Chin. Ann. Math.
34B(5), 2013, 667–682
DOI: 10.1007/s11401-013-0796-6

Chinese Annals of Mathematics, Series B

A Variational Finite Element Model for Large-Eddy Simulations of Turbulent Flows*

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Abstract The authors introduce a new Large Eddy Simulation model in a channel, based on the projection on finite element spaces as filtering operation in its variational form, for a given triangulation $\{\mathcal{T}_h\}_{h>0}$. The eddy viscosity is expressed in terms of the friction velocity in the boundary layer due to the wall, and is of a standard sub grid-model form outside the boundary layer. The mixing length scale is locally equal to the grid size. The computational domain is the channel without the linear sub-layer of the boundary layer. The no-slip boundary condition (or BC for short) is replaced by a Navier (BC) at the computational wall. Considering the steady state case, the authors show that the variational finite element model they have introduced, has a solution $(\mathbf{v}_h, p_h)_{h>0}$ that converges to a solution of the steady state Navier-Stokes equation with Navier BC.

Keywords Navier-Stokes equations, Turbulence modeling, Finite elements
2000 MR Subject Classification 76D05, 76F65, 65M60

1 Introduction

Numerical simulations of incompressible turbulent flows can not be performed from the evolutionary Navier-Stokes equations (or NSE for short),

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} - \nu \Delta \mathbf{v} + \nabla p = \mathbf{f}, \tag{1.1}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{1.2}$$

because of a great computational complexity due to the structure of the turbulence (see [30]). This is why various mathematical models derived from the NSE are used to simulate some features of turbulent flows, such as their statistical means or their large scales motions, this last way being known as "large-eddy simulation" (or LES for short), which is our concern in the present paper.

LES has attracted much attention these last two decades, especially because of the increasing of computational ressources, enabling to enlarge the range of scales that LES models might simulate. Basically, LES aims at computing filtered fields such as $\overline{\mathbf{v}} = G \star \mathbf{v}$, G being a smooth

Manuscript received June 6, 2013.
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*Project supported by the Spanish Government and European Union FEDER Grant (No. MTM2009-07719).

transfer function (see [25, 30–32]). The filtering operation also might be carried out by solving PDEs (see [6–7, 14, 16, 23–24])

Stresses that appear by filtering the nonlinear term $(\mathbf{v} \cdot \nabla)\mathbf{v}$ in the NSE, are considered to be diffusive, therefore often modeled by a turbulent diffusion term such as $-\nabla \cdot (\nu_t \nabla \overline{\mathbf{v}})$, where ν_t is an eddy viscosity. One challenge of the LES is the determination of ν_t.

In this paper, we study the case of a channel flow, periodic in the $x_1 - x_2$ axis for simplicity. The first idea is that the projection on finite element spaces, based on a given triangulation $\{\mathcal{T}_h\}_{h>0}$, is a natural filtering operation, so that we seek for $\mathbf{v}_h$ instead of $\overline{\mathbf{v}}$, where $\mathbf{v}_h$ is the projection of $\mathbf{v}$ on a suitable finite element space $\mathbf{W}_h$. The second idea is that one can specifically model the eddy viscosity on the boundary layer by means of wall laws.

Indeed, following Kolmogorov theory in [22], we consider the turbulence to be isotropic at scales small enough inside the flow domain. This assumption yields to take the eddy viscosity of a Kolmogorov-Prandtl-Smagorinsky form, $\nu_t = h^2 |D\mathbf{v}_h|$ [1], h being the mixing length, which is the standard sub-grid model (or SGM for short) (see [11]).

However, near the wall, turbulence is not isotropic and complexity is higher than far from the wall (see [29]), so that standard SGM cannot be used there. Usual methods proceed as follows.

First one uses the known structure of the turbulent boundary layer, as initially described by Von Kármán [21] and fully developed by Schlichting [34]. Basically, the boundary layer may be split into two sub-layers, the linear sub-layer where the mean velocity profile is linear, and next, the log sub-layer where the mean velocity profile is specified by a log function. Notice that one can consider more sophisticated models for the boundary layer (see [35]), nevertheless always involving a log law. In all cases, those models involve an essential quantity which is the friction velocity $u_\star$ (see (2.4) below).

Next, one splits the domain into two subdomains, the boundary layer, and the computational domain which is the domain's part not containing the boundary layer. One then uses nonlinear boundary conditions at boundaries of the computational domain such as wall laws (see in [26–28]).

Based on the fact that today more computational resources are available to increase accuracy for simulating the mean flow inside the log layer, we take as computational domain the domain's part without the linear sub-layer, using an eddy viscosity of the form $\nu_t = h u_\star$ inside the log layer, deduced from standard dimensional analysis (see [11, 26]).

To conclude the modeling process, it remains to: (i) specify how $u_\star$ is calculated, (ii) specify boundary conditions (BC) at computational domain boundaries, (iii) set the choice of the mixing length scale.

(i) We assume that log law holds inside the boundary layer. Thanks to invertibility of the nonlinear profile, we can define $u_\star$ as $u_\star(\mathbf{v}, \mathbf{x})$, that satisfies suitable estimates (see Subsection 2.2(iii) and (2.17)).

(ii) As the thickness of the linear sub-layer is very small compared to other scales involved in the problem, a Taylor expansion allows to deduce from the no-slip condition at the flow domain boundary a Navier BC at the computational walls (see (3.6) in Subsection 3.1(i)). This is as if the linear sub-layer would exert a friction over the log sub-layer.

[1] $D\mathbf{v}_h = (\frac{1}{2})(\nabla \mathbf{v}_h + \nabla \mathbf{v}_h^t)$.

(iii) The mesh yields natural numerical length scales h_K, where h_K is a diameter of any $K \in \mathcal{T}_h$. Therefore, one takes ν_t of the form $\nu_t = h_K^2 |D\mathbf{v}_h|$ on $K \in \mathcal{T}_h$ inside the computational domain, and $\nu_t = h_K u_\star(\mathbf{v}, \mathbf{x})$ on $K \in \mathcal{T}_h$ in the log layer (see Subsection 3.2(ii)).

Once this modeling process is completed, we get a model expressed in its variational form over finite element space $\mathbf{W}_h \times M_h$, as described in Subsection 3.2(iv). So far as we know, this model is totally new, and can be generalized to more complex and realistic geometries thanks to a careful differential geometry analysis, which is a work under progress.

We consider all over this paper the steady-state case, which is in coherence with the fact that in a permanent regime and for a developed turbulence, mean fields are steady, which is not in contradiction with the fact that fluctuations might be time dependent.

We prove that this variational problem has a solution $(\mathbf{v}_h, p_h) \in \mathbf{W}_h \times M_h$ (see Theorem 4.1) which converges to a solution $(\mathbf{v}, p)$ of the steady-state Navier Stokes equation (NSE) with Navier BC (see Theorem 4.2).

This paper is organized as follows. We start with general setting. Then we derive from the NSE a description of the boundary layer, introducing the friction velocity. We specify the computational domain and Navier BC, and next we perform the finite element setting and get the model. Finally, we state and prove Theorems 4.1–4.2.

2 General Framework

2.1 Channel flow

(i) Geometry, equations and boundary conditions.

Let Ω_f be a channel periodic in the x_1 axis and x_2 axis, of height $1 + 2d$ in the x_3-axis, for a small parameter $d \ll 1$,

$$\Omega_f = \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{T}_2 \times \mathbb{R}^3 \text{ s.t. } -d < x_3 < 1 + d\}, \tag{2.1}$$

where $\mathbb{T}_2$ is the two-dimensional torus defined by

$$\mathbb{T}_2 = \frac{\mathbb{R}^2}{\mathcal{T}_2}, \qquad \text{where } \mathcal{T}_2 = \frac{2\pi \nabla Z^2}{L},$$

and $L > 0$ is a given length scale. Let Γ_f denote

$$\Gamma_f = \{\mathbf{x} \in \mathbb{T}_2 \times \mathbb{R}^3 \text{ s.t. } x_3 = -d \text{ or } x_3 = 1 + d\}. \tag{2.2}$$

The steady-state Navier-Stokes equations with the no-slip boundary condition are as follows:

$$\begin{cases} (\mathbf{v} \cdot \nabla)\mathbf{v} - \nu \Delta \mathbf{v} + \nabla p = \mathbf{f} & \text{in } \Omega_f, \\ \nabla \cdot \mathbf{v} = 0 & \text{in } \Omega_f, \\ \mathbf{v} = \mathbf{0} & \text{on } \Gamma_f. \end{cases} \tag{2.3}$$

The source term $\mathbf{f}$ is a body force per mass unit, typically the gravity. Assuming $\mathbf{f} \in L^2(\Omega_f)^3 = \mathbf{L}^2(\Omega_f)$, we know that this equation has a solution $(\mathbf{v}, p) \in W^{2,\frac{3}{2}}(\Omega)^3 \times W^{1,\frac{3}{2}}(\Omega)$ (see [36]), whose norms are bounded by constants that only depend on ν, $\|\mathbf{f}\|_{0,2,\Omega_f}$ and d. Also, p is defined up to a constant. Uniqueness is known when $\frac{\|\mathbf{f}\|_{0,2,\Omega_f}}{\nu^2}$ is small enough.

(ii) Friction velocity.

Let $(\mathbf{v}, p)$ be any solution of (2.3). We still denote by $\mathbf{v}$ the trace of $\mathbf{v}$ on Γ_f. We deduce from trace theorems and Sobolev theorem that $\mathbf{v} \in W^{1,3}(\Gamma_f)^3 = \mathbf{W}^{1,3}(\Gamma_f)$. Therefore, it makes sense to consider $D\mathbf{v} \cdot \mathbf{n}$ on Γ_f, where $\mathbf{n}$ denotes the outward-pointing unit normal vector at Γ_f, $D\mathbf{v} = \left(\frac{1}{2}\right)(\nabla \mathbf{v} + \nabla \mathbf{v}^t)$. We split the vector $D\mathbf{v} \cdot \mathbf{n}$ into its tangential part and its normal part,

$$D\mathbf{v} \cdot \mathbf{n} = (D\mathbf{v} \cdot \mathbf{n})_\tau + ((D\mathbf{v} \cdot \mathbf{n}) \cdot \mathbf{n})\, \mathbf{n}. \tag{2.4}$$

Let $v_\star \in L^6(\Gamma_f)$ be defined on Γ_f by

$$v_\star = v_\star(\mathbf{v})(\mathbf{x}) = (\nu |(D\mathbf{v} \cdot \mathbf{n})_\tau(\mathbf{x})|)^{\frac{1}{2}}, \tag{2.5}$$

called the friction velocity associated to $\mathbf{v}$ at $\mathbf{x} \in \Gamma_f$.

2.2 Boundary layer description

(i) Length scale.

Condition of uniqueness to (2.3) is not satisfied in a steady-state turbulent regime. Let $\mathcal{S}$ be the set of solutions. According to [11], one can construct a probability measure μ on $\mathcal{S}$. We consider the following velocity friction $w_\star \in \mathbf{L}^6(\Gamma_f)$ defined by

$$w_\star = \int_{\mathcal{S}} v_\star(\mathbf{v}) \mathrm{d}\mu(\mathbf{v}). \tag{2.6}$$

We finally define the meanfriction velocity by

$$u_\star = \frac{1}{L} \|w_\star\|_{0,2,\Gamma_f} \in \mathbb{R}, \tag{2.7}$$

to which is associated the typical length scale λ that characterises the boundary layer,

$$\lambda = \frac{\nu}{u_\star}, \tag{2.8}$$

assuming $u_\star \neq 0$.

We conjecture that $u_\star \to \infty$ when $\|\mathbf{f}\|_{0,2,\Omega_f} \to \infty$.

(ii) Main assumption about the boundary layer structure.

We focus on the bottom of Ω_f, $\{x_3 = -d\}$, assuming that the boundary layer at the top $\{x_3 = 1 + d\}$ has a similar structure. According to experiments (see [34]), we assume that in the boundary layer, the mean fluid velocity is parallel to $\mathbf{e}_1$ and only depends on the variable x_3, which means $\mathbf{v}(\mathbf{x}) = v(x_3)\mathbf{e}_1$.

Notice that any plane P of the form $P = \{x_3 = h\}$ included in the boundary layer, and any vector $\mathbf{N}$ orthogonal to P being given, our assumption yields in particular $\mathbf{v} \cdot \mathbf{N} = 0$ at P.

(iii) Log law.

Experiments and suitable assumptions about turbulence (see [11, 34]) indicate that the boundary layer can be decomposed into two sub layers:

(1) near the boundary where the velocity profile v is linear (linear sub-layer),

(2) the next sub-layer specified by a log profile (log layer).

To be more specific, we introduce the dimensionless variable

$$z^+ = \frac{x_3}{\lambda}, \tag{2.9}$$

and we consider the following continuous function defined on $[0, z^+_{\max}]$ by

$$L(z^+) = \begin{cases} z^+, & \text{if } 0 \le z^+ \le z^+_0, \\ \dfrac{1}{\kappa} \log\Big(\dfrac{z^+}{z^+_0}\Big) + z^+_0, & \text{if } z^+_0 \le z^+ \le z^+_{\max}, \end{cases} \tag{2.10}$$

where $\kappa \approx 0,41$ is the Von Kármán constant. In practical calculations, one takes $z^+_0 \approx 20$, and $z^+_{\max} \approx 100$, that measures the thickness of the logarithmic boundary layer, taken to be equal to 100λ. According to experiments in [34], boundary layer thickness goes to zero as the Reynolds number goes to infinity.

The profile v in the boundary layer at the bottom of Ω_f is given by the formula

$$v(x_3) = u_\star L\Big(\frac{x_3}{\lambda}\Big). \tag{2.11}$$

A similar description applies to the boundary layer at the top of Ω_f, $\{z = 1 + d\}$.

(iv) Friction velocity expressed as a function of the velocity.

We still focus on the bottom. Any $x_3 > 0$ being given, let

$$F(\beta) = \beta L(\alpha\beta), \quad \alpha = \frac{x_3}{\nu}. \tag{2.12}$$

With this notation, (2.11) may be written as

$$v = F(u_\star), \tag{2.13}$$

thanks to (2.8).

Lemma 2.1 *Let $F : [0, +\infty) \to [0, +\infty)$ be defined by* (2.12)*. The function F is invertible, so that* (2.13) *can be written as $u_\star = F^{-1}(v)$ at each given x_3.*

Proof We observe that the function L satisfies

$$\lim_{x \to 0^+} \frac{L(x)}{x} = C_1, \tag{2.14}$$

$$\lim_{x \to \infty} \frac{L(x)}{\log x} = C_2, \tag{2.15}$$

where C_1 and C_2 are non-zero constants. As L is strictly increasing and continuous in $(0, +\infty)$, then F is strictly increasing and continuous in $(0, +\infty)$. Also, by (2.14), F is continuous at $\beta = 0$ with $F(0) = 0$. Moreover, by (2.15), $\lim\limits_{x \to \infty} F(x) = +\infty$. Then F is bijective from $[0, +\infty)$ onto $[0, +\infty)$, which yields the invertibility of F as claimed.

Lemma 2.2 *Denote $h = F^{-1}$. Then there exists a constant $C = C(x_3) > 0$, bounded, such that*

$$\forall\, \gamma > 0, \quad C(x_3)\,(1 + \gamma) \le h(\gamma) \le C(1 + \gamma), \quad C = \sup C(x_3). \tag{2.16}$$

Proof Then $h : [0, +\infty) \mapsto [0, +\infty)$ is bijective and continuous. Also,

$$\lim_{\gamma \to \infty} \frac{h(\gamma)}{\gamma} = \lim_{t \to \infty} \frac{t}{F(t)} = \lim_{t \to \infty} \frac{1}{L(\alpha t)} = \lim_{t \to \infty} \frac{1}{\log(\alpha t)} \frac{\log(\alpha t)}{L(\alpha t)} = 0.$$

The conclusion is a consequence of the continuity of h.

We deduce from Lemma 2.1, (2.16) and because top and bottom layers have the same structure, that the friction velocity can be calculated at each $\mathbf{x} \in \mathrm{BL}$ from the velocity $\mathbf{v}$, and satisfies the estimate

$$0 < u_\star = u_\star(\mathbf{v}, \mathbf{x}) \le C(1 + |\mathbf{v}|). \tag{2.17}$$

3 Turbulence Model

3.1 Geometry and meshing

(i) Calculation domain.

Here and hereafter, we assume that the boundary layer is included in the union of two strips:

$$\mathrm{BL} = \Big\{ -d \le x_3 \le \frac{D}{2} - d \Big\} \cup \Big\{ 1 + d - \frac{D}{2} \le x_3 \le 1 + d \Big\}, \tag{3.1}$$

where $d < D \ll 1$, with d being the order of the linear sub layer, D the thickness of the global boundary layer. Standard numerical simulations are carried out in a sub-domain of the flow domain that does not include the boundary layer at all, using a wall law (see [11, 27–28]) at artificial boundaries (walls). Our model includes the log layer, using a Navier BC based on a Taylor expansion as shown below.

The computational domain is

$$\Omega = \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{T}_2 \times \mathbb{R}^3 \text{ s.t. } 0 < x_3 < 1\}, \tag{3.2}$$

the artificial wall being defined by

$$\Gamma_w = \{\mathbf{x} \in \mathbb{T}_2 \times \mathbb{R}^3 \text{ s.t. } x_3 = 0 \text{ or } x_3 = 1\}. \tag{3.3}$$

(ii) Boundary conditions.

As above, we focus on the bottom layer. By a Taylor expension, we get

$$0 = v|_{x_3=-d} \approx v|_{x_3=0} - d\frac{\partial v}{\partial x_3}\Big|_{x_3=0}. \tag{3.4}$$

From the view point of the domain Ω, $v = \mathbf{v}_\tau|_{\Gamma_w}$, and $\frac{\partial}{\partial x_3} = -\frac{\partial}{\partial \mathbf{n}}$ at Γ_w, where $\mathbf{v}_\tau$ is the tangential part of $\mathbf{v}$, defined by

$$\mathbf{v} = \mathbf{v}_\tau + (\mathbf{v} \cdot \mathbf{n})\mathbf{n}, \tag{3.5}$$

by still denoting $\mathbf{v}$ the trace of $\mathbf{v}$ at Γ_w, so far no risk of confusion occurs. Therefore, by remarks in Subsection 2.2(ii) together with (3.4), we get

$$\mathbf{v} \cdot \mathbf{n}|_{\Gamma_w} = 0, \quad \frac{\partial \mathbf{v}_\tau}{\partial \mathbf{n}}\Big|_{\Gamma_w} = -\frac{1}{d}\mathbf{v}_\tau, \tag{3.6}$$

which is a Navier boundary condition at the artificial wall, that expresses in some sense that the linear sub-layer exerts a friction on the log layer. Hence, (2.3) becomes in Ω,

$$\begin{cases} (\mathbf{v} \cdot \nabla)\mathbf{v} - \nu \Delta \mathbf{v} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla \cdot \mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v} \cdot \mathbf{n} = \mathbf{0} & \text{on } \Gamma_w, \\ -\dfrac{\partial \mathbf{v}_\tau}{\partial \mathbf{n}} = \dfrac{1}{d}\mathbf{v}_\tau & \text{on } \Gamma_w. \end{cases} \tag{3.7}$$

Navier-Stokes equations with Navier boundary conditions was studied before (see [1–3, 10, 37]), and existence of a solution of (3.7) is already ensured.

(iii) Variational formulation.

Let us define the spaces

$$\mathbf{W}(\Omega) = \{\mathbf{w} \in \mathbf{H}^1(\Omega),\ \mathbf{w}\cdot\mathbf{n}|_{\Gamma_w} = 0\},$$
$$M(\Omega) = \Big\{q \in L^2(\Omega), \int_{\Omega_c} q\mathrm{d}\mathbf{x} = 0\Big\},$$

by reminding that $\mathbf{H}^1(\Omega) = H^1(\Omega)^3$. Strictly speaking, the space $M(\Omega)$ is isomorphic to the quotient space $L^2(\Omega)/\mathbb{R}$, endowed with the usual quotient norm

$$\|\dot{p}\|_M = \inf_{p\in\dot{p}} \|p\|_{0,2,\Omega}. \tag{3.8}$$

It also may be viewed as a closed subspace of $L^2(\Omega)$ endowed with the $L^2(\Omega)$ norm.

The space $\mathbf{W}(\Omega)$ is endowed with the $\mathbf{H}^1$ norm, denoted $\|\cdot\|_{1,2,\Omega}$. As a consequence of Korn's inequality, the following useful estimate holds:

$$\forall\, \mathbf{v} \in \mathbf{W}(\Omega), \quad \|\mathbf{v}\|_{1,2,\Omega} \le C(\|D\mathbf{v}\|_{0,2,\Omega} + \|\mathbf{v}\|_{0,2,\Gamma_w}), \tag{3.9}$$

the proof of which being carried out in [10].

Let a, b and G be the forms defined by

$$a(\mathbf{v},\mathbf{w}) = \nu\,(D\mathbf{v}, D\mathbf{w})_\Omega, \tag{3.10}$$
$$b(\mathbf{z};\mathbf{v},\mathbf{w}) = \frac{1}{2}[((\mathbf{z}\cdot\nabla)\,\mathbf{v},\mathbf{w})_\Omega - ((\mathbf{z}\cdot\nabla)\,\mathbf{w},\mathbf{v})_\Omega], \tag{3.11}$$
$$G(\mathbf{v},\mathbf{w}) = \frac{\nu}{d}(\mathbf{v}_\tau,\mathbf{w}_\tau)_{\Gamma_w} \tag{3.12}$$

for $\mathbf{z},\mathbf{v},\mathbf{w} \in \mathbf{H}^1(\Omega)$. Recall that when $\mathbf{z},\mathbf{v},\mathbf{w} \in \mathbf{W}(\Omega)$ and $\nabla\cdot\mathbf{z} = 0$, then $b(\mathbf{z};\mathbf{v},\mathbf{w}) = ((\mathbf{z}\cdot\nabla)\,\mathbf{v},\mathbf{w})_\Omega$, and $(\nabla\mathbf{z},\nabla\mathbf{w})_\Omega = (D\mathbf{z}, D\mathbf{w})_\Omega$. Also remark that when $\mathbf{v}\in\mathbf{b}$, then $\mathbf{v} = \mathbf{v}_\tau$ at Γ_w.

We say that a pair $(\mathbf{v},p) \in \mathbf{W}(\Omega)\times M(\Omega)$ is a weak solution of the boundary value problem (3.7) if it satisfies

$$\begin{cases} b(\mathbf{v};\mathbf{v},\mathbf{w}) + a(\mathbf{v},\mathbf{w}) - (p,\nabla\cdot\mathbf{w})_\Omega + G(\mathbf{v},\mathbf{w}) = \langle\mathbf{f},\mathbf{w}\rangle, \\ (\nabla\cdot\mathbf{v}, q)_\Omega = 0 \end{cases} \tag{3.13}$$

for any $(\mathbf{w},q) \in \mathbf{W}(\Omega)\times M(\Omega)$.

(iv) A priori estimate and existence result.

Assume $\mathbf{f} \in \mathbf{W}(\Omega)'$. Let $(\mathbf{v},p)$ be any solution of (3.13), and take $\mathbf{v} = \mathbf{w}$ in (3.13). From the standard formula $b(\mathbf{v};\mathbf{v},\mathbf{v}) = 0$ that holds since $\nabla\cdot\mathbf{v} = 0$ and $\mathbf{v}\cdot\mathbf{n} = 0$ at Γ_w, we get

$$\nu\|D\mathbf{v}\|_{0,2,\Omega} + \frac{\nu}{d}\|\mathbf{v}\|_{0,2,\Gamma_w} = \langle\mathbf{f},\mathbf{v}\rangle, \tag{3.14}$$

from where we deduce

$$\|\mathbf{v}\|_{1,2,\Omega} \le C\kappa^{-1}\|\mathbf{f}\|_{\mathbf{W}(\Omega)'}, \quad \kappa = \min\Big(\nu, \frac{\nu}{d}\Big), \tag{3.15}$$

by using (3.9).

3.2 Finite element setting

(i) Triangulation.

Let $D \subset \mathbb{R}^3$ denote the sample box $D = [0, L]^2 \times [0, 1]$. The computational domain Ω may be viewed as the periodic reproduction of D in the $x_1 - x_2$ axes. Let $\{\mathcal{T}_h\}_{(h>0)}$ be a regular familly of triangulation of D, compatible with the periodicity of the domain: The restriction of the grid to the planes $x_1 = 0$ and $x_1 = D$ is the same, and the restriction of the grid to the planes $x_2 = 0$ and $x_2 = D$ is the same. Reproducing this triangulation by periodicity, we get a regular triangulation of Ω, still denoted by $\{\mathcal{T}_h\}_{(h>0)}$.

In the following, for each $K \in \mathcal{T}_h$, $h_K = \mathrm{diam}(K)$ denotes the diameter of K, and

$$h = \max_{K \in \mathcal{T}_h} h_K. \tag{3.16}$$

(ii) Eddy viscosities.

We assume isotropy of the turbulence inside the domain defined by $\Omega_{\mathrm{in}} = \Omega \setminus \mathrm{BL}$, the boundary layer BL being defined by (3.1). This yields to consider in Ω_{in} the eddy viscosity $\nu_{t,\mathrm{in}}$ to be of the same form as in usual Sub-Grid-Models of Prandtl-Kolmogorov-Smagorinsky type, where following [11], we take in each K the length scale equal to h_K, leading to consider $\nu_{t,\mathrm{in}}$ to be of the form

$$\nu_{t,\mathrm{in}}(\mathbf{v}) = C_s^2 \sum_{K \in \mathcal{T}_h} h_K^2 \, \mathbf{1}_K |D\mathbf{v}|, \tag{3.17}$$

where $C_s > 0$ is an empirical constant, $\mathbf{1}_A$ denotes the characteristic function for any set A.

In the boundary layer part, $\Omega_w = \mathrm{BL} \cap \Omega$, turbulence is no longer isotropic and depends on the friction velocity. Taking again h_K as typical length scale and by a dimensional analysis argument in [11], we define the eddy viscosity $\nu_{t,w}$ in Ω_w by

$$\nu_{t,w}(\mathbf{v}) = C_w \sum_{K \in \mathcal{T}_h} h_K \, \mathbf{1}_K u_\star(\mathbf{v}, \mathbf{x}), \tag{3.18}$$

where $C_w > 0$ is an empirical constant and $u_\star$ is expressed in Subsection 2.2(iv).

Finally, the eddy viscosity we consider is of the form

$$\nu_t = \nu_t(\mathbf{v}) = \mathbf{1}_{\Omega_{\mathrm{in}}} \nu_{t,\mathrm{in}}(\mathbf{v}) + \mathbf{1}_{\Omega_w} \nu_{t,w}(\mathbf{v}). \tag{3.19}$$

(iii) Finite element spaces.

The model is a mixed formulation, based upon pairs of finite element spaces $(\mathbf{W}_h, M_h) \subset \mathbf{W}(\Omega) \times M(\Omega)$, associated to the family of regular triangulations $\{\mathcal{T}_h\}_{h>0}$ of Ω in the sense of Ciarlet [12]. We assume that the family of pairs of spaces $\{(\mathbf{W}_h, M_h)\}_{h>0}$ satisfies the following hypothesis:

Hypothesis 3.1 The family of spaces $\{\mathbf{W}_h \times M_h\}_{h>0}$ is an internal approximation of $\mathbf{W}(\Omega) \times M(\Omega)$: For all $(\mathbf{w}, p) \in \mathbf{W}(\Omega) \times M(\Omega)$, there exists a sequence $\{(\mathbf{v}_h, p_h)\}_{h>0}$ such that $(\mathbf{v}_h, p_h) \in \mathbf{W}_h \times M_h$, and

$$\lim_{h \to 0} (\|\mathbf{v} - \mathbf{v}_h\|_{1,2,\Omega} + \|p - p_h\|_{0,2,\Omega}) = 0.$$

Hypothesis 3.2 The family of pairs of spaces $\{(\mathbf{W}_h, M_h)\}_{h>0}$ satisfies the uniform discrete inf-sup condition : There exists a constant $\alpha > 0$ such that

$$\alpha \|q_h\|_{0,2,\Omega} \leq \sup_{\mathbf{w}_h \in \mathbf{W}_h} \frac{(\nabla \cdot \mathbf{w}_h, q_h)_\Omega}{\|\mathbf{w}_h\|_{1,2,\Omega}} \quad \text{for all } q_h \in M_h. \tag{3.20}$$

There is a wide literature about finite element spaces satisfying those properties (see [5, 9, 15], for instance).

(iv) The model.

Our LES model is expressed by the following variational problem:

$$\text{Find } (\mathbf{v}_h, p_h) \in \mathbf{W}_h \times M_h \text{ such that for all } (\mathbf{w}_h, q_h) \in \mathbf{W}_h \times M_h,$$

$$\begin{cases} b(\mathbf{v}_h; \mathbf{v}_h, \mathbf{w}_h) + a(\mathbf{v}_h, \mathbf{w}_h) + c(\mathbf{v}_h; \mathbf{w}_h) + \\ \quad G(\mathbf{v}_h, \mathbf{w}_h) - (p_h, \nabla \cdot \mathbf{w}_h)_\Omega = \langle \mathbf{f}, \mathbf{w}_h \rangle, \\ (\nabla \cdot \mathbf{v}_h, q_h)_\Omega = 0; \end{cases} \tag{3.21}$$

the form c being defined by

$$c(\mathbf{v}; \mathbf{w}) = (\nu_t(\mathbf{v})\, D\mathbf{v}, D\mathbf{w})_\Omega \quad \text{or} \quad c(\mathbf{v}; \mathbf{w}) = (\nu_{t,\text{in}}(\mathbf{v})\, D\mathbf{v}, D\mathbf{w})_{\Omega_\text{in}} + (\nu_{t,w}(\mathbf{v})\, \partial_3 \mathbf{v}, \partial_3 \mathbf{w})_{\Omega_w}. \tag{3.22}$$

The second expression neglects the tangential eddy viscosity in the boundary layer, which is very small compared to the normal one.

4 Analysis of the Model

4.1 Technical results

We state in this subsection some technical results concerning the eddy viscosities and the associated turbulent diffusion form c, that are needed by our analysis.

(i) L^∞ eddy viscosties estimates.

Lemma 4.1 *There exists a constant $C > 0$ depending only on the aspect ratio of the family of triangulations $\{\mathcal{T}_h\}_{h>0}$ such that*

$$\|\nu_t(\mathbf{v}_h)\|_{0,\infty,\Omega} \leq C\, h^{\frac{1}{2}}\, \|\mathbf{v}_h\|_{1,2,\Omega} \quad \textit{for all } \mathbf{v}_h \in \mathbf{W}_h. \tag{4.1}$$

Proof We start with the internal part of the eddy viscosity $\nu_{t,in}$. Consider $\mathbf{v}_h \in \mathbf{W}_h$. As $\nabla \mathbf{v}_h$ is piecewise continuous, there exists a $K \in \mathcal{T}_h$ such that

$$\|\nu_{t,\text{in}}(\mathbf{v}_h)\|_{0,\infty,\Omega} = \|\nu_{t,\text{in}}(\mathbf{v}_h)\|_{0,\infty,K} \leq C_S^2\, h_K^2\, \|\nabla \mathbf{v}_h\|_{0,\infty,K}.$$

By a standard finite element inverse estimate (see [4]),

$$\|\nabla \mathbf{v}_h\|_{0,\infty,K} \leq C\, h_K^{-\frac{3}{2}}\, \|\nabla \mathbf{v}_h\|_{0,2,K}$$

for some constant $C > 0$ depending only on the aspect ratio of the family of triangulations. Then,

$$\|\nu_{t,\text{in}}(\mathbf{v}_h)\|_{0,\infty,\Omega} \leq C C_S^2\, h_K^{2-\frac{3}{2}}\, \|\nabla \mathbf{v}_h\|_{0,2,K} \leq C C_S^2\, h^{\frac{1}{2}}\, \|\nabla \mathbf{v}_h\|_{0,2,\Omega}. \tag{4.2}$$

Next, we analyze the wall eddy diffusion $\nu_{t,w}$. There exists some element $K \in \mathcal{T}_h$ such that

$$\|\nu_{t,w}(\mathbf{v}_h)\|_{0,\infty,\Omega} = \|\nu_{t,w}(\mathbf{v}_h)\|_{0,\infty,K} \leq C_w\, h_K\, (1 + \|\mathbf{v}_h\|_{0,\infty,K}),$$

where in the last inequality we have used (2.5). Using the inverse estimate (see [4]), $\|\mathbf{v}_h\|_{0,\infty,K} \leq C\, h_K^{-\frac{1}{2}}\, \|\nabla \mathbf{v}_h\|_{0,2,K}$, we deduce

$$\|\nu_{t,w}(\mathbf{v}_h)\|_{0,\infty,\Omega} \leq C'\, C_w\, h^{\frac{1}{2}}\, \|\nabla \mathbf{v}_h\|_{0,2,\Omega} \quad \text{for some constant } C' > 0.$$

Combining this estimate with (4.2) and $\|\nabla \mathbf{v}_h\|_{0,2,\Omega} \leq \|\mathbf{v}_h\|_{1,2,\Omega}$, (4.1) follows.

(ii) Turbulent diffusion operator properties.

Lemma 4.2 *The form c defined by* (3.22) *satisfies the following properties:*
(i) *c is non-negative, in the sense that*

$$c(\mathbf{v}; \mathbf{v}) \geq 0 \quad \textit{for all } \mathbf{v} \in H^1(\Omega)^3.$$

(ii) *Assume that the family of triangulations* $\{\mathcal{T}_h\}_{h>0}$ *is regular. Then, for any* $\mathbf{v}_h, \mathbf{w}_h \in \mathbf{W}_h$,

$$|c(\mathbf{v}_h; \mathbf{w}_h)| \leq C\, h^{\frac{1}{2}}\, \|\mathbf{v}_h\|_{1,2,\Omega}^2 \|\mathbf{w}_h\|_{1,2,\Omega} \tag{4.3}$$

for some constant $C > 0$ *depending only on d,* Ω *and the aspect ratio of the family of triangulations.*

(iii) *Assume that the family of triangulations* $\{\mathcal{T}_h\}_{h>0}$ *is regular. Let* $\{\mathbf{v}_h\}_{h>0}$ *and* $\{\mathbf{w}_h\}_{h>0}$ *be two sequences such that* $\mathbf{v}_h, \mathbf{w}_h \in \mathbf{W}_h$. *Then, if both sequences are bounded in* $\mathbf{H}^1(\Omega)^d$,

$$\lim_{h \to 0} c(\mathbf{v}_h; \mathbf{w}_h) = 0. \tag{4.4}$$

Proof (1) Let $\mathbf{v} \in \mathbf{H}^1(\Omega)$. Then

$$c(\mathbf{v}; \mathbf{v}) = \int_\Omega \nu_t(\mathbf{v})\, |D\mathbf{v}|^2 \mathrm{d}\mathbf{x} \geq 0.$$

(2) By (4.1),

$$\begin{aligned} |c(\mathbf{v}_h; \mathbf{w}_h)| &\leq \|\nu_t(\mathbf{v}_h)\|_{0,\infty,\Omega}\, \|\mathbf{v}_h\|_{1,2,\Omega} \|\mathbf{w}_h\|_{1,2,\Omega} \\ &\leq C\, h^{\frac{1}{2}}\, \|\mathbf{v}_h\|_{1,2,\Omega}^2 \|\mathbf{w}_h\|_{1,2,\Omega}. \end{aligned}$$

(3) (4.4) directly follows from (4.3).

4.2 Existence result

(3.21) is a set of non-linear equations in finite dimension. These non-linearities are due to several effects: the convection operator, the eddy viscosity, and the wall-law boundary conditions. The space $\mathbf{W}(\Omega)$ is a closed sub-space of $\mathbf{H}^1(\Omega)$. Our main result is the following.

Theorem 4.1 *Let* $\{\mathcal{T}_h\}_{h>0}$ *be a regular family of triangulations of the domain* Ω. *Let* $\{(\mathbf{W}_h, M_h)\}_{h>0}$ *be a family of pairs of finite element spaces satisfying Hypotheses* 3.1–3.2.

Then for any $\mathbf{f} \in \mathbf{W}(\Omega)'$, *the variational problem* (3.21) *admits at least a solution, that satisfies the estimates*

$$\|\mathbf{v}_h\|_{1,2,\Omega} \leq C\kappa^{-1}\|\mathbf{f}\|_{\mathbf{W}(\Omega)'}, \quad \kappa = \min\Big(\nu, \frac{\nu}{d}\Big), \tag{4.5}$$

$$\|p_h\|_{0,2,\Omega} \leq C\kappa^{-1}\|\mathbf{f}\|_{\mathbf{W}(\Omega)'}\Big(\kappa^{-1}\|\mathbf{f}\|_{\mathbf{W}(\Omega)'}[1+h^{\frac{1}{2}}] + \nu + \frac{1}{d} + 1\Big), \tag{4.6}$$

where $C > 0$ *is a constant depending only on* d, Ω *and the aspect ratio of the family of triangulations.*

Proof We prove the existence of solution in two steps.

Step 1 Existence of the velocity.

Let us define the mapping $\Phi_h : \mathbf{W}_h \to \mathbf{W}'_h$ as follows: Given $\mathbf{z}_h \in \mathbf{W}_h$,

$$\langle \Phi_h(\mathbf{z}_h), \mathbf{w}_h\rangle = b(\mathbf{z}_h; \mathbf{z}_h, \mathbf{w}_h) + a(\mathbf{z}_h, \mathbf{w}_h) + c(\mathbf{z}_h; \mathbf{w}_h) + G(\mathbf{z}_h, \mathbf{w}_h) - \langle \mathbf{f}, \mathbf{w}_h\rangle$$

for any $\mathbf{w}_h \in \mathbf{W}_h$. This equation has a unique solution as its r.h.s. defines a linear bounded functional on $\mathbf{W}_h$. Moreover, the functional Φ_h is continuous as all functions that appear in its definition are continuous on the finite-dimensional space $\mathbf{W}_h$.

Consider the sub-space Z_h of $\mathbf{W}_h$ defined by

$$Z_h = \{\mathbf{w}_h \in \mathbf{W}_h \ \text{such that} \ (\nabla\cdot\mathbf{w}_h, q_h) = 0 \quad \text{for all} \ q_h \in M_h\}.$$

Z_h is a non-empty closed sub-space of $\mathbf{H}^1(\Omega)$. Then it is a Hilbert space endowed with the $\mathbf{H}^1(\Omega)$ norm. Let $\mathbf{z}_h \in Z_h$. Then, as $b(\mathbf{z}_h; \mathbf{z}_h, \mathbf{z}_h) = 0$ and c is non-negative,

$$\begin{aligned}\langle \Phi_h(\mathbf{z}_h), \mathbf{z}_h\rangle &\geq a(\mathbf{z}_h, \mathbf{z}_h) + G(\mathbf{z}_h, \mathbf{z}_h) - \langle \mathbf{f}, \mathbf{z}_h\rangle \\ &\geq \nu\,\|D(\mathbf{z}_h)\|^2_{0,2,\Omega} + \frac{\nu}{d}\|\mathbf{z}_h\|^2_{0,2,\Gamma_w} - \|\mathbf{f}\|_{\mathbf{W}(\Omega)'}\|\mathbf{z}_h\|_{1,2,\Omega} \\ &\geq \frac{C\kappa}{2}\|\mathbf{z}_h\|^2_{1,2,\Omega} - \frac{\|\mathbf{f}\|^2_{\mathbf{W}(\Omega)'}}{2C\kappa},\end{aligned}$$

where we have used (3.9) and Young's inequality. We deduce

$$\forall\, \mathbf{z}_h \in Z_h \ \text{such that} \ \|\mathbf{z}_h\|_{1,2,\Omega} = \frac{\|\mathbf{f}\|_{\mathbf{W}(\Omega)'}}{C\kappa}, \quad \langle \Phi_h(\mathbf{z}_h), \mathbf{z}_h\rangle_{H^1(\Omega)} \geq 0. \tag{4.7}$$

Consequently, by a classical variant of Brouwer's fixed point theorem (see [36]), the equation

$$b(\mathbf{v}_h; \mathbf{v}_h, \mathbf{w}_h) + a(\mathbf{v}_h, \mathbf{w}_h) + c(\mathbf{v}_h; \mathbf{w}_h) + G(\mathbf{v}_h, \mathbf{w}_h) = \langle \mathbf{f}, \mathbf{w}_h\rangle, \quad \forall \mathbf{w}_h \in Z_h \tag{4.8}$$

admits a solution $\mathbf{v}_h \in Z_h$ such that $\|\mathbf{v}_h\|_{1,2,\Omega} \leq \frac{\|\mathbf{f}\|_{\mathbf{W}(\Omega)'}}{C\kappa}$, which precisely is (4.5) by changing C in C^{-1}.

Step 2 Existence of the pressure.

Let the operator $\mathcal{G}_h : M_h \mapsto \mathbf{W}'_h$ be defined by

$$\forall q_h \in M_h, \quad \langle \mathcal{G}_h(q_h), \mathbf{v}_h\rangle = (\nabla\cdot\mathbf{v}_h, q_h)_\Omega \qquad \text{for all} \ \mathbf{v}_h \in \mathbf{W}_h.$$

Then $Z_h = \mathrm{Im}(\mathcal{G}_h)^\perp$. The discrete inf-sup condition (see Hypothesis 3.2) ensures that $\mathrm{Im}(\mathcal{G}_h)$ is closed, then $Z_h^\perp = \mathrm{Im}(\mathcal{G}_h)$. As $\mathbf{v}_h$ is a solution of (4.8), then $\Phi_h(\mathbf{v}_h) \in Z_h^\perp$. Consequently, there exists some discrete pressure p_h such that $\langle \Phi_h(\mathbf{v}_h), \mathbf{w}_h \rangle = (\nabla \cdot \mathbf{v}_h, p_h)_\Omega$, for all $\mathbf{w}_h \in \mathbf{W}_h$. Thus, the pair $(\mathbf{v}_h, p_h)$ solves (3.21). The estimate for the norm of the pressure is obtained via the discrete inf-sup condition (3.20),

$$\|p_h\|_{0,2,\Omega} \le \alpha^{-1} \|\Phi_h\|_{\mathbf{W}_h'}$$

for some constant $\alpha > 0$. By (4.3) and some standard estimates,

$$\langle \Phi_h(\mathbf{v}_h), \mathbf{w}_h \rangle \le C \left[\|\mathbf{v}_h\|^2_{1,2,\Omega}(1 + Ch^{\frac{1}{2}}) + \nu \|\mathbf{v}_h\|_{1,2,\Omega}\Big(1 + \frac{C}{d}\Big)\right] \|\mathbf{w}_h\|_{1,2,\Omega} + \|\mathbf{f}\|_{\mathbf{W}(\Omega)'} \|\mathbf{w}_h\|_{1,2,\Omega}.$$

Then, the pressure estimate (4.6) follows from the velocity estimate (4.5).

4.3 Convergence

We now prove the convergence of the solution provided by method (3.21) to a weak solution of the Navier-Stokes boundary value problem model (2.3).

Theorem 4.2 *Under the hypotheses of Theorem* 4.1, *the sequence* $\{(\mathbf{v}_h, p_h)\}_{h>0}$ *contains a sub-sequence strongly convergent in* $\mathbf{H}^1(\Omega)^2 \times L^2(\Omega)$ *to a weak solution* $(\mathbf{v}, p) \in \mathbf{W}(\Omega) \times L^2_0(\Omega)$ *of the steady Navier-Stokes equation* (2.3). *If this solution is unique, then the whole sequence converges to it.*

Proof The proof is divided into 7 steps.

(1) Extracting subsequences.

By (4.5)–(4.6), the sequence $\{(\mathbf{v}_h, p_h)\}_{h>0}$ is bounded in the space $\mathbf{W}(\Omega) \times L^2_0(\Omega)$ which is a Hilbert space. Therefore, this sequence contains a subsequence, that we denote in the same way, weakly convergent in $\mathbf{W}(\Omega) \times L^2_0(\Omega)$ to some pair $(\mathbf{v}, p)$. As the injection of $H^1(\Omega)$ in $L^q(\Omega)$ is compact for $1 \le q < 6$, we may assume that the subsequence is strongly convergent in $\mathbf{L}^q(\Omega)$ for $1 \le q < 6$, and so in particular in $\mathbf{L}^4(\Omega)$.

Also, the injection of $\mathbf{H}^{\frac{1}{2}}(\Gamma_w)$ into $L^2(\Gamma_w)$ is compact. Then we may assume that the sequence $\{\mathbf{v}_{h_{|\Gamma_w}}\}_{h>0}$ is strongly convergent to $\mathbf{v}_{|\Gamma_w}$ in $\mathbf{L}^2(\Gamma_w)$.

(2) Taking the limit in the diffusion terms.

Let $(\mathbf{w}, q) \in \mathbf{W}(\Omega) \times L^2_0(\Omega)$. By Hypothesis 3.1, there exists a sequence $\{(\mathbf{w}_h, q_h)\}_{h>0}$ such that $(\mathbf{w}_h, q_h) \in \mathbf{W}_h \times M_h$ which is strongly convergent in $\mathbf{H}^1(\Omega) \times L^2(\Omega)$ to $(\mathbf{w}, q)$.

As a is bilinear and continuous,

$$\lim_{h \to 0} a(\mathbf{v}_h, \mathbf{w}_h) = a(\mathbf{v}, \mathbf{w}). \tag{4.9}$$

Next, since the sequences $\{\mathbf{v}_h\}_{h>0}$ and $\{\mathbf{w}_h\}_{h>0}$ are bounded in $\mathbf{H}^1(\Omega)$, we deduce from Lemma 4.2,

$$\lim_{h \to 0} c(\mathbf{v}_h; \mathbf{w}_h) = 0. \tag{4.10}$$

Finally, it is straightforward to check that

$$\lim_{h \to 0} G(\mathbf{v}_h; \mathbf{w}_h) = 0. \tag{4.11}$$

(3) Taking the limit in the convective term.

We have

$$\begin{aligned}
&|(\mathbf{v}_h \cdot \nabla \mathbf{v}_h, \mathbf{w}_h)_\Omega - (\mathbf{v} \cdot \nabla \mathbf{v}, \mathbf{w})_\Omega| \\
&\leq |((\mathbf{v}_h - \mathbf{v}) \cdot \nabla \mathbf{v}_h, \mathbf{w}_h)_\Omega| + |(\mathbf{v} \cdot \nabla(\mathbf{v}_h - \mathbf{v}), \mathbf{w})_\Omega| + |(\mathbf{v} \cdot \nabla \mathbf{v}_h, \mathbf{w}_h - \mathbf{w})_\Omega| \\
&\leq \|\mathbf{v}_h - \mathbf{v}\|_{0,4,\Omega} \|\nabla \mathbf{v}_h\|_{0,2,\Omega} \|\mathbf{w}_h\|_{0,4,\Omega} \\
&\quad + \sum_{i,j=1}^{3} |(\partial_j(v_{hi} - v_i), v_j w_i)_\Omega| + \|\mathbf{v}\|_{0,4,\Omega} \|\nabla \mathbf{v}_h\|_{0,2,\Omega} \|\mathbf{w}_h - \mathbf{w}\|_{0,4,\Omega},
\end{aligned}$$

where we denote $\mathbf{v}_h = (v_{h1}, v_{h2}, v_{h3})$. All terms in the r.h.s. of the last inequality vanish in the limit because $\{\mathbf{v}_h\}_{h>0}$ is strongly convergent in $\mathbf{L}^4(\Omega)$, $\{\partial_i v_{hi}\}_{h>0}$ is weakly convergent in $L^2(\Omega)$ and $\{\mathbf{w}_h\}_{h>0}$ is strongly convergent in $\mathbf{H}^1(\Omega)$. Then,

$$\lim_{h\to 0}((\mathbf{v}_h \cdot \nabla \mathbf{v}_h), \mathbf{w}_h)_\Omega = ((\mathbf{v} \nabla \mathbf{v}), \mathbf{w})_\Omega. \tag{4.12}$$

Similarly, $\lim\limits_{h\to 0}((\mathbf{v}_h \cdot \nabla)\, \mathbf{w}_h, \mathbf{v}_h)_\Omega = ((\mathbf{v} \cdot \nabla \mathbf{w}), \mathbf{v})_\Omega$, and then

$$\lim_{h\to 0} b(\mathbf{v}_h; \mathbf{v}_h, \mathbf{w}_h) = b(\mathbf{v}; \mathbf{v}, \mathbf{w}).$$

(4) Taking the limit in the pressure terms.

Since $\{\nabla\cdot\mathbf{v}_h\}_{h>0}$ is weakly convergent in $L^2(\Omega)$ to $\nabla\cdot\mathbf{v}_h$ and $\{q_h\}_{h>0}$ is strongly convergent in $L^2(\Omega)$ to q,

$$\lim_{h\to 0}(\nabla \cdot \mathbf{v}_h, q_h)_\Omega = (\nabla \cdot \mathbf{v}, q)_\Omega.$$

Finally, we obviously have

$$\lim_{h\to 0}(p_h, \nabla \cdot \mathbf{w}_h)_\Omega = (p, \nabla \cdot \mathbf{w})_\Omega.$$

Consequently, the pair $(\mathbf{v}, q)$ is a weak solution of Navier-Stokes equations (3.13).

(5) Strong convergence of the velocities.

Set $\mathbf{w}_h = \mathbf{v}_h$ in (3.21). Then

$$\nu \|D\mathbf{v}_h\|_{0,2,\Omega}^2 + \frac{\nu}{d}\|\mathbf{v}_h\|_{0,2,\Gamma_w} = \langle \mathbf{f}, \mathbf{v}_h\rangle - c(\mathbf{v}_h; \mathbf{v}_h).$$

By Lemma 4.2(3), $\lim\limits_{h\to 0} c(\mathbf{v}_h; \mathbf{v}_h) = 0$. Therefore,

$$\lim_{h\to 0}\left(\nu \|D\mathbf{v}_h\|_{0,2,\Omega}^2 + \frac{\nu}{d}\|\mathbf{v}_h\|_{0,2,\Gamma_w}\right) = \langle \mathbf{f}, \mathbf{v}\rangle = \nu \|D\mathbf{v}\|_{0,2,\Omega}^2 + \frac{\nu}{d}\|\mathbf{v}\|_{0,2,\Gamma_w}^2,$$

where the last equality occurs because $(\mathbf{v}, q)$ is a weak solution of Navier-Stokes equations (3.13). As $\mathbf{W}(\Omega)$ is a Hilbert space and $\{\mathbf{v}_h\}_{h>0}$ is weakly convergent to $\mathbf{v}$, this proves the strong convergence, since

$$\mathbf{w} \to \left(\nu \|D\mathbf{w}\|_{0,2,\Omega}^2 + \frac{\nu}{d}\|\mathbf{w}\|_{0,2,\Gamma_w}^2\right)^{\frac{1}{2}} \tag{4.13}$$

is a norm equivalent to the $\mathbf{H}^1(\Omega)$ norm by (3.9).

(6) Strong convergence of the pressures.

We use the discrete inf-sup condition to estimate $\|p_h - p\|_{0,2,\Omega}$. There exists a sequence $\{P_h\}_{h>0}$ such that $P_h \in M_h$ for all $h > 0$ which is strongly convergent in $L^2_0(\Omega)$ to p. We shall show that $\lim_{h\to 0} \|p_h - P_h\|_{0,2,\Omega} = 0$. Let $\mathbf{w}_h \in \mathbf{W}_h$. We have

$$\begin{aligned}(p_h - P_h, \nabla\cdot\mathbf{w}_h) &= b(\mathbf{v}_h;\mathbf{v}_h,\mathbf{w}_h) - b(\mathbf{v};\mathbf{v},\mathbf{w}_h) + a(\mathbf{v}_h - \mathbf{v},\mathbf{w}_h) + c(\mathbf{v}_h;\mathbf{w}_h)\\ &\quad + G(\mathbf{v}_h - \mathbf{v},\mathbf{w}_h) + (p - P_h, \nabla\cdot\mathbf{w}_h).\end{aligned}$$

As

$$\begin{aligned}b(\mathbf{v}_h;\mathbf{v}_h,\mathbf{w}_h) - b(\mathbf{v};\mathbf{v},\mathbf{w}_h) &= b(\mathbf{v}_h;\mathbf{v}_h - \mathbf{v},\mathbf{w}_h) + b(\mathbf{v}_h - \mathbf{v};\mathbf{v},\mathbf{w}_h)\\ &\le C\,\|\mathbf{v}_h - \mathbf{v}\|_{1,2,\Omega}\,(\|\mathbf{v}_h\|_{1,2,\Omega} + \|\mathbf{v}\|_{1,2,\Omega}),\end{aligned}$$

using (4.3) and the continuity of a we deduce

$$\begin{aligned}(p_h - P_h, \nabla\cdot\mathbf{w}_h) &\le C\,[\|\mathbf{v}_h - \mathbf{v}\|_{1,2,\Omega}\,(\|\mathbf{v}_h\|_{1,2,\Omega} + \|\mathbf{v}\|_{1,2,\Omega}) + \nu\|D(\mathbf{v}_h - \mathbf{v})\|_{0,2,\Omega}\\ &\quad + h^{\frac12}\,\|\mathbf{v}_h\|^2_{1,2,\Omega} + \frac{\nu}{d}\,\|\mathbf{v}_h - \mathbf{v}\|_{0,2,\Gamma_w} + \|p - P_h\|_{0,2,\Omega}]\|\mathbf{w}_h\|_{1,2,\Omega}.\end{aligned}$$

As $\lim_{h\to 0} \|\mathbf{v}_h - \mathbf{v}\|_{1,2,\Omega} = 0$, then by Hypothesis 3.2, $\lim_{h\to 0} \|p_h - P_h\|_{0,2,\Omega} = 0$. Then p_h strongly converges to p in $L^2(\Omega)$.

(7) Uniqueness.

It remains to prove that if the Navier-Stokes equations (2.3) admit a unique solution $(\mathbf{v}, p)$, then the whole sequence $\{(\mathbf{v}_h, p)\}_{h>0}$ converges to it. This is a standard result that holds when compactness arguments are used, which is proved by reduction to absurdity: Assume that the whole sequence does not converge to $(\mathbf{v}, p_h)$. Then there exists a sub-sequence of $\{(\mathbf{v}_h, p_h)\}_{h>0}$ that lies outside some ball of $\mathbf{W}(\Omega)\times L^2_0(\Omega)$ with center $(\mathbf{v}, p)$. Then the preceding compactness argument proves that a sub-sequence of this sub-sequence would converge to the unique solution $(\mathbf{v}, p)$, what is absurd.

Acknowledgements R. Lewandowski thanks Professor Li Tatsien and the ISFMA at Fudan University, Shanghai, China, for their hospitality during spring 2012, where part of this work was performed. He also thanks E. Memin and the Fluminance team at INRIA Rennes for their support during the second term of the academic year 2012–2013. T. Chacón thanks his fellows of the Department of Differential Equations and Numerical Analysis of the University of Sevilla (particularly J. Casado, M. Gómez and A. Suárez) for supporting him during his stay at BCAM, where part of this work was prepared.

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