the unique support functional at any nonzero element $x \in X$ is $x^*(x) = (\mathrm{grad}\, \|\cdot\|)(x)$. It then follows from (3.1)–(3.2) that, on a smooth real Banach space X, the duality mapping corresponding to a gauge function φ is the mapping $J_\varphi : X \to X^*$ defined by

$$\begin{aligned} &J_\varphi 0_X = 0_{X^*}, \\ &J_\varphi x = \varphi(\|x\|)(\mathrm{grad}\, \|\cdot\|)(x) \quad \text{for all } x \neq 0_X. \end{aligned} \tag{3.9}$$

By the definition of J_φ and the metric properties (3.7)–(3.8), we infer that, for all $x, y \in X$,

$$\langle J_\varphi x - J_\varphi y, x - y\rangle \geq (\varphi(\|x\|) - \varphi(\|y\|))(\|x\| - \|y\|) \geq 0, \tag{3.10}$$

which shows that J_φ is monotone. From (3.3), we infer that J_φ is also bounded. For other properties of duality mappings, see [2; 6; 26, Proposition 32.22].

We now show how some geometric properties of a smooth Banach space may be expressed in terms of some specific properties of any duality mapping defined on such a space.

Theorem 3.1 *Let X be a real reflexive and smooth Banach space. Then X is strictly convex if and only if any duality mapping $J_\varphi : X \to X^*$ is injective.*

Proof Assume that X is strictly convex. Then J_φ is strictly monotone and thus it is injective (see [22]).

Conversely, assume that J_φ is injective. We will prove that J_φ is strictly monotone.

Assume on the contrary that there exist elements x and y in X, $x \neq y$, such that

$$\langle J_\varphi x - J_\varphi y, x - y\rangle = 0. \tag{3.11}$$

If one of these elements, say y, is the zero vector, then we conclude from (3.11) that

$$0 = \langle J_\varphi x, x\rangle = \varphi(\|x\|)\|x\| > 0,$$

a contradiction.

Assume that x and y are both nonzero. Since

$$0 = \langle J_\varphi x - J_\varphi y, x - y\rangle \geq (\varphi(\|x\|) - \varphi(\|y\|))(\|x\| - \|y\|) \geq 0,$$

we infer that

$$(\varphi(\|x\|) - \varphi(\|y\|))(\|x\| - \|y\|) = 0,$$

and thus that $\|x\| = \|y\|$ since φ is strictly increasing.

Rewriting (3.11) as

$$[\langle J_\varphi x, x\rangle - \langle J_\varphi x, y\rangle] + [\langle J_\varphi y, y\rangle - \langle J_\varphi y, x\rangle] = 0, \tag{3.12}$$

and observing that

$$\langle J_\varphi x, x\rangle - \langle J_\varphi x, y\rangle \geq 0$$

and

$$\langle J_\varphi y, y\rangle - \langle J_\varphi y, x\rangle \geq 0,$$

we infer from (3.12) that

$$\langle J_\varphi x, x\rangle - \langle J_\varphi x, y\rangle = 0, \tag{3.13}$$

$$\langle J_\varphi y, y\rangle - \langle J_\varphi y, x\rangle = 0. \tag{3.14}$$

From (3.14), it follows that

$$\langle J_\varphi y, x\rangle = \langle J_\varphi y, y\rangle = \|J_\varphi y\|\|y\| = \|J_\varphi y\|\|x\|.$$

Consequently, denoting by χ the canonical isomorphism between X and X^{**} (i.e., such that $\langle \chi(x), x^*\rangle = \langle x^*, x\rangle$ for all $x \in X$ and $x^* \in X^*$), we have

$$\Big\langle \frac{J_\varphi y}{\|J_\varphi y\|}, \frac{x}{\|x\|}\Big\rangle = 1 = \Big\langle \chi\Big(\frac{x}{\|x\|}\Big), \frac{J_\varphi y}{\|J_\varphi y\|}\Big\rangle = \Big\|\chi\Big(\frac{x}{\|x\|}\Big)\Big\|. \tag{3.15}$$

On the other hand, we derive from (3.4) that

$$\Big\langle \frac{J_\varphi x}{\|J_\varphi x\|}, \frac{x}{\|x\|}\Big\rangle = 1 = \Big\langle \chi\Big(\frac{x}{\|x\|}\Big), \frac{J_\varphi x}{\|J_\varphi x\|}\Big\rangle = \Big\|\chi\Big(\frac{x}{\|x\|}\Big)\Big\|. \tag{3.16}$$

Relations (3.15)–(3.16) then show that $\chi\big(\frac{x}{\|x\|}\big) \in X^{**}$ attains its norm at two different points of the unit ball in X^*, namely $\frac{J_\varphi y}{\|J_\varphi y\|}$ and $\frac{J_\varphi x}{\|J_\varphi x\|}$. This means that X^* is not strictly convex and, since X is supposed to be reflexive, it follows that X is not smooth (see [9, Corollary 2 in Chapter 2]), a contradiction.

Thus J_φ is strictly monotone. Consequently (see [22]), the space X is strictly convex.

Theorem 3.1 and Proposition 3.1 imply the next corollary.

Corollary 3.1 *Let X be a real reflexive and smooth Banach space. Then X is strictly convex if and only if the normalized duality mapping $J : X \to X^*$ on X is injective.*

Corollary 3.2 *Let X be a real reflexive and smooth Banach space, and let $J_\varphi : X \to X^*$ be a duality mapping on X that satisfies the condition* $(\mathrm{S})_+$ *of Theorem* 2.1*. Let $C \subset X$ be a nonempty, closed convex set, and let $K : C \to X^*$ be a demicontinuous operator with the following properties:*

(a) $J_\varphi^{-1}(Ku) = \{v \in X; J_\varphi v = Ku\} \subset C$ *for all* $u \in C$*;*

(b) $K(C)$ *is relatively compact.*

Then the equation

$$J_\varphi u = Ku$$

has a solution in C.

Proof It is known (see, e.g., [2]) that any duality mapping on a real reflexive and smooth Banach space is monotone, demicontinuous (or equivalently hemicontinuous) and coercive. Consequently, it suffices to apply Corollary 2.1 with $T = J_\varphi$.

Theorem 3.2 *Let X be a real, smooth, and strictly convex, reflexive Banach space. Then:*

(a) *Any duality mapping $J_\varphi : X \to X^*$ that satisfies the condition* $(\mathrm{S})_+$ *is bijective and has a continuous inverse.*

(b) *If, in addition, the norm of X is Fréchet differentiable, then any duality mapping J_φ : $X \to X^*$ that satisfies the condition* $(\mathrm{S})_+$ *is a homeomorphism of X onto X^*.*

Proof (a) Any duality mapping $J_\varphi : X \to X^*$ on a reflexive, smooth and strictly convex real Banach space X is a bijection of X onto X^*. Moreover (see, e.g., [12, Theorem 5; 13, Corollary 2.3; 26, Proposition 32.22(b)]),

$$J_\varphi^{-1} = \chi^{-1} J^*_{\varphi^{-1}},$$

where χ stands for the canonical isomorphism between X and X^{**}, and $J^*_{\varphi^{-1}} : X^* \to X^{**}$ is the duality mapping on X^* corresponding to the gauge function φ^{-1}.

In order to prove the continuity of J_φ^{-1}, let $x_n^* \to x^*$ in X^* as $n \to \infty$. As a duality mapping on a smooth and reflexive Banach space, $J^*_{\varphi^{-1}}$ is demicontinuous. Thus, $J_{\varphi^{-1}} x_n^* \rightharpoonup J_{\varphi^{-1}} x^*$ in X^{**} as $n \to \infty$. Consequently,

$$x_n = J_\varphi^{-1} x_n^* = \chi^{-1} J^*_{\varphi^{-1}} x_n^* \rightharpoonup \chi^{-1} J^*_{\varphi^{-1}} x^* = J_\varphi^{-1} x^* = x \quad \text{as } n \to \infty.$$

On the other hand,

$$J_\varphi x_n = x_n^* \to J_\varphi x = x^* \quad \text{as } n \to \infty.$$

Hence

$$\langle J_\varphi x_n, x_n - x \rangle \to 0 \quad \text{as } n \to \infty,$$

and, since J_φ satisfies the condition $(\mathrm{S})_+$, we conclude that $x_n \to x$ as $n \to \infty$. This shows that J_φ^{-1} is continuous.

(b) Any Fréchet-differentiable norm on a normed vector space X is necessarily of class $\mathcal{C}^1$ on $X - \{0\}$ (see [19, Lemma 2] and [23, p. 20]). Therefore, we infer from (3.9) that any duality mapping on a Banach space having a Fréchet-differentiable norm is continuous.

4 The Space U_{Γ_0}

In view of defining (see Subsection 4.2) the space U_{Γ_0}, we need first to review some properties of classical function spaces.

4.1 Lebesgue and Sobolev spaces with variable exponents

This section gathers various definitions and basic properties related to Lebesgue and Sobolev spaces with variable exponents needed through the paper. For proofs and references, see [14].

Given a function $p(\cdot) \in L^\infty(\Omega)$ that satisfies

$$1 \le p^- := \operatorname{ess}\inf_{x\in\Omega} p(x) \le p^+ := \operatorname{ess}\sup_{x\in\Omega} p(x),$$

the *Lebesgue space* $L^{p(\cdot)}(\Omega)$ *with variable exponent* $p(\cdot)$ is defined as

$$L^{p(\cdot)}(\Omega) := \Big\{ v : \Omega \to \mathbb{R}; v \text{ is d}x\text{-measurable and } \rho_{0,p(\cdot),\Omega}(v) := \int_\Omega |v(x)|^{p(x)} \mathrm{d}x < \infty \Big\},$$

where $\rho_{0,p(\cdot),\Omega}(v)$ is called the *convex modular* of v.

Often we shall omit to indicate the domain Ω and simply write $\rho_{0,p(\cdot)}$.

Given a function $q(\cdot) \in L^\infty(\Gamma)$ that satisfies

$$1 \le \operatorname{ess}\inf_{y\in\Gamma} q(y),$$

the *Lebesgue space* $L^{q(\cdot)}(\Gamma)$ *with variable exponent* $q(\cdot)$ is defined as

$$L^{q(\cdot)}(\Gamma) := \Big\{ v : \Gamma \to \mathbb{R}; v \text{ is d}\Gamma\text{-measurable and } \int_\Gamma |v(y)|^{q(y)} \mathrm{d}y < \infty \Big\}.$$

Theorem 4.1 *Let Ω be a domain in $\mathbb{R}^N$.*

(a) *Let $p(\cdot) \in L^\infty(\Omega)$ be such that $p^- \ge 1$. Equipped with the norm*

$$v \in L^{p(\cdot)}(\Omega) \to \|v\|_{0,p(\cdot)} := \inf\Big\{ \lambda > 0; \ \int_\Omega \Big| \frac{v(x)}{\lambda} \Big|^{p(x)} \mathrm{d}x \le 1 \Big\},$$

the space $L^{p(\cdot)}(\Omega)$ is a separable Banach space. If $p^- > 1$, the space $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive.

(b) *Let $p_1(\cdot) \in L^\infty(\Omega)$ and $p_2(\cdot) \in L^\infty(\Omega)$ be such that $p_1^- \ge 1$ and $p_2^- \ge 1$. Then*

$$L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$$

if and only if

$$p_1(x) \le p_2(x) \quad \textit{for almost all } x \in \Omega.$$

(c) *Given* $p(\cdot) \in L^\infty(\Omega)$ *such that* $p^- > 1$, *let* $p'(\cdot) \in L^\infty(\Omega)$ *be defined by*

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1 \quad \text{for almost all } x \in \Omega.$$

Then, given any function $u \in L^{p'(\cdot)}(\Omega)$, *the linear functional*

$$\ell : v \in L^{p(\cdot)}(\Omega) \to \int_\Omega u(x)v(x)\mathrm{d}x \in \mathbb{R}$$

is continuous; conversely, given any continuous-linear functional $\ell : L^{p(\cdot)}(\Omega) \to \mathbb{R}$, *there exists one, and only one, function* $u_\ell \in L^{p'(\cdot)}(\Omega)$ *such that*

$$\ell(v) = \int_\Omega u_\ell(x)v(x)\mathrm{d}x \quad \text{for all } v \in L^{p(\cdot)}(\Omega).$$

(d) *For any* $u \in L^{p(\cdot)}(\Omega)$ *with* $p(\cdot) \in L^\infty(\Omega)$ *satisfying* $p^- > 1$ *and* $v \in L^{p'(\cdot)}(\Omega)$,

$$\int_\Omega |u(x)v(x)|\mathrm{d}x \le \Big(\frac{1}{p^-} + \frac{1}{(p')^-}\Big)\|u\|_{0,p(\cdot)} \cdot \|v\|_{0,p'(\cdot)}. \tag{4.1}$$

The next theorem recapitulates the relations between the norm $\|\cdot\|_{0,p(\cdot)}$ and the convex modular $\rho_{0,p(\cdot)}$.

Theorem 4.2 *Let* $p(\cdot) \in L^\infty(\Omega)$ *be such that* $p^- \ge 1$ *and let* $u \in L^{p(\cdot)}(\Omega)$. *The following properties hold:*

(a) *If* $u \neq 0$, *then* $\|u\|_{0,p(\cdot)} = a$ *if and only if* $\rho_{0,p(\cdot)}(a^{-1}u) = 1$.

(b) $\|u\|_{0,p(\cdot)} < 1$ (*resp.* $= 1$, *or* > 1) *if and only if* $\rho_{0,p(\cdot)}(u) < 1$ (*resp.* $= 1$, *or* > 1).

(c) $\|u\|_{0,p(\cdot)} > 1$ *implies* $\|u\|^{p^-}_{0,p(\cdot)} \le \rho_{0,p(\cdot)}(u) \le \|u\|^{P^+}_{0,p(\cdot)}$.

(d) $\|u\|_{0,p(\cdot)} < 1$ *implies* $\|u\|^{p^+}_{0,p(\cdot)} \le \rho_{0,p(\cdot)}(u) \le \|u\|^{p^-}_{0,p(\cdot)}$.

(e) *Let* $u \in L^{p(\cdot)}(\Omega)$ *and* $u_k \in L^{p(\cdot)}(\Omega)$, $k = 1, 2, \cdots$. *Then the following properties are equivalent:*

(1) $\|u - u_k\|_{0,p(\cdot)} \to 0$ *as* $k \to \infty$;

(2) $\rho_{0,p(\cdot)}(u_k - u) \to 0$ *as* $k \to \infty$;

(3) (u_k) *converges to* u *in measure and* $\rho_{0,p(\cdot)}(u_k) \to \rho_{0,p(\cdot)}(u)$ *as* $k \to \infty$.

(f) *Let* $v \in L^{p(\cdot)}(\Omega)$. *If a measurable function* $u : \Omega \to \mathbb{R}$ *satisfies*

$$|u(x)| \le |v(x)| \quad \text{for almost all } x \in \Omega,$$

then $\rho_{0,p(\cdot)}(u) \le \rho_{0,p(\cdot)}(v)$. *Consequently,* $u \in L^{p(\cdot)}(\Omega)$ *and* $\|u\|_{0,p(\cdot)} \le \|v\|_{0,p(\cdot)}$.

(g) *Given a nonzero function* $u \in L^{p(\cdot)}(\Omega)$, *the function* $\lambda \to \rho_{0,p(\cdot)}(\frac{u}{\lambda})$ *is continuous and decreasing on the interval* $[1, \infty)$.

For the proof of (a)–(e), see [14]; for the proof of (f)–(g), see [20].

Corollary 4.1 *Let* $p \in L^\infty(\Omega)$ *with* $p^- \ge 1$. *A subset* $A \subset (L^{p(\cdot)}(\Omega), \|\cdot\|_{0,p(\cdot)})$ *is bounded if and only if the set* $\rho_{0,p(\cdot)}(A) := \{\rho_{0,p(\cdot)}(u); u \in A\}$ *is bounded in* $\mathbb{R}_+$.

Proof From Theorem 4.2(a), (c) and (d), it follows that

$$\rho_{0,p(\cdot)}(u) \le \max(\|u\|^{p^-}_{0,p(\cdot)}, \|u\|^{p^+}_{0,p(\cdot)}) \quad \text{for all } u \in L^{p(\cdot)}(\Omega), \tag{4.2}$$

which easily implies that, if $A \subset L^{p(\cdot)}(\Omega)$ is bounded, then $\rho_{0,p(\cdot)}(A)$ is bounded in $\mathbb{R}_+$.

From Theorem 4.2(a) and (d), it also follows that

$$\begin{aligned}
&\|u\|_{0,p(\cdot)} = 1 \text{ implies } \|u\|_{0,p(\cdot)} = \rho_{0,p(\cdot)}(u) = 1,\\
&\|u\|_{0,p(\cdot)} > 1 \text{ implies } \|u\|_{0,p(\cdot)} < \|u\|_{0,p(\cdot)}^{p^-} \le \rho_{0,p(\cdot)}(u),\\
&\|u\|_{0,p(\cdot)} < 1 \text{ implies } \|u\|_{0,p(\cdot)} < \rho_{0,p(\cdot)}(u) + 1.
\end{aligned}$$

Hence

$$\|u\|_{0,p(\cdot)} < \rho_{0,p(\cdot)}(u) + 1 \quad \text{for all } u \in L^{p(\cdot)}(\Omega). \tag{4.3}$$

Clearly, (4.3) implies that, if $\rho_{0,p(\cdot)}(u) \le M$, then $\|u\|_{0,p(\cdot)} < 1 + M$.

Remark 4.1 Corollary 4.1 was first proved in [27]. We simply remark that the inequalities (4.2)–(4.3) above are slightly stronger than the inequalities

$$\rho_{0,p(\cdot)}(u) \le (1 + \|u\|_{0,p(\cdot)})^{p^+} \quad \text{for all } u \in L^{p(\cdot)}(\Omega),$$

and the implication

$$\rho_{0,p(\cdot)}(u) \le M \text{ implies } \|u\|_{0,p(\cdot)} \le \max(1 + M, 2)$$

established in [27].

Given a function $p(\cdot) \in L^\infty(\Omega)$ that satisfies $p^- \ge 1$, the *Sobolev space* $W^{1,p(\cdot)}(\Omega)$ with variable exponent $p(\cdot)$ is defined as

$$W^{1,p(\cdot)}(\Omega) := \{v \in L^{p(\cdot)}(\Omega); \partial_i v \in L^{p(\cdot)}(\Omega), 1 \le i \le N\},$$

where, for each $1 \le i \le N$, ∂_i denotes the distributional derivative operator with respect to the i-th variable.

Theorem 4.3 *Let Ω be a domain in $\mathbb{R}^N$.*

(a) *Let $p(\cdot) \in L^\infty(\Omega)$ be such that $p^- \ge 1$. Equipped with the norm*

$$v \in W^{1,p(\cdot)}(\Omega) \to \|v\|_{1,p(\cdot)} := \|v\|_{0,p(\cdot)} + \sum_{i=1}^{N} \|\partial_i v\|_{0,p(\cdot)},$$

the space $W^{1,p(\cdot)}(\Omega)$ is a separable Banach space. If $p^- > 1$, the space $W^{1,p(\cdot)}(\Omega)$ is reflexive.

(b) *Let $p_1(\cdot) \in L^\infty(\Omega)$ with $p_1^- \ge 1$ and $p_2(\cdot) \in L^\infty(\Omega)$ with $p_2^- \ge 1$ be such that*

$$p_1(x) \le p_2(x) \quad \textit{for almost all } x \in \Omega.$$

Then

$$W^{1,p_2(\cdot)}(\Omega) \hookrightarrow W^{1,p_1(\cdot)}(\Omega).$$

(c) *Let $p(\cdot) \in \mathcal{C}(\overline{\Omega})$ be such that $p^- \ge 1$. Given any $x \in \overline{\Omega}$, let*

$$p^*(x) := \frac{Np(x)}{N - p(x)} \quad \textit{if } p(x) < N \quad \textit{and} \quad p^*(x) := \infty \quad \textit{if } p(x) \ge N,$$

and let there be given a function $q(\cdot) \in \mathcal{C}(\overline{\Omega})$ *that satisfies*

$$1 \leq q(x) < p^*(x) \quad \textit{for each } x \in \overline{\Omega}.$$

Then the following compact injection holds:

$$W^{1,p(\cdot)}(\Omega) \Subset L^{q(\cdot)}(\Omega),$$

so that, in particular,

$$W^{1,p(\cdot)}(\Omega) \Subset L^{p(\cdot)}(\Omega).$$

(d) *The function defined by*

$$v \in W^{1,p(\cdot)}(\Omega) \to \|v\|_{1,p(\cdot),\nabla} := \|v\|_{0,p(\cdot)} + \||\nabla v|\|_{0,p(\cdot)}$$

is a norm on $W^{1,p(\cdot)}(\Omega)$*, equivalent with the norm* $\|\cdot\|_{1,p(\cdot)}$.

Theorem 4.4 *Let* Ω *be a domain in* $\mathbb{R}^N$.

(a) *Let* $p(\cdot) \in L^\infty(\Omega)$ *be such that* $p^- \geq 1$. *Since* $W^{1,p(\cdot)}(\Omega) \hookrightarrow W^{1,1}(\Omega)$ *(see Theorem* 4.3(b)), *the trace* $\mathrm{tr}\, v$ *on* Γ *of any function* $v \in W^{1,p(\cdot)}(\Omega)$ *is a well-defined function in the space* $L^1(\Gamma)$.

(b) *Let there be given a function* $p(\cdot) \in \mathcal{C}(\overline{\Omega})$ *such that* $p^- > 1$. *Given any* $x \in \Gamma$*, let*

$$p^\partial(x) := \frac{(N-1)p(x)}{N-p(x)} \quad \textit{if } \; p(x) < N \quad \textit{and} \quad p^\partial(x) := \infty \quad \textit{if } \; p(x) \geq N,$$

and let there be given a function $q(\cdot) \in \mathcal{C}(\Gamma)$ *such that*

$$1 \leq q(x) < p^\partial(x) \quad \textit{for each } x \in \Gamma.$$

Then, $\mathrm{tr}\, v \in L^{q(\cdot)}(\Gamma)$ *for any function* $v \in W^{1,p(\cdot)}(\Omega)$*, and the trace operator*

$$\mathrm{tr} : W^{1,p(\cdot)}(\Omega) \to L^{q(\cdot)}(\Gamma)$$

defined in this fashion is compact. In particular, the trace operator

$$\mathrm{tr} : W^{1,p(\cdot)}(\Omega) \to L^{p(\cdot)}(\Gamma)$$

is compact.

4.2 Definition of the space U_{Γ_0}

This space is defined via the following theorem, which was the main result of [5].

Theorem 4.5 *Let* Ω *be a domain in* $\mathbb{R}^N$, $N \geq 2$, *let* Γ_0 *be a* $\mathrm{d}\Gamma$*-measurable subset of* $\Gamma = \partial\Omega$ *that satisfies* $\mathrm{d}\Gamma\text{-meas}\, \Gamma_0 > 0$, *let* $p(\cdot) \in \mathcal{C}(\overline{\Omega})$ *be such that* $p(x) > 1$ *for all* $x \in \overline{\Omega}$, *and let*

$$U_{\Gamma_0} := \{u \in (W^{1,p(\cdot)}(\Omega), \|\cdot\|_{1,p(\cdot),\nabla}); \mathrm{tr}\, u = 0 \textit{ on } \Gamma_0\}.$$

Then:

(a) *The space* U_{Γ_0} *is closed in* $(W^{1,p(\cdot)}(\Omega), \|\cdot\|_{1,p(\cdot),\nabla})$*; hence* $(U_{\Gamma_0}, \|\cdot\|_{1,p(\cdot),\nabla})$ *is a separable reflexive Banach space.*

(b) *The function*

$$u \in U_{\Gamma_0} \to \|u\|_{0,p(\cdot),\nabla} := \||\nabla u|\|_{0,p(\cdot)}$$

is a norm on U_{Γ_0}*, equivalent with the norm* $\|\cdot\|_{1,p(\cdot),\nabla}$.

(c) *The norm* $\|u\|_{0,p(\cdot),\nabla}$ *is Fréchet-differentiable, and the Fréchet derivative* $\|\cdot\|_{0,p(\cdot),\nabla}{}'$ *of this norm at any nonzero element* $u \in U_{\Gamma_0}$ *is given for any* $h \in U_{\Gamma_0}$ *by*

$$\langle \|\cdot\|'_{0,p(\cdot),\nabla}(u), h\rangle = \frac{\displaystyle\int_{\Omega\setminus\Omega_{0,u}} p(x)\frac{|\nabla u(x)|^{p(x)-2}\ \nabla u(x)\cdot\nabla h(x)}{\|u\|^{p(x)-1}_{0,p(\cdot),\nabla}}\mathrm{d}x}{\displaystyle\int_\Omega p(x)\frac{|\nabla u(x)|^{p(x)}}{\|u\|^{p(x)}_{0,p(\cdot),\nabla}}\mathrm{d}x}, \tag{4.4}$$

where $\Omega_{0,u} := \{x \in \Omega; |\nabla u(x)| = 0\}$.

(d) *If, in addition, the function* $p(\cdot) \in \mathcal{C}(\overline{\Omega})$ *satisfies* $p(x) \geq 2$ *for all* $x \in \overline{\Omega}$*, then the space* $(U_{\Gamma_0}, \|\cdot\|_{0,p(\cdot),\nabla})$ *is uniformly convex.*

Some comments are in order about this theorem.

We first recall that, in the classical case (i.e., when p is any real number that satisfies $p > 1$), the following equalities hold:

$$\begin{aligned} W_0^{1,p}(\Omega) :&= \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{1,p,\nabla}} = \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{0,p(\cdot),\nabla}} \\ &= \overset{0}{W}{}^{1,p}(\Omega) := \{v \in W^{1,p}(\Omega); \mathrm{tr}\, v = 0 \text{ on } \Gamma = \partial\Omega\}. \end{aligned}$$

In the case of a variable exponent $p(\cdot) \in \mathcal{C}(\overline{\Omega})$ with $p^- > 1$, the equality $W_0^{1,p}(\Omega) = \overset{0}{W}{}^{1,p}(\Omega)$ is replaced by the inclusion $W_0^{1,p(\cdot)}(\Omega) \subset \overset{0}{W}{}^{1,p(\cdot)}(\Omega)$, which may be strict, unless additional assumptions are imposed on the function $p(\cdot)$. Such an assumption is, for example,

$$|p(x) - p(y)| \leq \frac{C}{|\ln\|x-y\||}$$

for all $x, y \in \overline{\Omega}$ with $\|x - y\| < \frac{1}{2}$, where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^N$ (see [8, 14, 28], for more details).

Thus, in the case of a variable exponent $p(\cdot) \in \mathcal{C}(\overline{\Omega})$ with $p^- > 1$, one has

$$W_0^{1,p(\cdot)}(\Omega) \subset \overset{0}{W}{}^{1,p(\cdot)}(\Omega) \subset U_{\Gamma_0},$$

the last inclusion being strict, even if additional assumptions are imposed.

But the results of Theorem 4.5 (in particular, the Fréchet differentiability of the norm $\|\cdot\|_{0,p(\cdot),\nabla}$ on U_{Γ_0}) can be of interest, even if additional assumptions on the function $p(\cdot)$ need to be imposed, so as to ensure that the equality $W_0^{1,p(\cdot)}(\Omega) = \overset{0}{W}{}^{1,p(\cdot)}(\Omega)$ holds.

4.3 Duality mappings on the space $(U_{\Gamma_0}, \|\cdot\|_{0,p(\cdot),\nabla})$

The main goal of this section is to establish the following theorem.

Theorem 4.6 *Let* Ω *be a domain in* $\mathbb{R}^N$*,* $N \geq 2$*, and let a function* $p \in \mathcal{C}(\overline{\Omega})$ *be such that* $p(x) > 1$ *for all* $x \in \overline{\Omega}$*. Then:*

(a) *Any duality mapping on the space* $(U_{\Gamma_0}, \|\cdot\|_{0,p(\cdot),\nabla})$ *satisfies the condition* $(\mathrm{S})_+$ *of Theorem* 2.1.

(b) *Any duality mapping on the space* $(U_{\Gamma_0}, \|\cdot\|_{0,p(\cdot),\nabla})$ *is injective.*

Proof The proof rests on four lemmas.

Lemma 4.1 *Let* X *be a real Banach space, and let* $T : X \to X^*$ *be monotone. If*

$$u_n \rightharpoonup u \quad \text{and} \quad \limsup_{n\to\infty} \langle Tu_n, u_n - u\rangle \le 0, \tag{4.5}$$

then

$$\lim_{n\to\infty} \langle Tu_n, u_n - u\rangle = \lim_{n\to\infty} \langle Tu_n - Tu, u_n - u\rangle = 0. \tag{4.6}$$

Proof The monotonicity of T and the assumption (4.5) together imply that

$$\begin{aligned} 0 &\le \liminf_{n\to\infty} \langle Tu_n - Tu, u_n - u\rangle \le \limsup_{n\to\infty} \langle Tu_n - Tu, u_n - u\rangle \\ &= \limsup_{n\to\infty} \langle Tu_n, u_n - u\rangle \le 0, \end{aligned}$$

and thus (4.6) follows.

Lemma 4.2 *Let* X *be a smooth real Banach space, and let* $J_\varphi : X \to X^*$ *be a duality mapping on* X*. Let* u *be a nonzero element in* X*, and let* $(u_n) \subset X$ *be a sequence such that*

$$u_n \rightharpoonup u \quad \text{and} \quad \limsup_{n\to\infty} \langle J_\varphi u_n, u_n - u\rangle \le 0.$$

Then

(a) *the sequence* $(J_\varphi u_n)$ *is bounded;*

(b) $\lim_{n\to\infty} \langle J_\varphi u_n, u_n - u\rangle = \lim_{n\to\infty} \langle J_\varphi u_n - J_\varphi u, u_n - u\rangle = 0$;

(c) $\|u_n\| \to \|u\|$ *as* $n \to \infty$;

(d) $\frac{u_n}{\|u_n\|} \rightharpoonup \frac{u}{\|u\|}$ *as* $n \to \infty$

and

$$\lim_{n\to\infty} \left\langle J_\varphi u_n, \frac{u_n}{\|u_n\|} - \frac{u}{\|u\|} \right\rangle = \lim_{n\to\infty} \left\langle J_\varphi u_n - J_\varphi u, \frac{u_n}{\|u_n\|} - \frac{u}{\|u\|} \right\rangle = 0.$$

Proof (a) Since the sequence (u_n) is weakly convergent, it is bounded. Since $\|J_\varphi u_n\| = \varphi(\|u_n\|)$, it follows that the sequence $(J_\varphi u_n)$ is also bounded.

(b) Since J_φ is monotone, the result follows by Lemma 4.1.

(c) Since the sequence $(\|u_n\|)$ is bounded, it suffices to show that $\|u\|$ is the unique cluster point to the sequence $(\|u_n\|)$. Let l be a cluster point to the sequence $(\|u_n\|)$, and let us consider a subsequence, also denoted by $(\|u_n\|)$, such that $\|u_n\| \to l$ as $n \to \infty$. We have (see (3.10))

$$\langle J_\varphi u_n - J_\varphi u, u_n - u\rangle \ge (\varphi(\|u_n\|) - \varphi(\|u\|))(\|u_n\| - \|u\|) \ge 0.$$

Letting $n \to \infty$ and taking into account the result of (b), we obtain

$$(\varphi(l) - \varphi(\|u\|))(l - \|u\|) = 0.$$

Since φ is strictly increasing, this implies that $l = \|u\|$.

(d) Since $u_n \rightharpoonup u$ and $\|u_n\| \to \|u\|$ as $n \to \infty$, it is easily seen that $\frac{u_n}{\|u_n\|} \rightharpoonup \frac{u}{\|u\|}$ as $n \to \infty$. Moreover

$$\Big\langle J_\varphi u_n, \frac{u_n}{\|u_n\|} - \frac{u}{\|u\|} \Big\rangle = \frac{1}{\|u_n\|} \langle J_\varphi u_n, u_n - u\rangle + \Big(\frac{1}{\|u_n\|} - \frac{1}{\|u\|} \Big) \langle J_\varphi u_n, u\rangle.$$

From (b) and (c), it follows that

$$\frac{1}{\|u_n\|} \langle J_\varphi u_n, u_n - u\rangle \to 0 \quad \text{as } n \to \infty,$$

and from (a) and (c), it follows that

$$\Big(\frac{1}{\|u_n\|} - \frac{1}{\|u\|} \Big) \langle J_\varphi u_n, u\rangle \to 0 \quad \text{as } n \to \infty.$$

Consequently,

$$\Big\langle J_\varphi u_n, \frac{u_n}{\|u_n\|} - \frac{u}{\|u\|} \Big\rangle \to 0 \quad \text{as } n \to \infty.$$

Since $\frac{u_n}{\|u_n\|} \rightharpoonup \frac{u}{\|u\|}$ as $n \to \infty$, this last relation is equivalent to

$$\lim_{n\to\infty} \Big\langle J_\varphi u_n - J_\varphi u, \frac{u_n}{\|u_n\|} - \frac{u}{\|u\|} \Big\rangle = 0.$$

Remark 4.2 Recall that a Banach space is said to possess the *Kadeč-Klee property* (see, e.g., [7, p. 146], where this property is called the "H-property", [9, Theorem 4 in Chapter 2], and [18]) if

$$u_n \rightharpoonup u \text{ and } \|u_n\| \to \|u\| \text{ imply } u_n \to u.$$

We derive the following corollary from Lemma 4.2(c).

Corollary 4.2 *Any duality mapping on a smooth real Banach space having the Kadeč-Klee property satisfies the condition* $(\mathrm{S})_+$.

Remark 4.3 Locally uniformly convex Banach spaces possess the Kadeč-Klee property (see [9, Theorem 4(iii) in Chapter 2]). In virtue of Theorem 4.5(d), if $p \in C(\overline{\Omega})$ satisfies $p(x) \geq 2$ for all $x \in \overline{\Omega}$, then the space $(U_{\Gamma_0}, \|\cdot\|_{0,p(\cdot),\nabla})$ is uniformly convex. In particular, $(U_{\Gamma_0}, \|\cdot\|_{0,p(\cdot),\nabla})$ possesses the Kadeč-Klee property and, by Corollary 4.2, any duality mapping on $(U_{\Gamma_0}, \|\cdot\|_{0,p(\cdot),\nabla})$ satisfies the condition $(\mathrm{S})_+$. The significance of Theorem 4.6 is now clear: It allows to extend the preceding result to the more general case, where

$$p \in \mathcal{C}(\overline{\Omega}) \quad \text{and} \quad p(x) > 1 \quad \text{for all } x \in \overline{\Omega}.$$

The following technical preliminary will be useful for proving Theorem 4.6.

Lemma 4.3 (see [17]) (a) *If* $p \in (1,2)$, *then for any* $z, y \in \mathbb{R}^N$,

$$(|z| + |y|)^{2-p} (|z|^{p-2} z - |y|^{p-2} y) \cdot (z - y) \geq |z - y|^2.$$

(b) *If* $p \in [2, \infty)$, *then for any* $z, y \in \mathbb{R}^N$,

$$(|z|^{p-2} z - |y|^{p-2} y) \cdot (z - y) \geq k(p) |z - y|^p,$$

where $k(p) = \min\{2^{-1-p}, 5^{2-p}\}$.

We are now in a position to give the proof of Theorem 4.6(a). Let $J_\varphi : U_{\Gamma_0} \to U_{\Gamma_0}^*$ be a duality mapping. We need to prove that

$$u_n \rightharpoonup u \text{ in } (U_{\Gamma_0}, \|\cdot\|_{0,p(\cdot),\nabla}) \quad \text{and} \quad \limsup_{n\to\infty} \langle J_\varphi u_n, u_n - u\rangle \le 0$$

imply that

$$u_n \to u \text{ in } (U_{\Gamma_0}, \|\cdot\|_{0,p(\cdot),\nabla}).$$

If $u = 0_{U_{\Gamma_0}}$, then the relations $u_n \rightharpoonup 0_{U_{\Gamma_0}}$ and $\limsup_{n\to\infty} \langle J_\varphi u_n, u_n\rangle \le 0$ and Lemma 4.2(c) together imply that $\|u_n\|_{0,p(\cdot),\nabla} \to \|u\|_{0,p(\cdot),\nabla} = 0$; hence $u_n \to 0_{U_{\Gamma_0}}$. Assume next that $u \neq 0_{U_{\Gamma_0}}$. It then suffices to prove that

$$u_n \rightharpoonup u \quad \text{and} \quad \limsup_{n\to\infty} \langle J_\varphi u_n, u_n - u\rangle \le 0$$

imply

$$\rho_{0,p(\cdot)}\Big(\Big|\nabla\Big(\frac{u_n}{\|u_n\|_{0,p(\cdot),\nabla}} - \frac{u}{\|u\|_{0,p(\cdot),\nabla}}\Big)\Big|\Big) \to 0 \quad \text{as } n \to \infty. \tag{4.7}$$

Indeed, assuming that (4.7) holds, we infer from Theorem 4.2(e) that

$$\Big\|\Big|\nabla\Big(\frac{u_n}{\|u_n\|_{0,p(\cdot),\nabla}} - \frac{u}{\|u\|_{0,p(\cdot),\nabla}}\Big)\Big|\Big\|_{0,p(\cdot)} = \Big\|\frac{u_n}{\|u_n\|_{0,p(\cdot),\nabla}} - \frac{u}{\|u\|_{0,p(\cdot),\nabla}}\Big\|_{0,p(\cdot),\nabla} \to 0$$

as $n \to \infty$.

On the other hand, it follows from the relations $u_n \rightharpoonup u$ and $\limsup_{n\to\infty} \langle J_\varphi u_n, u_n - u\rangle \le 0$, and from Lemma 4.2(c), that $\|u_n\|_{0,p(\cdot),\nabla} \to \|u\|_{0,p(\cdot),\nabla}$.

Since

$$\begin{aligned}
\|u_n - u\|_{0,p(\cdot),\nabla} &= \|u_n\|_{0,p(\cdot),\nabla}\Big\|\frac{u_n}{\|u_n\|_{0,p(\cdot),\nabla}} - \frac{u}{\|u_n\|_{0,p(\cdot),\nabla}}\Big\|_{0,p(\cdot),\nabla} \\
&\le \|u_n\|_{0,p(\cdot),\nabla}\Big[\Big(\Big\|\frac{u_n}{\|u_n\|_{0,p(\cdot),\nabla}} - \frac{u}{\|u\|_{0,p(\cdot),\nabla}}\Big\|_{0,p(\cdot),\nabla}\Big) \\
&\qquad + \Big|\frac{1}{\|u\|_{0,p(\cdot),\nabla}} - \frac{1}{\|u_n\|_{0,p(\cdot),\nabla}}\Big|\|u\|_{0,p(\cdot),\nabla}\Big]
\end{aligned}$$

and since the sequence (u_n) is bounded, it follows that

$$\|u_n - u\|_{0,p(\cdot),\nabla} \to 0 \quad \text{as } n \to \infty.$$

Thus proving (a) reduces to showing that (4.7) holds, as we now show. For any nonzero v and w in U_{Γ_0}, let

$$B(v,w) := \int_{\Omega\setminus\Omega_{0,v}} p(x)\frac{|\nabla v(x)|^{p(x)-2}\ \nabla v(x)\cdot\Big(\frac{\nabla v(x)}{\|v\|_{0,p(\cdot),\nabla}} - \frac{\nabla w(x)}{\|w\|_{0,p(\cdot),\nabla}}\Big)}{\|v\|_{0,p(\cdot),\nabla}^{p(x)-1}}\mathrm{d}x,$$

or equivalently (see (3.9) and (4.4)),

$$B(v,w) = \Big(\int_\Omega p(x)\frac{|\nabla v(x)|^{p(x)}}{\|v\|_{0,p(\cdot),\nabla}^{p(x)}}\mathrm{d}x\Big) \times \frac{1}{\varphi(\|v\|_{0,p(\cdot),\nabla})}\Big\langle J_\varphi v, \frac{v}{\|v\|_{0,p(\cdot),\nabla}} - \frac{w}{\|w\|_{0,p(\cdot),\nabla}}\Big\rangle. \tag{4.8}$$

Since

$$p^- \le \int_\Omega p(x)\frac{|\nabla u_n(x)|^{p(x)}}{\|u_n\|_{0,p(\cdot),\nabla}^{p(x)}}\mathrm{d}x \le p^+,$$
$$\varphi(\|u_n\|_{0,p(\cdot),\nabla}) \to \varphi(\|u\|_{0,p(\cdot),\nabla})$$

and (see Lemma 4.2(d))

$$\Big\langle J_\varphi u_n, \frac{u_n}{\|u_n\|_{0,p(\cdot),\nabla}} - \frac{u}{\|u\|_{0,p(\cdot),\nabla}}\Big\rangle \to 0 \quad \text{as } n \to \infty,$$

it is clear that

$$B(u_n,u) = \Big(\int_\Omega p(x)\frac{|\nabla u_n(x)|^{p(x)}}{\|u_n\|_{0,p(\cdot),\nabla}^{p(x)}}\mathrm{d}x\Big)\frac{1}{\varphi(\|u_n\|_{0,p(\cdot),\nabla})} \times \Big\langle J_\varphi u_n, \frac{u_n}{\|u_n\|_{0,p(\cdot),\nabla}} - \frac{u}{\|u\|_{0,p(\cdot),\nabla}}\Big\rangle \to 0 \quad \text{as } n \to \infty.$$

By Lemma 4.2(d) again, we also infer that

$$B(u,u_n) = -\Big(\int_\Omega p(x)\frac{|\nabla u(x)|^{p(x)}}{\|u\|_{0,p(\cdot),\nabla}^{p(x)}}\mathrm{d}x\Big)\frac{1}{\varphi(\|u\|_{0,p(\cdot),\nabla})} \times \Big\langle J_\varphi u, \frac{u_n}{\|u_n\|_{0,p(\cdot),\nabla}} - \frac{u}{\|u\|_{0,p(\cdot),\nabla}}\Big\rangle \to 0 \quad \text{as } n \to \infty.$$

Consequently,

$$\lim_{n\to\infty} |B(u_n,u) + B(u,u_n)| = 0. \tag{4.9}$$

The definition of B (see (4.8)) shows that the relation (4.9) can be rewritten as

$$\lim_{n\to\infty}\int_\Omega [w_n(x) - v_n(x)]\mathrm{d}x = 0,$$

where

$$w_n(x) := p(x)\frac{|\nabla u_n(x)|^{p(x)-2}\ \nabla u_n(x)\cdot\Big(\frac{\nabla u_n(x)}{\|u_n\|_{0,p(\cdot),\nabla}} - \frac{\nabla u(x)}{\|u\|_{0,p(\cdot),\nabla}}\Big)}{\|u_n\|_{0,p(\cdot),\nabla}^{p(x)-1}}, \quad \text{if } x \in \Omega\backslash\Omega_{0,u_n},$$
$$w_n(x) := 0, \quad \text{if } x \in \Omega_{0,u_n}$$

and

$$v_n(x) := p(x)\frac{|\nabla u(x)|^{p(x)-2}\ \nabla u(x)\cdot\Big(\frac{\nabla u_n(x)}{\|u_n\|_{0,p(\cdot),\nabla}} - \frac{\nabla u(x)}{\|u\|_{0,p(\cdot),\nabla}}\Big)}{\|u\|_{0,p(\cdot),\nabla}^{p(x)-1}}, \quad \text{if } \Omega\backslash\Omega_{0,u},$$
$$v_n(x) := 0, \quad \text{if } x \in \Omega_{0,u}.$$

We now show that

$$w_n(x) - v_n(x) \geq 0 \quad \text{for almost all } x \in \Omega. \tag{4.10}$$

Indeed, for $x \in \Omega \backslash (\Omega_{0,u_n} \cup \Omega_{0,u})$,

$$\begin{aligned} & w_n(x) - v_n(x) \\ = & \, p(x)\Big(\Big|\frac{|\nabla u_n(x)|}{\|u_n\|_{0,p(\cdot),\nabla}}\Big|^{p(x)-2}\frac{\nabla u_n(x)}{\|u_n\|_{0,p(\cdot),\nabla}} - \Big|\frac{|\nabla u(x)|}{\|u\|_{0,p(\cdot),\nabla}}\Big|^{p(x)-2}\frac{\nabla u(x)}{\|u\|_{0,p(\cdot),\nabla}}\Big) \\ & \cdot \Big(\frac{\nabla u_n(x)}{\|u_n\|_{0,p(\cdot),\nabla}} - \frac{\nabla u(x)}{\|u\|_{0,p(\cdot),\nabla}}\Big), \end{aligned}$$

and the inequality (4.10) follows by Lemma 4.3 applied with

$$z := \frac{\nabla u_n(x)}{\|u_n\|_{0,p(\cdot),\nabla}} \quad \text{and} \quad y := \frac{\nabla u(x)}{\|u\|_{0,p(\cdot),\nabla}}.$$

Let $x \in \Omega_{0,u_n} \cup \Omega_{0,u} = (\Omega_{0,u_n} \backslash \Omega_{0,u}) \cup (\Omega_{0,u} \backslash \Omega_{0,u_n}) \cup (\Omega_{0,u_n} \cap \Omega_{0,u})$. If $x \in \Omega_{0,u_n} \backslash \Omega_{0,u}$, then

$$w_n(x) - v_n(x) = p(x)\frac{|\nabla u(x)|^{p(x)}}{\|u\|^{p(x)}_{0,p(\cdot),\nabla}} \geq 0;$$

if $x \in \Omega_{0,u} \backslash \Omega_{0,u_n}$, then

$$w_n(x) - v_n(x) = p(x)\frac{|\nabla u_n(x)|^{p(x)}}{\|u_n\|^{p(x)}_{0,p(\cdot),\nabla}} \geq 0;$$

if $x \in \Omega_{0,u_n} \cap \Omega_{0,u}$, then

$$w_n(x) - v_n(x) = 0.$$

We are now in a position to show that (4.7) holds. To this end, let

$$\Omega_1 := \{x \in \Omega; 1 < p(x) < 2\} \quad \text{and} \quad \Omega_2 = \{x \in \Omega; 2 \leq p(x)\}.$$

It then suffices to show that

$$\lim_{n\to\infty} \int_{\Omega_1} \Big|\frac{\nabla u_n(x)}{\|u_n\|_{0,p(\cdot),\nabla}} - \frac{\nabla u(x)}{\|u\|_{0,p(\cdot),\nabla}}\Big|^{p(x)} \mathrm{d}x = 0, \tag{4.11}$$

and that

$$\lim_{n\to\infty} \int_{\Omega_2} \Big|\frac{\nabla u_n(x)}{\|u_n\|_{0,p(\cdot),\nabla}} - \frac{\nabla u(x)}{\|u\|_{0,p(\cdot),\nabla}}\Big|^{p(x)} \mathrm{d}x = 0.$$

Applying Lemma 4.3(a) with $z := \frac{\nabla u_n(x)}{\|u_n\|_{0,p(\cdot),\nabla}}$ and $y := \frac{\nabla u(x)}{\|u\|_{0,p(\cdot),\nabla}}$, we obtain

$$\begin{aligned} & (w_n(x) - v_n(x))\Big(\frac{|\nabla u_n(x)|}{\|u_n\|_{0,p(\cdot),\nabla}} + \frac{|\nabla u(x)|}{\|u\|_{0,p(\cdot),\nabla}}\Big)^{2-p(x)} \\ \geq & \, p^-\Big|\frac{\nabla u_n(x)}{\|u_n\|_{0,p(\cdot),\nabla}} - \frac{\nabla u(x)}{\|u\|_{0,p(\cdot),\nabla}}\Big|^2 \quad \text{for } x \in \Omega_1. \end{aligned}$$

Hence it follows that

$$\begin{aligned}
&\int_{\Omega_1}\Big|\frac{\nabla u_n(x)}{\|u_n\|_{0,p(\cdot),\nabla}}-\frac{\nabla u(x)}{\|u\|_{0,p(\cdot),\nabla}}\Big|^{p(x)}\mathrm{d}x\\
&=\int_{\Omega_1}\Big|\frac{\nabla u_n(x)}{\|u_n\|_{0,p(\cdot),\nabla}}-\frac{\nabla u(x)}{\|u\|_{0,p(\cdot),\nabla}}\Big|^{2\frac{p(x)}{2}}\mathrm{d}x\\
&\le\int_{\Omega_1}\Big(\frac{1}{p^-}\Big)^{\frac{p(x)}{2}}\Big(\frac{|\nabla u_n(x)|}{\|u_n\|_{0,p(\cdot),\nabla}}+\frac{|\nabla u(x)|}{\|u\|_{0,p(\cdot),\nabla}}\Big)^{(2-p(x))\frac{p(x)}{2}}(w_n(x)-v_n(x))^{\frac{p(x)}{2}}\mathrm{d}x\\
&\le\Big(\frac{1}{p^-}\Big)^{\frac{p^-}{2}}\int_{\Omega_1}\Big(\frac{|\nabla u_n(x)|}{\|u_n\|_{0,p(\cdot),\nabla}}+\frac{|\nabla u(x)|}{\|u\|_{0,p(\cdot),\nabla}}\Big)^{(2-p(x))\frac{p(x)}{2}}(w_n(x)-v_n(x))^{\frac{p(x)}{2}}\mathrm{d}x.
\end{aligned}$$

Since

$$\begin{aligned}
&\Big(\frac{|\nabla u_n|}{\|u_n\|_{0,p(\cdot),\nabla}}+\frac{|\nabla u|}{\|u\|_{0,p(\cdot),\nabla}}\Big)^{(2-p(\cdot))\frac{p(\cdot)}{2}}\in L^{\frac{2}{2-p(\cdot)}}(\Omega_1),\\
&(w_n-v_n)^{\frac{p(\cdot)}{2}}\in L^{\frac{2}{p(\cdot)}}(\Omega_1)
\end{aligned}$$

and since

$$\Big(\frac{2}{2-p(x)}\Big)^{-1}+\Big(\frac{2}{p(x)}\Big)^{-1}=1,$$

it follows from Theorem 4.1(d) that

$$\begin{aligned}
&\int_{\Omega_1}\Big|\frac{\nabla u_n(x)}{\|u_n\|_{0,p(\cdot),\nabla}}-\frac{\nabla u(x)}{\|u\|_{0,p(\cdot),\nabla}}\Big|^{p(x)}\mathrm{d}x\\
&\le C_1\Big\|\Big(\frac{|\nabla u_n|}{\|u_n\|_{0,p(\cdot),\nabla}}+\frac{|\nabla u|}{\|u\|_{0,p(\cdot),\nabla}}\Big)^{(2-p(\cdot))\frac{p(\cdot)}{2}}\Big\|_{0,\frac{2}{2-p(\cdot)},\Omega_1}\|(w_n-v_n)^{\frac{p(\cdot)}{2}}\|_{0,\frac{2}{p(\cdot)},\Omega_1},
\end{aligned}$$

with

$$C_1:=\Big(\frac{1}{p^-}\Big)^{\frac{p^-}{2}}\Big(\frac{1}{\big(\frac{2}{2-p(\cdot)}\big)^-_{\Omega_1}}+\frac{1}{\big(\frac{2}{p(\cdot)}\big)^-_{\Omega_1}}\Big).$$

To conclude that (4.11) holds, we now show that

$$\|(w_n-v_n)^{\frac{p(\cdot)}{2}}\|_{0,\frac{2}{p(\cdot)},\Omega_1}\to 0\quad\text{ as } n\to\infty, \tag{4.12}$$

and that there exists a constant C_2 such that

$$\Big\|\Big(\frac{|\nabla u_n|}{\|u_n\|_{0,p(\cdot),\nabla}}+\frac{|\nabla u|}{\|u\|_{0,p(\cdot),\nabla}}\Big)^{(2-p(\cdot))\frac{p(\cdot)}{2}}\Big\|_{0,\frac{2}{2-p(\cdot)},\Omega_1}\le C_2. \tag{4.13}$$

Indeed,

$$\rho_{0,\frac{2}{p(\cdot)},\Omega_1}((w_n-v_n)^{\frac{p(\cdot)}{2}})=\int_{\Omega_1}(w_n(x)-v_n(x))\mathrm{d}x,$$

and, since

$$0\le\int_{\Omega_1}(w_n(x)-v_n(x))\mathrm{d}x\le\int_{\Omega}(w_n(x)-v_n(x))\mathrm{d}x\to 0\quad\text{ as } n\to\infty,$$

it follows that

$$\rho_{0,\frac{2}{p(\cdot)},\Omega_1}((w_n-v_n)^{\frac{p(\cdot)}{2}})\to 0\quad\text{ as } n\to\infty.$$

Equivalently, (4.12) holds. On the other hand,

$$\begin{aligned}
&\rho_{0,\frac{2}{2-p(\cdot)},\Omega_1}\Big(\Big(\frac{|\nabla u_n|}{\|u_n\|_{0,p(\cdot),\nabla}}+\frac{|\nabla u|}{\|u\|_{0,p(\cdot),\nabla}}\Big)^{(2-p(\cdot))\frac{p(\cdot)}{2}}\Big)\\
&=\rho_{0,p(\cdot),\Omega_1}\Big(\frac{|\nabla u_n|}{\|u_n\|_{0,p(\cdot),\nabla}}+\frac{|\nabla u|}{\|u\|_{0,p(\cdot),\nabla}}\Big)\\
&\le\rho_{0,p(\cdot),\Omega}\Big(\frac{|\nabla u_n|}{\|u_n\|_{0,p(\cdot),\nabla}}+\frac{|\nabla u|}{\|u\|_{0,p(\cdot),\nabla}}\Big)\\
&\le 2^{p^+-1}\Big(\rho_{0,p(\cdot),\Omega}\Big(\frac{|\nabla u_n|}{\|u_n\|_{0,p(\cdot),\nabla}}\Big)+\rho_{0,p(\cdot),\Omega}\Big(\frac{|\nabla u|}{\|u\|_{0,p(\cdot),\nabla}}\Big)\Big)=2^{p^+}.
\end{aligned}$$

Hence we infer from Corollary 4.1 that (4.13) holds with $C_2 := 2^{p^+}+1$.

By Lemma 4.3(b) applied with $z := \frac{\nabla u_n(x)}{\|u_n\|_{0,p(\cdot),\nabla}}$ and $y := \frac{\nabla u(x)}{\|u\|_{0,p(\cdot),\nabla}}$, we infer that

$$w_n(x)-v_n(x)\ge p^-k_1\Big|\frac{\nabla u_n(x)}{\|u_n\|_{0,p(\cdot),\nabla}}-\frac{\nabla u(x)}{\|u\|_{0,p(\cdot),\nabla}}\Big|^{p(x)},\quad \text{if } x\in\Omega_2,$$

where $k_1 := \min(2^{-1-p^+}, 5^{2-p^+})$. We thus conclude that

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\Omega_2}\Big|\frac{\nabla u_n(x)}{\|u_n\|_{0,p(\cdot),\nabla}}-\frac{\nabla u(x)}{\|u\|_{0,p(\cdot),\nabla}}\Big|^{p(x)}\mathrm{d}x\\
&\le (p^-k_1)^{-1}\lim_{n\to\infty}\int_{\Omega_2}(w_n(x)-v_n(x))\mathrm{d}x\\
&\le (p^-k_1)^{-1}\lim_{n\to\infty}\int_{\Omega}(w_n(x)-v_n(x))\mathrm{d}x=0.
\end{aligned}$$

Thus, relation (4.7) is proved and, as already observed, it follows from (4.7) that $u_n \to u$ as $n \to \infty$.

We now prove Theorem 4.6(b). To this end, we need the following preliminary result.

Lemma 4.4 *Let $u, v \in U_{\Gamma_0}$ be such that*

$$\|u\|_{0,p(\cdot),\nabla}=\|v\|_{0,p(\cdot),\nabla}=1 \tag{4.14}$$

and

$$(\mathrm{grad}\,\|\cdot\|_{0,p(\cdot),\nabla})(u)=(\mathrm{grad}\,\|\cdot\|_{0,p(\cdot),\nabla})(v). \tag{4.15}$$

Then $u = v$.

Proof From (3.7)–(3.8) and (4.14)–(4.15), we infer that

$$\langle(\mathrm{grad}\,\|\cdot\|_{0,p(\cdot),\nabla})(u),v\rangle=\langle(\mathrm{grad}\,\|\cdot\|_{0,p(\cdot),\nabla})(v),v\rangle=\|v\|_{0,p(\cdot),\nabla}=1. \tag{4.16}$$

Taking into account the formula (4.4), we can rewrite (4.16) as

$$\int_{\Omega\setminus\Omega_{0,u}}p(x)[|\nabla u(x)|^{p(x)-2}\ \nabla u(x)\cdot(\nabla u(x)-\nabla v(x))]\mathrm{d}x=0, \tag{4.17}$$

and, exchanging u with v, we also get

$$\int_{\Omega\setminus\Omega_{0,v}} p(x)[|\nabla v(x)|^{p(x)-2}\nabla v(x)\cdot(\nabla u(x)-\nabla v(x))]\mathrm{d}x = 0, \tag{4.18}$$

Substracting (4.18) from (4.17), we get

$$\int_\Omega p(x)[f(x)-g(x)]\mathrm{d}x = 0, \tag{4.19}$$

where

$$f(x) := |\nabla u(x)|^{p(x)-2}\nabla u(x)\cdot(\nabla u(x)-\nabla v(x)), \quad \text{if } x\in\Omega\setminus\Omega_{0,u},$$
$$f(x) := 0, \quad \text{if } x\in\Omega_{0,u}$$

and

$$g(x) := |\nabla v(x)|^{p(x)-2}\nabla v(x)\cdot(\nabla u(x)-\nabla v(x)), \quad \text{if } x\in\Omega\setminus\Omega_{0,v},$$
$$g(x) := 0, \quad \text{if } x\in\Omega_{0,v}.$$

Similarly, we can show that

$$f(x)-g(x)\geq 0 \quad \text{for almost all } x\in\Omega, \tag{4.20}$$

by means of a proof analogous to that of inequality (4.10) (the details are left to the reader). Combining (4.19)–(4.20), we then get

$$0 = \int_\Omega p(x)[f(x)-g(x)]\mathrm{d}x \geq p^- \int_\Omega [f(x)-g(x)]\mathrm{d}x \geq 0,$$

which implies that

$$\int_\Omega [f(x)-g(x)]\mathrm{d}x = 0.$$

We next show that

$$\rho_{0,p(\cdot)}(|\nabla u-\nabla v|) = \int_\Omega |\nabla u(x)-\nabla v(x)|^{p(x)}\mathrm{d}x = 0. \tag{4.21}$$

Let

$$\Omega_1 := \{x\in\Omega;\ 1<p(x)<2\} \quad \text{and} \quad \Omega_2 := \{x\in\Omega;\ 2\leq p(x)\}.$$

It is sufficient to show that

$$\int_{\Omega_1} |\nabla u(x)-\nabla v(x)|^{p(x)}\mathrm{d}x = 0, \tag{4.22}$$

and that

$$\int_{\Omega_2} |\nabla u(x)-\nabla v(x)|^{p(x)}\mathrm{d}x = 0. \tag{4.23}$$

By applying Lemma 4.3(a) with $z := \nabla u(x)$ and $y := \nabla v(x)$, we obtain

$$[f(x)-g(x)](|\nabla u(x)|+|\nabla v(x)|)^{2-p(x)} \geq |\nabla u(x)-\nabla v(x)|^2 \quad \text{for } x\in\Omega_1.$$

Hence it follows that

$$\begin{aligned}&\int_{\Omega_1} |\nabla u(x) - \nabla v(x)|^{p(x)} \mathrm{d}x \\ &= \int_{\Omega_1} |\nabla u(x) - \nabla v(x)|^{2\frac{p(x)}{2}} \mathrm{d}x \\ &\le \int_{\Omega_1} (|\nabla u(x)| + |\nabla v(x)|)^{(2-p(x))\frac{p(x)}{2}} [f(x) - g(x)]^{\frac{p(x)}{2}} \mathrm{d}x.\end{aligned}$$

Since the functions

$$\begin{aligned}&x \in \Omega_1 \mapsto (|\nabla u(x)| + |\nabla v(x)|)^{(2-p(x))\frac{p(x)}{2}}, \\ &x \in \Omega_2 \mapsto [f(x) - g(x)]^{\frac{p(x)}{2}}\end{aligned}$$

belong to the spaces $L^{\frac{2}{2-p(\cdot)}}(\Omega_1)$ and $L^{\frac{2}{p(\cdot)}}(\Omega_1)$, respectively, and since

$$\Big(\frac{2}{2-p(x)}\Big)^{-1} + \Big(\frac{2}{p(x)}\Big)^{-1} = 1,$$

it follows that

$$\begin{aligned}&\int_{\Omega_1} |\nabla u(x) - \nabla v(x)|^{p(x)} \mathrm{d}x \\ &\le C_1 \|(|\nabla u| + |\nabla v|)^{(2-p(\cdot))\frac{p(\cdot)}{2}}\|_{0,\frac{2}{2-p(\cdot)},\Omega_1} \|(f-g)^{\frac{p(\cdot)}{2}}\|_{0,\frac{2}{p(\cdot)},\Omega_1}\end{aligned}$$

with

$$C_1 := \Big(\frac{1}{p^-}\Big)^{\frac{p^-}{2}} \Big(\frac{1}{\big(\frac{2}{2-p(\cdot)}\big)^-_{\Omega_1}} + \frac{1}{\big(\frac{2}{p(\cdot)}\big)^-_{\Omega_1}}\Big).$$

To conclude that (4.22) holds, we now show that

$$\|(f-g)^{\frac{p(\cdot)}{2}}\|_{0,\frac{2}{p(\cdot)},\Omega_1} = 0. \tag{4.24}$$

Indeed, one has

$$\rho_{0,\frac{2}{p(\cdot)},\Omega_1}((f-g)^{\frac{p(\cdot)}{2}}) = \int_{\Omega_1} (f(x) - g(x)) \mathrm{d}x,$$

and, since

$$0 \le \int_{\Omega_1} (f(x) - g(x)) \mathrm{d}x \le \int_{\Omega} (f(x) - g(x)) \mathrm{d}x = 0,$$

it follows that

$$\rho_{0,\frac{2}{p(\cdot)},\Omega_1}((f-g)^{\frac{p(\cdot)}{2}}) = 0,$$

or equivalently, (4.24) holds.

Lemma 4.3(b) applied with $z := \nabla u(x)$ and $y := \nabla v(x)$ shows that

$$f(x) - g(x) \ge k_1 |\nabla u(x) - \nabla v(x)|^{p(x)}, \quad \text{if } x \in \Omega_2,$$

where $k_1 := \min(2^{-1-p^+}, 5^{2-p^+})$. Consequently,

$$\begin{aligned}&\int_{\Omega_2} |\nabla u(x) - \nabla v(x)|^{p(x)} \mathrm{d}x \\ &\le k_1^{-1} \int_{\Omega_2} (f(x) - g(x)) \mathrm{d}x \\ &\le k_1^{-1} \int_{\Omega} (f(x) - g(x)) \mathrm{d}x = 0.\end{aligned}$$

Thus, relation (4.23) is proved.

Clearly, (4.21) is a direct consequence of (4.22)–(4.23). On the other hand, (4.21) is equivalent to

$$\|u - v\|_{0,p(\cdot),\nabla} = 0,$$

which means that $u = v$. This completes the proof.

We are now in a position to give the proof of Theorem 4.6(b). By Proposition 3.1, it is sufficient to prove that the normalized duality mapping is injective on U_{Γ_0}.

So, let $u, v \in U_{\Gamma_0}$ be such that $Ju = Jv$. If $Ju = Jv = 0$, it is easily seen that $u = v = 0$. Assume that $Ju = Jv \neq 0$. Hence $u \neq 0$, $v \neq 0$, $\|u\|_{0,p(\cdot),\nabla} = \|v\|_{0,p(\cdot),\nabla}$, and

$$(\mathrm{grad}\,\|\cdot\|_{0,p(\cdot),\nabla})(u) = (\mathrm{grad}\,\|\cdot\|_{0,p(\cdot),\nabla})(v). \tag{4.25}$$

Since (see [25, Lemma 2.5])

$$(\mathrm{grad}\,\|\cdot\|)(\alpha w) = \mathrm{sign}\,\alpha(\mathrm{grad}\,\|\cdot\|)(w), \quad \alpha \neq 0,\ w \neq 0,$$

it follows from (4.25) that

$$(\mathrm{grad}\,\|\cdot\|_{0,p(\cdot),\nabla})\Big(\frac{u}{\|u\|_{0,p(\cdot),\nabla}}\Big) = (\mathrm{grad}\,\|\cdot\|_{0,p(\cdot),\nabla})\Big(\frac{v}{\|v\|_{0,p(\cdot),\nabla}}\Big).$$

By Lemma 4.4, we then conclude that $u = v$.

Corollary 4.3 *Let Ω be a domain in $\mathbb{R}^N$, $N \geq 2$, and let $p \in \mathcal{C}(\overline{\Omega})$.*

(a) *If $p(x) > 1$ for all $x \in \overline{\Omega}$, then $(U_{\Gamma_0}, \|\cdot\|_{0,p(\cdot),\nabla})$ is strictly convex.*

(b) *If $p(x) \geq 2$ for all $x \in \overline{\Omega}$, then $(U_{\Gamma_0}, \|\cdot\|_{0,p(\cdot),\nabla})$ is uniformly convex.*

Proof (a) Since $(U_{\Gamma_0}, \|\cdot\|_{0,p(\cdot),\nabla})$ is reflexive and smooth (see Theorem 4.5(a)–(c)) and since any duality mapping on $(U_{\Gamma_0}, \|\cdot\|_{0,p(\cdot),\nabla})$ is injective (see Theorem 4.6(b)), the result follows by Theorem 3.1.

(b) This is nothing but Theorem 4.5(d).

Corollary 4.4 *Let Ω be a domain in $\mathbb{R}^N$, $N \geq 2$, and let $p \in \mathcal{C}(\overline{\Omega})$ be such that $p(x) > 1$ for all $x \in \overline{\Omega}$. Then any duality mapping on U_{Γ_0} is a homeomorphism.*

Proof Since the space $(U_{\Gamma_0}, \|\cdot\|_{0,p(\cdot),\nabla})$ is reflexive (see Theorem 4.5(a)) and strictly convex (see Corollary 4.3(a)), since the norm $\|\cdot\|_{0,p(\cdot),\nabla}$ is Fréchet differentiable (see Theorem 4.5(c)), and since any duality mapping on U_{Γ_0} satisfies the condition $(S)_+$ (see Theorem 4.6(a)), the assertion follows by Theorem 3.2(b).

5 Existence Results for Operators Equations Involving Duality Mappings and Nemytskij Operators on the Space U_{Γ_0}

5.1 An estimate concerning the Nemytskij operators between Lebesgue spaces with variable exponents

Theorem 5.1 *Let $\Omega \subset \mathbb{R}^N$ be a domain, let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying the growth condition*

$$|f(x,s)| \leq C_1 |s|^{\frac{q(x)}{q'(x)}} + a(x) \quad \text{for almost all } x \in \Omega \text{ and all } s \in \mathbb{R}, \tag{5.1}$$

where $C_1 > 0$ is a constant, $q \in L^\infty(\Omega)$ with $q^- > 1$,

$$\frac{1}{q(x)} + \frac{1}{q'(x)} = 1 \quad \text{for almost all } x \in \Omega,$$

and $a \in L^{q'(\cdot)}(\Omega)$ with $a(x) \geq 0$ for almost all $x \in \Omega$, and let $N_f : L^{q(\cdot)}(\Omega) \to L^{q'(\cdot)}(\Omega)$ be the Nemytskij operator generated by f, i.e.,

$$(N_f v)(x) = f(x, v(x)) \quad \text{for all } v \in L^{q(\cdot)}(\Omega) \text{ and almost all } x \in \Omega. \tag{5.2}$$

Then the following estimate holds:

$$\|N_f v\|_{0,q'(\cdot)} \leq C_1 \max\{\|v\|_{0,q(\cdot)}^{q^- -1}, \|v\|_{0,q(\cdot)}^{q^+ -1}\} + \|a\|_{0,q'(\cdot)} \tag{5.3}$$

for all $v \in L^{q(\cdot)}(\Omega)$.

The proof rests essentially on the following lemma.

Lemma 5.1 *Let $p \in L^\infty(\Omega)$ be such that $p^- \geq 1$, and let $r \in L^\infty(\Omega)$ be such that $(rp)^- \geq 1$. Then for any $\mathrm{d}x$-measurable function u on Ω satisfying $|u(\cdot)|^{r(\cdot)} \in L^{p(\cdot)}(\Omega)$, the following inequality holds:*

$$\||u(\cdot)|^{r(\cdot)}\|_{0,p(\cdot)} \leq \max\{\|u\|_{0,r(\cdot)p(\cdot)}^{r^-}, \|u\|_{0,r(\cdot)p(\cdot)}^{r^+}\}. \tag{5.4}$$

Proof We first remark that $r(\cdot)p(\cdot) \in L^\infty(\Omega)$ and that the assumptions that u is $\mathrm{d}x$-measurable on Ω and that $|u(\cdot)|^{r(\cdot)} \in L^{p(\cdot)}(\Omega)$ together imply that $u \in L^{r(\cdot)p(\cdot)}(\Omega)$. We now prove that

$$\|u\|_{0,r(\cdot)p(\cdot)} \geq 1 \text{ implies } \||u(\cdot)|^{r(\cdot)}\|_{0,p(\cdot)} \leq \|u\|_{0,r(\cdot)p(\cdot)}^{r^+}, \tag{5.5}$$

which is equivalent (see Theorem 4.2) to proving that

$$\rho_{0,p(\cdot)}\Big(\frac{|u(\cdot)|^{r(\cdot)}}{\|u\|_{0,r(\cdot)p(\cdot)}^{r^+}}\Big) \leq 1. \tag{5.6}$$

In order to prove that (5.6) holds, we will use the simple technique already used for proving inequality (3.16) in [10]. Since $r^+ - r(x) \geq 0$ and $\|u\|_{0,r(\cdot)p(\cdot)} \geq 1$, it easily follows that

$$\frac{|u(x)|^{r(x)}}{\|u\|_{0,r(\cdot)p(\cdot)}^{r^+}} = \Big(\frac{|u(x)|^{r(x)}}{\|u\|_{0,r(\cdot)p(\cdot)}^{r(x)}}\Big)\frac{1}{\|u\|_{0,r(\cdot)p(\cdot)}^{r^+ - r(x)}} \leq \frac{|u(x)|^{r(x)}}{\|u\|_{0,r(\cdot)p(\cdot)}^{r(x)}},$$

which in turn implies that

$$\rho_{0,p(\cdot)}\Big(\frac{|u(\cdot)|^{r(\cdot)}}{\|u\|_{0,r(\cdot)p(\cdot)}^{r^+}}\Big) \leq \rho_{0,p(\cdot)}\Big(\frac{|u(\cdot)|^{r(\cdot)}}{\|u\|_{0,r(\cdot)p(\cdot)}^{r(\cdot)}}\Big) = \rho_{0,r(\cdot)p(\cdot)}\Big(\frac{|u(\cdot)|}{\|u\|_{0,r(\cdot)p(\cdot)}}\Big) = 1.$$

Thus, the inequality (5.6) holds. Consequently, the inequality (5.5) holds too.

By using a similar technique, one can prove that

$$\|u\|_{0,r(\cdot)p(\cdot)} \leq 1 \text{ implies } \||u(\cdot)|^{r(\cdot)}\|_{0,p(\cdot)} \leq \|u\|_{0,r(\cdot)p(\cdot)}^{r^-}. \tag{5.7}$$

Clearly, (5.5) and (5.7) together imply that the inequality (5.4) holds.

Remark 5.1 Under the more restrictive assumptions that $p \in C(\overline{\Omega})$ with $p^- > 1$ and $r \in L^\infty(\Omega)$ with $r^- > 1$, the result of Lemma 5.1 was also obtained in [4, Lemma 1.4] (the continuity of $p(\cdot)$ was needed for using the mean value theorem in proving Lemma 1.4 in [4]).

Proof of Theorem 5.1 We can now prove Theorem 5.1. First, we note that, by virtue of [14, Theorem 1.16], N_f is well-defined as a Nemytskij operator from $L^{q(\cdot)}(\Omega)$ into $L^{q'(\cdot)}(\Omega)$, which is continuous and bounded. From (5.1)–(5.2), we infer that

$$\|N_f v\|_{0,q'(\cdot)} \le C_1 \||v(\cdot)|^{\frac{q(\cdot)}{q'(\cdot)}}\|_{0,q'(\cdot)} + \|a\|_{0,q'(\cdot)}.$$

Applying (5.4) with $u(\cdot) = v(\cdot)$, $r(\cdot) = \frac{q(\cdot)}{q'(\cdot)} = q(\cdot) - 1$ and $p(\cdot) = q'(\cdot)$, we obtain

$$\||v(\cdot)|^{\frac{q(\cdot)}{q'(\cdot)}}\|_{0,q'(\cdot)} \le \max\{\|v\|_{0,q(\cdot)}^{q^- -1}, \|v\|_{0,q(\cdot)}^{q^+ -1}\},$$

from which the estimate (5.3) follows.

Remark 5.2 If $q(\cdot)$ is a constant function, then $q^- = q^+ = q$, and so the estimate (5.3) becomes

$$\|N_f v\|_{L^{q'}(\Omega)} \le C_1 \|v\|_{L^q(\Omega)}^{q-1} + \|a\|_{L^{q'}(\Omega)},$$

a well-known property of Nemytskij operators acting between classical Lebesgue spaces $L^q(\Omega)$ and $L^{q'}(\Omega)$ (see [16]).

6 The Main Existence Result

Theorem 6.1 *Let Ω be a domain in $\mathbb{R}^N$ $(N \ge 2)$, let $p \in \mathcal{C}(\overline{\Omega})$ and $q \in \mathcal{C}(\overline{\Omega})$ be two functions such that $p^- > 1$, $q^- > 1$, and*

$$q(x) < p^*(x) := \frac{Np(x)}{N - p(x)} \quad \text{if } p(x) < N \quad \text{and} \quad p^*(x) := \infty \quad \text{if } p(x) \ge N,$$

and let there be given a Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ that satisfies the growth condition:

$$|f(x,s)| \le C_1 |s|^{\frac{q(x)}{q'(x)}} + a(x) \quad \text{for almost all } x \in \Omega \text{ and all } s \in \mathbb{R},$$

with $\frac{1}{q(x)} + \frac{1}{q'(x)} = 1$, $a \in L^{q'(\cdot)}(\Omega)$, $a(x) \ge 0$ for almost all $x \in \Omega$, for some constant $C_1 > 0$. Let

$$N_f : L^{q(\cdot)}(\Omega) \to L^{q'(\cdot)}(\Omega), \quad (N_f u)(x) = f(x, u(x)) \quad \text{for almost all } x \in \Omega$$

denote the Nemytskij operator generated by f.

Then, for any gauge function φ which possesses the property that $\frac{\varphi(t)}{t^{q^+ -1}} \to \infty$ as $t \to \infty$, the solution set of the equation

$$J_\varphi u = N_f u \tag{6.1}$$

is a nonempty and compact subset of U_{Γ_0}.

Proof First we need to explain what is meant by a solution of equation (6.1). By Theorem 4.3(c) and Theorem 4.5(a)–(b), the compact inclusion $(U_{\Gamma_0}, \|\cdot\|_{1,p(\cdot),\nabla}) \Subset (L^{q(\cdot)}(\Omega), \|\cdot\|_{0,q(\cdot)})$

holds. Let ι be the compact injection of U_{Γ_0} into $L^{q(\cdot)}(\Omega)$, and let $\iota^* : L^{q'(\cdot)}(\Omega) \to (U_{\Gamma_0})^*$ be its adjoint, in the sense that $\iota^* v = v \circ \iota$ for all $v \in L^{q'(\cdot)}(\Omega)$. Therefore, ι^* is also compact and $\|\iota\| = \|\iota^*\|$.

A solution of equation (6.1) is an element $u \in U_{\Gamma_0}$ that satisfies

$$J_\varphi u = (\iota^* N_f \iota) u \quad \text{ in } (U_{\Gamma_0})^*. \tag{6.2}$$

In the sequel, three methods for proving the existence of such a solution of equation (6.1) will be described.

The *first method* is based on Corollary 3.2. Since U_{Γ_0} is reflexive and smooth (see Theorem 4.5), any duality mapping on U_{Γ_0} satisfies the condition $(\mathrm{S})_+$ (see Theorem 4.6), and $K = (\iota^* N_f \iota) : U_{\Gamma_0} \to U^*_{\Gamma_0}$ is compact. So, it is sufficient to prove the existence of a closed convex set $C \subset U_{\Gamma_0}$ such that the assumptions (a)–(b) of Corollary 3.2 are satisfied.

Let v and u in U_{Γ_0} be such that

$$J_\varphi v = (\iota^* N_f \iota) u. \tag{6.3}$$

Taking the norm in both sides of this equality and taking into account the estimate (5.3), we get

$$\begin{aligned}
\varphi(\|v\|) &\le \|\iota\| \|N_f(\iota u)\| \\
&\le \|\iota\| [C_1 \max\{\|\iota u\|^{q^- -1}_{0,q(\cdot)}, \|\iota u\|^{q^+ -1}_{0,q(\cdot)}\} + \|a\|_{0,q'(\cdot)}] \\
&\le \|\iota\| [C_1 \{\|\iota\|^{q^- -1} \|u\|^{q^- -1}_{0,q(\cdot)} + \|\iota\|^{q^+ -1} \|u\|^{q^+ -1}_{0,q(\cdot)}\} + \|a\|_{0,q'(\cdot)}] \\
&= C_1 \|\iota\|^{q^+} \|u\|^{q^+ -1}_{0,q(\cdot)} + C_1 \|\iota\|^{q^-} \|u\|^{q^- -1}_{0,q(\cdot)} + \|a\|_{0,q'(\cdot)} \|\iota\|.
\end{aligned} \tag{6.4}$$

Since $\frac{\varphi(t)}{t^{q^+ -1}} \to \infty$ as $t \to \infty$, there exists a constant $R > 0$ such that

$$\varphi(t) - (C_1 \|\iota\|^{q^+} t^{q^+ -1} + C_1 \|\iota\|^{q^-} t^{q^- -1} + \|a\|_{0,q'(\cdot)} \|\iota\|) > 0 \quad \text{ for all } t \ge R. \tag{6.5}$$

Taking into account (6.5), we deduce that, if v and u in U_{Γ_0} satisfy (6.3) with $\|u\| \le R$, then

$$\varphi(\|v\|) \le C_1 \|\iota\|^{q^+} R^{q^+ -1} + C_1 \|\iota\|^{q^-} R^{q^- -1} + \|a\|_{0,q'(\cdot)} \|\iota\| < \varphi(R),$$

which implies that $\|v\| < R$. Thus, the assumption (a) in Corollary 3.2 is satisfied by letting $C := \overline{B}_{U_{\Gamma_0}}(0, R) = \{u \in U_{\Gamma_0}; \|u\|_{0,p(\cdot),\nabla} \le R\}$. Moreover, since $K = \iota^* N_f \iota$ is compact, the set $K(\overline{B}_{U_{\Gamma_0}}(0, R))$ is relatively compact. Thus the assumption (b) in Corollary 3.2 is also satisfied. We conclude that Corollary 3.2 applies with $C := \overline{B}_{U_{\Gamma_0}}(0, R)$ and $K := \iota^* N_f \iota$. Consequently, there exists $u \in \overline{B}_{U_{\Gamma_0}}(0, R)$ satisfying (6.2).

It remains to prove that the solution set of equation (6.2), viz.,

$$\mathcal{S}(J_\varphi, K) := \{u \in U_{\Gamma_0};\, J_\varphi u = Ku, K = \iota^* N_f \iota\},$$

which is not empty, is compact.

To this end, we first observe that $\mathcal{S}(J_\varphi, K) \subset \overline{B}_{U_{\Gamma_0}}(0, R)$. Indeed, if some $u \in U_{\Gamma_0}$ satisfies (6.2), then (6.4) implies that

$$\varphi(\|u\|) \le C_1 \|\iota\|^{q^+} \|u\|^{q^+ -1}_{0,q(\cdot)} + C_1 \|\iota\|^{q^-} \|u\|^{q^- -1}_{0,q(\cdot)} + \|a\|_{0,q'(\cdot)} \|\iota\|,$$

which, by virtue of (6.5), implies that $\|u\| < R$.

Second,

$$\mathcal{S}(J_\varphi, K) = \{u \in \overline{B}_{U_{\Gamma_0}}(0,R); u = Tu, T = J_\varphi^{-1}K\}, \tag{6.6}$$

i.e., $\mathcal{S}(J_\varphi, K)$ is the set of fixed points of T. Indeed, since any duality mapping on U_{Γ_0} is a homeomorphism (see Corollary 4.4), the equality (6.6) clearly holds. Also notice that, since K is compact and J_φ^{-1} is continuous, T is compact and the inclusion $T(\overline{B}_{U_{\Gamma_0}}(0,R)) \subset \overline{B}_{U_{\Gamma_0}}(0,R)$ holds. Indeed, let $u \in \overline{B}_{U_{\Gamma_0}}(0,R)$ and let $v = Tu$, or equivalently, $J_\varphi v = Ku$. From the above, it thus follows that $\|v\| < R$. Now a standard argument shows that $\mathcal{S}(J_\varphi, K)$ is a compact set.

The *second method* is based on the Schauder fixed point theorem: As we have already seen, the solution set in U_{Γ_0} of the equation (6.2) coincides with the fixed point set $\mathrm{Fix}(T)$ of the operator $T = J_\varphi^{-1}K$ with $K = \iota^* N_f \iota$, so that $\mathrm{Fix}(T) \subset \overline{B}_{U_{\Gamma_0}}(0,R)$ with R defined by (6.5). Moreover, the operator $T : U_{\Gamma_0} \to U_{\Gamma_0}$ is compact and $T(\overline{B}_{U_{\Gamma_0}}(0,R)) \subset \overline{B}_{U_{\Gamma_0}}(0,R)$. Using Schauder's fixed point theorem, we thus conclude that $\mathrm{Fix}(T)$ is nonempty, compact and contained in $\overline{B}_{U_{\Gamma_0}}(0,R)$.

The *third method* is based on some fundamental properties of the Leray-Schauder degree: The notations are as above, and we begin by showing that

$$\mathcal{B} := \{u \in U_{\Gamma_0}; \text{there exists } t \in [0,1] \text{ such that } u = tTu\} \subset B_{U_{\Gamma_0}}(0,R). \tag{6.7}$$

Since for $t = 0$ the only solution of the equation $u = tTu$ is $u = 0$, the problem reduces to that of establishing the inclusion

$$\{u \in U_{\Gamma_0}; \text{there exists } t \in (0,1] \text{ such that } u = tTu\} \subset B_{U_{\Gamma_0}}(0,R).$$

So, let $u \in U_{\Gamma_0}$ satisfy

$$u = tTu = tJ_\varphi^{-1}(\iota^* N_f \iota)u \quad \text{for some } t \in (0,1],$$

or equivalently,

$$J_\varphi\Big(\frac{u}{t}\Big) = (\iota^* N_f \iota)u.$$

From the estimates (6.4), we then get

$$\begin{aligned}\varphi(\|u\|_{0,p(\cdot),\nabla}) &\le \varphi\Big(\frac{\|u\|_{0,p(\cdot),\nabla}}{t}\Big) = \Big\|J_\varphi\big(\frac{u}{t}\big)\Big\| = \|(\iota^* N_f \iota)u\| \\ &\le C_1\|\iota\|^{q^+}\|u\|_{0,q(\cdot)}^{q^+-1} + C_1\|\iota\|^{q^-}\|u\|_{0,q(\cdot)}^{q^--1} + \|a\|_{0,q'(\cdot)}\|\iota\|,\end{aligned}$$

and thus, using the definition of R (see (6.5)), we conclude that $\|u\| < R$.

The a priori estimate (6.7), which is uniform with respect to $t \in [0,1]$, and the homotopy invariance property of the Leray-Schauder degree together give

$$d_{LS}(I - (tT), B_{U_{\Gamma_0}}(0,R), 0) = d_{LS}(I, B_{U_{\Gamma_0}}(0,R), 0) = 1 \quad \text{for all } t \in [0,1],$$

where I stands for the identity over U_{Γ_0}.

We then deduce that, for any $t \in [0,1]$, $\mathrm{Fix}(tT)$ is nonempty, compact and contained in $B_{U_{\Gamma_0}}(0,R)$. In particular, this is thus true for $t = 1$.

Remark 6.1 It is known (see [12]) that, in the usual Sobolev spaces $(W_0^{1,p}(\Omega), \|\cdot\|_{0,p,\nabla})$, $p > 1$, where $\|u\|_{0,p,\nabla} = \||\nabla u|\|_{0,p}$, the *p-Laplacian operator*

$$\Delta_p : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^*, \Delta_p u := \frac{\partial}{\partial x_i}\Big(|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\Big)$$

may be equivalently defined as

$$-\Delta_p u := J_{(p-1)}u \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

where $J_{(p-1)}$ stands for the duality mapping on $W_0^{1,p}(\Omega)$ corresponding to the gauge function φ defined by $\varphi(t) = t^{p-1}$, $t \geq 0$. This property allows us to define a natural extension of Δ_p from $(W_0^{1,p}(\Omega), \|\cdot\|_{0,p,\nabla})$ into $(W_0^{1,p(\cdot)}(\Omega), \|\cdot\|_{0,p(\cdot),\nabla})$, where $p(\cdot) \in C(\overline{\Omega})$ satisfies $p(x) > 1$ for all $x \in \overline{\Omega}$, namely, if φ is a gauge function, we define the $(\varphi, p(\cdot))$-Laplacian operator on $(W_0^{1,p(\cdot)}(\Omega), \|\cdot\|_{0,p(\cdot),\nabla})$ as the operator

$$\Delta_{(\varphi,p(\cdot))} : W_0^{1,p(\cdot)}(\Omega) \to (W_0^{1,p(\cdot)}(\Omega))^*$$

defined by $-\Delta_{(\varphi,p(\cdot))}u := J_\varphi u$ for all $u \in W_0^{1,p(\cdot)}(\Omega)$, where J_φ stands for the duality mapping on $(W_0^{1,p(\cdot)}(\Omega), \|\cdot\|_{0,p(\cdot),\nabla})$ corresponding to the gauge function φ.

It is obvious that, under the assumptions allowing to define the space $(U_{\Gamma_0}, \|\cdot\|_{0,p(\cdot),\nabla})$, this last definition makes sense on the larger space $(U_{\Gamma_0}, \|\cdot\|_{0,p(\cdot),\nabla}) \supset (W_0^{1,p(\cdot)}(\Omega), \|\cdot\|_{0,p(\cdot),\nabla})$. Thus, we will call $(\varphi, p(\cdot))$-*Laplacian operator* the operator

$$\Delta_{(\varphi,p(\cdot))} : (U_{\Gamma_0}, \|\cdot\|_{0,p(\cdot),\nabla}) \to (U_{\Gamma_0})^*$$

defined by

$$-\Delta_{(\varphi,p(\cdot))}u := J_\varphi u \quad \text{for all } u \in U_{\Gamma_0},$$

where J_φ stands for the duality mapping on $(U_{\Gamma_0}, \|\cdot\|_{0,p(\cdot),\nabla})$ corresponding to the gauge function φ.

In the light of this last definition, the existence result of Theorem 6.1 reads as follows: Under the assumptions of Theorem 6.1, the solution set of the boundary value problem

$$\begin{aligned} &-\Delta_{(\varphi,p(\cdot))}u = f(x,u) \quad \text{in } \Omega, \\ &u\,|_{\Gamma_0} = 0, \quad \Gamma_0 \subset \partial\Omega = \Gamma, \quad \mathrm{d}\Gamma - \mathrm{meas}\Gamma_0 > 0 \end{aligned}$$

is a nonempty and compact subset of U_{Γ_0}.

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