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On the Ratio Between 2-Domination and Total Outer-Independent Domination Numbers of Trees*

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Abstract A 2-dominating set of a graph G is a set D of vertices of G such that every vertex of $V(G) \setminus D$ has at least two neighbors in D. A total outer-independent dominating set of a graph G is a set D of vertices of G such that every vertex of G has a neighbor in D, and the set $V(G) \setminus D$ is independent. The 2-domination (total outer-independent domination, respectively) number of a graph G is the minimum cardinality of a 2-dominating (total outer-independent dominating, respectively) set of G. We investigate the ratio between 2-domination and total outer-independent domination numbers of trees.

Keywords 2-Domination, Total domination, Total outer-independent domination, Tree
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1 Introduction

Let $G = (V, E)$ be a graph. By the neighborhood of a vertex v of G, we mean the set $N_G(v) = \{u \in V(G)\colon uv \in E(G)\}$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a subset of $V(G)$ is independent if there is no edge between any two vertices of this set. We denote the path on n vertices by P_n. By a star we mean a connected graph in which exactly one vertex has a degree greater than one. Let T be a tree, and let v be a vertex of T. We say that v is adjacent to a path P_n if there is a neighbor of v, say x, such that the subtree resulting from T by removing the edge vx and which contains the vertex x as a leaf, is a path P_n.

A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) \setminus D$ has a neighbor in D, while it is a 2-dominating set (abbreviated as 2DS) of G if every vertex of $V(G) \setminus D$ has at least two neighbors in D. The domination (2-domination, respectively) number of a graph G, denoted by $\gamma(G)$ ($\gamma_2(G)$, respectively), is the minimum cardinality of a dominating (2-dominating, respectively) set of G. A 2-dominating set of G of the minimum cardinality is called a $\gamma_2(G)$-set. Note that 2-domination is a type of multiple domination in which each vertex, which is not in the dominating set, is dominated at least k times for a fixed positive integer k. Multiple domination was introduced by Fink and Jacobson [12], and was further

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studied for example in [4–5, 14–15, 20, 22]. For a comprehensive survey of domination in graphs, see [16–17].

A subset $D \subseteq V(G)$ is a total dominating set of G if every vertex of G has a neighbor in D. The total domination number of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. Total domination in graphs was introduced by Cockayne, Dawes and Hedetniemi [7], and was further studied for example in [1–3, 8–11, 13, 18–19, 23–24].

A subset $D \subseteq V(G)$ is a total outer-independent dominating set (abbreviated as TOIDS) of G if every vertex of G has a neighbor in D, and the set $V(G) \setminus D$ is independent. The total outer-independent domination number of G, denoted by $\gamma_t^{oi}(G)$, is the minimum cardinality of a total outer-independent dominating set of G. A total outer-independent dominating set of G of the minimum cardinality is called a $\gamma_t^{oi}(G)$-set. The study of total outer-independent domination in graphs was initiated in [21].

The authors of [6] gave the upper bounds on the ratios of several domination parameters in trees.

We investigate the ratio between 2-domination and total outer-independent domination numbers of trees.

2 Results

Since the one-vertex graph does not have a total outer-independent dominating set, in this paper, by a tree we mean only a connected graph with no cycle, which has at least two vertices.

We begin with the following three straightforward observations.

Observation 2.1 Every support vertex of a graph G is in every $\gamma_t^{oi}(G)$-set.

Observation 2.2 For every connected graph G of diameter at least three, there exists a $\gamma_t^{oi}(G)$-set that contains no leaf.

Observation 2.3 Every leaf of a graph G is in every $\gamma_2(G)$-set.

Let T be a tree. Let us observe that the ratio $\frac{\gamma_2(T)}{\gamma_t^{oi}(T)}$ is not bounded above, as attaching a new vertex to any support vertex increases the 2-domination number but not affecting the total outer-independent domination number.

We show that $\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} \geq \frac{3}{4}$, for every tree T. For the purpose of characterizing the trees attaining this bound, we introduce a family $\mathcal{T}$ of trees $T = T_k$ that can be obtained as follows. Let $T_1 \in \{P_2, P_3\}$. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by one of the following operations.

(1) Operation $\mathcal{O}_1$: Attach a path P_6 by joining one of its leaves to a vertex of T_k adjacent to a path P_6.

(2) Operation $\mathcal{O}_2$: Attach a path P_6 by joining one of its leaves to a vertex of T_k adjacent to a support vertex of degree two.

(3) Operation $\mathcal{O}_3$: Attach a path P_6 by joining one of its leaves to any leaf of $T_k \neq P_2$.

Now we prove that for every tree of the family $\mathcal{T}$, the ratio between the 2-domination number plus one and the total outer-independent domination number plus two equals three fourths.

Lemma 2.1 *If $T \in \mathcal{T}$, then* $\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} = \frac{3}{4}$.

Proof We use induction on the number k of operations performed to construct the tree T. If $T = T_1 \in \{P_2, P_3\}$, then $\gamma_2(T) = \gamma_t^{oi}(T) = 2$. We have $\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} = \frac{3}{4}$. Let $k \geq 2$ be an integer. Assume that the result is true for every tree $T' = T_k$ of the family $\mathcal{T}$ constructed by $k-1$ operations. Let $T = T_{k+1}$ be a tree of the family $\mathcal{T}$ constructed by k operations.

First assume that T is obtained from T' operation $\mathcal{O}_1$. We denote by x the vertex to which P_6 is attached. Let $v_1v_2v_3v_4v_5v_6$ be the attached path. Let v_1 be joined to x. Let $abcdef$ denote a path P_6 adjacent to x and different from $v_1v_2v_3v_4v_5v_6$. Let x and a be adjacent. Let us observe that there exists a $\gamma_2(T')$-set that contains the vertices d, b and x. Let D' be such a set. It is easy to observe that $D' \cup \{v_2, v_4, v_6\}$ is a 2DS of the tree T. Thus $\gamma_2(T) \leq \gamma_2(T')+3$. Now let us observe that there exists a $\gamma_2(T)$-set that does not contain the vertices v_5, v_3 and v_1. Let D be such a set. By Observation 2.3 we have $v_6 \in D$. No one of the vertices v_2 and v_4 has a neighbor in D, and thus $v_2, v_4 \in D$. Observe that $D \setminus \{v_2, v_4, v_6\}$ is a 2DS of the tree T'. Therefore, $\gamma_2(T') \leq \gamma_2(T) - 3$. This implies that $\gamma_2(T) = \gamma_2(T') + 3$. Now let D' be any $\gamma_t^{oi}(T')$-set. It is easy to observe that $D' \cup \{v_1, v_2, v_4, v_5\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 4$. Now let us observe that there exists a $\gamma_t^{oi}(T)$-set that does not contain the vertices v_3 and c, or any leaf. Let D be such a set. By Observation 2.1 we have $v_5 \in D$. Each one of the vertices b, v_2 and v_5 has to have a neighbor in D, and thus $a, v_1, v_4 \in D$. We have that $v_2 \in D$ as the set $V(T) \setminus D$ is independent. Let us observe that $D \setminus \{v_1, v_2, v_4, v_5\}$ is a TOIDS of the tree T' as the vertex x has a neighbor in $D \setminus \{v_1, v_2, v_4, v_5\}$. Therefore, $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 4$. This implies that $\gamma_t^{oi}(T) = \gamma_t^{oi}(T') + 4$. Now we get

$$\begin{aligned}\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} &= \frac{\gamma_2(T')+1+3}{\gamma_t^{oi}(T')+6} = \Big(\frac{3(\gamma_t^{oi}(T')+2)}{4} + 3\Big)/(\gamma_t^{oi}(T')+6) \\ &= \Big(\frac{3\gamma_t^{oi}(T')}{4} + \frac{9}{2}\Big)/(\gamma_t^{oi}(T')+6) = \frac{3}{4}.\end{aligned}$$

Now assume that T is obtained from T' by operation $\mathcal{O}_2$. We denote by x the vertex to which is attached P_6. Let $v_1v_2v_3v_4v_5v_6$ be the attached path. Let v_1 be joined to x. Let a denote a support vertex of degree two adjacent to x. We denote by b the leaf adjacent to a. Let us observe that there exists a $\gamma_2(T')$-set that contains the vertex x. Let D' be such a set. It is easy to observe that $D' \cup \{v_2, v_4, v_6\}$ is a 2DS of the tree T. Thus $\gamma_2(T) \leq \gamma_2(T') + 3$. In the same way as when considering the operation $\mathcal{O}_1$, we conclude that $\gamma_2(T') \leq \gamma_2(T) - 3$. This implies that $\gamma_2(T) = \gamma_2(T') + 3$. Now let us observe that there exists a $\gamma_t^{oi}(T)$-set that does not contain the vertex v_3, or any leaf. Let D be such a set. By Observation 2.1 we have $v_5, a \in D$. Each one of the vertices v_2 and v_5 has to have a neighbor in D, and thus $v_1, v_4 \in D$. We have $v_2 \in D$ as the set $V(T) \setminus D$ is independent. Let us observe that $D \setminus \{v_1, v_2, v_4, v_5\}$ is a TOIDS of the tree T' as the vertex x has a neighbor in $D \setminus \{v_1, v_2, v_4, v_5\}$. Therefore, $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 4$. In the same way as when considering the operation $\mathcal{O}_1$, we conclude that $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 4$. This implies that $\gamma_t^{oi}(T) = \gamma_t^{oi}(T') + 4$. Now we get

$$\begin{aligned}\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} &= \frac{\gamma_2(T')+1+3}{\gamma_t^{oi}(T')+6} = \Big(\frac{3(\gamma_t^{oi}(T')+2)}{4} + 3\Big)/(\gamma_t^{oi}(T')+6) \\ &= \Big(\frac{3\gamma_t^{oi}(T')}{4} + \frac{9}{2}\Big)/(\gamma_t^{oi}(T')+6) = \frac{3}{4}.\end{aligned}$$

Now assume that T is obtained from T' by operation $\mathcal{O}_3$. We denote by x the leaf to which P_6 is attached. Let $v_1v_2v_3v_4v_5v_6$ be the attached path. Let v_1 be joined to x. We denote by y the neighbor of x other than v_1. Let D' be any $\gamma_2(T')$-set. By Observation 2.3 we have $x \in D'$. It is easy to observe that $D' \cup \{v_2, v_4, v_6\}$ is a 2DS of the tree T. Thus $\gamma_2(T) \le \gamma_2(T') + 3$. In the same way as when considering the operation $\mathcal{O}_1$, we conclude that $\gamma_2(T') \le \gamma_2(T) - 3$. This implies that $\gamma_2(T) = \gamma_2(T') + 3$. Now let us observe that there exists a $\gamma_t^{oi}(T)$-set that does not contain the vertices v_6, v_3 and x. Let D be such a set. By Observation 2.1 we have $v_5 \in D$. Each one of the vertices v_2 and v_5 has to have a neighbor in D, and thus $v_1, v_4 \in D$. We have $v_2, y \in D$ as the set $V(T) \setminus D$ is independent. Let us observe that $D \setminus \{v_1, v_2, v_4, v_5\}$ is a TOIDS of the tree T' as the vertex x has a neighbor in $D \setminus \{v_1, v_2, v_4, v_5\}$. Therefore, $\gamma_t^{oi}(T') \le \gamma_t^{oi}(T) - 4$. In the same way as when considering the operation $\mathcal{O}_1$, we conclude that $\gamma_t^{oi}(T) \le \gamma_t^{oi}(T') + 4$. This implies that $\gamma_t^{oi}(T) = \gamma_t^{oi}(T') + 4$. Now we get

$$\begin{aligned}\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} &= \frac{\gamma_2(T')+1+3}{\gamma_t^{oi}(T')+6} = \Big(\frac{3(\gamma_t^{oi}(T')+2)}{4}+3\Big)/(\gamma_t^{oi}(T')+6)\\ &= \Big(\frac{3\gamma_t^{oi}(T')}{4}+\frac{9}{2}\Big)/(\gamma_t^{oi}(T')+6) = \frac{3}{4}.\end{aligned}$$

Now we establish the main result, i.e., a lower bound on the ratio between the 2-domination number of a tree plus one and its total outer-independent domination number plus two, together with a characterization of the extremal trees.

Theorem 2.1 *If T is a tree, then $\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} \ge \frac{3}{4}$ with equality if and only if $T \in \mathcal{T}$.*

Proof Let n mean the number of vertices of the tree T. We proceed by induction on this number. If $\mathrm{diam}(T) = 1$, then $T = P_2 \in \mathcal{T}$. By Lemma 2.1 we have $\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} = \frac{3}{4}$. Now assume that $\mathrm{diam}(T) = 2$. Thus T is a star. We have $\gamma_2(T) = n-1$ and $\gamma_t^{oi}(T) = 2$. We get

$$\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} = \frac{n}{4} \ge \frac{3}{4}.$$

If $\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} = \frac{3}{4}$, then $n = 3$. Consequently, $T = P_3 \in \mathcal{T}$.

Now assume that $\mathrm{diam}(T) \ge 3$. Thus the order n of the tree T is at least four. We obtain the result by induction on the number n. Assume that the theorem is true for every tree T' of the order $n' < n$.

First assume that some support vertex of T, say x, is adjacent to at least three leaves. Let y be a leaf adjacent to x. Let $T' = T - y$. Let D' be any $\gamma_t^{oi}(T')$-set. By Observation 2.1 we have $x \in D'$. It is easy to see that D' is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \le \gamma_t^{oi}(T')$. Now let D be any $\gamma_2(T)$-set. By Observation 2.3 we have $y \in D$. Let us observe that $D \setminus \{y\}$ is a 2DS of the tree T' as the vertex x has at least two neighbors in $D \setminus \{y\}$. Therefore, $\gamma_2(T') \le \gamma_2(T) - 1$. Now we get

$$\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} \ge \frac{\gamma_2(T')+2}{\gamma_t^{oi}(T')+2} > \frac{\gamma_2(T')+1}{\gamma_t^{oi}(T')+2} \ge \frac{3}{4}.$$

Henceforth, we can assume that every support vertex of T is adjacent to at most two leaves.

We now root T at a vertex r of the maximum eccentricity $\mathrm{diam}(T)$. Let t be a leaf at the maximum distance from r, v be the parent of t, and u be the parent of v in the rooted tree. If

$\mathrm{diam}(T) \geq 4$, then let w be the parent of u. If $\mathrm{diam}(T) \geq 5$, then let d be the parent of w. If $\mathrm{diam}(T) \geq 6$, then let e be the parent of d. If $\mathrm{diam}(T) \geq 7$, then let f be the parent of e. By T_x let us denote the subtree induced by a vertex x and its descendants in the rooted tree T.

First assume that $d_T(v) = 3$. We denote by a the leaf adjacent to v and different from t. Let $T' = T - T_v$. Let D' be any $\gamma_t^{oi}(T')$-set. If $u \in D'$, then it is easy to see that $D' \cup \{v\}$ is a TOIDS of the tree T. Now assume that $u \notin D$. Let us observe that $D' \cup \{u, v\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 2$. Now let us observe that there exists a $\gamma_2(T)$-set that does not contain the vertex v. Let D be such a set. By Observation 2.3 we have $t, a \in D$. Observe that $D \setminus \{t, a\}$ is a 2DS of the tree T'. Therefore, $\gamma_2(T') \leq \gamma_2(T) - 2$. Now we get

$$\begin{aligned}\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} &\geq \frac{\gamma_2(T')+1+2}{\gamma_t^{oi}(T')+4} \geq \Big(\frac{3(\gamma_t^{oi}(T')+2)}{4}+2\Big)/(\gamma_t^{oi}(T')+4)\\ &= \Big(\frac{3\gamma_t^{oi}(T')}{4}+\frac{7}{2}\Big)/(\gamma_t^{oi}(T')+4) = \Big(\frac{3}{4}\Big)\Big(\gamma_t^{oi}(T')+\frac{14}{3}\Big)/(\gamma_t^{oi}(T')+4) > \frac{3}{4}.\end{aligned}$$

Now assume that $d_T(v) = 2$. First assume that among the children of u there is a support vertex, say x, different from v. Let $T' = T - T_v$. Let D' be a $\gamma_t^{oi}(T')$-set that contains no leaf. The vertex x has to have a neighbor in D', and thus $u \in D'$. It is easy to see that $D' \cup \{v\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 1$. Now let us observe that there exists a $\gamma_2(T)$-set that does not contain the vertex v. Let D be such a set. By Observation 2.3 we have $t \in D$. Observe that $D \setminus \{t\}$ is a 2DS of the tree T'. Therefore, $\gamma_2(T') \leq \gamma_2(T) - 1$. Now we get

$$\begin{aligned}\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} &\geq \frac{\gamma_2(T')+1+1}{\gamma_t^{oi}(T')+3} \geq \Big(\frac{3(\gamma_t^{oi}(T')+2)}{4}+1\Big)/(\gamma_t^{oi}(T')+3)\\ &= \Big(\frac{3\gamma_t^{oi}(T')}{4}+\frac{5}{2}\Big)/(\gamma_t^{oi}(T')+3) = \Big(\frac{3}{4}\Big)\Big(\gamma_t^{oi}(T')+\frac{10}{3}\Big)/(\gamma_t^{oi}(T')+3) > \frac{3}{4}.\end{aligned}$$

Now assume that some child of u, say x, is a leaf. Let $T' = T - x$. Let D' be a $\gamma_t^{oi}(T')$-set that contains no leaf. The vertex v has to have a neighbor in D', and thus $u \in D'$. It is easy to see that D' is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T')$. Now let us observe that there exists a $\gamma_2(T)$-set that contains the vertex u. Let D be such a set. By Observation 2.3 we have $x \in D$. It is easy to observe that $D \setminus \{x\}$ is a 2DS of the tree T'. Therefore, $\gamma_2(T') \leq \gamma_2(T) - 1$. Now we get

$$\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} \geq \frac{\gamma_2(T')+2}{\gamma_t^{oi}(T')+2} > \frac{\gamma_2(T')+1}{\gamma_t^{oi}(T')+2} \geq \frac{3}{4}.$$

Now assume that $d_T(u) = 2$. Let k be a child of w different from u. Let us observe that it suffices to consider only the possibility when $d_T(k) \leq 2$. First assume that $d_T(w) \geq 4$. Let l be a child of w different from u and k. Let us observe that it suffices to consider only the possibility when $d_T(l) \leq 2$. Let $T' = T - T_u$. Let D' be any $\gamma_t^{oi}(T')$-set. It is easy to observe that $D' \cup \{u, v\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 2$. Let us observe that there exists a $\gamma_2(T)$-set that contains the vertex u. Let D be such a set. By Observation 2.3 we have $t \in D$. The set D is minimal, and thus $v \notin D$. If $w \in D$, then it is easy to observe that $D \setminus \{u, t\}$ is a 2DS of the tree T'. Now assume that $w \notin D$. Thus both vertices k and l belong to the set D as each of them has at most one neighbor in the set D. Let us observe that $D \setminus \{u,t\}$ is a 2DS of the tree T' as the vertex w has at least two neighbors in $D \setminus \{u,t\}$. Therefore, $\gamma_2(T') \le \gamma_2(T) - 2$. Now we get

$$\begin{aligned}\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} &\ge \frac{\gamma_2(T')+1+2}{\gamma_t^{oi}(T')+4} \ge \Big(\frac{3(\gamma_t^{oi}(T')+2)}{4}+2\Big)/(\gamma_t^{oi}(T')+4)\\ &= \Big(\frac{3\gamma_t^{oi}(T')}{4}+\frac{7}{2}\Big)/(\gamma_t^{oi}(T')+4) = \Big(\frac{3}{4}\Big)\Big(\gamma_t^{oi}(T')+\frac{14}{3}\Big)/(\gamma_t^{oi}(T')+4) > \frac{3}{4}.\end{aligned}$$

Now assume that $d_T(w) = 3$. First assume that the distance of w to the most distant vertex of T_k is three. It suffices to consider only the possibility when T_k is a path P_3, say klm. Let $T' = T - T_w$. Let D' be any $\gamma_t^{oi}(T')$-set. It is easy to observe that $D' \cup \{w,u,v,k,l\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \le \gamma_t^{oi}(T') + 5$. Now let us observe that there exists a $\gamma_2(T)$-set that does not contain the vertices v, l and w. Let D be such a set. By Observation 2.3 we have $t, m \in D$. None of the vertices u and k has a neighbor in the set D, and thus $u, k \in D$. Observe that $D \setminus \{u,t,k,m\}$ is a 2DS of the tree T'. Therefore, $\gamma_2(T') \le \gamma_2(T) - 4$. Now we get

$$\begin{aligned}\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} &\ge \frac{\gamma_2(T')+1+4}{\gamma_t^{oi}(T')+7} \ge \Big(\frac{3(\gamma_t^{oi}(T')+2)}{4}+4\Big)/(\gamma_t^{oi}(T')+7)\\ &= \Big(\frac{3\gamma_t^{oi}(T')}{4}+\frac{11}{2}\Big)/(\gamma_t^{oi}(T')+7) = \Big(\frac{3}{4}\Big)\Big(\gamma_t^{oi}(T')+\frac{22}{3}\Big)/(\gamma_t^{oi}(T')+7) > \frac{3}{4}.\end{aligned}$$

Now assume that the distance of w to the most distant vertex of T_k is two. Thus k is a support vertex of degree two. We denote by l the leaf adjacent to k. Let $T' = T - T_u$. Let D' be any $\gamma_t^{oi}(T')$-set. It is easy to see that $D' \cup \{u,v\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \le \gamma_t^{oi}(T')+2$. Now let us observe that there exists a $\gamma_2(T)$-set that contains the vertices u and w. Let D be such a set. By Observation 2.3 we have $t \in D$. The set D is minimal, and thus $v \notin D$. It is easy to observe that $D \setminus \{u,t\}$ is a 2DS of the tree T'. Therefore, $\gamma_2(T') \le \gamma_2(T)-2$. Now we get

$$\begin{aligned}\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} &\ge \frac{\gamma_2(T')+1+2}{\gamma_t^{oi}(T')+4} \ge \Big(\frac{3(\gamma_t^{oi}(T')+2)}{4}+2\Big)/(\gamma_t^{oi}(T')+4)\\ &= \Big(\frac{3\gamma_t^{oi}(T')}{4}+\frac{7}{2}\Big)/(\gamma_t^{oi}(T')+4) = \Big(\frac{3}{4}\Big)\Big(\gamma_t^{oi}(T')+\frac{14}{3}\Big)/(\gamma_t^{oi}(T')+4) > \frac{3}{4}.\end{aligned}$$

Now assume that k is a leaf. Let $T' = T - T_w$. Let D' be any $\gamma_t^{oi}(T')$-set. It is easy to observe that $D' \cup \{w,u,v\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \le \gamma_t^{oi}(T') + 3$. Now let us observe that there exists a $\gamma_2(T)$-set that does not contain the vertices v and w. Let D be such a set. By Observation 2.3 we have $t, k \in D$. The vertex u has no neighbor in the set D, and thus $u \in D$. Observe that $D \setminus \{u,t,k\}$ is a 2DS of the tree T'. Therefore, $\gamma_2(T') \le \gamma_2(T) - 3$. Now we get

$$\begin{aligned}\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} &\ge \frac{\gamma_2(T')+1+3}{\gamma_t^{oi}(T')+5} \ge \frac{\frac{3(\gamma_t^{oi}(T')+2)}{4}+3}{\gamma_t^{oi}(T')+5}\\ &= \Big(\frac{3\gamma_t^{oi}(T')}{4}+\frac{9}{2}\Big)/(\gamma_t^{oi}(T')+5) = \Big(\frac{3}{4}\Big)\frac{\gamma_t^{oi}(T')+6}{\gamma_t^{oi}(T')+5} > \frac{3}{4}.\end{aligned}$$

If $d_T(w) = 1$, then $T = P_4$. We get $\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} = \frac{4}{4} > \frac{3}{4}$, a contradiction. Now assume that $d_T(w) = 2$. First assume that there is a child of d other than w, say k, such that the distance of

d to the most distant vertex of T_k is four. It suffices to consider only the possibility when T_k is a path P_4, say $klmp$. Let $T' = T - T_w$. Let us observe that there exists a $\gamma_t^{oi}(T')$-set that does not contain the vertex k. Let D' be such a set. The set $V(T') \setminus D'$ is independent, and thus $d \in D'$. It is easy to see that $D' \cup \{u, v\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 2$. Now let us observe that there exists a $\gamma_2(T)$-set that does not contain the vertices v and w. Let D be such a set. By Observation 2.3 we have $t \in D$. The vertex u has no neighbor in D, and thus $u \in D$. Observe that $D \setminus \{u, t\}$ is a 2DS of the tree T'. Therefore, $\gamma_2(T') \leq \gamma_2(T) - 2$. Now we get

$$\begin{aligned}\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} &\geq \frac{\gamma_2(T')+1+2}{\gamma_t^{oi}(T')+4} \geq \Big(\frac{3(\gamma_t^{oi}(T')+2)}{4}+2\Big)/(\gamma_t^{oi}(T')+4)\\ &= \Big(\frac{3\gamma_t^{oi}(T')}{4}+\frac{7}{2}\Big)/(\gamma_t^{oi}(T')+4) = \Big(\frac{3}{4}\Big)\Big(\gamma_t^{oi}(T')+\frac{14}{3}\Big)/(\gamma_t^{oi}(T')+4) > \frac{3}{4}.\end{aligned}$$

Now assume that there is a child of d, say k, such that the distance of d to the most distant vertex of T_k is three. It suffices to consider only the possibility when T_k is a path P_3, say klm. Let $T' = T - T_v$. Let D' be a $\gamma_t^{oi}(T')$-set that contains no leaf. By Observation 2.1 we have $w \in D'$. Each one of the vertices w and l has to have a neighbor in D', and thus $d, k \in D'$. Let us observe that $D' \setminus \{w\} \cup \{u, v\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 1$. Now let us observe that there exists a $\gamma_2(T)$-set that does not contain the vertex v. Let D be such a set. By Observation 2.3 we have $t \in D$. Observe that $D \setminus \{t\}$ is a 2DS of the tree T'. Therefore, $\gamma_2(T') \leq \gamma_2(T) - 1$. Now we get

$$\begin{aligned}\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} &\geq \frac{\gamma_2(T')+1+1}{\gamma_t^{oi}(T')+3} \geq \Big(\frac{3(\gamma_t^{oi}(T')+2)}{4}+1\Big)/(\gamma_t^{oi}(T')+3)\\ &= \Big(\frac{3\gamma_t^{oi}(T')}{4}+\frac{5}{2}\Big)/(\gamma_t^{oi}(T')+3) = \Big(\frac{3}{4}\Big)\Big(\gamma_t^{oi}(T')+\frac{10}{3}\Big)/(\gamma_t^{oi}(T')+3) > \frac{3}{4}.\end{aligned}$$

Now assume that there is a child of d, say k, such that the distance of d to the most distant vertex of T_k is two. Thus k is a support vertex of degree two. We denote by l the leaf adjacent to k. Let $T' = T - T_k$. Let us observe that there exists a $\gamma_t^{oi}(T')$-set that does not contain the vertex w. Let D' be such a set. The set $V(T') \setminus D'$ is independent, and thus $d \in D'$. It is easy to see that $D' \cup \{k\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 1$. Now let us observe that there exists a $\gamma_2(T)$-set that does not contain the vertex k. Let D be such a set. By Observation 2.3 we have $l \in D$. Observe that $D \setminus \{l\}$ is a 2DS of the tree T'. Therefore, $\gamma_2(T') \leq \gamma_2(T) - 1$. Now we get

$$\begin{aligned}\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} &\geq \frac{\gamma_2(T')+1+1}{\gamma_t^{oi}(T')+3} \geq \Big(\frac{3(\gamma_t^{oi}(T')+2)}{4}+1\Big)/(\gamma_t^{oi}(T')+3)\\ &= \Big(\frac{3\gamma_t^{oi}(T')}{4}+\frac{5}{2}\Big)/(\gamma_t^{oi}(T')+3) = \Big(\frac{3}{4}\Big)\Big(\gamma_t^{oi}(T')+\frac{10}{3}\Big)/(\gamma_t^{oi}(T')+3) > \frac{3}{4}.\end{aligned}$$

Now assume that some child of d, say k, is a leaf. Let $T' = T - k$. Let us observe that there exists a $\gamma_t^{oi}(T')$-set that does not contain the vertex w. Let D' be such a set. The set $V(T') \setminus D'$ is independent, and thus $d \in D'$. It is easy to see that D' is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T')$. Now let us observe that there exists a $\gamma_2(T)$-set that contains the vertices u and d. Let D be such a set. By Observation 2.3 we have $k \in D$. It is easy to observe that $D \setminus \{k\}$ is a 2DS of the tree T'. Therefore, $\gamma_2(T') \leq \gamma_2(T) - 1$. Now we get

$$\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} \geq \frac{\gamma_2(T')+2}{\gamma_t^{oi}(T')+2} > \frac{\gamma_2(T')+1}{\gamma_t^{oi}(T')+2} \geq \frac{3}{4}.$$

Now assume that $d_T(d) = 2$. First assume that there is a child of e other than d, say k, such that the distance of e to the most distant vertex of T_k is five. It suffices to consider only the possibility when T_k is a path P_5, say $klmpq$. Let $T' = T - T_v$. Let us observe that there exists a $\gamma_t^{oi}(T')$-set that does not contain the vertex l, or any leaf. Let D' be such a set. By Observation 2.1 we have $w \in D'$. Each one of the vertices w and k has to have a neighbor in D', and thus $d, e \in D'$. Let us observe that $D' \setminus \{w\} \cup \{u, v\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 1$. Now let us observe that there exists a $\gamma_2(T)$-set that does not contain the vertex v. Let D be such a set. By Observation 2.3 we have $t \in D$. Observe that $D \setminus \{t\}$ is a 2DS of the tree T'. Therefore, $\gamma_2(T') \leq \gamma_2(T) - 1$. Now we get

$$\begin{aligned}\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} &\geq \frac{\gamma_2(T')+1+1}{\gamma_t^{oi}(T')+3} \geq \Big(\frac{3(\gamma_t^{oi}(T')+2)}{4}+1\Big)/(\gamma_t^{oi}(T')+3)\\ &= \Big(\frac{3\gamma_t^{oi}(T')}{4}+\frac{5}{2}\Big)/(\gamma_t^{oi}(T')+3) = \Big(\frac{3}{4}\Big)\Big(\gamma_t^{oi}(T')+\frac{10}{3}\Big)/(\gamma_t^{oi}(T')+3) > \frac{3}{4}.\end{aligned}$$

Now assume that there is a child of e, say k, such that the distance of e to the most distant vertex of T_k is four. It suffices to consider only the possibility when T_k is a path P_4, say $klmp$. Let $T' = T - T_k$. Let us observe that there exists a $\gamma_t^{oi}(T')$-set that does not contain the vertex w. Let D' be such a set. The vertex d has to have a neighbor in D', and thus $e \in D'$. It is easy to see that $D' \cup \{l, m\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 2$. Now let us observe that there exists a $\gamma_2(T)$-set that does not contain the vertices m and k. Let D be such a set. By Observation 2.3 we have $p \in D$. The vertex l has no neighbor in the set D, and thus $l \in D$. Observe that $D \setminus \{l, p\}$ is a 2DS of the tree T'. Therefore, $\gamma_2(T') \leq \gamma_2(T) - 2$. Now we get

$$\begin{aligned}\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} &\geq \frac{\gamma_2(T')+1+2}{\gamma_t^{oi}(T')+4} \geq \Big(\frac{3(\gamma_t^{oi}(T')+2)}{4}+2\Big)/(\gamma_t^{oi}(T')+4)\\ &= \Big(\frac{3\gamma_t^{oi}(T')}{4}+\frac{7}{2}\Big)/(\gamma_t^{oi}(T')+4) = \Big(\frac{3}{4}\Big)\Big(\gamma_t^{oi}(T')+\frac{14}{3}\Big)/(\gamma_t^{oi}(T')+4) > \frac{3}{4}.\end{aligned}$$

Now assume that there is a child of e, say k, such that the distance of e to the most distant vertex of T_k is two. Thus k is a support vertex of degree two. We denote by l the leaf adjacent to k. Let $T' = T - T_k$. Let us observe that there exists a $\gamma_t^{oi}(T')$-set that does not contain the vertex w. Let D' be such a set. The vertex d has to have a neighbor in D', and thus $e \in D'$. It is easy to see that $D' \cup \{k\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 1$. Now let us observe that there exists a $\gamma_2(T)$-set that does not contain the vertex k. Let D be such a set. By Observation 2.3 we have $l \in D$. Observe that $D \setminus \{l\}$ is a 2DS of the tree T'. Therefore, $\gamma_2(T') \leq \gamma_2(T) - 1$. Now we get

$$\begin{aligned}\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} &\geq \frac{\gamma_2(T')+1+1}{\gamma_t^{oi}(T')+3} \geq \Big(\frac{3(\gamma_t^{oi}(T')+2)}{4}+1\Big)/(\gamma_t^{oi}(T')+3)\\ &= \Big(\frac{3\gamma_t^{oi}(T')}{4}+\frac{5}{2}\Big)/(\gamma_t^{oi}(T')+3) = \Big(\frac{3}{4}\Big)\Big(\gamma_t^{oi}(T')+\frac{10}{3}\Big)/(\gamma_t^{oi}(T')+3) > \frac{3}{4}.\end{aligned}$$

Now assume that some child of e, say k, is a leaf. Let $T' = T - T_v$. Let D' be a $\gamma_t^{oi}(T')$-set that contains no leaf. By Observation 2.1 we have $w, e \in D'$. The vertex w has to have a neighbor in D', and thus $d \in D'$. Let us observe that $D' \setminus \{w\} \cup \{u, v\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 1$. Now let us observe that there exists a $\gamma_2(T)$-set that does not contain the vertex v. Let D be such a set. By Observation 2.3 we have $t \in D$. Observe that $D \setminus \{t\}$ is a 2DS of the tree T'. Therefore, $\gamma_2(T') \leq \gamma_2(T) - 1$. Now we get

$$\begin{aligned}\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} &\geq \frac{\gamma_2(T')+1+1}{\gamma_t^{oi}(T')+3} \geq \Big(\frac{3(\gamma_t^{oi}(T')+2)}{4}+1\Big)/(\gamma_t^{oi}(T')+3)\\ &= \Big(\frac{3\gamma_t^{oi}(T')}{4}+\frac{5}{2}\Big)/(\gamma_t^{oi}(T')+3) = \Big(\frac{3}{4}\Big)\Big(\gamma_t^{oi}(T')+\frac{10}{3}\Big)/(\gamma_t^{oi}(T')+3) > \frac{3}{4}.\end{aligned}$$

Now assume that the distance of e to the most distant vertex of T_k is three. It suffices to consider only the possibility when T_k is a path P_3, say klm. First assume that $d_T(e) \geq 4$. Let p denote a child of e different from d and k. It suffices to consider only the possibility when T_p is a path P_3, say pqs. Let $T' = T - T_k$. Let D' be any $\gamma_t^{oi}(T')$-set. It is easy to see that $D' \cup \{k, l\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi} \leq \gamma_t^{oi}(T') + 2$. Now let us observe that there exists a $\gamma_2(T)$-set that contains the vertices u, d, k and p. Let D be such a set. By Observation 2.3 we have $m \in D$. The set D is minimal, and thus $l \notin D$. Let us observe that $D \setminus \{k, m\}$ is a 2DS of the tree T' as the vertex e has at least two neighbors in $D \setminus \{k, m\}$. Therefore, $\gamma_2(T') \leq \gamma_2(T) - 2$. Now we get

$$\begin{aligned}\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} &\geq \frac{\gamma_2(T')+1+2}{\gamma_t^{oi}(T')+4} \geq \Big(\frac{3(\gamma_t^{oi}(T')+2)}{4}+2\Big)/(\gamma_t^{oi}(T')+4)\\ &= \Big(\frac{3\gamma_t^{oi}(T')}{4}+\frac{7}{2}\Big)/(\gamma_t^{oi}(T')+4) = \Big(\frac{3}{4}\Big)\Big(\gamma_t^{oi}(T')+\frac{14}{3}\Big)/(\gamma_t^{oi}(T')+4) > \frac{3}{4}.\end{aligned}$$

Now assume that $d_T(e) = 3$. Let $T' = T - T_e$. Let D' be any $\gamma_t^{oi}(T')$-set. It is easy to observe that $D' \cup \{e, d, u, v, k, l\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 6$. Now let us observe that there exists a $\gamma_2(T)$-set that does not contain the vertices v, w, l and e. Let D be such a set. By Observation 2.3 we have $t, m \in D$. None of the vertices u, d and k has a neighbor in the set D, and thus $u, d, k \in D$. Observe that $D \setminus \{d, u, t, k, m\}$ is a 2DS of the tree T'. Therefore, $\gamma_2(T') \leq \gamma_2(T) - 5$. Now we get

$$\begin{aligned}\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} &\geq \frac{\gamma_2(T')+1+5}{\gamma_t^{oi}(T')+8} \geq \Big(\frac{3(\gamma_t^{oi}(T')+2)}{4}+5\Big)/(\gamma_t^{oi}(T')+8)\\ &= \Big(\frac{3\gamma_t^{oi}(T')}{4}+\frac{13}{2}\Big)/(\gamma_t^{oi}(T')+8) = \Big(\frac{3}{4}\Big)\Big(\gamma_t^{oi}(T')+\frac{26}{3}\Big)/(\gamma_t^{oi}(T')+8) > \frac{3}{4}.\end{aligned}$$

Now assume that $d_T(e) = 2$. First assume that there is a child of f other than e, say k, such that the distance of f to the most distant vertex of T_k is six. It suffices to consider only the possibility when T_k is a path P_6, say $klmpqs$. Let $T' = T - T_e$. Let D' be any $\gamma_t^{oi}(T')$-set. It is easy to observe that $D' \cup \{e, d, u, v\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 4$. Now let us observe that there exists a $\gamma_2(T)$-set that does not contain the vertices v, w and e. Let D be such a set. By Observation 2.3 we have $t \in D$. None of the vertices u and d has a neighbor in the set D, and thus $u, d \in D$. Observe that $D \setminus \{d, u, t\}$ is a 2DS of the tree T'.

Therefore, $\gamma_2(T') \le \gamma_2(T) - 3$. Now we get

$$\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} \ge \frac{\gamma_2(T')+1+3}{\gamma_t^{oi}(T')+6} \ge \Big(\frac{3(\gamma_t^{oi}(T')+2)}{4}+3\Big)/(\gamma_t^{oi}(T')+6)$$
$$= \Big(\frac{3\gamma_t^{oi}(T')}{4}+\frac{9}{2}\Big)/(\gamma_t^{oi}(T')+6) = \frac{3}{4}.$$

If $\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} = \frac{3}{4}$, then $\frac{\gamma_2(T')+1}{\gamma_t^{oi}(T')+2} = \frac{3}{4}$.

By the inductive hypothesis, we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_1$. Thus $T \in \mathcal{T}$.

Now assume that there is a child of f, say k, such that the distance of f to the most distant vertex of T_k is five. It suffices to consider only the possibility when T_k is a path P_5, say $klmpq$. Let $T' = T - T_v - T_k$. Let us observe that there exists a $\gamma_t^{oi}(T')$-set that does not contain the vertex e, or any leaf. Let D' be such a set. By Observation 2.1 we have $w \in D'$. The vertex w has to have a neighbor in D', and thus $d \in D'$. We have $f \in D'$ as the set $V(T') \setminus D'$ is independent. Let us observe that $D' \setminus \{w\} \cup \{e,u,v,k,m,p\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \le \gamma_t^{oi}(T')+5$. Now let us observe that there exists a $\gamma_2(T)$-set that contains the vertices u, d, f, m and k. Let D be such a set. By Observation 2.3 we have $t,q \in D$. The set D is minimal, and thus $v,p,l \notin D$. It is easy to observe that $D \setminus \{t,k,m,q\}$ is a 2DS of the tree T'. Therefore, $\gamma_2(T') \le \gamma_2(T) - 4$. Now we get

$$\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} \ge \frac{\gamma_2(T')+1+4}{\gamma_t^{oi}(T')+7} \ge \Big(\frac{3(\gamma_t^{oi}(T')+2)}{4}+4\Big)/(\gamma_t^{oi}(T')+7)$$
$$= \Big(\frac{3\gamma_t^{oi}(T')}{4}+\frac{11}{2}\Big)/(\gamma_t^{oi}(T')+7) = \Big(\frac{3}{4}\Big)\Big(\gamma_t^{oi}(T')+\frac{22}{3}\Big)/(\gamma_t^{oi}(T')+7) > \frac{3}{4}.$$

Now assume that there is a child of f, say k, such that the distance of f to the most distant vertex of T_k is four. It suffices to consider only the possibility when T_k is path P_4, say $klmp$. Let $T' = T - T_w - T_k$. Let D' be a $\gamma_t^{oi}(T')$-set that contains no leaf. By Observation 2.1 we have $e \in D'$. The vertex e has to have a neighbor in D', and thus $f \in D'$. It is easy to observe that $D' \cup \{d,u,v,l,m\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \le \gamma_t^{oi}(T')+5$. Now let us observe that there exists a $\gamma_2(T)$-set that does not contain the vertices v, w, m and k. Let D be such a set. By Observation 2.3 we have $t,p \in D$. None of the vertices u and l has a neighbor in the set D, and thus $u,l \in D$. Observe that $D \setminus \{u,t,l,p\}$ is a 2DS of the tree T'. Therefore, $\gamma_2(T') \le \gamma_2(T) - 4$. Now we get

$$\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} \ge \frac{\gamma_2(T')+1+4}{\gamma_t^{oi}(T')+7} \ge \Big(\frac{3(\gamma_t^{oi}(T')+2)}{4}+4\Big)/(\gamma_t^{oi}(T')+7)$$
$$= \Big(\frac{3\gamma_t^{oi}(T')}{4}+\frac{11}{2}\Big)/(\gamma_t^{oi}(T')+7) = \Big(\frac{3}{4}\Big)\Big(\gamma_t^{oi}(T')+\frac{22}{3}\Big)/(\gamma_t^{oi}(T')+7) > \frac{3}{4}.$$

Now assume that there is a child of f, say k, such that the distance of f to the most distant vertex of T_k is three. It suffices to consider only the possibility when T_k is a path P_3, say klm. Let $T' = T - T_k$. Let D' be any $\gamma_t^{oi}(T')$-set. It is easy to see that $D' \cup \{k,l\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \le \gamma_t^{oi}(T')+2$. Now let us observe that there exists a $\gamma_2(T)$-set that contains the vertices u, d, f and k. Let D be such a set. By Observation 2.3 we have $m \in D$.

The set D is minimal, and thus $l \notin D$. It is easy to observe that $D \setminus \{k, m\}$ is a 2DS of the tree T'. Therefore, $\gamma_2(T') \leq \gamma_2(T) - 2$. Now we get

$$\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} \geq \frac{\gamma_2(T')+1+2}{\gamma_t^{oi}(T')+4} \geq \Big(\frac{3(\gamma_t^{oi}(T')+2)}{4}+2\Big)/(\gamma_t^{oi}(T')+4)$$
$$= \Big(\frac{3\gamma_t^{oi}(T')}{4}+\frac{7}{2}\Big)/(\gamma_t^{oi}(T')+4) = \Big(\frac{3}{4}\Big)\Big(\gamma_t^{oi}(T')+\frac{14}{3}\Big)/(\gamma_t^{oi}(T')+4) > \frac{3}{4}.$$

Now assume that there is a child of f, say k, such that the distance of f to the most distant vertex of T_k is two. Thus k is a support vertex of degree two. Let $T' = T - T_e$. Similarly, as noted earlier, we conclude that $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 4$ and $\gamma_2(T') \leq \gamma_2(T) - 3$. Now we get

$$\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} \geq \frac{\gamma_2(T')+1+3}{\gamma_t^{oi}(T')+6} \geq \Big(\frac{3(\gamma_t^{oi}(T')+2)}{4}+3\Big)/(\gamma_t^{oi}(T')+6)$$
$$= \Big(\frac{3\gamma_t^{oi}(T')}{4}+\frac{9}{2}\Big)/(\gamma_t^{oi}(T')+6) = \frac{3}{4}.$$

If $\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} = \frac{3}{4}$, and then $\frac{\gamma_2(T')+1}{\gamma_t^{oi}(T')+2} = \frac{3}{4}$. By the inductive hypothesis, we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$.

Now assume that some child of f, say k, is a leaf. Let $T' = T - k$. Let D' be any $\gamma_t^{oi}(T')$-set. It $f \in D'$, then it is easy to see that D' is a TOIDS of the tree T. Now assume that $f \notin D'$. Let us observe that $D' \cup \{f\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 1$. Now let us observe that there exists a $\gamma_2(T)$-set that contains the vertices u, d and f. Let D be such a set. By Observation 2.3 we have $k \in D$. It is easy to observe that $D \setminus \{k\}$ is a 2DS of the tree T'. Therefore, $\gamma_2(T') \leq \gamma_2(T) - 1$. Now we get

$$\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} \geq \frac{\gamma_2(T')+1+1}{\gamma_t^{oi}(T')+3} \geq \Big(\frac{3(\gamma_t^{oi}(T')+2)}{4}+1\Big)/(\gamma_t^{oi}(T')+3)$$
$$= \Big(\frac{3\gamma_t^{oi}(T')}{4}+\frac{5}{2}\Big)/(\gamma_t^{oi}(T')+3) = \Big(\frac{3}{4}\Big)\Big(\gamma_t^{oi}(T')+\frac{10}{3}\Big)/(\gamma_t^{oi}(T')+3) > \frac{3}{4}.$$

Now assume that $d_T(f) = 2$. Let $T' = T - T_e$. Similarly, as stated earlier, we conclude that $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 4$ and $\gamma_2(T') \leq \gamma_2(T) - 3$. Now we get

$$\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} \geq \frac{\gamma_2(T')+1+3}{\gamma_t^{oi}(T')+6} \geq \Big(\frac{3(\gamma_t^{oi}(T')+2)}{4}+3\Big)/(\gamma_t^{oi}(T')+6)$$
$$= \Big(\frac{3\gamma_t^{oi}(T')}{4}+\frac{9}{2}\Big)/(\gamma_t^{oi}(T')+6) = \frac{3}{4}.$$

If $\frac{\gamma_2(T)+1}{\gamma_t^{oi}(T)+2} = \frac{3}{4}$, and then $\frac{\gamma_2(T')+1}{\gamma_t^{oi}(T')+2} = \frac{3}{4}$. By the inductive hypothesis, we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_3$. Thus $T \in \mathcal{T}$.

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