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Uniqueness Theorems for Meromorphic Mappings in Several Complex Variables into $P^N(\mathbb{C})$ with Two Families of Moving Targets*

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Abstract The authors prove some uniqueness theorems for meromorphic mappings in several complex variables into the complex projective space $P^N(\mathbb{C})$ with two families of moving targets, and the results obtained improve some earlier work.

Keywords Meromorphic mapping, Moving target, Uniqueness theorem, Value distribution theory
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1 Introduction

Nevanlinna [4] began the study of the uniqueness problem of meromorphic functions on the complex plane, known as the famous four-value theorem and five-value theorem. Since then, there have been a number of papers working towards this kind of problems. In 1975, Fujimoto [3] extended Nevanlinna's results to meromorphic mappings of $\mathbb{C}^n$ into $P^N(\mathbb{C})$ and proved the following theorem.

Theorem 1.1 (see [3]) *Let H_i, $1 \le i \le q$, be q hyperplanes in $P^N(\mathbb{C})$ located in the general position, and let f and g be two nonconstant meromorphic mappings of $\mathbb{C}^n$ into $P^N(\mathbb{C})$ with $f(\mathbb{C}^n) \not\subset H_i$ and $g(\mathbb{C}^n) \not\subset H_i$ such that $\nu_{(f,H_i)}(z) = \nu_{(g,H_i)}(z)$ for $1 \le i \le q$.*

(i) *If $q = 3N+1$, then there is a linear transform L of $P^N(\mathbb{C})$ such that $L(f) = g$.*

(ii) *If $q = 3N+2$, then $f \equiv g$.*

Motivated by the accomplishment of the second main theorem of the value distribution theory for moving targets (e.g. [8–9]), and using the idea in Fujimoto [3], Tu [11] proved the following result related to moving targets.

Theorem 1.2 (see [11]) *Let $f, g : \mathbb{C}^n \to P^N(\mathbb{C})$ be two nonconstant meromorphic mappings, and let $\{a_i\}_{i=1}^q$ be "small" (with respect to f) meromorphic mappings of $\mathbb{C}^n$ into $P^N(\mathbb{C})$*

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in the general position such that f and g are linearly nondegenerate over $\widetilde{\mathcal{R}}(\{a_i\}_{i=1}^q)$. Assume that

(i) $\nu_{(f,a_i)}(z) = \nu_{(g,a_i)}(z)$ *for* $1 \leq i \leq q$,

(ii) $\dim\{z \in \mathbb{C}^n;\ (f(z), a_i(z)) = (f(z), a_j(z)) = 0\} \leq n-2$ *for* $1 \leq i < j \leq q$,

(iii) $f(z) = g(z)$ *on* $\bigcup_{j=1}^{q}\{z \in \mathbb{C}^n;\ (f(z), a_j(z)) = 0\}$.

Then

(1) *If $q = 3N+1$, there is a matrix L with its elements l_{ij} in $\widetilde{\mathcal{R}}(\{a_i\}_{i=1}^q)$ such that $L(f) = g$.*

(2) *If $q = 3N+2$, then $f \equiv g$.*

For the case of moving targets without counting multiplicity, Thai and Quang [10] proved the result as follows.

Theorem 1.3 (see [10]) *Let $f, g : \mathbb{C}^n \to P^N(\mathbb{C})$ be two meromorphic mappings, and let $\{a_i\}_{i=1}^q$ be "small" (with respect to f) meromorphic mappings of $\mathbb{C}^n$ into $P^N(\mathbb{C})$ in the general position such that f and g are linearly nondegenerate over $\mathcal{R}(\{a_i\}_{i=1}^q)$. Assume that*

(i) $\nu^1_{(f,a_i)}(z) = \nu^1_{(g,a_i)}(z)$ *for* $1 \leq i \leq q$,

(ii) $\dim\{z \in \mathbb{C}^n;\ (f(z), a_i(z)) = (f(z), a_j(z)) = 0\} \leq n-2$ *for* $1 \leq i < j \leq q$,

(iii) $f(z) = g(z)$ *on* $\bigcup_{j=1}^{q}\{z \in \mathbb{C}^n;\ (f(z), a_j(z)) = 0\}$.

If $q = 2N^2 + 4N$ and $N \geq 2$, then $f \equiv g$.

In 2009, Chen and Yan [1] obtained a sharp result of the uniqueness problem of meromorphic mappings related to a family of hyperplanes as follows.

Theorem 1.4 (see [1]) *Let $f(z)$, $g(z)$ be meromorphic mappings of $\mathbb{C}^n$ into $P^N(\mathbb{C})$ such that $\nu^1_{(f,H_j)}(z) = \nu^1_{(g,H_j)}(z)$ for $2N+3$ hyperplanes H_j located in the general position. If f and g are linearly nondegenerate, then $f \equiv g$.*

Recently, Dethloff, Quang and Tan [2] introduced uniqueness problems for meromorphic mappings related to two families of hyperplanes. Inspired by these developments and using some ideas in Chen and Yan [1], in this paper we will extend Theorems 1.2 and 1.3 to the case of meromorphic mappings related to two families of moving hyperplanes, and our results improve on some earlier work.

2 Preliminaries and Our Results

Let $F(z)$ $(\not\equiv 0)$ be an entire function on $\mathbb{C}^n$. For $a \in \mathbb{C}^n$, set $F(z) = \sum_{m=0}^{\infty} P_m(z-a)$, where the term $P_m(z)$ is either identically zero or a homogeneous polynomial of degree m. The number $\nu_F(a) := \min\{m; P_m \not\equiv 0\}$ is said to be the zero-multiplicity of F at a. For an integer $M > 0$, define $\nu_F^M(a) = \min\{\nu_F(a), M\}$. Set $|\nu_F| := \overline{\{z \in \mathbb{C}^n; \nu_F(z) \neq 0\}}$. Let φ be a nonzero meromorphic function on $\mathbb{C}^n$. For each $a \in \mathbb{C}^n$, we choose nonzero holomorphic functions F and G on a neighborhood U of a such that $\varphi = \frac{F}{G}$ on U and $\dim(F^{-1}(0) \cap G^{-1}(0)) \leq n-2$, and we define $\nu_\varphi = \nu_F - \nu_G$, which is independent of the choices of F and G.

For $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$ we set $\|z\| = (|z_1|^2 + \cdots + |z_n|^2)^{\frac{1}{2}}$. For $r > 0$, define

$$B(r) = \{z \in \mathbb{C}^n;\ \ \|z\| < r\}, \quad (r) = \{z \in \mathbb{C}^n;\ \ \|z\| = r\}.$$

Let $d = \partial + \overline{\partial}$ and $d^c = (4\pi\sqrt{-1})^{-1}(\partial - \overline{\partial})$. We write

$$v(z) = (dd^c\|z\|^2)^{n-1}, \quad \sigma(z) = d^c \log \|z\|^2 \wedge (dd^c \log \|z\|^2)^{n-1}$$

for $z \in \mathbb{C}^n \setminus \{0\}$.

Let $f : \mathbb{C}^n \to P^N(\mathbb{C})$ be a meromorphic mapping. We take holomorphic functions $f_0, f_1, \cdots, f_N$ on $\mathbb{C}^n$ such that $I_f := \{z \in \mathbb{C}^n;\ \ f_0(z) = f_1(z) = \cdots = f_N(z) = 0\}$ is of dimension at most $n-2$ and $f(z) = (f_0(z), f_1(z), \cdots, f_N(z))$ on $\mathbb{C}^n \setminus I_f$ in terms of homogeneous coordinates on $P^N(\mathbb{C})$. We call such a representation $f = (f_0, f_1, \cdots, f_N)$ a reduced representation of f. Since our notation is often independent of the choice of reduced representations, we shall identify f with its reduced representations in this paper. Set $\|f\| = (|f_0|^2 + \cdots + |f_N|^2)^{\frac{1}{2}}$. The order function of f is given by

$$T(r, f) = \int_{S(r)} \log \|f\|\sigma - \int_{S(1)} \log \|f\|\sigma$$

for $r > 1$.

A meromorphic mapping $a : \mathbb{C}^n \to P^N(\mathbb{C})$ is "small" with respect to the meromorphic mapping f of $\mathbb{C}^n$ into $P^N(\mathbb{C})$ if $T(r, a) = o(T(r, f))$ as $r \to +\infty$. Let $a = (a_0, a_1, \cdots, a_N)$ be a reduced representation of a. We define

$$m_{(f,a)}(r) = \int_{S(r)} \log \frac{\|f\|\|a\|}{|(f, a)|}\sigma - \int_{S(1)} \log \frac{\|f\|\|a\|}{|(f, a)|}\sigma$$

and

$$N_{(f,a)}(r) = \int_{S(r)} \log |(f, a)|\sigma - \int_{S(1)} \log |(f, a)|\sigma$$

for $r > 1$, where $(f, a) := \sum\limits_{i=0}^{N} a_i f_i$. Then

$$N_{(f,a)}(r) = \int_1^r \frac{n(t)}{t^{2n-1}} \mathrm{d}t,$$

where

$$n(t) := \begin{cases} \displaystyle\int_{|\nu_{(f,a)}| \cap B(t)} \nu_{(f,a)}(z)v, & n \geq 2, \\ \displaystyle\sum_{|z| \leq t} \nu_{(f,a)}(z), & n = 1. \end{cases}$$

For a positive integer M, define

$$N^M_{(f,a)}(r) = \int_1^r \frac{n^M(t)}{t^{2n-1}} \mathrm{d}t,$$

where

$$n^M(t) := \begin{cases} \displaystyle\int_{|\nu_{(f,a)}| \cap B(t)} \nu^M_{(f,a)}(z)v, & n \geq 2, \\ \displaystyle\sum_{|z| \leq t} \nu^M_{(f,a)}(z), & n = 1. \end{cases}$$

If F is a meromorphic function on $\mathbb{C}^n$ and $a \in \mathbb{C} \cup \{\infty\}$, then we adopt the standard notation for $m_F(r,a)$, $N_F(r,a)$, etc. Thus we have

$$N_{(f,a)}(r) = N_{(f,a)}(r,0)$$

for two meromorphic mappings f, a of $\mathbb{C}^n$ into $P^N(\mathbb{C})$. If $(f,a) \not\equiv 0$, then the first main theorem for moving targets in the value distribution theory states

$$T(r,f) + T(r,a) = m_{(f,a)}(r) + N_{(f,a)}(r)$$

for $r > 1$.

For a closed subset A of an analytic subset of $\mathbb{C}^n$, we define

$$N_A(r) = \int_1^r \frac{n(t)}{t^{2n-1}} \mathrm{d}t,$$

where

$$n(t) := \begin{cases} \displaystyle\int_{A \cap B(t)} v, & n \geq 2, \\ \sharp(A \cap B(t)), & n = 1. \end{cases}$$

For any $q \geq N+1$, let $a_1, \cdots, a_q$ be q "small" meromorphic mappings of $\mathbb{C}^n$ into $P^N(\mathbb{C})$ with reduced representations $a_j = (a_{j0}, a_{j1}, \cdots, a_{jN})$, $j = 1, \cdots, q$. We say that $a_1, \cdots, a_q$ are located in a general position if for any $1 \leq j_0 < j_1 < \cdots < j_N \leq q$, $\det(a_{j_k l}) \not\equiv 0$. Let $\mathcal{M}_n$ be the field (over $\mathbb{C}$) of all meromorphic functions on $\mathbb{C}^n$. Let $\mathcal{R}(\{a_i\}_{i=1}^q) \subset \mathcal{M}_n$ be the smallest subfield over $\mathbb{C}$ which contains $\mathbb{C}$ and all $\frac{a_{jk}}{a_{jl}}$ with $a_{jl} \not\equiv 0$, where $1 \leq j \leq q$ and $0 \leq k, l \leq N$. Define $\widetilde{\mathcal{R}}(\{a_i\}_{i=1}^q) \subset \mathcal{M}_n$ by the smallest subfield over $\mathbb{C}$ which contains all $h \in \mathcal{M}_n$ with $h^k \in \mathcal{R}(\{a_i\}_{i=1}^q)$ for some positive integer k. For two groups of meromorphic mappings $\{a_j, b_j\}_{j=1}^q$ of $\mathbb{C}^n$ into $P^N(\mathbb{C})$ with reduced representations $a_j = (a_{j0}, a_{j1}, \cdots, a_{jN})$ and $b_j = (b_{j0}, b_{j1}, \cdots, b_{jN})$, $j = 1, \cdots, q$, we denote $\mathcal{R}(\{a_i, b_i\}_{i=1}^q)$ as the smallest subfield over $\mathbb{C}$ which contains $\mathbb{C}$ and all $\frac{a_{jk}}{a_{jl}}, \frac{b_{jk}}{b_{jl}}$ with $a_{jl} \not\equiv 0$ and $b_{jl} \not\equiv 0$, where $1 \leq j \leq q$ and $0 \leq k, l \leq N$. Similarly, we can define $\widetilde{\mathcal{R}}(\{a_i, b_i\}_{i=1}^q)$.

Our main results are stated as follows.

Theorem 2.1 *Let $f, g, a_i, b_i : \mathbb{C}^n \to P^N(\mathbb{C})$ be meromorphic mappings $(i = 1, 2, \cdots, q)$. Suppose that $\{a_i\}_{i=1}^q$ are "small" (with respect to f) and located in the general position, and that $\{b_i\}_{i=1}^q$ are "small" (with respect to g) and located in the general position such that f and g are linearly nondegenerate over $\widetilde{\mathcal{R}}(\{a_i, b_i\}_{i=1}^q)$. For any reduced representations $a_i = (a_{i0}, \cdots, a_{iN})$ and $b_i = (b_{i0}, \cdots, b_{iN})$ $(i = 1, 2, \cdots, q)$, we may assume $a_{i0} \not\equiv 0$ and $b_{i0} \not\equiv 0$ $(i = 1, 2, \cdots, q)$ by changing the homogeneous coordinate system of $P^N(\mathbb{C})$. Let $\widetilde{a}_i = \frac{a_i}{a_{i0}}$ and $\widetilde{b}_i = \frac{b_i}{b_{i0}}$ $(i = 1, 2, \cdots, q)$. Assume that*

(i) $\nu_{(f,\widetilde{a}_i)}(z) = \nu_{(g,\widetilde{b}_i)}(z)$ *for* $1 \leq i \leq q$,

(ii) $\dim\{z \in \mathbb{C}^n;\ (f(z), a_i(z)) = (f(z), a_j(z)) = 0\} \leq n-2$ *for* $1 \leq i < j \leq q$,

(iii) $\frac{(f,\widetilde{a}_i)}{(g,\widetilde{b}_i)} = \frac{(f,\widetilde{a}_j)}{(g,\widetilde{b}_j)}$ *on* $\bigcup_{\substack{k=1 \\ k \neq i,j}}^{q} \{z \in \mathbb{C}^n;\ (f(z), a_k(z)) = 0\}$ *for* $1 \leq i < j \leq q$.

Then

(1) *if* $q = 3N+1$, *there exists a matrix* L *with its elements* l_{ij} *in* $\widetilde{\mathcal{R}}(\{a_i, b_i\}_{i=1}^{3N+1})$ *such that* $L(f) = g$;

(2) *if* $q = 3N+2$, *there exist* $\{i_1, \cdots, i_{N+1}\} \subset \{1, \cdots, q\}$ *such that* $\frac{(f,\widetilde{a}_{i_1})}{(g,\widetilde{b}_{i_1})} \equiv \cdots \equiv \frac{(f,\widetilde{a}_{i_{N+1}})}{(g,\widetilde{b}_{i_{N+1}})}$.

When $a_i(z) \equiv b_i(z)$ $(i = 1, 2, \cdots, q)$, the above theorem yields the following corrollary (i.e., Theorem 2.1 implies Theorem 1.2).

Corollary 2.1 *Let* $f, g, a_i : \mathbb{C}^n \to P^N(\mathbb{C})$ *be meromorphic mappings* $(i = 1, 2, \cdots, q)$. *Suppose that* $\{a_i\}_{i=1}^q$ *are "small"* (*with respect to* f) *located in general position such that* f *and* g *are linearly nondegenerate over* $\widetilde{\mathcal{R}}(\{a_i\}_{i=1}^q)$. *Assume that*

(i) $\nu_{(f,a_i)}(z) = \nu_{(g,a_i)}(z)$ *for* $1 \le i \le q$.

(ii) $\dim\{z \in \mathbb{C}^n;\ (f(z), a_i(z)) = (f(z), a_j(z)) = 0\} \le n-2$ *for* $1 \le i < j \le q$.

(iii) $f(z) = g(z)$ *on* $\bigcup_{k=1}^{q} \{z \in \mathbb{C}^n;\ (f(z), a_k(z)) = 0\}$.

Then

(1) *if* $q = 3N+1$, *there exists a matrix* L *with its elements* l_{ij} *in* $\widetilde{\mathcal{R}}(\{a_i\}_{i=1}^{3N+1})$ *such that* $L(f) = g$;

(2) *if* $q = 3N+2$, *there exist* $\{i_1, \cdots, i_{N+1}\} \subset \{1, \cdots, q\}$ *such that* $\frac{(f,a_{i_1})}{(g,a_{i_1})} \equiv \cdots \equiv \frac{(f,a_{i_{N+1}})}{(g,a_{i_{N+1}})}$, *which immediately gives* $f \equiv g$.

Proof For any reduced representations of a_j $(j = 1, 2, \cdots, q)$, let $b_j = a_j$. It is easy to see that (i) implies $\nu_{(f,\widetilde{a}_i)}(z) = \nu_{(g,\widetilde{b}_i)}(z)$ for $1 \le i \le q$, and (iii) implies $\frac{(f,\widetilde{a}_i)}{(g,\widetilde{b}_i)} = \frac{(f,\widetilde{a}_j)}{(g,\widetilde{b}_j)}$ on $\bigcup_{\substack{k=1\\k\neq i,j}}^{q} \{z \in \mathbb{C}^n;\ (f(z), a_k(z)) = 0\}$ for $1 \le i < j \le q$. Then, by the proof of Theorem 2.1, we have that

(1) if $q = 3N+1$, there exists a matrix L with its elements l_{ij} in $\widetilde{\mathcal{R}}(\{a_i\}_{i=1}^{3N+1})$ such that $L(f) = g$;

(2) if $q = 3N+2$, there exist $\{i_1, \cdots, i_{N+1}\} \subset \{1, \cdots, q\}$ such that $\frac{(f,a_{i_1})}{(g,a_{i_1})} \equiv \cdots \equiv \frac{(f,a_{i_{N+1}})}{(g,a_{i_{N+1}})}$.

Define $h := \frac{(f,a_{i_1})}{(g,a_{i_1})} \equiv \cdots \equiv \frac{(f,a_{i_{N+1}})}{(g,a_{i_{N+1}})}$. Since $\{a_{i_k}\}_{k=1}^{N+1}$ are located in a general position, we obtain $f(z) = h(z)g(z)$ on $\mathbb{C}^n$ as their reduced representations. This means $f(z) \equiv g(z)$ as meromorphic mappings from $\mathbb{C}^n$ to $P^N(\mathbb{C})$. The proof of Corollary 2.1 is finished.

Theorem 2.2 *Let* $f, g, a_i, b_i : \mathbb{C}^n \to P^N(\mathbb{C})$ *be meromorphic mappings* $(i = 1, 2, \cdots, q)$. *Suppose that* $\{a_i\}_{i=1}^q$ *are "small"* (*with respect to* f) *and located in the general position, and that* $\{b_i\}_{i=1}^q$ *are "small"* (*with respect to* g) *and located in the general position such that* f *and* g *are linearly nondegenerate over* $\mathcal{R}(\{a_i, b_i\}_{i=1}^q)$. *For any reduced representations* $a_i = (a_{i0}, \cdots, a_{iN})$ *and* $b_i = (b_{i0}, \cdots, b_{iN})$ $(i = 1, 2, \cdots, q)$, *we may assume* $a_{i0} \not\equiv 0$ *and* $b_{i0} \not\equiv 0$ $(i = 1, 2, \cdots, q)$ *by changing the homogeneous coordinate system of* $P^N(\mathbb{C})$. *Let* $\widetilde{a}_i = \frac{a_i}{a_{i0}}$ *and* $\widetilde{b}_i = \frac{b_i}{b_{i0}}$ $(i = 1, 2, \cdots, q)$. *Assume that*

(i) $\nu^1_{(f,\widetilde{a}_i)}(z) = \nu^1_{(g,\widetilde{b}_i)}(z)$ *for* $1 \le i \le q$,

(ii) $\dim\{z \in \mathbb{C}^n;\ (f(z), a_i(z)) = (f(z), a_j(z)) = 0\} \le n-2$ *for* $1 \le i < j \le q$,

(iii) $\frac{(f,\widetilde{a}_i)}{(g,\widetilde{b}_i)}=\frac{(f,\widetilde{a}_j)}{(g,\widetilde{b}_j)}$ *on* $\bigcup\limits_{\substack{k=1\\k\neq i,j}}^{q}\{z\in\mathbb{C}^n;\ \ (f(z),a_k(z))=0\}$ *for* $1\le i<j\le q$.

Then

If $q=2N^2+2N+3$, *then there exist* $\{i_1,\cdots,i_{N+1}\}\subset\{1,\cdots,q\}$ *such that*

$$\frac{(f,\widetilde{a}_{i_1})}{(g,\widetilde{b}_{i_1})}\equiv\cdots\equiv\frac{(f,\widetilde{a}_{i_{N+1}})}{(g,\widetilde{b}_{i_{N+1}})},$$

which immediately means that there exists a matrix L with its elements l_{ij} in $\mathcal{R}(\{a_i,b_i\}_{i=1}^q)$ such that $L(f)=g$.

When $a_i(z)\equiv b_i(z)$ $(i=1,2,\cdots,q)$, the above theorem yields the following corrollary.

Corollary 2.2 *Let $f,g,a_i:\mathbb{C}^n\to P^N(\mathbb{C})$ be meromorphic mappings $(i=1,2,\cdots,q)$. Suppose that $\{a_i\}_{i=1}^q$ are "small" (with respect to f) and located in the general position such that f and g are linearly nondegenerate over $\mathcal{R}(\{a_i\}_{i=1}^q)$. Assume that*

(i) $\nu^1_{(f,a_i)}(z)=\nu^1_{(g,a_i)}(z)$ *for* $1\le i\le q$,

(ii) $\dim\{z\in\mathbb{C}^n;\ \ (f(z),a_i(z))=(f(z),a_j(z))=0\}\le n-2$ *for* $1\le i<j\le q$,

(iii) $f(z)=g(z)$ *on* $\bigcup\limits_{k=1}^{q}\{z\in\mathbb{C}^n;\ \ (f(z),a_k(z))=0\}$.

If $q=2N^2+2N+3$, *then there exist* $\{i_1,\cdots,i_{N+1}\}\subset\{1,\cdots,q\}$ *such that*

$$\frac{(f,a_{i_1})}{(g,a_{i_1})}\equiv\cdots\equiv\frac{(f,a_{i_{N+1}})}{(g,a_{i_{N+1}})},$$

which immediately gives $f\equiv g$.

Remark 2.1 If $N=1$, then $2N^2+2N+3=7$ (cf. Corollary in p. 2702 of Ru [7]). If $N\ge 2$, then $2N^2+2N+3<2N^2+4N$. Thus Corollary 2.2 improves Theorem 1.3.

From the proof of Corollary 2.1, the proof of Corollary 2.2 is obvious.

3 Some Lemmas and Propositions

To prove our results, we need some preparations.

Proposition 3.1 (see [7]) *Assume that f and $\{a_i\}_{i=1}^q$ $(q\ge N+1)$ are meromorphic mappings of $\mathbb{C}^n$ into $P^N(\mathbb{C})$ such that $\{a_i\}_{i=1}^q$ are in the general position and f is linearly nondegenerate over $\mathcal{R}(\{a_i\}_{i=1}^q)$. Then*

$$\frac{q}{N+2}T(r,f)\le\sum_{j=1}^{q}N^N_{(f,a_j)}(r)+o(T(r,f))+O\Big(\max_{1\le i\le q}T(r,a_i)\Big)||,$$

where "||" means that the estimate holds for all large r outside a set of finite Lebesgue measures.

Lemma 3.1 (see [6]) *Let a,b be meromorphic functions on $\mathbb{C}^n$. Then*

$$T\Big(r,\frac{a}{b}\Big)\le T(r,a)+T(r,b)+O(1).$$

Lemma 3.2 *Let f, g be non-zero meromorphic functions of $\mathbb{C}^n$ with $\nu_f^1(z) = \nu_g^1(z)$. Then* $\min\{\nu_f(z), \nu_g(z)\} \geq \nu_f^N(z) + \nu_g^N(z) - N\nu_g^1(z)$.

Proof Notice that $\nu_g^N(z) - N\nu_g^1(z) \leq 0$.

(i) If $\nu_f(z) \geq N$ and $\nu_g(z) \geq N$, then $\min\{\nu_f(z), \nu_g(z)\} \geq N = \nu_f^N(z) \geq \nu_f^N(z) + \nu_g^N(z) - N\nu_g^1(z)$.

(ii) If $\nu_f(z) \geq N$ and $\nu_g(z) < N$, then $\min\{\nu_f(z), \nu_g(z)\} = \nu_g(z) = \nu_g^N(z) + N - N = \nu_f^N(z) + \nu_g^N(z) - N\nu_g^1(z)$.

(iii) If $\nu_f(z) < N$ and $\min\{\nu_f(z), \nu_g(z)\} = \nu_f(z)$, then $\min\{\nu_f(z), \nu_g(z)\} \geq \nu_f^N(z) \geq \nu_f^N(z) + \nu_g^N(z) - N\nu_g^1(z)$.

(iv) If $\nu_f(z) < N$ and $\min\{\nu_f(z), \nu_g(z)\} = \nu_g(z)$, then $\min\{\nu_f(z), \nu_g(z)\} \geq \nu_g^N(z) = \nu_g^N(z) + N - N\nu_g^1(z) \geq \nu_g^N(z) + \nu_f^N(z) - N\nu_g^1(z)$.

The proof of Lemma 3.2 is completed.

Lemma 3.3 (see [10]) *Let f, a_1, a_2 be meromorphic mappings of $\mathbb{C}^n$ into $P^N(\mathbb{C})$ with reduced representations $a_j = (a_{j0}, \cdots, a_{jN})$ such that a_j is small with respect to f and $(f, a_j) \not\equiv 0$ for $j = 1, 2$. Then*

$$T\Big(r, \frac{(f, \widetilde{a}_1)}{(f, \widetilde{a}_2)}\Big) \leq T(r, f) + o(T(r, f)),$$

where $\widetilde{a}_j = \frac{a_j}{a_{jk_0}}$ for some $a_{jk_0} \not\equiv 0$, $j = 1, 2$.

Let G be a torsion free Abelian group and $A = (a_1, \cdots, a_q)$ be a q-tuple of elements a_i in G. Let $q \geq r > s > 1$. We say that the q-tuple A has the property $(P_{r,s})$ if any r elements $a_{l(1)}, \cdots, a_{l(r)}$ in A satisfy the condition that for any given $i_1, \cdots, i_s$ $(1 \leq i_1 < \cdots < i_s \leq r)$, there exist $j_1, \cdots, j_s$ $(1 \leq j_1 < \cdots < j_s \leq r)$ with $\{i_1, \cdots, i_s\} \neq \{j_1, \cdots, j_s\}$ such that $a_{l(i_1)} \cdots a_{l(i_s)} = a_{l(j_1)} \cdots a_{l(j_s)}$.

Proposition 3.2 (see [3]) *Let G be a torsion free Abelian group and $A = (a_1, \cdots, a_q)$ be a q-tuple of elements a_i in G. If A has the property $(P_{r,s})$ for some r, s with $q \geq r > s > 1$, then there exist $i_1, \cdots, i_{q-r+2}$ with $1 \leq i_1 < \cdots < i_{q-r+2} \leq q$ such that $a_{i_1} = a_{i_2} = \cdots = a_{i_{q-r+2}}$.*

We also need the following two theorems that can be found in [11].

Proposition 3.3 (see [11]) *Suppose that $h_0, h_1, \cdots, h_m$ $(m \geq 2)$ are nowhere vanishing entire functions on $\mathbb{C}^n$ and $b_0, b_1, \cdots, b_m$ are nonzero meromorphic functions on $\mathbb{C}^n$ with $T(r, \frac{b_i}{b_j}) = o(T(r, h_{rst})) + O(1)||(0 \leq i < j \leq m)$ for $0 \leq r, s, t \leq m$ with $r \neq s$, $s \neq t$, $t \neq r$, where $h_{rst} := (h_r, h_s, h_t)$ is a holomorphic mapping of $\mathbb{C}^n$ into $P^2(\mathbb{C})$. Assume that $b_0h_0 + b_1h_1 + \cdots + b_mh_m = 0$. Then there exists a decomposition of indices $\{0, 1, \cdots, m\} = I_1 \cup I_2 \cup \cdots \cup I_l$ such that*

(i) *every I_k contains at least two indices,*

(ii) *for $i, j \in I_k$, $\frac{b_ih_i}{b_jh_j}$ is constant,*

(iii) *for $i \in I_p$ and $j \in I_q$ $(p \neq q)$, $\frac{b_ih_i}{b_jh_j}$ is not constant,*

(iv) *for every I_k, $\sum\limits_{j \in I_k} b_jh_j = 0$.*

Proposition 3.4 (see [11]) *Assume that f and $\{a_i\}_{i=1}^q$ $(q \geq N+1)$ are meromorphic mappings of $\mathbb{C}^n$ into $P^N(\mathbb{C})$ such that $\{a_i\}_{i=1}^q$ are in the general position and "small" with respect to f. If f is not linearly degenerate over $\mathcal{R}(\{a_i\}_{i=1}^q)$, then, for any $\varepsilon > 0$, there exists a positive integer M such that*

$$(q-N-1-\varepsilon)T(r,f) \leq \sum_{j=1}^{q} N^M_{(f,a_j)}(r) + o(T(r,f))||.$$

4 Proof of the Main Results

First, we have the following proposition.

Proposition 4.1 *Under the same assumption as in Theorem* 2.1 *or Theorem* 2.3*, we have that*

(i) $N_{a_{j0}}(r) = o(T(r,f))$, $N_{b_{j0}}(r) = o(T(r,g))$ *for* $j = 1, 2, \cdots, q$ *and* $N_D(r) = o(T(r,f)) + o(T(r,g))$, *where* $D = \bigcup\limits_{j=1}^{q} \{z \in \mathbb{C}^n;\ a_{j0}(z) = 0 \text{ or } b_{j0}(z) = 0\}$,

(ii) $T(r,f) = O(T(r,g))||$.

Proof (i) Let $H = \{z = [z_o : \cdots : z_N];\ z_0 = 0\}$ be a hyperplane in $P^N(\mathbb{C})$. By the first main theorem, we have $N_{a_{j0}}(r) = N_{(a_j,H)}(r) \leq T(r,a_j) = o(T(r,f))$. Similarly, $N_{b_{j0}}(r) = o(T(r,g))$. Since $N_D(r) \leq \sum\limits_{j=1}^{q} (N_{a_{j0}}(r) + N_{b_{j0}}(r))$, it is easy to get $N_D(r) = o(T(r,f)) + o(T(r,g))$.

(ii) By Proposition 3.1, we have

$$\begin{aligned}
\frac{q}{N+2}T(r,g) &\leq \sum_{j=1}^{q} N^N_{(g,b_j)}(r) + o(T(r,g)) + O\Big(\max_{1\leq i\leq q} T(r,b_i)\Big)|| \\
&\leq \sum_{j=1}^{q} N \cdot N^1_{(g,\widetilde{b}_j)}(r) + N_D(r) + o(T(r,g)) + o(T(r,f))|| \\
&= \sum_{j=1}^{q} N \cdot N^1_{(f,\widetilde{a}_j)}(r) + N_D(r) + o(T(r,g)) + o(T(r,f))|| \\
&\leq \sum_{j=1}^{q} N \cdot N^1_{(f,a_j)}(r) + 2N_D(r) + o(T(r,g)) + o(T(r,f))|| \\
&\leq qNT(r,f) + o(T(r,f)) + o(T(r,g))||.
\end{aligned}$$

Hence, we have $T(r,g) \leq O(T(r,f))||$. Similarly, $T(r,f) \leq O(T(r,g))||$.

Proposition 4.1 in [11] in the key part in the proof of Theorem 1.2. By modifying the proof of Proposition 4.1 in [11], we can get a generalization of Proposition 4.1 in [11] to the case of two families of moving targets as follows.

Proposition 4.2 *Under the same assumption as in Theorem* 2.1*, define* $h_i := \frac{(f,\widetilde{a}_i)}{(g,\widetilde{b}_i)}$, $i = 1, \cdots, q$. *Then there exist* i_k $(1 \leq k \leq q-2N)$ *with* $1 \leq i_1 < \cdots < i_{q-2N} \leq q$ *such that* $\frac{h_{i_u}}{h_{i_v}} \in \widetilde{\mathcal{R}}(\{a_i,b_i\}_{i=1}^q)$ *for* $1 \leq u < v \leq q-2N$.

Proof From Proposition 4.1, we have $T(r,a_i)=o(T(r,g))||$ and $T(r,b_i)=o(T(r,f))||$. Take $2N+2$ moving targets $\{a_j\}_{j=1}^{2N+2}$. Let $\widetilde{a}_{ik}=\frac{a_{ik}}{a_{i0}}$ and $\widetilde{b}_{ik}=\frac{b_{ik}}{b_{i0}}$, and then we have $N_{\widetilde{a}_{ik}}(r)=o(T(r,f))$ and $N_{\widetilde{b}_{ik}}(r)=o(T(r,g))$, $i=1,2,\cdots,q$, $k=0,1,\cdots,N$. Since $h_i=\frac{(f,\widetilde{a}_i)}{(g,\widetilde{b}_i)}$ and $\nu_{(f,\widetilde{a}_i)}(z)=\nu_{(g,\widetilde{b}_i)}(z)$, h_i is a nowhere vanishing entire function of $\mathbb{C}^n$. By the definition,

$$\sum_{k=0}^{N}\widetilde{a}_{ik}f_k-h_i\sum_{k=0}^{N}\widetilde{b}_{ik}g_k=0,\quad i=1,\cdots,2N+2.$$

Therefore

$$\det(\widetilde{a}_{i0},\cdots,\widetilde{a}_{iN},\widetilde{b}_{i0}h_i,\cdots,\widetilde{b}_{iN}h_i;\ \ 1\le i\le 2N+2)=0.$$

Let $\mathcal{I}$ be the set of all combinations $I=(i_1,\cdots,i_{N+1})$ with $1\le i_1<\cdots<i_{N+1}\le 2N+2$ of indices $1,2,\cdots,2N+2$. For any $I=(i_1,\cdots,i_{N+1})\in\mathcal{I}$, define

$$\{I\}:=\{i_1,\cdots,i_{N+1}\},\quad h_I:=h_{i_1}\cdots h_{i_{N+1}}$$

and

$$\begin{aligned}A_I:=&(-1)^{\frac{(N+1)(N+2)}{2}+i_1+\cdots+i_{N+1}}\det(\widetilde{a}_{i_rl};\ 1\le r\le N+1,0\le l\le N)\\ &\times\det(\widetilde{b}_{j_pl};\ 1\le p\le N+1,0\le l\le N),\end{aligned}$$

where $J=(j_1,\cdots,j_{N+1})\in\mathcal{I}$ such that $\{I\}\cup\{J\}=\{1,2,\cdots,2N+2\}$. Then we have

$$\sum_{I\in\mathcal{I}}A_Ih_I=0,$$

where $A_I\not\equiv 0$ by $\{a_i\}_{i=1}^q$ and $\{b_i\}_{i=1}^q$ are in the general position. For any $I,J\in\mathcal{I}$, we have $\frac{A_I}{A_J}\in\mathcal{R}\ (\{a_i,b_i\}_{i=1}^{2N+2})$.

By (ii) and (iii), we have $\frac{h_p(z)}{h_s(z)}=1$ for $z\in\bigcup\limits_{\substack{j=1\\ j\ne p,s}}^{2N+2}\{z\in\mathbb{C}^n;\ \ (f(z),a_j(z))=0\}$ outside an analytic set of dimension $\le n-2$. Then, for distinct $I,J\in\mathcal{I}$, we have

$$N_{\frac{h_I}{h_J}}(r,1)\ge\sum_{k\notin\{I\}\Delta\{J\}}N^1_{(f,a_k)}(r),$$

where $\{I\}\Delta\{J\}=\{I\}\cup\{J\}-\{I\}\cap\{J\}$. For distinct $I,J,K\in\mathcal{I}$, set $h_{IJK}:=(h_I,h_J,h_K)$ as a holomorphic mapping of $\mathbb{C}^n$ into $P^2(\mathbb{C})$. Then, by (5.2.29) in [5], we have

$$\begin{aligned}3T_{h_{IJK}}(r)&\ge T\Big(r,\frac{h_I}{h_J}\Big)+T\Big(r,\frac{h_J}{h_K}\Big)+T\Big(r,\frac{h_K}{h_I}\Big)+O(1)\\ &\ge N_{\frac{h_I}{h_J}}(r,1)+N_{\frac{h_J}{h_K}}(r,1)+N_{\frac{h_K}{h_I}}(r,1)+O(1)\\ &\ge\Big(\sum_{k\notin\{I\}\Delta\{J\}}+\sum_{k\notin\{J\}\Delta\{K\}}+\sum_{k\notin\{K\}\Delta\{I\}}\Big)N^1_{(f,a_k)}(r)+o(T(r,f))\\ &\ge\sum_{k=1}^{2N+2}N^1_{(f,a_k)}(r)+o(T(r,f))\\ &\ge\frac{1}{M}\sum_{k=1}^{2N+2}N^M_{(f,a_k)}(r)+o(T(r,f))\end{aligned}$$

$$\geq \frac{1}{M}(N+1-\varepsilon)T(r,f)+o(T(r,f))||$$
$$\geq \frac{N}{M}T(r,f)+o(T(r,f))||,$$

where $0<\varepsilon<\frac{1}{2}$ and M are given by Proposition 3.7. Thus $T\big(r,\frac{A_P}{A_Q}\big)=o(T(r,h_{IJK}))||$ for any $P,Q,I,J,K\in\mathcal{I}$ with $P\neq Q$, $I\neq J$, $J\neq K$ and $K\neq I$. Therefore, for any $I\in\mathcal{I}$, by Proposition 3.6 there exists $J\in\mathcal{I}$ with $I\neq J$ such that $A_Ih_I=cA_Jh_J$ for a nonzero constant c. So $\frac{h_I}{h_J}=c\frac{A_J}{A_I}\in\mathcal{R}(\{a_i,b_i\}_{i=1}^{2N+2})$.

Let $\mathcal{H}^*$ be the Abelian multiplication group of all nowhere vanishing entire functions on $\mathbb{C}^n$. Define $\mathcal{T}\subset\mathcal{H}^*$ by the smallest subgroup which contains all $f\in\mathcal{H}^*$ with $f^k\in\mathcal{R}(\{a_i,b_i\}_{i=1}^{q})$ for some positive integer k. So we have $\mathcal{H}^*\cap\mathcal{R}(\{a_i,b_i\}_{i=1}^{q})\subset\mathcal{T}\subset\widetilde{\mathcal{R}}(\{a_i,b_i\}_{i=1}^{q})$. Then the multiplication group $G:=\mathcal{H}^*/\mathcal{T}$ is a torsion free Abelian group, and the q-tuple of elements in G represented by $(h_1,\cdots,h_q)$ has the property $(P_{2N+2,N+1})$ by the above argument. Define $f_i\sim f_j$ if $\frac{f_i}{f_j}\in\widetilde{\mathcal{R}}(\{a_i,b_i\}_{i=1}^{q})$ for $f_i,f_j\in\mathcal{H}^*$. Then by Proposition 3.2 we finish the proof.

Proof of Theorem 2.1 Define $h_i:=\frac{(f,\widetilde{a}_i)}{(g,\widetilde{b}_i)}$, $i=1,\cdots,q$. By Proposition 4.2 and a suitable change of the reduced representations, we may assume that $h_1,\cdots,h_{q-2N}\in\widetilde{\mathcal{R}}(\{a_i,b_i\}_{i=1}^{q})$. Put $A:=(\widetilde{a}_{ij})_{1\leq i\leq N+1,\,0\leq j\leq N}$, $B:=(\widetilde{b}_{ij})_{1\leq i\leq N+1,\,0\leq j\leq N}$, and $H:=\mathrm{diag}(h_1,\cdots,h_{N+1})$. By the assumption of Theorem 2.1 we have $|A(z)|\not\equiv 0$, $|B(z)|\not\equiv 0$ and $|H(z)|\not\equiv 0$.

(1) If $q=3N+1$, then $h_1,\cdots,h_{N+1}\in\widetilde{\mathcal{R}}(\{a_i,b_i\}_{i=1}^{q})$. By the definition of h_i, we have

$$A\begin{pmatrix} f_0\\ f_1\\ \vdots\\ f_N\end{pmatrix}=HB\begin{pmatrix} g_0\\ g_1\\ \vdots\\ g_N\end{pmatrix}.$$

Taking $L=B^{-1}H^{-1}A$, we get $L(f)=g$.

(2) If $q=3N+2$, then $h_{N+2}\in\widetilde{\mathcal{R}}(\{a_i,b_i\}_{i=1}^{q})$, and

$$(\widetilde{a}_{(N+2)0},\cdots,\widetilde{a}_{(N+2)N})\begin{pmatrix} f_0\\ f_1\\ \vdots\\ f_N\end{pmatrix}=h_{N+2}(\widetilde{b}_{(N+2)0},\cdots,\widetilde{b}_{(N+2)N})\begin{pmatrix} g_0\\ g_1\\ \vdots\\ g_N\end{pmatrix}.$$

Therefore,

$$(\widetilde{a}_{(N+2)0},\cdots,\widetilde{a}_{(N+2)N})\begin{pmatrix} f_0\\ f_1\\ \vdots\\ f_N\end{pmatrix}=h_{N+2}(\widetilde{b}_{(N+2)0},\cdots,\widetilde{b}_{(N+2)N})B^{-1}H^{-1}A\begin{pmatrix} f_0\\ f_1\\ \vdots\\ f_N\end{pmatrix}.$$

Since f is not linearly degenerate over $\widetilde{\mathcal{R}}(\{a_i,b_i\}_{i=1}^{3N+2})$, we have

$$(\widetilde{a}_{(N+2)0},\cdots,\widetilde{a}_{(N+2)N})=h_{N+2}(\widetilde{b}_{(N+2)0},\cdots,\widetilde{b}_{(N+2)N})B^{-1}H^{-1}A.$$

Let $c=(c_1,\cdots,c_{N+1}):=\widetilde{a}_{N+2}A^{-1}$ and $d=(d_1,\cdots,d_{N+1}):=\widetilde{b}_{N+2}B^{-1}$. Then $\widetilde{a}_{N+2}=cA$ and $\widetilde{b}_{N+2}=dB$. The above equation becomes $cA=h_{N+2}dBB^{-1}H^{-1}A$, and hence $cH=h_{N+2}d$, which means $c_ih_i=d_ih_{N+2}$, $i=1,\cdots,N+1$.

We know that $\frac{h_i(z)}{h_{N+2}(z)} = 1$ on $\bigcup_{\substack{k=1\\k\neq i,N+2}}^{q} \{z \in \mathbb{C}^n;\ (f(z), a_k(z)) = 0\}$. If $\frac{h_i(z)}{h_{N+2}(z)} \not\equiv 1$, we have

$$T\Big(r, \frac{h_i}{h_{N+2}}\Big) \geq N_{\frac{h_i}{h_{N+2}}}(r,1) \geq \sum_{\substack{k=1\\k\neq i,N+1}}^{q} N^1_{(f,a_k)}(r)$$
$$\geq \frac{1}{N} \sum_{\substack{k=1\\k\neq i,N+1}}^{q} N^N_{(f,a_k)}(r) \geq \frac{q-2}{N(N+2)} T(r,f) + o(T(r,f))||.$$

From the definitions of c and d, we have $T(r, c_i) = o(T(r,f))$ and $T(r, d_i) = o(T(r,g))$. Thus by Lemma 3.2,

$$\frac{3}{N+2} T(r,f) + o(T(r,f)) \leq T\Big(r, \frac{h_i}{h_{N+2}}\Big) = T\Big(r, \frac{d_i}{c_i}\Big)$$
$$\leq T(r, d_i) + T(r, c_i) + O(1) = o(T(r,f))||,$$

which is a contradiction.

So, $\frac{h_i(z)}{h_{N+2}(z)} \equiv 1$, and hence, $h_i(z) \equiv h_{N+2}(z)$ for $i = 1, \cdots, N+1$. Then $Af = h_{N+2}Bg$, which implies

$$\frac{(f, \widetilde{a}_1)}{(g, \widetilde{b}_1)} \equiv \cdots \equiv \frac{(f, \widetilde{a}_{N+1})}{(g, \widetilde{b}_{N+1})}.$$

This completes the proof of Theorem 2.1.

Proof of Theorem 2.2 Let $T(r) = T(r,f) + T(r,g)$. Define

$$D = \bigcup_{i=1}^{q} \{z \in \mathbb{C}^n;\ a_{i0}(z) = 0 \text{ or } b_{i0}(z) = 0\}$$

and by Proposition 4.2 we have $N_D(r) = o(T(r,f))$. By changing indices if necessary, we assume that

$$\frac{(f,\widetilde{a}_1)}{(g,\widetilde{b}_1)} \equiv \frac{(f,\widetilde{a}_2)}{(g,\widetilde{b}_2)} \equiv \cdots \equiv \frac{(f,\widetilde{a}_{k_1})}{(g,\widetilde{b}_{k_1})} \not\equiv \frac{(f,\widetilde{a}_{k_1+1})}{(g,\widetilde{b}_{k_1+1})} \equiv \cdots \equiv \frac{(f,\widetilde{a}_{k_2})}{(g,\widetilde{b}_{k_2})} \not\equiv \cdots$$
$$\not\equiv \frac{(f,\widetilde{a}_{k_{s-1}+1})}{(g,\widetilde{b}_{k_{s-1}+1})} \equiv \cdots \equiv \frac{(f,\widetilde{a}_{k_s})}{(g,\widetilde{b}_{k_s})},$$

where $k_s = q$. Suppose that the theorem is not true and the number of each group is at most N. We define a map $\sigma : \{1, \cdots, q\} \to \{1, \cdots, q\}$ by $\sigma(i) = i + N$ if $i \leq q - N$ and $\sigma(i) = i + N - q$ if $i > q - N$. Obviously, σ is bijective, and $|\sigma(i) - i| \geq N$. Thus $\frac{(f,\widetilde{a}_i)}{(g,\widetilde{b}_i)}$ and $\frac{(f,\widetilde{a}_{\sigma(i)})}{(g,\widetilde{b}_{\sigma(i)})}$ belong to different groups, and so

$$\Phi_i = \frac{(f,\widetilde{a}_i)}{(f,\widetilde{a}_{\sigma(i)})} - \frac{(g,\widetilde{b}_i)}{(g,\widetilde{b}_{\sigma(i)})} = \frac{a_{\sigma(i)0}(f,a_i)}{a_{i0}(f,a_{\sigma(i)})} - \frac{b_{\sigma(i)0}(g,b_i)}{b_{i0}(g,b_{\sigma(i)})} \not\equiv 0$$

for $1 \leq i \leq q$.

For any z_0 in $\{z \in \mathbb{C}^n;\ (f(z), a_i(z)) = 0\} \backslash D$, $(f(z_0), a_i(z_0)) = 0$, $a_{i0}(z_0) \neq 0$ and $b_{i0}(z_0) \neq 0$. Then $(f(z_0), \widetilde{a}_i(z_0)) = 0$ and by (i) $(g(z_0), \widetilde{b}_i(z_0)) = 0$, which gives $(g(z_0), b_i(z_0)) = 0$. Hence, z_0 is a zero of Φ_i. By Lemma 3.3, we have

$$\nu_{\Phi_i}(z_0) \geq \min\{\nu_{(f,a_i)}(z_0), \nu_{(g,b_i)}(z_0)\} \geq \nu^N_{(f,a_i)}(z_0) + \nu^N_{(g,b_i)}(z_0) - N\nu^1_{(g,b_i)}(z_0).$$

For any z_0 in $\{z \in \mathbb{C}^n;\ (f(z), a_j(z)) = 0\} \setminus D$ $(j \neq i, \sigma(i))$, by the condition (iii), z_0 is also a zero of Φ_i. Since $N_D(r) = o(T(r, f))$,

$$\sum_{\substack{j=1 \\ j\neq i,\sigma(i)}}^{q} N^1_{(f,a_j)}(r) + N^N_{(f,a_i)}(r) + N^N_{(g,b_i)}(r) - NN^1_{(g,b_i)}(r) - o(T(r,f)) \leq N_{\Phi_i}(r).$$

On the other hand, with Lemma 3.4,

$$\begin{aligned}
N_{\Phi_i}(r) &\leq T(r, \Phi_i) \\
&= N(r, \nu^\infty_{\Phi_i}) + m(r, \Phi_i) + O(1) \\
&\leq N(r, \nu^\infty_{\Phi_i}) + m\Big(r, \frac{(f, \widetilde{a}_i)}{(f, \widetilde{a}_{\sigma(i)})}\Big) + m\Big(r, \frac{(g, \widetilde{b}_i)}{(g, \widetilde{b}_{\sigma(i)})}\Big) + O(1) \\
&\leq N(r, \nu^\infty_{\Phi_i}) + T\Big(r, \frac{(f, \widetilde{a}_i)}{(f, \widetilde{a}_{\sigma(i)})}\Big) + T\Big(r, \frac{(g, \widetilde{b}_i)}{(g, \widetilde{b}_{\sigma(i)})}\Big) \\
&\quad - N_{a_{i0}(f,a_{\sigma(i)})}(r) - N_{b_{i0}(g,b_{\sigma(i)})}(r) + o(T(r,f)) \\
&\leq T(r) + N(r, \nu^\infty_{\Phi_i}) - N_{(f,a_{\sigma(i)})}(r) - N_{(g,b_{\sigma(i)})}(r) + o(T(r,f)).
\end{aligned}$$

For any z_0 in $\{z \in \mathbb{C}^n;\ (f(z), a_{\sigma(i)}(z)) = 0\} \setminus D$, we also have $(g(z_0), b_{\sigma(i)}(z_0)) = 0$ as above. So z_0 is also a pole of Φ_i and by Lemma 3.3, we have

$$\begin{aligned}
&\nu_{(f,a_{\sigma(i)})}(z_0) + \nu_{(g,b_{\sigma(i)})}(z_0) - \nu^\infty_{\Phi_i}(z_o) \\
&\geq \nu_{(f,a_{\sigma(i)})}(z_0) + \nu_{(g,b_{\sigma(i)})}(z_0) - \max\{\nu_{(f,a_{\sigma(i)})}(z_0), \nu_{(g,b_{\sigma(i)})}(z_0)\} \\
&= \min\{\nu_{(f,a_{\sigma(i)})}(z_0), \nu_{(g,b_{\sigma(i)})}(z_0)\} \\
&\geq \nu^N_{(f,a_{\sigma(i)})}(z_0) + \nu^N_{(g,b_{\sigma(i)})}(z_0) - N\nu^1_{(g,b_{\sigma(i)})}(z_0),
\end{aligned}$$

which implies

$$N_{(f,a_{\sigma(i)})}(r) + N_{(g,b_{\sigma(i)})}(r) - N^\infty_{\Phi_i}(r) - o(T(r,f)) \geq N^N_{(f,a_{\sigma(i)})}(r) + N^N_{(g,b_{\sigma(i)})}(r) - NN^1_{(g,b_{\sigma(i)})}(r).$$

Therefore,

$$N_{\Phi_i}(r) \leq T(r) - N^N_{(f,a_{\sigma(i)})}(r) - N^N_{(g,b_{\sigma(i)})}(r) + NN^1_{(g,b_{\sigma(i)})}(r) + o(T(r,f)).$$

Together with both sides of $N_{\Phi_i}(r)$, we have

$$\begin{aligned}
&\sum_{\substack{j=1 \\ j\neq i,\sigma(i)}}^{q} N^1_{(f,a_j)}(r) + N^N_{(f,a_i)}(r) + N^N_{(g,b_i)}(r) - NN^1_{(g,b_i)}(r) \\
&\leq T(r) - N^N_{(f,a_{\sigma(i)})}(r) - N^N_{(g,b_{\sigma(i)})}(r) + NN^1_{(g,b_{\sigma(i)})}(r) + o(T(r,f)).
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_{\substack{j=1 \\ j\neq i,\sigma(i)}}^{q} N^1_{(f,a_j)}(r) + (N^N_{(f,a_i)}(r) + N^N_{(f,a_{\sigma(i)})}(r)) + (N^N_{(g,b_i)}(r) + N^N_{(g,b_{\sigma(i)})}(r)) \\
&\leq T(r) + N(N^1_{(g,b_i)}(r) + N^1_{(g,b_{\sigma(i)})}(r)) + o(T(r,f)).
\end{aligned}$$

Summing up over $1 \le i \le q$, we have

$$\begin{aligned}&(q-2)\sum_{j=1}^{q} N^{1}_{(f,a_j)}(r) + 2\Big(\sum_{j=1}^{q} N^{N}_{(f,a_j)}(r) + \sum_{j=1}^{q} N^{N}_{(g,b_j)}(r)\Big)\\ \le\ & qT(r) + 2N\sum_{j=1}^{q} N^{1}_{(g,b_j)}(r) + o(T(r,f)).\end{aligned}$$

Similarly, we have

$$\begin{aligned}&(q-2)\sum_{j=1}^{q} N^{1}_{(g,b_j)}(r) + 2\Big(\sum_{j=1}^{q} N^{N}_{(f,a_j)}(r) + \sum_{j=1}^{q} N^{N}_{(g,b_j)}(r)\Big)\\ \le\ & qT(r) + 2N\sum_{j=1}^{q} N^{1}_{(f,a_j)}(r) + o(T(r,g)).\end{aligned}$$

Hence, we get

$$\begin{aligned}&(q-2-2N)\Big(\sum_{j=1}^{q} N^{1}_{(f,a_j)}(r) + \sum_{j=1}^{q} N^{1}_{(g,b_j)}(r)\Big) + 4\Big(\sum_{j=1}^{q} N^{N}_{(f,a_j)}(r) + \sum_{j=1}^{q} N^{N}_{(g,b_j)}(r)\Big)\\ \le\ & 2qT(r) + o(T(r)),\end{aligned}$$

which implies

$$\Big(\frac{q-2-2N}{N} + 4\Big)\Big(\sum_{j=1}^{q} N^{N}_{(f,a_j)}(r) + \sum_{j=1}^{q} N^{N}_{(g,b_j)}(r)\Big) \le 2qT(r) + o(T(r)).$$

Therefore,

$$\Big(\frac{q-2-2N}{N} + 4\Big)\frac{q}{N+2}T(r) + o(T(r)) \le 2qT(r)||,$$

which contradicts $q = 2N^2 + 2N + 3$. The proof of Theorem 2.2 is completed.

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