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On the Diophantine Equation $x^2 - kxy + y^2 + lx = 0$*

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Abstract Let l be a given nonzero integer. The authors give an explicit characterization of the positive integer k that makes the Diophantine equation $x^2 - kxy + y^2 + lx = 0$ have infinitely many positive integer solutions (x, y).

Keywords Quadratic Diophantine equation, Solvability, Counting solutions
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1 Introduction

Let $\mathbb{Z}, \mathbb{N}$ be the sets of all integers and positive integers respectively. For a given nonzero integer l, we let $K(l)$ be the subset of $\mathbb{N}$ such that for any $k \in K(l)$, the Diophantine equation

$$x^2 - kxy + y^2 + lx = 0, \quad x, y \in \mathbb{N} \tag{1.1}$$

has infinitely many solutions (x, y). When l is a positive integer, Marlewski and Zarzycki [3] proved $K(1) = \{3\}$, while Yuan and Hu [5] proved $K(2) = \{3, 4\}$ and $K(4) = \{3, 4, 6\}$.

In this paper, we will give a necessary and sufficient condition for $k \in K(l)$ with a nonzero integer l.

Theorem 1.1 *If l is a positive integer, then the necessary and sufficient condition for $k \in K(l)$ is $k \geq 3$ and that l has a positive factor l_1 such that the equation*

$$X^2 - (k^2 - 4)Y^2 = -\frac{4l_1}{\delta^2}, \quad \delta \in \{1, 2\}, \quad X, Y \in \mathbb{N}, \quad \gcd(X, Y) = 1 \tag{1.2}$$

has solutions (X, Y).

Corollary 1.1 *If l is a positive integer, then $k \leq 4l + 2$ for any $k \in K(l)$.*

When l is a negative integer, Keskin [2] proved that $k \in K(-1)$ for any integer $k > 3$. In this paper, we will completely solve this case as follows.

Theorem 1.2 *If l is a negative integer, then the necessary and sufficient condition for $k \in K(l)$ is $k > 1$.*

2 Some Lemmas

Let $D > 0$ be a non-square integer and r be a nonzero integer.

Lemma 2.1 (see [1, Theorem 11.4.2]) *If the equation*

$$X^2 - DY^2 = r, \quad X, Y \in \mathbb{Z}, \quad \gcd(X, Y) = 1 \tag{2.1}$$

has a solution (X, Y), then it has infinitely many solutions.

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Lemma 2.2 (see [1, Theorem 10.8.2]) *If $|r| < \sqrt{D}$ and (X, Y) is a solution of equation* (2.1)*, then $\frac{|X|}{|Y|}$ is a convergent of $\sqrt{D}$.*

Let $\frac{p_m}{q_m}$ be the m-th convergent of $\sqrt{D}$ and

$$s_m = p_m^2 - Dq_m^2. \tag{2.2}$$

Since D is a non-square integer, it is well-known that the Pell equation

$$u^2 - Dv^2 = 1, \quad u, v \in \mathbb{N} \tag{2.3}$$

has solutions (u, v). According to Lemma 2.2, $\frac{u}{v}$ is a convergent of $\sqrt{D}$.

Lemma 2.3 (see [4, Chapter 6]) *Let $R(D)$ denote the subset of $\mathbb{Z}$ such that for any $r \in R(D)$, $0 < |r| < \sqrt{D}$ and the equation* (2.1) *has solutions. Further, letting (u_1, v_1) be the least solution of the equation* (2.3) *and*

$$\frac{u_1}{v_1} = \frac{p_n}{q_n}, \quad n \in \mathbb{N} \cup \{0\}, \tag{2.4}$$

then, $R(D) \subseteq \{s_0, s_1, \cdots, s_n\}$, where s_i $(i = 0, 1, \cdots, n)$ is the same as in (2.2)*.*

Lemma 2.4 *Let k (≥ 3) be an integer, and then*

$$R(k^2 - 4) = \begin{cases} \{-1, 1\}, & if \;\; k = 3, \\ \{-k + 2, 1, 4\}, & if \;\; k > 3 \text{ and } 2 \nmid k, \\ \{-3, 1\}, & if \;\; k = 4, \\ \{1, 4\}, & if \;\; k > 4 \text{ and } 2 \mid k. \end{cases} \tag{2.5}$$

Proof Let $D = k^2 - 4$, and then D is a positive non-square integer.

If $k = 3$, then $D = 5$ and the smallest solution to the equation (2.3) corresponds to $(u_1, v_1) = (9, 4)$. Notice that $\frac{p_0}{q_0} = \frac{1}{2}, \frac{p_1}{q_1} = \frac{9}{4}$, so the number n satisfying (2.4) is 1. From (2.2) we get $s_0 = -1, s_1 = 1$, which leads to $R(5) = \{-1, 1\}$ according to Lemma 2.3.

If $k > 3$ and $2 \nmid k$, we have $(u_1, v_1) = \left(\frac{k^3-3k}{2}, \frac{k^2-1}{2}\right)$ and the number n satisfying (2.4) is 5. Thus, we get

$$\begin{aligned} &\frac{p_0}{q_0} = \frac{k-1}{1}, \quad \frac{p_1}{q_1} = \frac{k}{1}, \quad \frac{p_2}{q_2} = \frac{\frac{k^2-k-2}{2}}{\frac{k-1}{2}}, \quad \frac{p_3}{q_3} = \frac{k^2-2}{k}, \\ &\frac{p_4}{q_4} = \frac{\frac{k^3-2k^2-3k+4}{2}}{\frac{k^2-2k-1}{2}}, \quad \frac{p_5}{q_5} = \frac{\frac{k^3-3k}{2}}{\frac{k^2-1}{2}}. \end{aligned} \tag{2.6}$$

From (2.2) and (2.6), we get $s_0 = -2k+5, s_1 = 4, s_2 = -k+2, s_3 = 4, s_4 = -2k+5$ and $s_5 = 1$, which leads to $R(k^2 - 4) = \{-k + 2, 1, 4\}$ according to Lemma 2.3.

In a similar way, we can prove that Lemma 2.4 remains true when $2 \mid k$.

3 Proof of the Main Results

Proof of Theorem 1.1 When $l > 0$ and $k \in K(l)$, if $k \leq 2$, we have $x^2 - kxy + y^2 + lx = (x - y)^2 + lx > 0$ for any positive integers x and y, which enables us to conclude that $k \geq 3$.

Let (x, y) be a solution to equation (1.1) and $d_1 = \gcd(x, y)$, and then

$$x = d_1x_1, \quad y = d_1y_1, \quad x_1, y_1 \in \mathbb{N}, \quad \gcd(x_1, y_1) = 1. \tag{3.1}$$

By substituting (3.1) into (1.1), we get

$$x_1^2 - kx_1y_1 + y_1^2 + \frac{lx_1}{d_1} = 0. \tag{3.2}$$

It follows from (3.2) that

$$d_1 \mid lx_1. \tag{3.3}$$

Let $d_2 = \gcd(d_1, x_1)$, and then

$$d_1 = d_2d_3, \quad x_1 = d_2x_2, \quad d_3, x_2 \in \mathbb{N}, \quad \gcd(d_3, x_2) = 1. \tag{3.4}$$

We infer from (3.3)–(3.4) that $d_3 \mid l$. Let

$$l = d_3l_1, \quad l_1 \in \mathbb{N}. \tag{3.5}$$

By substituting (3.4)–(3.5) into (3.2), we get

$$(d_2x_2)^2 - k(d_2x_2)y_1 + y_1^2 + l_1x_2 = 0. \tag{3.6}$$

It follows from (3.6) that $x_2 \mid y_1^2$. From (3.1) and (3.4) we know that $(x_1, y_1) = \gcd(d_2x_2, y_1) = 1$, and thus $x_2 = 1$. It follows from (3.6) that

$$d_2^2 - kd_2y_1 + y_1^2 = -l_1, \tag{3.7}$$

which leads to

$$(2d_2 - ky_1)^2 - (k^2 - 4)y_1^2 = -4l_1. \tag{3.8}$$

Since $\gcd(d_2, y_1) = 1$, we have

$$\gcd(2d_2 - ky_1, y_1) = \begin{cases} 1, & \text{if } \ 2 \nmid y_1, \\ 2, & \text{if } \ 2 \mid y_1. \end{cases} \tag{3.9}$$

From (3.8)–(3.9), we infer that for $l > 0$, the necessary condition for $k \in K(l)$ is $k \geq 3$ and that l has a positive factor l_1 such that the equation (1.2) has solutions (X, Y).

Conversely, if $k \geq 3$ and the equation (1.2) has the solution (X, Y), we let

$$x = ab, \quad y = ac, \tag{3.10}$$

where

$$a = \frac{l}{l_1}b, \quad b = \frac{\delta}{2}(|X| + k|Y|), \quad c = \delta|Y|. \tag{3.11}$$

From (1.2) and (3.11) we know that (x, y) is a solution of the equation (1.1). Noticing that $k^2 - 4$ is a non-square positive integer, it is clear by Lemma 2.1 that if the equation (1.2) has a solution (X, Y), then the equation (1.1) has infinite many solutions. Thus, we get the sufficient condition for $k \in K(l)$.

Proof of Corollary 1.1 Supposing that $k \in K(l)$, we have that $k \geq 3$ and the equation (1.2) has solutions according to Theorem 1.1. If $\left|-\frac{4l_1}{\delta^2}\right| < \sqrt{k^2 - 4}$, the definition of $R(k^2 - 4)$ implies that $-\frac{4l_1}{\delta^2} \in R(k^2 - 4)$. By applying Lemma 2.4, we get $-\frac{4l_1}{\delta^2} \in \{-1, -3, -k + 2\}$, which leads to $4l \geq \frac{4l_1}{\delta^2} \geq k - 2$. If $\left|-\frac{4l_1}{\delta^2}\right| > \sqrt{k^2 - 4}$, we can also get $4l \geq \left|-\frac{4l_1}{\delta^2}\right| > \sqrt{k^2 - 4} \geq k - 2$. Thus $k \leq 4l + 2$.

Proof of Theorem 1.2 Now, we first discuss the set $K(-1)$. If the equation

$$x^2 - xy + y^2 - x = 0, \quad x, y \in \mathbb{N} \tag{3.12}$$

has a solution (x, y), then

$$(x-y)^2 + x(y-1) = 0. \tag{3.13}$$

Thus, we get $x = y = 1$, which implies that the equation (3.12) has only the solution $(x, y) = (1, 1)$ and so $1 \notin K(-1)$.

Since the equation

$$x^2 - 2xy + y^2 - x = 0, \quad x, y \in \mathbb{N} \tag{3.14}$$

has infinitely many solutions $(x, y) = (a^2, a^2 + a)$ with $a \in \mathbb{N}$, we have $2 \in K(-1)$.

Let (U, V) be a solution to the equation

$$U^2 - 5V^2 = 4, \quad U, V \in \mathbb{N}, \quad 2 \nmid UV \tag{3.15}$$

and let

$$x = a^2, \quad y = aV, \quad a = \frac{1}{2}(U + 3V). \tag{3.16}$$

It is not hard to see from (3.15) and (3.16) that (x, y) is a solution of the equation

$$x^2 - 3xy + y^2 - x = 0, \quad x, y \in \mathbb{N}, \tag{3.17}$$

and it is known that the equation (3.15) has infinitely many solutions (U, V) (see [1, Theorem 11.4.4]), so equation (3.17) has infinitely many solutions (x, y) too. Thus $3 \in K(-1)$. Using $k \in K(-1)$ with $k > 3$, which is the result of Keskin [2], we get that the necessary and sufficient condition for $k \in K(-1)$ is $k > 1$.

For a general negative integer l, let (x, y) be a solution of the equation

$$x^2 - kxy + y^2 - x = 0, \quad x, y \in \mathbb{N} \tag{3.18}$$

and let

$$x' = -lx, \quad y' = -ly. \tag{3.19}$$

Obviously x' and y' are positive integers. From (3.18)–(3.19) we can easily get $x'^2 - kx'y' + y'^2 + lx' = 0$. Thus (x', y') is a solution of the equation

$$x'^2 - kx'y' + y'^2 + lx' = 0, \quad x', y' \in \mathbb{N}. \tag{3.20}$$

From the above argument, we get $K(l) = K(-1)$ for any negative integer l.

References

[1] Hua, L. K., Introduction to Number Theory, Springer-Verlag, Berlin, 1982.

[2] Keskin, R., Solutions of some quadratic Diophantine equations, *Comput. Math. Appl.*, **60**(8), 2010, 2225–2230.

[3] Marlewski, A. and Zarzycki, P., Infinitely many positive solutions of the Diophantine equation $x^2 - kxy + y^2 + x = 0$, *Comput. Math. Appl.*, **47**(1), 2004, 115–121.

[4] Perron, O., Die Lehre von den Kettenbrüchen, Teubner., Leipzig, 1929.

[5] Yuan, P. Z. and Hu, Y. Z., On the Diophantine equation $x^2 - kxy + y^2 + lx = 0,\ l \in \{1, 2, 4\}$, *Comput. Math. Appl.*, **61**(3), 2011, 573–577.