Chin. Ann. Math.
34B(5), 2013, 683–690
DOI: 10.1007/s11401-013-0795-7

Chinese Annals of Mathematics, Series B

A Note on the Essential Norm of Composition Operators from $H^p(B_N)$ to $H^q(B_N)$*

Zhihua CHEN[1] Liangying JIANG[2] Qiming YAN[3]

Abstract The authors give an upper bound of the essential norms of composition operators between Hardy spaces of the unit ball in terms of the counting function in the higher dimensional value distribution theory defined by Professor S. S. Chern. The sufficient condition for such operators to be bounded or compact is also given.

Keywords Essential norm, Composition operator, Hardy space
2000 MR Subject Classification 47B38, 47B33, 32H02

1 Introduction

Let D $(= B_1)$ denote the unit disc of $\mathbb{C}$ and let φ be a holomorphic function on D with $\varphi(D) \subset D$. Then $C_\varphi f = f \circ \varphi$ defines a composition operator C_φ on the space of holomorphic functions in D.

In 1987, J. Shapiro [1] expressed the essential norm of the composition operators C_φ : $H^2(D) \to H^2(D)$ in terms of the Nevanlinna counting function of these φ as follows.

Theorem A *Let $\|C_\varphi\|_{\rm e}$ denote the essential norm of C_φ, regarded as an operator on $H^2(D)$. Then*

$$\|C_\varphi\|_{\rm e}^2 = \limsup_{|w|\to 1^-} \frac{N_\varphi(w)}{(-\log|w|)},$$

where $N_\varphi(w) = \sum\limits_{z\in\varphi^{-1}(w)} -\log|z|$ is the Nevanlinna counting function and $\varphi^{-1}(w)$ denotes the sequence of φ-preimages of w, each point being repeated in the sequence according to its multiplicity. In particular, C_φ is compact on $H^2(D)$ if and only if

$$\lim_{|w|\to 1^-} \frac{N_\varphi(w)}{(-\log|w|)} = 0.$$

In [2], Luo and Li extended Shapiro's result to the case of a composition operator C_φ : $H^p(D) \to H^q(D)$ with $1 < p \le q < \infty$.

Manuscript received September 10, 2012. Revised January 11, 2013.
[1]Department of Mathematics, Tongji University, Shanghai 200092, China. E-mail: zzzhhc@tongji.edu.cn
[2]Department of Applied Mathematics, Shanghai Finance University, Shanghai 201209, China.
E-mail: liangying1231@163.com
[3]Department of Mathematics, Tongji University, Shanghai 200092, China.
E-mail: yan_qiming@hotmail.com
*Project supported by the National Natural Science Foundation of China (Nos. 11171255, 11101279) and the Natural Science Foundation of Shanghai (No. 13ZR1444100).

Theorem B *Let φ be a holomorphic self-map of D. For $1 < p \le q < \infty$, $C_\varphi : H^p(D) \to H^q(D)$ is the composition operator induced by φ. If C_φ is bounded, then there exist constants C_1 and C_2 such that*

$$C_1 \limsup_{|w|\to 1^-} \frac{N_\varphi(w)}{(-\log|w|)^{\frac{q}{p}}} \le \|C_\varphi\|_{\mathrm{e}}^q \le C_2 \limsup_{|w|\to 1^-} \frac{N_\varphi(w)}{(-\log|w|)^{\frac{q}{p}}}.$$

Recently, for the case of several complex variables, Chen, Jiang and Yan [3] gave an upper bound of the essential norm of a composition operator on $H^2(B_N)$, which involves the counting function in the higher dimensional value distribution theory defined by Professor S. S. Chern [4].

Theorem C *Let $\varphi(z) = (\varphi_1(z), \cdots, \varphi_N(z)) : B_N \to B_N$ be a holomorphic map and let C_φ be the composition operator on $H^2(B_N)$. Assume that $a \le \Omega_\varphi(z) \le b$ on B_N with $a, b \in \mathbb{R}^+$, where $\Omega_\varphi = \frac{\|(\frac{\partial\varphi}{\partial z})\|^2}{|\det(\frac{\partial\varphi}{\partial z})|^2}$, $(\frac{\partial\varphi}{\partial z})$ is the Jacobi matrix of the map φ and $\|(\frac{\partial\varphi}{\partial z})\|^2 = \sum\limits_{i,j=1}^{N} |\frac{\partial\varphi_i}{\partial z_j}|^2$. If*

$$\limsup_{|w|\to 1^-} \frac{N_\varphi(w)}{1-|w|} = A < +\infty,$$

then C_φ is a bounded operator and the essential norm

$$\|C_\varphi\|_{\mathrm{e}}^2 \le 2b(N-1) \limsup_{|w|\to 1^-} \frac{N_\varphi(w)}{1-|w|}.$$

Furthermore, C_φ is compact on $H^2(B_N)$ if $\lim\limits_{|w|\to 1^-} \frac{N_\varphi(w)}{1-|w|} = 0$. Here,

$$N_\varphi(w) = \frac{1}{2N-2} \sum_{z\in\varphi^{-1}(w)} \Big(\frac{1}{|z|^{2N-2}} - 1 \Big)$$

for $w \in B_N \setminus \{\varphi(0)\}$.

In this paper, we will extend Theorem C to the composition operator $C_\varphi : H^p(B_N) \to H^q(B_N)$ for $1 \le p \le q < \infty$. Our main result is stated as follows.

Theorem 1.1 *Let $\varphi(z) = (\varphi_1(z), \cdots, \varphi_N(z)) : B_N \to B_N$ be a holomorphic map. For $1 \le p \le q < \infty$, $C_\varphi : H^p(B_N) \to H^q(B_N)$ is the composition operator induced by φ. Assume that $a \le \Omega_\varphi(z) \le b$ on B_N with $a, b \in \mathbb{R}^+$. If*

$$\limsup_{|w|\to 1^-} \frac{N_\varphi(w)}{(1-|w|)^{\frac{Nq}{p}-(N-1)}} = A < +\infty,$$

then C_φ is a bounded operator and there exists a positive constant C such that

$$\|C_\varphi\|_{\mathrm{e}}^q \le C \limsup_{|w|\to 1^-} \frac{N_\varphi(w)}{(1-|w|)^{\frac{Nq}{p}-(N-1)}}.$$

Corollary 1.1 *For $1 \le p \le q < \infty$, $C_\varphi : H^p(B_N) \to H^q(B_N)$ is compact if*

$$\lim_{|w|\to 1^-} \frac{N_\varphi(w)}{(1-|w|)^{\frac{Nq}{p}-(N-1)}} = 0.$$

2 Some Notations and Lemmas

Denote by $B_N(r) = \{z \in \mathbb{C}^N : |z| < r\}$ the ball of $\mathbb{C}^N$ with radius r. Let $B_N = B_N(1)$ be the unit ball and $rB_N = B_N(r)$. Set $\partial B_N(r) = \{z \in \mathbb{C}^N : |z| = r\}$. Let $\mathrm{d}\tau$ be the Euclidean volume element of $\mathbb{C}^N = \mathbb{R}^{2N}$ and $\mathrm{d}\sigma$ be the induced volume element on ∂B_N.

Let $H(B_N)$ denote the space of all holomorphic functions in B_N. For each p $(0 < p < \infty)$, the Hardy space $H^p(B_N)$ is defined by

$$H^p(B_N) = \Big\{ f \in H(B_N) : \sup_{0<r<1} \int_{\partial B_N} |f(r\xi)|^p \mathrm{d}\sigma(\xi) < \infty \Big\}$$

and

$$\|f\|_p^p = \sup_{0<r<1} \int_{\partial B_N} |f(r\xi)|^p \mathrm{d}\sigma(\xi),$$

where $\mathrm{d}\sigma(\xi) = \frac{(N-1)!}{2\pi^N}\mathrm{d}\sigma$ is the normalized Lebesgue measure on ∂B_N.

Lemma 2.1 (i) *For $0 < p < \infty$, $f \in H^p(B_N)$ and $w \in B_N$, we have*

$$|f(w)| \le \frac{C_1\|f\|_p}{(1-|w|)^{\frac{N}{p}}},$$

where C_1 is independent of f.

(ii) *For $1 \le p < \infty$, $f \in H^p(B_N)$ and $w \in B_N$, we have*

$$|\mathrm{grad} f(w)| \le \frac{C_2\|f\|_p}{(1-|w|)^{\frac{N}{p}+1}},$$

where C_2 is independent of f.

Proof (i) The proof can be found in [5]. Here we give another simple proof as follows.

For $f \in H^p(B_N)$, set $f_r(w) = f(rw)$. Then $|f_r(w)|^p$ is a subharmonic function. Let

$$\begin{aligned} u_r(w) &:= \int_{\partial B_N} \frac{|f_r(\xi)|^p (1-|w|^2)^N}{|1-\langle w,\xi\rangle|^{2N}} \mathrm{d}\sigma(\xi) \le \frac{2^N}{(1-|w|)^N} \int_{\partial B_N} |f_r(\xi)|^p \mathrm{d}\sigma(\xi) \\ &\le \frac{2^N}{(1-|w|)^N} \|f\|_p^p. \end{aligned}$$

We have that $u_r(w)$ is harmonic on B_N and $\lim\limits_{\substack{w\to\xi \\ w\in B_N}} = |f_r(\xi)|^p$ for $\xi \in \partial B_N$. Thus,

$$|f_r(w)|^p \le \sup_{\xi\in\partial B_N} |f_r(\xi)|^p \le \frac{2^N}{(1-|w|)^N} \|f\|_p^p$$

for $w \in B_N$, i.e.,

$$|f_r(w)| \le \frac{2^{\frac{N}{p}}}{(1-|w|)^{\frac{N}{p}}} \|f\|_p.$$

Letting $r \to 1^-$, we get

$$|f(w)| \le \frac{C_1\|f\|_p}{(1-|w|)^{\frac{N}{p}}}.$$

(ii) For $0<r<1$ and $w\in B_N$, set $f_r(w)=f(rw)$. Then we have

$$\frac{\partial}{\partial w_j}f_r(w)=\int_{\partial B_N}\frac{f(r\xi)}{(1-\langle w,\xi\rangle)^{N+1}}\cdot N\overline{\xi_j}\mathrm{d}\sigma(\xi),\quad 1\le j\le N.$$

If $p=1$, then

$$\begin{aligned}\Big|\frac{\partial}{\partial w_j}f_r(w)\Big| &\le N\int_{\partial B_N}|f(r\xi)|\mathrm{d}\sigma(\xi)\cdot\frac{1}{(1-|w|)^{N+1}}\\ &\le\frac{N\|f\|_1}{(1-|w|)^{N+1}}.\end{aligned}$$

If $p>1$, by Hölder's inequality, we have

$$\begin{aligned}\Big|\frac{\partial}{\partial w_j}f_r(w)\Big| &\le N\int_{\partial B_N}\frac{|f(r\xi)|}{|1-\langle w,\xi\rangle|^{N+1}}\mathrm{d}\sigma(\xi)\\ &\le N\Big(\int_{\partial B_N}|f(r\xi)|^p\mathrm{d}\sigma(\xi)\Big)^{\frac{1}{p}}\Big(\int_{\partial B_N}\Big(\frac{1}{|1-\langle w,\xi\rangle|^{N+1}}\Big)^q\mathrm{d}\sigma(\xi)\Big)^{\frac{1}{q}}\\ &\le\frac{CN\|f\|_p}{(1-|w|)^{\frac{N}{p}+1}},\end{aligned}$$

where the last inequality is provided by [6, Theorem 1.12].

Hence, letting $r\to 1^-$, we get

$$|\mathrm{grad}f(w)|\le\frac{C_2\|f\|_p}{(1-|w|)^{\frac{N}{p}+1}}.$$

Assume that φ is a holomorphic self-map of B_N and $C_\varphi: H^p(B_N)\to H^q(B_N)$ $(1\le p\le q<\infty)$ is the composition operator with norm

$$\|C_\varphi\|=\sup_{f\neq 0}\frac{\|f\circ\varphi\|_q}{\|f\|_p}.$$

In order to estimate $\|C_\varphi\|$, we need the following well-known Green formula.

Lemma 2.2 (Green Formula) *Let U and V be C^2 real functions on $\overline{D}\subset\mathbb{R}^m$, where D is a domain with smooth boundary ∂D. Then*

$$\int_D(U\triangle V-V\triangle U)\mathrm{d}\tau=\int_{\partial D}\Big(U\frac{\partial V}{\partial n}-V\frac{\partial U}{\partial n}\Big)\mathrm{d}\sigma,$$

where $\mathrm{d}\tau$ is the volume form on $\mathbb{R}^m$, $\mathrm{d}\sigma$ is the induced volume form on ∂D and $\frac{\partial V}{\partial n}\big(\frac{\partial U}{\partial n}\big)$ is the outward normal derivative of $V(U)$ on ∂D.

Lemma 2.3 *Let $(w_1,\cdots,w_N)=\varphi(z_1,\cdots,z_N)$ be a holomorphic self-map of B_N with $N>1$. Assume that $a\le\Omega_\varphi(z)\le b$ on B_N with $a,b\in\mathbb{R}^+$. If $f(w)\in H(B_N)$, then*

$$\|f\circ\varphi\|_p^p\le bp^2\frac{(N-1)!}{2\pi^N}\int_{B_N}|f(w)|^{p-2}|\mathrm{grad}f(w)|^2N_\varphi(w)\mathrm{d}\tau_w+|f(\varphi(0))|^p.\tag{2.1}$$

Proof Using the Green formula for $U=\frac{1}{|z|^{2N-2}}-\frac{1}{r^{2N-2}}$ and $V=|f\circ\varphi|^p$ on $B_N(r)\backslash B_N(\varepsilon)$ with $r\to 1^-$ and $\varepsilon\to 0^+$, we obtain

$$\|f\circ\varphi\|_p^p=p^2\frac{(N-2)!}{4\pi^N}\int_{B_N}\Big(\frac{1}{|z|^{2N-2}}-1\Big)|f(\varphi(z))|^{p-2}\mathrm{grad}f\cdot\Big(\frac{\partial\varphi}{\partial z}\Big)\cdot\overline{\Big(\frac{\partial\varphi}{\partial z}\Big)}^{\mathrm{T}}\cdot\overline{\mathrm{grad}f}^{\mathrm{T}}\mathrm{d}\tau_z + |f(\varphi(0))|^p. \tag{2.2}$$

By

$$\mathrm{grad}f\cdot\Big(\frac{\partial\varphi}{\partial z}\Big)\cdot\overline{\Big(\frac{\partial\varphi}{\partial z}\Big)}^{\mathrm{T}}\cdot\overline{\mathrm{grad}f}^{\mathrm{T}}\le|\mathrm{grad}f|^2\sum_{i,j=1}^N\Big|\frac{\partial\varphi_i}{\partial z_j}\Big|^2,$$

we have

$$\begin{aligned}\|f\circ\varphi\|_p^p&\le p^2\frac{(N-2)!}{4\pi^N}\int_{B_N}\Big(\frac{1}{|z|^{2N-2}}-1\Big)|f(\varphi(z))|^{p-2}|\mathrm{grad}f|^2\Omega_\varphi(z)\Big|\det\Big(\frac{\partial\varphi}{\partial z}\Big)\Big|^2\mathrm{d}\tau_z\\&\quad+|f(\varphi(0))|^p\\&\le bp^2\frac{(N-2)!}{4\pi^N}\int_{B_N}\Big(\frac{1}{|z|^{2N-2}}-1\Big)|f(\varphi(z))|^{p-2}|\mathrm{grad}f|^2\Big|\det\Big(\frac{\partial\varphi}{\partial z}\Big)\Big|^2\mathrm{d}\tau_z\\&\quad+|f(\varphi(0))|^p\\&=bp^2\frac{(N-1)!}{2\pi^N}\int_{B_N}|f(w)|^{p-2}|\mathrm{grad}f|^2N_\varphi(w)\mathrm{d}\tau_w+|f(\varphi(0))|^p.\end{aligned}$$

As $\varphi(z)\equiv z$, then by (2.2), we get

$$\begin{aligned}\|f\|_p^p&=|f(0)|^p+p^2\frac{(N-2)!}{4\pi^N}\int_{B_N}\Big(\frac{1}{|z|^{2N-2}}-1\Big)|f|^{p-2}|\mathrm{grad}f|^2\mathrm{d}\tau_z\\&=|f(0)|^p+p^2\frac{(N-2)!}{4\pi^N}\int_{B_N}\Big(\frac{1-|w|^{2N-2}}{|w|^{2N-2}}\Big)|f|^{p-2}|\mathrm{grad}f|^2\mathrm{d}\tau_w\\&\ge p^2\frac{(N-2)!}{4\pi^N}\int_{B_N}|f|^{p-2}|\mathrm{grad}f|^2(1-|w|)\mathrm{d}\tau_w.\end{aligned}$$

Hence, we have

$$\int_{B_N}|f|^{p-2}|\mathrm{grad}f|^2(1-|w|)\mathrm{d}\tau_w\le\frac{4\pi^N}{(N-2)!p^2}\|f\|_p^p. \tag{2.3}$$

As $f=w_i$ $(1\le i\le N)$, then $f\circ\varphi=\varphi_i$. By (2.2) and $p=2$, we have

$$\int_{B_n}N_\varphi(w)\mathrm{d}\tau_w\le\frac{\pi^N}{2a\cdot(N-1)!}. \tag{2.4}$$

For a bounded operator $C_\varphi:H^p(B_N)\to H^q(B_N)$ $(1\le p\le q<\infty)$, the essential norm $\|C_\varphi\|_{\mathrm{e}}$ of C_φ is defined to be the distance from C_φ to the set of the compact operators $K: H^p(B_N)\to H^q(B_N)$, namely,

$$\|C_\varphi\|_{\mathrm{e}}:=\inf\{\|C_\varphi-K\|:K\text{ is a compact operator}\}.$$

For $1\le p<\infty$, we define $K_n:=C_{\varphi_n}$ for any $n\ge 2$, where $\varphi_n=\frac{n-1}{n}z$. It is easy to deduce that $\|K_n\|\le 1$ and K_n is compact on every $H^p(B_N)$ (see [7]). Put $R_n:=I-K_n$ for $n\ge 2$, and in a way similar to the proof of Proposition 2.3 in [7], we have

$$\frac12\limsup_{n\to\infty}\|R_nC_\varphi\|\le\|C_\varphi\|_{\mathrm{e}}\le\liminf_{n\to\infty}\|R_nC_\varphi\|. \tag{2.5}$$

Remark 2.1 The equation (3.3) in the proof of Theorem 3.1 in [7] is not correct. In fact, one should use the above inequality to estimate $\|C_\varphi\|_{\mathrm{e}}$.

3 The Proof of Theorem 1.1

Throughout this section, C denotes a positive constant independent of f and φ, whose value may change from one occurrence to the next.

Firstly, we show that $C_\varphi : H^p(B_N) \to H^q(B_N)$ $(1 \le p \le q < \infty)$ is bounded.

If

$$\limsup_{|w|\to 1^-} \frac{N_\varphi(w)}{(1-|w|)^{\frac{Nq}{p}-(N-1)}} = A,$$

then, for any fixed $\varepsilon > 0$, we can find an r with $\frac{1}{2} \le r < 1$, such that

$$\frac{N_\varphi(w)}{(1-|w|)^{\frac{Nq}{p}-(N-1)}} < A + \varepsilon,$$

where $|w| \ge R$.

By (2.1), we obtain

$$\begin{aligned}
\|f\circ\varphi\|_q^q &\le |f(\varphi(0))|^q + bq^2\frac{(N-1)!}{2\pi^N}\int_{B_N}|f(w)|^{q-2}|\mathrm{grad} f|^2 N_\varphi(w)\mathrm{d}\tau_w \\
&= |f(\varphi(0))|^q + bq^2\frac{(N-1)!}{2\pi^N}\int_{B_N\setminus rB_N}|f(w)|^{q-2}|\mathrm{grad} f|^2 N_\varphi(w)\mathrm{d}\tau_w \\
&\quad + bq^2\frac{(N-1)!}{2\pi^N}\int_{rB_N}|f(w)|^{q-2}|\mathrm{grad} f|^2 N_\varphi(w)\mathrm{d}\tau_w \\
&= |f(\varphi(0))|^q + bq^2\frac{(N-1)!}{2\pi^N}\int_{B_N\setminus rB_N}|f(w)|^{q-p}|f(w)|^{p-2}|\mathrm{grad} f|^2 N_\varphi(w)\mathrm{d}\tau_w \\
&\quad + bq^2\frac{(N-1)!}{2\pi^N}\int_{rB_N}|f(w)|^{q-2}|\mathrm{grad} f|^2 N_\varphi(w)\mathrm{d}\tau_w
\end{aligned}$$

for any $f \in H^p(B_N)$

By Lemma 2.1(i), we have

$$|f(\varphi(0))|^q \le \frac{C\|f\|_p^q}{(1-|\varphi(0)|)^{\frac{Nq}{p}}}$$

and

$$\begin{aligned}
& bq^2\frac{(N-1)!}{2\pi^N}\int_{B_N\setminus rB_N}|f(w)|^{q-p}|f(w)|^{p-2}|\mathrm{grad} f|^2 N_\varphi(w)\mathrm{d}\tau_w \\
\le{}& C\|f\|_p^{q-p}\int_{B_N\setminus rB_N}|f(w)|^{p-2}|\mathrm{grad} f|^2(1-|w|)\cdot\frac{N_\varphi(w)}{(1-|w|)^{\frac{Nq}{p}-(N-1)}}\mathrm{d}\tau_w \\
\le{}& C\cdot(A+\varepsilon)\|f\|_p^{q-p}\int_{B_N\setminus rB_N}|f(w)|^{p-2}|\mathrm{grad} f|^2(1-|w|)\mathrm{d}\tau_w \\
\le{}& C\|f\|_p^q,
\end{aligned}$$

where the last inequality is provided by (2.3).

By (ii) of Lemma 2.1, we have

$$\begin{aligned}
& bq^2 \frac{(N-1)!}{2\pi^N} \int_{rB_N} |f(w)|^{q-2} |\mathrm{grad} f|^2 N_\varphi(w) \mathrm{d}\tau_w \\
\leq\, & C \cdot \frac{\|f\|_p^q}{(1-r)^{\frac{N}{p}(q-4)+2N+2}} \int_{rB_N} N_\varphi(w) \mathrm{d}\tau_w \\
\leq\, & C \|f\|_p^q,
\end{aligned}$$

where the last inequality is provided by (2.4).

Hence, we have

$$\|f \circ \varphi\|_q^q \leq C \|f\|_p^q,$$

i.e., C_φ is a bounded operator.

Next, we estimate the essential norm of C_φ by (2.5).

By (2.1), for any $f \in H^p(B_N)$ with $\|f\|_p = 1$ and $\frac{1}{2} \leq r < 1$, we have

$$\begin{aligned}
\|R_n C_\varphi f\|_q^q &\leq |R_n f(\varphi(0))|^q + bq^2 \frac{(N-1)!}{2\pi^N} \int_{B_N} |R_n f(w)|^{q-2} |\mathrm{grad} R_n f|^2 N_\varphi(w) \mathrm{d}\tau_w \\
&= |R_n f(\varphi(0))|^q + bq^2 \frac{(N-1)!}{2\pi^N} \int_{B_N \setminus rB_N} |R_n f(w)|^{q-2} |\mathrm{grad} R_n f|^2 N_\varphi(w) \mathrm{d}\tau_w \\
&\quad + bq^2 \frac{(N-1)!}{2\pi^N} \int_{rB_N} |R_n f(w)|^{q-2} |\mathrm{grad} R_n f|^2 N_\varphi(w) \mathrm{d}\tau_w.
\end{aligned} \tag{3.1}$$

Since $\|K_n\| \leq 1$ and $R_n = I - K_n$, we have $\|R_n\| \leq 2$. Thus, by Lemma 2.1(i) and (2.3),

$$\begin{aligned}
& bq^2 \frac{(N-1)!}{2\pi^N} \int_{B_N \setminus rB_N} |R_n f(w)|^{q-2} |\mathrm{grad} R_n f|^2 N_\varphi(w) \mathrm{d}\tau_w \\
\leq\, & C \|R_n f\|_p^{q-p} \int_{B_N \setminus rB_N} |R_n f(w)|^{p-2} |\mathrm{grad} R_n f|^2 (1-|w|) \cdot \frac{N_\varphi(w)}{(1-|w|)^{\frac{Nq}{p}-(N-1)}} \mathrm{d}\tau_w \\
\leq\, & C \|f\|_p^q \cdot \sup_{r \leq |w| < 1} \frac{N_\varphi(w)}{(1-|w|)^{\frac{Nq}{p}-(N-1)}} \leq C \cdot \sup_{r \leq |w| < 1} \frac{N_\varphi(w)}{(1-|w|)^{\frac{Nq}{p}-(N-1)}}.
\end{aligned} \tag{3.2}$$

On the other hand, we have, for $w \in rB_N$,

$$\begin{aligned}
|R_n f(w)| &= \Big| f(w) - f\Big(\frac{n-1}{n} w\Big) \Big| = \Big| \int_{\frac{n-1}{n}}^1 \frac{\mathrm{d} f(tw)}{\mathrm{d}t} \mathrm{d}t \Big| \\
&= \Big| \int_{\frac{n-1}{n}}^1 \sum_{j=1}^N w_j \frac{\partial f}{\partial w_j}(tw) \mathrm{d}t \Big| \\
&\leq \frac{1}{n} \sum_{j=1}^N \sup_{z \in rB_N} \Big| \frac{\partial f}{\partial w_j}(z) \Big|.
\end{aligned}$$

For all $z \in rB_N$ and any fixed $\delta \in (r, 1)$,

$$\Big| \frac{\partial^m f}{\partial w^m}(z) \Big| \leq C \cdot \sup\{|f(z)| : |z| \leq \delta\},$$

where $m = (m_1, \cdots, m_N)$ is a multi-index (see the proof of Lemma 2.4 in [6]).

Hence, by Lemma 2.1(i),

$$\begin{aligned}|R_nf(w)| &\le \frac{1}{n}\sum_{j=1}^{N}\sup_{z\in rB_N}\Big|\frac{\partial f}{\partial w_j}(z)\Big| \le \frac{C}{n}\sup\Big\{|f(z)|:|z|\le\frac{1+r}{2}\Big\}\\ &\le \frac{C}{n}\cdot\frac{\|f\|_p}{(1-r)^{\frac{N}{p}}} \le \frac{C}{n(1-r)^{\frac{N}{p}}}\end{aligned}$$

for $w\in rB_N$.

Similarly, we have, for $1\le k\le N$ and $w\in rB_N$,

$$\Big|\frac{\partial}{\partial w_k}R_nf(w)\Big| = \Big|\int_{\frac{n-1}{n}}^{1}\sum_{j=1}^{N}\Big(\delta_{kj}\frac{\partial f}{\partial w_j}(tw)+tw_j\frac{\partial^2 f}{\partial w_j\partial w_k}(tw)\Big)\mathrm{d}t\Big| \le \frac{C}{n(1-r)^{\frac{N}{p}}}.$$

Hence,

$$\begin{aligned}&bq^2\frac{(N-1)!}{2\pi^N}\int_{rB_N}|R_nf(w)|^{q-2}|\mathrm{grad}R_nf|^2N_\varphi(w)\mathrm{d}\tau_w\\ &\le\Big(\frac{C}{n(1-r)^{\frac{N}{p}}}\Big)^q\int_{rB_N}N_\varphi(w)\mathrm{d}\tau_w\to 0\end{aligned}\tag{3.3}$$

and $|R_nf(\varphi(0))|\to 0$ as $n\to\infty$.

Combining (2.5) and (3.1)–(3.3) and letting $n\to\infty$, we get

$$\|C_\varphi\|_{\mathrm{e}}^q \le C\cdot\sup_{r\le|w|<1}\frac{N_\varphi(w)}{(1-|w|)^{\frac{Nq}{p}-(N-1)}}.$$

Let $r\to 1^-$. Then

$$\|C_\varphi\|_{\mathrm{e}}^q \le C\cdot\limsup_{|w|\to 1^-}\frac{N_\varphi(w)}{(1-|w|)^{\frac{Nq}{p}-(N-1)}}.$$

References

[1] Shapiro, J. H., The essential norm of a composition operator, *Ann. Math.*, **125**, 1987, 375–404.

[2] Luo, L. and Li, K., Essential norms of composition operators between Hardy space of the unit disc, *Chin. Ann. Math.*, **32B**(2), 2011, 209–214.

[3] Chen, Z. H., Jiang, L. Y. and Yan, Q. M., An upper bound of the essential norm of a composition operator on $H^2(B_n)$, *Chin. Ann. Math.*, **33**B(6), 2012, 841–856.

[4] Chern, S. S., The integrated form of the first main theorem for complex analytic mappings in several complex variables, *Ann. Math.*, **71**(3), 1960, 536–551.

[5] Rudin, W., Function Theory in the Unit Ball of $\mathbb{C}^n$, Springer-Verlag, New York, 1980.

[6] Zhu, K., Spaces of Holomorphic Functions in the Unit Ball, Springer-Verlag, New York, 2005.

[7] Charpentier, S., Essential norm of composition operators on the Hardy space H^1 and the weighted Bergman spaces A^p_α on the ball, *Arch. Math.*, **98**, 2012, 327–340.