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Existence of Classical Solutions to a Stationary Simplified Quantum Energy-Transport Model in 1-Dimensional Space*

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Abstract The existence of classical solutions to a stationary simplified quantum energy-transport model for semiconductor devices in 1-dimensional space is proved. The model consists of a nonlinear elliptic third-order equation for the electron density, including a temperature derivative, an elliptic nonlinear heat equation for the electron temperature, and the Poisson equation for the electric potential. The proof is based on an exponential variable transformation and the Leray-Schauder fixed-point theorem.

Keywords Quantum energy-transport model, Stationary solutions, Existence
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1 Introduction and Main Results

The quantum energy-transport model reads as (see [1])

$$n_t + \operatorname{div}\Big[\frac{\varepsilon^2}{6} n\nabla\Big(\frac{\triangle\sqrt{n}}{\sqrt{n}}\Big) - \nabla(nT) + n\nabla V\Big] = 0, \tag{1.1}$$

$$-\operatorname{div}(k(n,T)\nabla T) = \frac{n}{\tau}(T_L(x) - T), \tag{1.2}$$

$$\lambda^2 \triangle V = n - C(x), \tag{1.3}$$

where the electron density n, the electron temperature T and the self-consistent electric potential V are unknown variables, the doping profile $C(x)$ models fixed background charges in the semiconductor crystal, the lattice temperature $T_L(x)$ is a given function, the scaled Planck constant $\varepsilon > 0$, the energy relaxation time $\tau > 0$ and the Debye length $\lambda > 0$ are physical parameters, and the heat conductivity $k(n,T)$ is often taken as $k(n,T) = nT$ (see [2–3]). The model (1.1)–(1.3) can be derived from the quantum hydrodynamic equations after a diffusive rescaling and a relaxation-time limit (see [1]). In [1], Jüngel and Milišić proved the global existence of weak solutions to (1.1)–(1.3) with $k(n,T) = n$ under periodic boundary conditions. Recently, the semiclassical limit of solutions to (1.1)–(1.3) with $k(n,T) = n$ has been performed in [4].

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In this paper, we will study the classical solutions to the stationary model of (1.1)–(1.3) with $k(n,T)=n$ in 1-dimensional space

$$\frac{\varepsilon^2}{6}n\Big(\frac{(\sqrt{n})_{xx}}{\sqrt{n}}\Big)_x-(nT)_x+nV_x=J_0, \tag{1.4}$$

$$-(nT_x)_x=\frac{n}{\tau}(T_L(x)-T), \tag{1.5}$$

$$\lambda^2V_{xx}=n-C(x),\quad \text{in } (0,1), \tag{1.6}$$

where J_0 is the current density. We choose the following conditions:

$$n(0)=n(1)=1,\quad n_x(0)=n_x(1)=0,\quad T(0)=T_0,\quad T_x(0)=T_x(1)=0, \tag{1.7}$$

$$V(0)=V_0=-\frac{\varepsilon^2}{6}(\sqrt{n})_{xx}(0)+T_0. \tag{1.8}$$

The boundary condition (1.8) can be interpreted as a Dirichlet condition for the Bohm potential $\frac{(\sqrt{n})_{xx}}{\sqrt{n}}$ at $x=0$.

Our main results are stated as follows.

Theorem 1.1 *Let $C(x),T_L(x)\in L^\infty(0,1)$ with $C(x)>0,\ 0<m_L\le T_L(x)\le M_L$ for $x\in(0,1)$. Then there exists a classical solution (n,T,V) to (1.4)–(1.8), such that $n(x)\ge \mathrm{e}^{-M}>0$ for $x\in(0,1)$, where M is the solution to*

$$M=\sqrt{\frac{\mathrm{e}^{2M}}{\tau m_L^2}(M_L-m_L)M_L+\frac{2(\mathrm{e}^{-1}+\|C(x)\log C(x)\|_{L^\infty(0,1)})}{\lambda^2 m_L}}. \tag{1.9}$$

2 Proof of the Results

The main idea of this paper is to reformulate (1.4) and (1.6) as an elliptic fourth-order equation and use the exponential variable $n=\mathrm{e}^u$. This method was used to prove the existence of classical solutions to the stationary quantum Navier-Stokes equations (see [5]). But to the authors' knowledge, it is the first time for us to use this method to deal with the quantum energy-transport model.

As in [5], dividing (1.4) by n, differentiating the resulting equation with respect to x, and using the Poisson equation (1.6), we obtain

$$\frac{\varepsilon^2}{6}\Big(\frac{(\sqrt{n})_{xx}}{\sqrt{n}}\Big)_{xx}-T_{xx}-[(\log n)_xT]_x+\frac{n-C(x)}{\lambda^2}=J_0\Big(\frac{1}{n}\Big)_x. \tag{2.1}$$

The electrostatic potential can be recovered from (1.4), after division by n and integration (noticing that the integration constant vanishes due to the boundary conditions (1.7)–(1.8)):

$$V(x)=-\frac{\varepsilon^2}{6}\frac{(\sqrt{n})_{xx}}{\sqrt{n}}(x)+T(x)+\int_0^x(\log n)_x(s)T(s)\mathrm{d}s+J_0\int_0^x\frac{\mathrm{d}s}{n(s)}. \tag{2.2}$$

Introducing the exponential variable $n=\mathrm{e}^u$ and observing that

$$\frac{(\sqrt{n})_{xx}}{\sqrt{n}}=\frac{(\mathrm{e}^{\frac{u}{2}})_{xx}}{\mathrm{e}^{\frac{u}{2}}}=\frac{1}{2}\Big(u_{xx}+\frac{u_x^2}{2}\Big),$$

we can write (2.1), (1.5), (2.2) and (1.7)–(1.8) as following respectively:

$$\frac{\varepsilon^2}{12}\Big(u_{xx}+\frac{u_x^2}{2}\Big)_{xx}-T_{xx}-(u_xT)_x+\frac{\mathrm{e}^u-C(x)}{\lambda^2}=J_0(\mathrm{e}^{-u})_x, \tag{2.3}$$

$$-(\mathrm{e}^uT_x)_x=\frac{\mathrm{e}^u}{\tau}(T_L(x)-T), \tag{2.4}$$

$$V(x)=-\frac{\varepsilon^2}{12}\Big(u_{xx}+\frac{u_x^2}{2}\Big)(x)+T(x)+\int_0^x u_x(s)T(s)\mathrm{d}s+J_0\int_0^x \mathrm{e}^{-u(s)}\mathrm{d}s, \tag{2.5}$$

$$u(0)=u(1)=0,\quad u_x(0)=u_x(1)=0,\quad T(0)=T_0,\quad T_x(0)=T_x(1)=0, \tag{2.6}$$

$$V(0)=V_0=-\frac{\varepsilon^2}{12}u_{xx}(0)+T_0. \tag{2.7}$$

It is easy to prove that problems (1.4)–(1.8) and (2.3)–(2.7) are equivalent for classical solutions if $n>0$ in $(0,1)$. We solve problem (2.3)–(2.7) as follows.

As usual, we call $(u,T)\in H_0^2(0,1)\times H^1(0,1)$ a weak solution to (2.3)–(2.4) and (2.6) if for all $(\psi,\varphi)\in H_0^2(0,1)\times H^1(0,1)$ there holds

$$\begin{aligned}&\frac{\varepsilon^2}{12}\int_0^1\Big(u_{xx}+\frac{u_x^2}{2}\Big)\psi_{xx}\mathrm{d}x+\int_0^1 T_x\psi_x\mathrm{d}x+\int_0^1 u_xT\psi_x\mathrm{d}x\\ =&-\frac{1}{\lambda^2}\int_0^1(\mathrm{e}^u-C(x))\psi\mathrm{d}x-J_0\int_0^1\mathrm{e}^{-u}\psi_x\mathrm{d}x\end{aligned} \tag{2.8}$$

and

$$\int_0^1\mathrm{e}^uT_x\varphi_x\mathrm{d}x=\frac{1}{\tau}\int_0^1\mathrm{e}^u(T_L(x)-T)\varphi\mathrm{d}x. \tag{2.9}$$

We consider (2.8) and the following truncated problem:

$$\int_0^1\mathrm{e}^{u_M}T_x\varphi_x\mathrm{d}x=\frac{1}{\tau}\int_0^1\mathrm{e}^{u_M}(T_L(x)-T)\varphi\mathrm{d}x, \tag{2.10}$$

where $M>0$ is the constant defined in (1.9) and $u_M=\min\{M,\ \max\{-M,u\}\}$. The following lemma is the key a priori estimate of this paper.

Lemma 2.1 *Let* $(u,T)\in H_0^2(0,1)\times H^1(0,1)$ *be a solution to* (2.8) *and* (2.10). *Under the assumptions of Theorem* 1.1, *there holds*

$$\begin{aligned}&\frac{\varepsilon^2}{12}\|u_{xx}\|^2_{L^2(0,1)}+\frac{m_L}{2}\|u_x\|^2_{L^2(0,1)}\\ \le&\frac{\mathrm{e}^{2M}}{2\tau m_L}(M_L-m_L)M_L+\lambda^{-2}(\mathrm{e}^{-1}+\|C(x)\log C(x)\|_{L^\infty(0,1)}).\end{aligned} \tag{2.11}$$

In particular, there follows

$$\|u\|_{L^\infty(0,1)}\le M,\quad \|T_x\|_{L^2(0,1)}\le\mathrm{e}^M\sqrt{\frac{(M_L-m_L)M_L}{\tau}}, \tag{2.12}$$

where $M>0$ *is the constant defined in* (1.9).

Proof First, with the test function $\varphi=(T-M_L)^+=\max\{0,T-M_L\}\in H^1(0,1)$ in (2.10), we infer that

$$\int_0^1\mathrm{e}^{u_M}(T-M_L)_x^{+2}\mathrm{d}x=\frac{1}{\tau}\int_0^1\mathrm{e}^{u_M}(T_L(x)-T)(T-M_L)^+\mathrm{d}x\le 0,$$

where we have used the assumption $T_L(x) \le M_L$ for $x \in (0,1)$. This implies that $(T-M_L)^+ = 0$ and hence $T \le M_L$ in $(0,1)$. Similarly, with the test function $\varphi = (T-m_L)^- = \min\{0, T-m_L\} \in H^1(0,1)$ in (2.10), we obtain $T \ge m_L > 0$ in $(0,1)$.

Next, we employ the test function $\varphi = T$ in (2.10) as follows:

$$\int_0^1 \mathrm{e}^{u_M} T_x^2 \mathrm{d}x = \frac{1}{\tau}\int_0^1 \mathrm{e}^{u_M}(T_L(x)-T)T\mathrm{d}x \le \frac{\mathrm{e}^M}{\tau}(M_L-m_L)M_L.$$

This inequality and

$$\mathrm{e}^{-M}\int_0^1 T_x^2\mathrm{d}x \le \int_0^1 \mathrm{e}^{u_M}T_x^2\mathrm{d}x$$

imply

$$\int_0^1 T_x^2\mathrm{d}x \le \frac{\mathrm{e}^{2M}}{\tau}(M_L-m_L)M_L. \tag{2.13}$$

Then, using the test function $\psi = u \in H_0^2(0,1)$ in (2.8), we have

$$\begin{aligned}&\frac{\varepsilon^2}{12}\int_0^1 u_{xx}^2\mathrm{d}x + \frac{\varepsilon^2}{24}\int_0^1 u_x^2 u_{xx}\mathrm{d}x + \int_0^1 T u_x^2\mathrm{d}x\\ &= -\int_0^1 T_x u_x\mathrm{d}x - \frac{1}{\lambda^2}\int_0^1 u(\mathrm{e}^u - C(x))\mathrm{d}x - J_0\int_0^1 \mathrm{e}^{-u}u_x\mathrm{d}x.\end{aligned} \tag{2.14}$$

Due to the boundary conditions (2.6), the second integral on the left-hand side and the third integral on the right-hand side of (2.14) vanish

$$\int_0^1 u_x^2 u_{xx}\mathrm{d}x = \frac{1}{3}[u_x^3(1)-u_x^3(0)] = 0, \quad \int_0^1 \mathrm{e}^{-u}u_x\mathrm{d}x = \mathrm{e}^{-u(0)} - \mathrm{e}^{-u(1)} = 0.$$

Since $T \ge m_L > 0$ in $(0,1)$,

$$\int_0^1 T u_x^2\mathrm{d}x \ge m_L\int_0^1 u_x^2\mathrm{d}x.$$

It is not difficult to see that $\mathrm{e}^{-1} + \|C(x)\log C(x)\|_{L^\infty(0,1)}$ is an upper bound for the function $u \mapsto -u(\mathrm{e}^u - C(x))$, $u \in \mathbb{R}$, for any $x \in (0,1)$. Here we use the assumption that $C(x)$ is positive. Therefore,

$$-\frac{1}{\lambda^2}\int_0^1 u(\mathrm{e}^u - C(x))\mathrm{d}x \le \lambda^{-2}(\mathrm{e}^{-1} + \|C(x)\log C(x)\|_{L^\infty(0,1)}).$$

By the Young inequality and (2.13),

$$-\int_0^1 T_x u_x\mathrm{d}x \le \frac{m_L}{2}\int_0^1 u_x^2\mathrm{d}x + \frac{1}{2m_L}\int_0^1 T_x^2\mathrm{d}x \le \frac{m_L}{2}\int_0^1 u_x^2\mathrm{d}x + \frac{\mathrm{e}^{2M}}{2\tau m_L}(M_L-m_L)M_L.$$

The above estimates yield (2.11).

Finally, from the Pincaré-Sobolev estimate and (2.11),

$$\begin{aligned}\|u\|_{L^\infty(0,1)} &\le \|u_x\|_{L^2(0,1)}\\ &\le M = \sqrt{\frac{\mathrm{e}^{2M}}{\tau m_L^2}(M_L-m_L)M_L + \frac{2(\mathrm{e}^{-1} + \|C(x)\log C(x)\|_{L^\infty(0,1)})}{\lambda^2 m_L}}.\end{aligned}$$

This inequality and (2.13) yield (2.12).

Now we use the Leray-Schauder fixed-point theorem to prove the existence of a solution to problem (2.8)–(2.9).

Lemma 2.2 *Under the assumptions of Theorem* 1.1*, there exists a solution* $(u,T) \in H_0^2(0,1) \times H^1(0,1)$ *to* (2.8)–(2.9).

Proof For given $w \in H_0^1(0,1)$, let $T \in H^1(0,1)$ be the unique solution to

$$\int_0^1 \mathrm{e}^{w_M} T_x \varphi_x \mathrm{d}x = \frac{1}{\tau} \int_0^1 (\mathrm{e}^{w_M} - C(x))\varphi \mathrm{d}x$$

with test functions $\varphi \in H^1(0,1)$. As in Lemma 2.1, we can get $0 < m_L \le T \le M_L$. Then we consider the following linear problem with test functions $\psi \in H_0^2(0,1)$:

$$\begin{aligned} &\frac{\varepsilon^2}{12} \int_0^1 u_{xx}\psi_{xx}\mathrm{d}x + \frac{\sigma\varepsilon^2}{24} \int_0^1 w_x^2 \psi_{xx}\mathrm{d}x + \sigma \int_0^1 T_x\psi_x \mathrm{d}x + \int_0^1 T u_x \psi_x \mathrm{d}x \\ = &-\frac{\sigma}{\lambda^2} \int_0^1 (\mathrm{e}^w - C(x))\psi \mathrm{d}x - \sigma J_0 \int_0^1 \mathrm{e}^{-w}\psi_x \mathrm{d}x, \end{aligned} \tag{2.15}$$

where $\sigma \in [0,1]$. We define the bilinear form

$$a(u,\psi) = \frac{\varepsilon^2}{12} \int_0^1 u_{xx}\psi_{xx}\mathrm{d}x + \int_0^1 T u_x \psi_x \mathrm{d}x \tag{2.16}$$

and the linear functional

$$\begin{aligned} F(\psi) = &-\frac{\sigma\varepsilon^2}{24} \int_0^1 w_x^2 \psi_{xx}\mathrm{d}x - \sigma \int_0^1 T_x \psi_x \mathrm{d}x \\ &- \frac{\sigma}{\lambda^2} \int_0^1 (\mathrm{e}^w - C(x))\psi \mathrm{d}x - \sigma J_0 \int_0^1 \mathrm{e}^{-w}\psi_x \mathrm{d}x. \end{aligned} \tag{2.17}$$

Since the bilinear form $a(u,\psi)$ is continuous and coercive on $H_0^2(0,1) \times H_0^2(0,1)$ for $0 < m_L \le T \le M_L$, and the linear functional $F(\psi)$ is continuous on $H_0^2(0,1)$, we can apply the Lax-Milgram theorem to obtain the existence of a solution $u \in H_0^2(0,1)$ to (2.15). Thus, the operator

$$S: \ H_0^1(0,1) \times [0,1] \to H_0^1(0,1), \quad (w,\sigma) \mapsto u$$

is well-defined. Moreover, it is continuous and compact, since the embedding $H_0^2(0,1) \hookrightarrow H_0^1(0,1)$ is compact. Furthermore, $S(w,0) = 0$. Following the steps of the proof of Lemma 2.1, we can show that $\|u\|_{H_0^2(0,1)} \le$ const., satisfying $S(u,\sigma) = u$ for all $(u,\sigma) \in H_0^1(0,1) \times [0,1]$. Therefore, the existence of a fixed point u with $S(u,1) = u$ follows from the Leray-Schauder fixed-point theorem. This gives a solution (u,T) to (2.8) and (2.10), and, in fact, also to (2.8)–(2.9), since $\|u\|_{L^\infty(0,1)} \le M$.

With Lemma 2.2, we can obtain the existence of a solution to (2.3)–(2.7).

Theorem 2.1 *Under the assumptions of Theorem* 1.1*, there exists a solution* $(u,T,V) \in H^4(0,1) \times H^2(0,1) \times H^2(0,1)$ *to* (2.3)–(2.7).

Proof Let (u,T) be a weak solution to (2.8)–(2.9) or (2.3)–(2.4) and (2.6). Since $u \in H_0^2(0,1)$, there hold $u_x \in L^\infty(0,1)$ and $u_x^2 \in H_0^1(0,1)$. The equation (2.4) is equivalent to

$$-T_{xx} = u_x T_x + \frac{1}{\tau}(T_L(x) - T) \tag{2.18}$$

for classical solutions, since $e^u \geq e^{-M} > 0$ for $\|u\|_{L^\infty} \leq M$, and this yields $T_{xx} \in L^2(0,1)$ by using $u_x \in L^\infty(0,1)$ and $T_x \in L^2(0,1)$. Then, from (2.3), $u_x^2 \in H_0^1(0,1)$ and $T_{xx} \in L^2(0,1)$, we infer $u_{xxxx} \in H^{-1}(0,1)$. Hence, there exists a $w \in L^2(0,1)$, such that $w_x = u_{xxxx}$. This implies $u_{xxx} = w + \text{const.} \in L^2(0,1)$, and by (2.3) and $T_{xx} \in L^2(0,1)$, we have $u_{xxxx} \in L^2(0,1)$. This implies $u \in H^4(0,1)$, and from (2.5) and the regularity of $u,\ T$, there follows the regularity of V.

Proof of Theorem 1.1 Since $u \in H^4(0,1)$, $\|u\|_{L^\infty(0,1)} \leq M$ and $n = e^u$, we have $n \in H^4(0,1)$ and $n(x) \geq e^{-M} > 0$ for $x \in (0,1)$. The equivalence of problems (1.4)–(1.8) and (2.3)–(2.7) provides the existence of a classical solution (n,T,V) to (1.4)–(1.8) by Theorem 2.1.

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