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Curvature Estimates of Hypersurfaces in the Minkowski Space*

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Abstract A class of curvature estimates of spacelike admissible hypersurfaces related to translating solitons of the higher order mean curvature flow in the Minkowski space is obtained, which may offer an idea to study an open question of the existence of hypersurfaces with the prescribed higher mean curvature in the Minkowski space.

Keywords Minkowski space, Hypersurface, Curvature estimates, Translating solitons
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1 Introduction and Main Results

In this paper, we are interested in curvature estimates of spacelike admissible hypersurfaces $\mathbb{M}$ in the Minkowski space $\mathbb{R}^{n,1}$, which is the space $\mathbb{R}^n \times \mathbb{R}$ equipped with the metric

$$\mathrm{d}s^2 = \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_n^2 - \mathrm{d}x_{n+1}^2. \tag{1.1}$$

We assume that

$$\mathbb{M} = \text{Graph } u = \{(x, u(x)) \mid x \in \Omega \subset \mathbb{R}^n\}$$

for a spacelike function u, i.e., $\sup\limits_{\Omega} |Du| \le \theta < 1$. Then the upward (future directed) unit normal of $\mathbb{M}$ is

$$\nu = \frac{(Du, 1)}{\sqrt{1 - |Du|^2}} = w(Du, 1),$$

where we denote $w = \frac{1}{\sqrt{1-|Du|^2}}$. The Minkowski metric (1.1) restricted to $\mathbb{M}$ defines a Riemannian metric on $\mathbb{M}$, which in the standard coordinates on $\mathbb{R}^{n,1}$ is given by

$$g_{ij} = \delta_{ij} - D_i u D_j u, \quad 1 \le i, j \le n. \tag{1.2}$$

The inverse of the metric is

$$g^{ij} = \delta_{ij} + \frac{D_i u D_j u}{1 - |Du|^2}. \tag{1.3}$$

The second fundamental form of $\mathbb{M}$ is given by

$$h_{ij} = \frac{D_{ij} u}{\sqrt{1 - |Du|^2}} = w D_{ij} u. \tag{1.4}$$

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The principal curvatures $\lambda_1, \lambda_2, \cdots, \lambda_n$ of $\mathbb{M}$ are the eigenvalues of $[h_{ij}]$ relative to $[g_{ij}]$. Then the k-th Weingarten curvature at $x \in \mathbb{M}$ is defined as

$$\sigma_k(\lambda_1, \lambda_2, \cdots, \lambda_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}.$$

The classical geometric problem is to find hypersurfaces with the prescribed k-th Weingarten curvature ψ, which in turn poses a fundamental question on nonlinear partial differential equations

$$\sigma_k(u) = \psi. \tag{1.5}$$

This subject was studied in [2–4, 7–8, 11–12, 14, 25].

Following the ideas from [5–6, 20, 23] etc, let us define the k-admissible spacelike hypersurfaces as following.

Definition 1.1 *For $1 \leq k \leq n$, let Γ_k be a cone in $\mathbb{R}^n$ determined by*

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : \sigma_l(\lambda) > 0,\ l = 1, 2, \cdots, k\}.$$

A smooth hypersurface $\mathbb{M}$ is called k-admissible if $\mathbb{M}$ is spacelike and at every point $X \in \mathbb{M}$, $(\lambda_1, \lambda_2, \cdots, \lambda_n) \in \Gamma_k$.

The corresponding problem in the Euclidean context was extensively studied by various authors. We refer to [5–6, 14, 17, 22] and the references therein for related works.

In the Minkowski space or general Lorentzian spaces, Gerhardt [8, 11], Bartnik [2] and Bartnik-Simon [3] at first solved the Dirichlet problem for the prescribed mean curvature, and Delanoe [7] worked for the prescribed Gauss-Kronecker curvature equation (see also [14]). For the scalar curvature, see [4, 12, 25]. However, the existence of an admissible solution is open about other $3 \leq k < n$. The key point is the lack of some suitable C^2 a prior estimates for admissible solutions. Gerhardt [10, 12–13] has some important results in his curvature class, which excludes σ_k for $1 < k < n$. On the other hand, Guan and Spruck [15–16] have made important progress in hypersurfaces of constant curvature in the hyperbolic space.

In this paper, we consider a special case in which ψ depends on w. Our motivation has three parts. Firstly, this is reasonable from the Einstein equation as in Gerhardt [12] because $w = \langle \nu, E_{n+1} \rangle$. Secondly, a family of spacelike embeddings $X_t = X(,t) : \mathbb{R}^n \to \mathbb{R}^{n,1}$ with corresponding hypersurfaces $M_t = (x, V(x,t))$ satisfies the evolution equations

$$\frac{\partial V}{\partial t} = \sqrt{1 - |DV|^2}(\sigma_k^{\frac{1}{k}} - f(x, V)). \tag{1.6}$$

The solutions of (1.6) which moves by vertical translation are called translating solitons. Therefore, a translating soliton of (1.6) is characterized by $V(x,t) = u(x) + t$, where $u : \mathbb{R}^n \to \mathbb{R}$ is an initial spacelike hypersurface satisfying

$$\sigma_k^{\frac{1}{k}} = f(X) + \frac{1}{\sqrt{1 - |Du|^2}}. \tag{1.7}$$

Lastly, we believe that our methods may provide an idea to study Bayard's open question of the existence of hypersurfaces with the prescribed higher mean curvature in the Minkowski space (see [4]), even though our problem does not contain the situation of a constant ψ. Moreover, we shall study the flow (1.6) with some methods here in the future. In a word, we consider the curvature estimates of a class of equations

$$\begin{cases} \sigma_k = \psi(x,u,w), & x \in \Omega, \\ u = \varphi, & x \in \partial\Omega. \end{cases} \tag{1.8}$$

We have the following maximum principle for the curvature.

Theorem 1.1 *Suppose that $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$ is a spacelike, admissible solution of* (1.8), $0 < \psi \in C^\infty(\overline{\Omega})$ *and that $\psi^{\frac{1}{k}}(X,w)$ is convex in w and satisfies*

$$\frac{\partial \psi^{\frac{1}{k}}(X,w)}{\partial w} \cdot w \geq \psi^{\frac{1}{k}}(X,w) \quad \textit{for fixed } X \in \mathbb{M}. \tag{1.9}$$

Then the second fundamental form A of graph u satisfies

$$\sup_{\Omega} |A| \leq C\Big(1 + \sup_{\partial\Omega} |A|\Big), \tag{1.10}$$

where C depends only on n, $\|\varphi\|_{C^1(\overline{\Omega})}$, $\|\psi\|_{C^2\left(\overline{\Omega}\times\left[\inf\limits_{\partial\Omega} u, \sup\limits_{\partial\Omega} u\right]\times\mathbb{R}\right)}$.

Remark 1.1 There are some examples for $\psi(x,u,w)$:
(1) $\psi(x,u,w) = w^p g(x,u)$ for any $p \geq k$;
(2) $\psi(x,u,w) = \mathrm{e}^{pw} g(x,u)$ for any $p \geq k$.
It has been noticed that the exponential function was applied in [21].

We will also derive an interior curvature bound in the case that φ is affine and satisfies the strict inequality (1.9).

Theorem 1.2 *Suppose that $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$ is a spacelike, admissible solution of* (1.8), $0 < \psi \in C^\infty(\overline{\Omega})$ *and that $\psi^{\frac{1}{k}}(X,w)$ is convex in w and satisfies*

$$\frac{\partial \psi^{\frac{1}{k}}(X,w)}{\partial w} \cdot w > \psi^{\frac{1}{k}}(X,w) \quad \textit{for fixed } X \in \mathbb{M}. \tag{1.11}$$

In addition, suppose that $\Omega \subset \mathbb{R}^n$ is C^2 and uniformly convex, and that φ is spacelike and affine. If $u \in C^4(\Omega)$ is a spacelike, admissible solution of (1.5)*, then*

$$\sup_{\Omega'} |A| \leq C(\Omega') \tag{1.12}$$

for any $\Omega' \subset\subset \Omega$, where $C(\Omega')$ depends only on n, θ, Ω, $\operatorname{dist}(\Omega', \partial\Omega)$, $\|\varphi\|_{C^1(\overline{\Omega})}$, and

$$\|\psi\|_{C^2\left(\overline{\Omega}\times\left[\inf\limits_{\partial\Omega} u, \sup\limits_{\partial\Omega} u\right]\times\mathbb{R}\right)}.$$

With the above curvature estimates, the existence of the Dirichlet problem is obtained from Bayard's lower order and boundary C^2 estimates in [4].

Theorem 1.3 *Suppose that Ω is a smooth bounded domain of $\mathbb{R}^n$ and is strictly convex, while ψ is a smooth positive function and is convex in w satisfying*

$$\frac{\partial \psi^{\frac{1}{k}}(x,u,w)}{\partial w} w \geq \psi^{\frac{1}{k}}(x,u,w) \quad \textit{for fixed } (x,u) \in \Omega \times \mathbb{R}. \tag{1.13}$$

Then for any spacelike, affine function φ, there is a smooth admissible hypersurface $\mathbb{M}$ with the prescribed curvature ψ and boundary data φ.

2 Some Elementary Formulas

The standard basis of $\mathbb{R}^{n,1}$ will be denoted by $\epsilon_1, \epsilon_2, \cdots, \epsilon_{n+1}$, and the components of the position vector X in this basis will be denoted by $X_1, X_2, \cdots, X_{n+1}$. We choose an orthonormal frame such that $e_1, e_2, \cdots, e_n$ are tangent to M and ν is normal. The second fundamental form of M is given by

$$h_{ij} = \langle D_{e_i}\nu, e_j\rangle. \tag{2.1}$$

For any equation

$$F(A) = f, \tag{2.2}$$

some fundamental formulas are well-known for hypersurfaces in $\mathbb{R}^{n,1}$ as [24].

Lemma 2.1 *For any* $\alpha = 1, \cdots, n+1$,

$$\nabla_i \nu_\alpha = \nabla_i \langle \nu, \epsilon_\alpha\rangle = h_{il}\langle e_l, \epsilon_\alpha\rangle, \tag{2.3}$$

$$\nabla_j \nabla_i X_\alpha = \nabla_j \nabla_i \langle X, \epsilon_\alpha\rangle = h_{ij}\nu_\alpha, \tag{2.4}$$

$$R_{ijkl} = -(h_{ik}h_{jl} - h_{il}h_{jk}), \tag{2.5}$$

$$F^{ij}\nabla_j\nabla_i\nu_\alpha = \nu_\alpha F^{ij}h_{im}h_{jm} + \langle \nabla f, \epsilon_{n+1}\rangle. \tag{2.6}$$

Proof We only prove formula (2.6),

$$\begin{aligned}
\nu_\alpha &= \langle \nu, \epsilon_\alpha\rangle,\\
D_{e_i}\nu_\alpha &= \langle D_{e_i}\nu, \epsilon_\alpha\rangle\\
&= \langle D_{e_i}\nu, e_l\rangle\langle e_l, \epsilon_\alpha\rangle\\
&= h_{il}\langle e_l, \epsilon_\alpha\rangle, \qquad (2.7)\\
\nabla_i\nabla_j\nu_\alpha &= D_{e_i}D_{e_j}\nu_\alpha - \Gamma_{ij}^m D_{e_m}\nu_\alpha\\
&= D_{e_i}h_{jl}\langle e_l, \epsilon_\alpha\rangle + h_{jl}\langle D_{e_i}e_l, \epsilon_\alpha\rangle - \Gamma_{ij}^m D_{e_m}\nu_\alpha\\
&= D_{e_i}h_{jl}\langle e_l, \epsilon_\alpha\rangle + h_{jl}\Gamma_{il}^m\langle e_m, \epsilon_\alpha\rangle + \nu_\alpha h_{il}h_{jl} - \Gamma_{ij}^m D_{e_m}\nu_\alpha\\
&= \nabla_l h_{ij}\langle e_l, \epsilon_\alpha\rangle + \nu_\alpha h_{il}h_{jl}, \qquad (2.8)
\end{aligned}$$

where we have used $D_{e_i}e_l = \langle D_{e_i}e_l, e_p\rangle e_p - \langle D_{e_i}e_l, \nu\rangle\nu$ in the Minkowski space. In particular, when $\alpha = n+1$, for $w = -\nu_{n+1}$,

$$\nabla_i\nabla_j w = \nabla_l h_{ij}\langle e_l, \epsilon_{n+1}\rangle + wh_{il}h_{jl}. \tag{2.9}$$

Using the Codazzi equations, the Gauss equations (2.5) and the standard formula for commuting covariant derivatives, we have the following identities.

Lemma 2.2 *The* σ_1 *and* h_{ab} *satisfy*

$$\begin{aligned}
F^{ij}\nabla_i\nabla_j\sigma_1 &= -F^{ij,\ pq}\nabla_\alpha h_{ij}\nabla_\alpha h_{pq} + F^{ij}h_{jm}h_{im}\sigma_1\\
&\quad - F^{ij}h_{ij}h_{\alpha m}h_{m\alpha} + \nabla_\alpha\nabla_\alpha f, \qquad (2.10)\\
F^{ij}\nabla_i\nabla_j h_{ab} &= -F^{ij,\ pq}\nabla_a h_{ij}\nabla_b h_{pq} + F^{ij}h_{jm}h_{im}h_{ab}\\
&\quad - F^{ij}h_{ij}h_{am}h_{mb} + \nabla_a\nabla_b f. \qquad (2.11)
\end{aligned}$$

3 Curvature Estimates

3.1 Proof of Theorem 1.1

We are now ready to prove Theorems 1.1. We write (1.8) in the form

$$F(A)=\sigma_k^{\frac{1}{k}}(A)=\psi^{\frac{1}{k}}(X,w)=f(X,w)\quad\text{for any } X\in\mathbb{M}. \tag{3.1}$$

Proof of Theorem 1.1 We now consider the function

$$W=\sigma_1(A),$$

which achieves the maximal value at $X_0\in\mathbb{M}$. If the maximum is assumed on $\partial\Omega$, we are through. We choose the frame $e_1,e_2,\cdots e_n,\nu$ at X_0 such that $e_1,e_2,\cdots e_n\in T_{X_0}\mathbb{M}$ at X_0 and (h_{ij}) is diagonal at X_0 with eigenvalues $h_{11}\geq h_{22}\geq\cdots\geq h_{nn}$.

We have that for each $i=1,\cdots,n$,

$$\nabla_i\sigma_1=0\quad\text{at } X_0. \tag{3.2}$$

Therefore, at X_0,

$$\begin{aligned}0&\geq F^{ii}\nabla_i\nabla_i\sigma_1\\&=-F^{ij,\,pq}\nabla_l h_{ij}\nabla_l h_{pq}+F^{ij}h_{im}h_{mj}\sigma_1-F^{ij}h_{ij}|A|^2+\Delta f.\end{aligned} \tag{3.3}$$

Since f is convex in w, owing to (2.7)–(2.8) and Gaussian formula

$$\nabla_i\nabla_j X_l=h_{ij}\nu_l, \tag{3.4}$$

we have

$$\begin{aligned}\Delta f&=\frac{\partial^2 f}{\partial X_\alpha\partial X_\beta}\nabla_l X_\alpha\nabla_l X_\beta+2\frac{\partial^2 f}{\partial X_\alpha\partial w}\nabla_l X_\alpha\nabla_l w\\&\quad+\frac{\partial^2 f}{\partial w^2}|\nabla w|^2+\frac{\partial f}{\partial X_\alpha}\Delta X_\alpha+\frac{\partial f}{\partial w}\Delta w\\&\geq\frac{\partial f}{\partial w}\Delta w+\frac{\partial^2 f}{\partial w^2}|\nabla w|^2-C_1\sigma_1-C_2\\&\geq\frac{\partial f}{\partial w}w|A|^2+\frac{\partial f}{\partial w}\nabla_l\sigma_1\langle e_l,\epsilon_{n+1}\rangle-C_1\sigma_1-C_2.\end{aligned}$$

Inserting this into (3.3), we have

$$\begin{aligned}0&\geq F^{ii}\nabla_i\nabla_i\sigma_1\\&\geq-F^{ij,\,pq}\nabla_l h_{ij}\nabla_l h_{pq}+F^{ij}h_{im}h_{mj}\sigma_1\\&\quad+\Big(\frac{\partial f}{\partial w}w-f\Big)|A|^2-C\sigma_1\\&\geq F^{ij}h_{im}h_{mj}\sigma_1-C\sigma_1,\end{aligned} \tag{3.5}$$

where we have used (1.9) and the concavity of F. On the other hand,

$$\begin{aligned}F^{ij}h_{im}h_{mj}&=\frac{1}{k}\sigma_k^{\frac{1}{k}-1}[\sigma_k\sigma_1-(k+1)\sigma_{k+1}]\\&\geq\frac{1}{n}\sigma_k^{\frac{1}{k}}\sigma_1,\end{aligned} \tag{3.6}$$

where the last inequality is from the Newton inequalities for $\sigma_{k+1} > 0$,

$$\frac{\sigma_{k+1}}{C_n^{k+1}} \frac{\sigma_{k-1}}{C_n^{k-1}} \leq \Big(\frac{\sigma_k}{C_n^k}\Big)^2.$$

Then σ_1 is bounded from (3.5). We have proved Theorem 1.1.

3.2 Proof of Theorem 1.2

We have thought that the proofs of Theorems 1.1–1.2 are identical, however, that is not the case. The main difference is that the first-order derivative terms of σ_1 and w appear in the processes of the proofs. We apply the construction of a suitable auxiliary function of [22, 25] to deal with them

$$F(A) = \sigma_k^{\frac{1}{k}}(A) = g(\lambda) = f(X, w) \quad \text{for any } X \in \mathbb{M}. \tag{3.7}$$

Set

$$\sigma_k^{\frac{1}{k}}(\lambda_1, \cdots, \lambda_n) = g(\lambda_1, \cdots, \lambda_n), \tag{3.8}$$

$$F^{ij} = \frac{\partial F}{\partial h_{ij}}, \quad F^{ij,pq} = \frac{\partial^2 F}{\partial h_{ij} \partial h_{pq}}, \tag{3.9}$$

$$\operatorname{tr} F^{ij} = \sum_{i=1}^n F^{ii}, \quad g_i = \frac{\partial g}{\partial \lambda_i}. \tag{3.10}$$

Firstly, we list a useful lemma which is stated in a book of [13], see also [1, 22, 25].

Lemma 3.1 *For any symmetric matrix $\eta = [\eta_{ij}]$, we have*

$$F^{ij,pq} \eta_{ij} \eta_{pq} = \sum_{i,j} \frac{\partial^2 g}{\partial \lambda_i \lambda_j} \eta_{ii} \eta_{jj} + \sum_{i \neq j} \frac{g_i - g_j}{\lambda_i - \lambda_j} \eta_{ij}^2. \tag{3.11}$$

The second term on the right-hand side is nonpositive if g is concave, and it is interpreted as the limit if $\lambda_i = \lambda_j$.

Proof Let $\eta = \varphi - u$, as observed in Remark 1.2 of [22], $\eta > 0$, in Ω. We now consider the function

$$W = \eta^\beta \exp(\Phi(w)) h_{ab} \xi_a \xi_b,$$

which achieves the maximal value at $X_0 \in \mathbb{M}$, where $\beta \geq 1$ and a function Φ are to be determined. We may choose without loss of generality the frame $e_1 = \xi$, $e_2, \cdots e_n, \nu$ such that $e_1, e_2, \cdots e_n \in T_{X_0}\mathbb{M}$ so that $\nabla_{e_i} e_j = 0$ at X_0 for all $i, j = 1, \cdots, n$, and (h_{ij}) is diagonal at X_0 with eigenvalues $h_{11} \geq h_{22} \geq \cdots \geq h_{nn}$. We have that for each $i = 1, \cdots, n$,

$$\beta \frac{\nabla_i \eta}{\eta} + \Phi' \nabla_i w + \frac{\nabla_i h_{11}}{h_{11}} = 0 \quad \text{at } X_0, \tag{3.12}$$

$$\begin{aligned} &\beta \Big\{ \frac{\nabla_i \nabla_j \eta}{\eta} - \frac{\nabla_i \eta \nabla_j \eta}{\eta^2} \Big\} + \Phi'' \nabla_i w \nabla_j w + \Phi' \nabla_i \nabla_j w \\ &+ \frac{\nabla_i \nabla_j h_{11}}{h_{11}} - \frac{\nabla_i h_{11} \nabla_j h_{11}}{h_{11}^2} \geq 0 \quad \text{at } X_0. \end{aligned} \tag{3.13}$$

Therefore, using Lemma 2.2 and making contraction,

$$\begin{aligned}
0 \geq\ & \beta F^{ij}\Big\{\frac{\nabla_i\nabla_j\eta}{\eta}-\frac{\nabla_i\eta\nabla_j\eta}{\eta^2}\Big\}+\Phi''F^{ij}\nabla_i w\nabla_j w+\Phi' F^{ij}\nabla_i\nabla_j w\\
&+F^{ij}\frac{\nabla_i\nabla_j h_{11}}{h_{11}}-F^{ij}\frac{\nabla_i h_{11}\nabla_j h_{11}}{h_{11}^2}\\
=\ & \beta F^{ij}\Big\{\frac{\nabla_i\nabla_j\eta}{\eta}-\frac{\nabla_i\eta\nabla_j\eta}{\eta^2}\Big\}+\Phi''F^{ij}\nabla_i w\nabla_j w+\Phi' F^{ij}\nabla_i\nabla_j w\\
&-fh_{11}+F^{ij}h_{im}h_{jm}+\frac{\nabla_1\nabla_1 f}{h_{11}}\\
&-\frac{1}{h_{11}}F^{ij,pq}\nabla_1 h_{ij}\nabla_1 h_{pq}-F^{ij}\frac{\nabla_i h_{11}\nabla_j h_{11}}{h_{11}^2}.
\end{aligned}\tag{3.14}$$

We also find that

$$F^{ii}\nabla_i\nabla_i w = wF^{ii}h_{im}h_{mi}+\nabla_l f\langle e_l,\epsilon_{n+1}\rangle.\tag{3.15}$$

Consequently,

$$\begin{aligned}
0 \geq\ & \beta F^{ij}\Big\{\frac{\nabla_i\nabla_j\eta}{\eta}-\frac{\nabla_i\eta\nabla_j\eta}{\eta^2}\Big\}+\Phi''F^{ij}\nabla_i w\nabla_j w+\Phi'\nabla_l f\langle e_l,\epsilon_{n+1}\rangle\\
&-fh_{11}+(\Phi' w+1)F^{ij}h_{im}h_{jm}+\frac{\nabla_1\nabla_1 f}{h_{11}}\\
&-\frac{1}{h_{11}}F^{ij,pq}\nabla_1 h_{ij}\nabla_1 h_{pq}-F^{ij}\frac{\nabla_i h_{11}\nabla_j h_{11}}{h_{11}^2}.
\end{aligned}\tag{3.16}$$

Since f is convex in w, owing to (2.7) and (2.8),

$$\begin{aligned}
\nabla_1 f &= \frac{\partial f}{\partial X_\alpha}\nabla_l X_\alpha+\frac{\partial f}{\partial w}\nabla_1 w,\\
\nabla_1\nabla_1 f &= \frac{\partial^2 f}{\partial X_\alpha\partial X_\beta}\nabla_1 X_\alpha\nabla_1 X_\beta+2\frac{\partial^2 f}{\partial X_\alpha\partial w}\nabla_1 X_\alpha\nabla_1 w\\
&\quad+\frac{\partial^2 f}{\partial w^2}|\nabla_1 w|^2+\frac{\partial f}{\partial X_\alpha}\nabla_1\nabla_1 X_\alpha+\frac{\partial f}{\partial w}\nabla_1\nabla_1 w\\
&\geq \frac{\partial f}{\partial w}\nabla_1\nabla_1 w - C_1 h_{11}-C_2\\
&= \frac{\partial f}{\partial w}(wh_{11}^2+\nabla_l h_{11}\langle e_l,\epsilon_{n+1}\rangle)-C_1h_{11}-C_2.
\end{aligned}$$

Inserting this into (3.16),

$$\begin{aligned}
0 \geq\ & \beta F^{ij}\Big\{\frac{\nabla_i\nabla_j\eta}{\eta}-\frac{\nabla_i\eta\nabla_j\eta}{\eta^2}\Big\}+\Phi''F^{ij}\nabla_i w\nabla_j w+\Phi'\nabla_l f\langle e_l,\epsilon_{n+1}\rangle\\
&+\Big(\frac{\partial f}{\partial w}w-f\Big)h_{11}+(\Phi' w+1)F^{ij}h_{im}h_{jm}+\frac{\partial f}{\partial w}\frac{\nabla_l h_{11}\langle e_l,\epsilon_{n+1}\rangle}{h_{11}}\\
&-\frac{1}{h_{11}}F^{ij,pq}\nabla_1 h_{ij}\nabla_1 h_{pq}-F^{ij}\frac{\nabla_i h_{11}\nabla_j h_{11}}{h_{11}^2}-C,
\end{aligned}\tag{3.17}$$

where we assume that h_{11} is sufficiently large, otherwise, theorem 1.2 holds.

Next, we assume that the affine function φ has been extended to be constant in the ϵ_{n+1} direction. It is easy to calculate by using Gaussian formula

$$F^{ij}\nabla_i\nabla_j\eta=\Big(\sum_{\alpha=1}^n\frac{\partial\varphi}{\partial X_\alpha}\nu_\alpha-\nu_{n+1}\Big)F^{ij}h_{ij}\geq -C.\tag{3.18}$$

With (3.17) and (3.18), we have that, at X_0

$$
\begin{aligned}
0 \geq & -\frac{C\beta}{\eta} - \beta F^{ij}\frac{\nabla_i\eta\nabla_j\eta}{\eta^2} + \Phi'' F^{ij}\nabla_i w\nabla_j w + \Phi'\nabla_l f\langle e_l, \epsilon_{n+1}\rangle \\
& + \Big(\frac{\partial f}{\partial w}w - f\Big)h_{11} + (\Phi' w + 1)F^{ij}h_{im}h_{jm} + \frac{\partial f}{\partial w}\frac{\nabla_l h_{11}\langle e_l, \epsilon_{n+1}\rangle}{h_{11}} \\
& - \frac{1}{h_{11}}F^{ij,pq}\nabla_1 h_{ij}\nabla_1 h_{pq} - F^{ij}\frac{\nabla_i h_{11}\nabla_j h_{11}}{h_{11}^2} - C. \qquad (3.19)
\end{aligned}
$$

We now estimate the remaining terms in (3.19), and consider two cases.

Case 1 There is a positive constant ζ to be determined such that

$$h_{nn} \leq -\zeta h_{11}. \qquad (3.20)$$

Using the critical point condition (3.12),

$$
\begin{aligned}
F^{ij}\frac{\nabla_i h_{11}\nabla_j h_{11}}{h_{11}^2} &= F^{ij}\Big(\beta\frac{\nabla_i\eta}{\eta} + \Phi'\nabla_i w\Big)\Big(\beta\frac{\nabla_j\eta}{\eta} + \Phi'\nabla_j w\Big) \\
&\leq (1+\delta^{-1})\beta^2 F^{ij}\frac{\nabla_i\eta\nabla_j\eta}{\eta^2} + (1+\delta)\Phi'^2 F^{ij}\nabla_i w\nabla_j w \qquad (3.21)
\end{aligned}
$$

for any $\delta > 0$. Since $|\nabla\eta| \leq C$,

$$F^{ij}\frac{\nabla_i\eta\nabla_j\eta}{\eta^2} \leq C\frac{\operatorname{tr} F^{ij}}{\eta^2}. \qquad (3.22)$$

Therefore, at X_0 we have

$$
\begin{aligned}
0 \geq & -\frac{C\beta}{\eta} - C[\beta + (1+\delta^{-1})\beta^2]\frac{\operatorname{tr} F^{ij}}{\eta^2} \\
& + [\Phi'' - (1+\delta)\Phi'^2]F^{ij}\nabla_i w\nabla_j w \\
& + \Big(\frac{\partial f}{\partial w}w - f\Big)h_{11} + (\Phi' w + 1)F^{ij}h_{im}h_{jm} \\
& + \frac{\partial f}{\partial w}\frac{\nabla_l h_{11}\langle e_l, \epsilon_{n+1}\rangle}{h_{11}} + \Phi'\nabla_l f\langle e_l, \epsilon_{n+1}\rangle - C, \qquad (3.23)
\end{aligned}
$$

where we also have used the concavity of $F(A)$. On the other hand, from (3.12), the last two terms are bounded from below

$$
\begin{aligned}
& \frac{\partial f}{\partial w}\frac{\nabla_l h_{11}\langle e_l, \epsilon_{n+1}\rangle}{h_{11}} + \Phi'\nabla_l f\langle e_l, \epsilon_{n+1}\rangle \\
&= \Big(\Phi'\nabla_l f - \beta\frac{\partial f}{\partial w}\frac{\nabla_i\eta}{\eta} - \frac{\partial f}{\partial w}\Phi'\nabla_i w\Big)\langle e_l, \epsilon_{n+1}\rangle \\
&= \Big(\Phi'\frac{\partial f}{\partial X_\alpha}\nabla_l X_\alpha - \beta\frac{\partial f}{\partial w}\frac{\nabla_i\eta}{\eta}\Big)\langle e_l, \epsilon_{n+1}\rangle \\
&\geq -\frac{C\beta}{\eta} - C, \qquad (3.24)
\end{aligned}
$$

and therefore

$$
\begin{aligned}
0 \geq & -\frac{C\beta}{\eta} - C[\beta + (1+\delta^{-1})\beta^2]\frac{\operatorname{tr} F^{ij}}{\eta^2} \\
& + [\Phi'' - (1+\delta)\Phi'^2]F^{ij}\nabla_i w\nabla_j w \\
& + \Big(\frac{\partial f}{\partial w}w - f\Big)h_{11} + (\Phi' w + 1)F^{ij}h_{im}h_{jm} - C. \qquad (3.25)
\end{aligned}
$$

We note that from (2.7),

$$F^{ij}\nabla_i w\nabla_j w = F^{ij}h_{il}h_{jm}\langle e_l,\ \epsilon_{n+1}\rangle\langle e_m,\ \epsilon_{n+1}\rangle \le F^{ij}h_{il}h_{jm},$$

and then we would like to take a function Φ allowing

$$\Phi'' - (1+\delta)\Phi'^2 \le 0. \tag{3.26}$$

We know that there is a positive constant $a > 2$ depending only on $\sup\limits_{\Omega}|Du|$ such that

$$\frac{a}{2} \ge w = \frac{1}{\sqrt{1-|D^2u|}} > 1.$$

Let us take

$$\Phi(t) = -\log(a-t),$$

so we have (3.26) and

$$\Phi' w + 1 + \Phi'' - (1+\delta)\Phi'^2 \ge \frac{1}{2} \quad \text{for } \delta \le \frac{3a^2}{8}.$$

From (3.25), together with

$$F^{ij}h_{im}h_{jm} = F^{ii}h_{ii}^2 \ge \frac{\zeta^2}{n}h_{11}^2 \operatorname{tr} F^{ij},$$

which follows from (3.20) and the fact $F^{nn} \ge \frac{1}{n}\operatorname{tr} F^{ij}$, we have that, at X_0,

$$\begin{aligned} 0 \ge{}& -\frac{C\beta}{\eta} - C[\beta + (1+\delta^{-1})\beta^2]\frac{\operatorname{tr} F^{ij}}{\eta^2} \\ &+ \Big(\frac{\partial f}{\partial w}w - f\Big)h_{11} + \frac{\zeta^2}{2n}h_{11}^2 \operatorname{tr} F^{ij} - C, \end{aligned} \tag{3.27}$$

which implies an upper bound

$$\eta h_{11} \le \frac{C(\beta)}{\zeta} \quad \text{at } X_0,$$

since

$$\operatorname{tr} F^{ij} = \frac{(n-k-1)\sigma_{k-1}}{kf^{k-1}} \ge \frac{n-k+1}{k} > 0.$$

Case 2 We now assume that

$$h_{nn} \ge -\zeta h_{11}. \tag{3.28}$$

Since $h_{11} \ge h_{22} \ge \cdots \ge h_{nn}$, we have

$$h_{ii} \ge -\zeta h_{11} \quad \text{for all } i = 1, \cdots, n. \tag{3.29}$$

For a positive constant τ, assume to be 4, we partition $\{1, \cdots, n\}$ into

$$I = \{j : g^{jj} \le 4g^{11}\}, \quad J = \{j : g^{jj} > 4g^{11}\},$$

where g^{jj} is evaluated at $\lambda(X_0)$. Then for each $j \in I$, by (3.12), we have,

$$\begin{aligned} g_j \frac{|\nabla_i h_{11}|^2}{h_{11}^2} &= g_j \Big(\beta \frac{\nabla_i \eta}{\eta} + \Phi' \nabla_i w\Big)^2 \\ &\le (1+\delta^{-1})\beta^2 g_j \frac{|\nabla_i \eta|^2}{\eta^2} + (1+\delta)\Phi'^2 g_j |\nabla_i w|^2 \end{aligned} \tag{3.30}$$

for any $\delta > 0$. For each $j \in J$, we have

$$\begin{aligned} \beta g_j \frac{|\nabla_i \eta|^2}{\eta^2} &= \beta^{-1} g_j \Big(\frac{\nabla_i h_{11}}{h_{11}^2} + \Phi' \nabla_i w\Big)^2 \\ &\le \frac{1+\delta}{\beta}\Phi'^2 g_j |\nabla_i w|^2 + \frac{1+\delta^{-1}}{\beta} g_j \frac{|\nabla_i h_{11}|^2}{h_{11}^2} \end{aligned} \tag{3.31}$$

for any $\delta > 0$. Consequently,

$$\begin{aligned} &\beta \sum_{j=1}^n g_j \frac{|\nabla_i \eta|^2}{\eta^2} + \sum_{j=1}^n g_j \frac{|\nabla_i h_{11}|^2}{h_{11}^2} \\ &\le [\beta + (1+\delta^{-1})\beta^2] \sum_{j \in I} g_j \frac{|\nabla_i \eta|^2}{\eta^2} + (1+\delta)\Phi'^2 \sum_{j\in I} g_j |\nabla_i w|^2 \\ &\quad + \frac{1+\delta}{\beta}\Phi'^2 \sum_{j \in J} g_j |\nabla_i w|^2 + [1 + (1+\delta^{-1})\beta^{-1}] \sum_{j \in J} g_j \frac{|\nabla_i h_{11}|^2}{h_{11}^2} \\ &\le 4n[\beta + (1+\delta^{-1})\beta^2] g_1 \frac{|\nabla_i \eta|^2}{\eta^2} + (1+\delta)(1+\beta^{-1})\Phi'^2 \sum_{j=1}^n g_j |\nabla_i w|^2 \\ &\quad + [1 + (1+\delta^{-1})\beta^{-1}] \sum_{j\in J} g_j \frac{|\nabla_i h_{11}|^2}{h_{11}^2}. \end{aligned} \tag{3.32}$$

With this estimate and (3.19), the following inequality holds at X_0:

$$\begin{aligned} 0 \ge& -\frac{C\beta}{\eta} - 4n[\beta + (1+\delta^{-1})\beta^2] g_j \frac{|\nabla_i \eta|^2}{\eta^2} \\ &+ [\Phi'' - (1+\delta)(1+\beta^{-1})\Phi'^2] g_j |\nabla_j w|^2 + \Phi' \nabla_l f \langle e_l, \epsilon_{n+1}\rangle \\ &+ \Big(\frac{\partial f}{\partial w} w - f\Big) h_{11} + (\Phi' w + 1) F^{ij} h_{im} h_{jm} + \frac{\partial f}{\partial w} \frac{\nabla_l h_{11} \langle e_l, \epsilon_{n+1}\rangle}{h_{11}} \\ &- \frac{1}{h_{11}} F^{ij,pq} \nabla_1 h_{ij} \nabla_1 h_{pq} - [1 + (1+\delta^{-1})\beta^{-1}] \sum_{j \in J} g_j \frac{|\nabla_i h_{11}|^2}{h_{11}^2} - C. \end{aligned} \tag{3.33}$$

Then as Case 1, we have that for an appropriate selection of Φ,

$$\begin{aligned} 0 \ge& -\frac{C\beta}{\eta} - C(\beta + \beta^2)\frac{g_1}{\eta^2} + \frac{1}{2n} g_1 h_{11}^2 + \Big(\frac{\partial f}{\partial w} w - f\Big) h_{11} - C \\ &- \frac{1}{h_{11}} F^{ij,pq} \nabla_1 h_{ij} \nabla_1 h_{pq} - [1 + C\beta^{-1}] \sum_{j\in J} g_j \frac{|\nabla_i h_{11}|^2}{h_{11}^2}. \end{aligned} \tag{3.34}$$

We claim that

$$-\frac{1}{h_{11}} F^{ij,pq} \nabla_1 h_{ij} \nabla_1 h_{pq} - [1 + C\beta^{-1}] \sum_{j \in J} g_j \frac{|\nabla_i h_{11}|^2}{h_{11}^2} \ge 0. \tag{3.35}$$

If the claim (3.35) holds, then from (3.34) we have

$$\Big(\frac{\partial f}{\partial w}w - f\Big)h_{11} + \frac{1}{2n}g_1 h_{11}^2 \le C\Big(1 + \frac{1}{\eta} + \frac{g_1}{\eta^2}\Big),$$

from which we again conclude a bound for ηh_{11} at X_0 due to condition (1.11).

We now prove the claim. Using the concavity of g, Lemma 3.1 and the Codazzi equations, we see that

$$-\frac{1}{h_{11}}F^{ij,pq}\nabla_1 h_{ij}\nabla_1 h_{pq} \ge -\frac{2}{h_{11}}\sum_{j\in J}\frac{g_1 - g_j}{\lambda_1 - \lambda_j}|\nabla_j h_{11}|^2.$$

We then need to show that

$$-\frac{2(g_1 - g_j)}{h_{11}(\lambda_1 - \lambda_j)} \ge (1 + C\beta^{-1})\frac{g_j}{h_{11}^2} \quad \text{for each } j \in J,$$

provided that β is sufficiently large. This was indicated on p. 247 in [22] for $\zeta = \frac{1}{5}$ in (3.20). So Theorem 1.2 is proved.

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