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Recognizing Finite Groups Through Order and Degree Patterns*

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Abstract The degree pattern of a finite group G associated with its prime graph has been introduced by Moghaddamfar in 2005 and it is proved that the following simple groups are uniquely determined by their order and degree patterns: All sporadic simple groups, the alternating groups A_p ($p \geq 5$ is a twin prime) and some simple groups of the Lie type. In this paper, the authors continue this investigation. In particular, the authors show that the symmetric groups S_{p+3}, where $p+2$ is a composite number and $p+4$ is a prime and $97 < p \in \pi(1000!)$, are 3-fold OD-characterizable. The authors also show that the alternating groups A_{116} and A_{134} are OD-characterizable. It is worth mentioning that the latter not only generalizes the results by Hoseini in 2010 but also gives a positive answer to a conjecture by Moghaddamfar in 2009.

Keywords Prime graph, Degree pattern, Degree of a vertex
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1 Introduction

Throughout this paper, G will represent a finite group. We use $\pi_e(G)$ to denote the set of orders of its elements and by $\pi(G)$ the set of prime divisors of $|G|$. One of the well-known simple graphs associated with G is the prime graph (or Gruenberg-Kegel graph) denoted by $\Gamma(G)$ (see [1]). This graph is constructed as follows: The vertex set of this graph is $\pi(G)$, and two distinct vertices p, q are joined by an edge if and only if $pq \in \pi_e(G)$. In this case, we write $p \sim q$. The number of connected components of $\Gamma(G)$ is denoted as $t(G)$ and the connected components of $\Gamma(G)$ as $\pi_i = \pi_i(G)\,(i = 1, 2, \cdots, t(G))$. When $|G|$ is even, we suppose that $2 \in \pi_1(G)$. We also denote by $\pi(n)$ the set of all primes dividing n, where n is a positive integer.

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In this article, we also use the following symbols. For a finite group G, the socle of G is defined as the subgroup generated by the minimal normal subgroup of G, denoted as $\mathrm{Soc}(G)$. $\mathrm{Syl}_p(G)$ denotes the set of all Sylow p-subgroups of G, where $p \in \pi(G)$, and P_r denotes a Sylow r-subgroup of G for $r \in \pi(G)$. Moreover, the symmetric and alternating groups of degree n are denoted by S_n and A_n, respectively. Let p be a prime, and we use $\mathrm{Exp}(m, p)$ to denote the exponent of the largest power of a prime p in the factorization of a positive integer m (> 1). All further unexplained symbols and notations are standard and can be found, for instance, in [2].

Definition 1.1 (cf. [3]) *Let G be a finite group and $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where p_is are primes and α_is are integers. For $p \in \pi(G)$, let $\deg(p) := |\{q \in \pi(G) \mid p \sim q\}|$, which we call the degree of p. We also define $D(G) := (\deg(p_1), \deg(p_2), \cdots, \deg(p_k))$, where $p_1 < p_2 < \cdots < p_k$. We call $D(G)$ the degree pattern of G.*

Definition 1.2 (cf. [3]) *A group M is called k-fold OD-characterizable if there exist exactly k non-isomorphic groups G such that (1) $|G| = |M|$ and (2) $D(G) = D(M)$. Moreover, a 1-fold OD-characterizable group is simply called an OD-characterizable group.*

In a series of articles (cf. [3–13]), it was shown that many finite almost simple groups are k-fold OD-characterizable, where $k \geq 1$. For convenience, we point out some of these results, which are included in the following propositions.

Proposition 1.1 *A finite group G is OD-characterizable if G is one of the following groups:*

(1) *The alternating groups A_p, A_{p+1} and A_{p+2}, where p is a prime.*

(2) *The alternating groups A_{p+3}, where p is a prime and $7 \neq p \in \pi(100!)$.*

(3) *All finite almost simple K_3-groups except $\mathrm{Aut}(A_6)$ and $\mathrm{Aut}(U_4(2))$.*

(4) *The symmetric groups S_p and S_{p+1}, where p is a prime.*

(5) *All finite simple K_4-groups except A_{10}.*

(6) *All finite simple $C_{2,2}$-groups.*

(7) *The simple groups of the Lie type $L_2(q)$, $L_3(q)$, $U_3(q)$, ${}^2B_2(q)$ and ${}^2G_2(q)$ for a certain prime power q.*

(8) *All sporadic simple groups and their automorphism groups except $\mathrm{Aut}(J_2)$ and $\mathrm{Aut}(M^cL)$.*

(9) *The almost simple groups of $\mathrm{Aut}(F_4(2))$, $\mathrm{Aut}(O_{10}^+(2))$ and $\mathrm{Aut}(O_{10}^-(2))$.*

Proposition 1.2 *A finite group G is 3-fold OD-characterizable if G is one of the following groups:*

(1) *The almost simple groups of $U_3(5) \cdot 3$ and $U_6(2) \cdot 3$.*

(2) *The symmetric groups of S_{16}, S_{22}, S_{26}, S_{27} and S_{28}.*

Proposition 1.3 *Let G be a finite group with $|G| = |S_{10}|$ and $D(G) = D(S_{10})$. Then G is 8-fold OD-characterizable.*

2 Main Results

According to Propositions 1.2–1.3, we see that the symmetric groups S_{16}, S_{22}, S_{26}, S_{27} and S_{28} are 3-fold OD-characterizable, and S_{10} is 8-fold OD-characterizable. Proposition 1.1(4) says that the symmetric groups S_p and S_{p+1}, where p is a prime, are OD-characterizable. Now, omitting the symmetric groups S_p and S_{p+1}, there remains the following groups:

$$S_9, \quad S_{10}, \quad S_{15}, \quad S_{16}, \quad S_{21}, \quad S_{22}, \quad S_{25}, \quad S_{26}, \quad S_{27}, \quad S_{28}, \quad \cdots . \tag{2.1}$$

In addition, as mentioned in Proposition 1.3, the symmetric group S_{10} is 8-fold OD-characterizable, and is the first symmetric group in the series of symmetric groups found to be not OD-characterizable. By [1], it is easy to see that all groups in (2.1) have connected prime graphs. By these facts, we see that it is like a puzzle to investigate how many-fold OD-characterization of symmetric groups, many special cases appear. In this paper, we will prove that the symmetric groups S_{p+3}, where $p+2$ is a composite number and $p+4$ is a prime and $97 < p \in \pi(1000!)$, are 3-fold OD-characterizable. In other words, we will prove the following theorem.

Theorem 2.1 *All symmetric groups S_{p+3}, where $p+2$ is a composite number, $p+4$ is a prime and $97 < p \in \pi(1000!)$, are 3-fold OD-characterizable.*

As we have mentioned already, the symmetric groups S_p, S_{p+1}, where p is a prime number, are OD-characterizable (see Proposition 1.1(4)) and the symmetric groups of S_{16}, S_{22}, S_{26}, S_{27} and S_{28} are 3-fold OD-characterizable (see Proposition 1.2(2)). On the other hand, Proposition 1.3 says that the symmetric group S_{10} is 8-fold OD-characterizable, and S_{10} is the first symmetric group which has not been considered OD-characterizable. Up to now, we have not found a symmetric group S_n ($n \neq p$, $p+1$), except S_{10}, which is not 3-fold OD-characterizable. Hence, we put forward the following open problem.

Open Problem Are the symmetric groups S_n ($n \neq p$, $p+1$), except S_{10}, 3-fold OD-characterizable?

In what follows, we will focus attention on the following alternating groups: A_{116} and A_{134}. In this article, we will show the following result.

Theorem 2.2 *The alternating groups A_{116} and A_{134} are OD-characterizable.*

By Proposition 1.1(1), we see that the alternating groups A_p, A_{p+1} and A_{p+2}, where p is a prime, are OD-characterizable. Proposition 1.1(2) says that all alternating groups A_{p+3}, where p is a prime and $7 \neq p \in \pi(100!)$, are OD-characterizable. In fact, Theorem 2.2 and Proposition 1.2(1)–(3) imply the following corollary.

Corollary 2.1 *Let A_n be an alternating group of degree n. Assume that one of the following conditions is fulfilled:*

(1) *$n = p$, $p+1$ or $p+2$, where p is a prime.*

(2) $n = p + 3$, *where* $7 \neq p \in \pi(136!)$.

Then A_n *is OD-characterizable.*

3 Preliminaries

In this section, we consider some results which will be applied for our further investigations.

Lemma 3.1 (cf. [14]) *The group* S_n (*or* A_n) *has an element of order* $m = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_s^{\alpha_s}$, *where* $p_1, p_2, \cdots, p_s$ *are distinct primes and* $\alpha_1, \alpha_2, \cdots \alpha_s$ *are natural numbers, if and only if* $p_1^{\alpha_1} + p_2^{\alpha_2} + \cdots + p_s^{\alpha_s} \leq n$ (*or* $p_1^{\alpha_1} + p_2^{\alpha_2} + \cdots + p_s^{\alpha_s} \leq n$ *for* m *odd, and* $p_1^{\alpha_1} + p_2^{\alpha_2} + \cdots + p_s^{\alpha_s} \leq n - 2$ *for* m *even*).

As an immediate corollary of Lemma 3.1, we have the following lemma.

Lemma 3.2 *Let* A_n (*or* S_n) *be an alternating group* (*or a symmetric group*) *of degree* n. *Then the following assertions hold:*

(1) *Let* $p, q \in \pi(A_n)$ *be odd primes. Then* $p \sim q$ *if and only if* $p + q \leq n$.

(2) *Let* $p \in \pi(A_n)$ *be an odd prime. Then* $2 \sim p$ *if and only if* $p + 4 \leq n$.

(3) *Let* $p, q \in \pi(S_n)$. *Then* $p \sim q$ *if and only if* $p + q \leq n$.

Lemma 3.3 (cf. [15]) *Let* G *be a finite solvable group, all of whose elements are of the prime power order. Then* $|\pi(G)| \leq 2$.

Lemma 3.4 *Let* A_{p+3} *be an alternating group of degree* $p + 3$, *where* p *is a prime and* $p + 2$ *is a composite number. Suppose that* $|\pi(A_{p+3})| = d$. *Then the following assertions hold*:

(1) $\deg(2) = d - 2$. *Particularly,* $2 \sim r$ *for each* $r \in \pi(A_{p+3}) \backslash \{p\}$.

(2) $\deg(3) = d - 1$, i.e., $3 \sim r$ *for each* $r \in \pi(A_{p+3})$.

(3) $\deg(p) = 1$. *In other words,* $p \sim r$, *where* $r \in \pi(A_{p+3})$, *if and only if* $r = 3$.

(4) $\mathrm{Exp}(|A_{p+3}|, 2) = \sum\limits_{i=1}^{\infty} \left[\frac{p+3}{2^i}\right] - 1$. *In particular,* $\mathrm{Exp}(|A_{p+3}|, 2) < p + 3$.

(5) $\mathrm{Exp}(|A_{p+3}|, r) = \sum\limits_{i=1}^{\infty} \left[\frac{p+3}{r^i}\right]$ *for each* $r \in \pi(A_{p+3}) \backslash \{2\}$. *Furthermore,* $\mathrm{Exp}(|A_{p+3}|, r) < \frac{p-1}{2}$, *where* $5 \leq r \in \pi(A_{p+3})$. *Particularly, if* $r > [\frac{p+3}{2}]$, *then* $\mathrm{Exp}(|A_{p+3}|, r) = 1$.

Proof By Lemma 3.2, we have $2 \nsim p$. Obviously, $r + 4 \leq p + 3$ for each $r \in \pi(A_{p+3}) \backslash \{p\}$, so it follows that $2 \sim r$ and thus $\deg(2) = d - 2$. For the same reason, we have $\deg(3) = d - 1$. For $r \in \pi(A_{p+3}) \backslash \{2, p\}$, by Lemma 3.2, it is easy to see that $p \sim r$ if and only if $p + r \leq p + 3$. Hence $r = 3$ and $\deg(p) = 1$.

Till now we have proved that (1)–(3) hold. Next, we prove that (4) and (5) also hold.

By the definition of Gaussian integer function, we have that

$$\begin{aligned}\mathrm{Exp}(|A_{p+3}|,2) &= \sum_{i=1}^{\infty}\Big[\frac{p+3}{2^i}\Big]-1\\ &= \Big(\Big[\frac{p+3}{2}\Big]+\Big[\frac{p+3}{2^2}\Big]+\Big[\frac{p+3}{2^3}\Big]+\cdots\Big)-1\\ &\le \Big(\frac{p+3}{2}+\frac{p+3}{2^2}+\frac{p+3}{2^3}+\cdots\Big)-1\\ &= (p+3)\Big(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\Big)-1=p+2.\end{aligned}$$

Hence $\mathrm{Exp}(|A_{p+3}|,2) < p+3$.

For the same reason as above, we can prove $\mathrm{Exp}(|A_{p+3}|,r) < \frac{p-1}{2}$, where $5 \le r \in \pi(A_{p+3})$. Clearly, if $r > \big[\frac{p+3}{2}\big]$, then we have $\mathrm{Exp}(|A_{p+3}|,r) = 1$. This completes the proof of Lemma 3.4.

Similarly, we can prove the following Lemma 3.5.

Lemma 3.5 *Let S_{p+3} be an alternating group of degree $p+3$, where p is a prime and $p+2$ is a composite number. Suppose that $|\pi(S_{p+3})| = k$. Then the following assertions hold*:

(1) $\deg(2) = k-1$. *Particularly, $2 \sim r$ for each $r \in \pi(S_{p+3})$.*

(2) $\deg(3) = k-1$, i.e., *$3 \sim r$ for each $r \in \pi(S_{p+3})$.*

(3) $\deg(p) = 2$. *In other words, $p \sim r$, where $r \in \pi(S_{p+3})$, if and only if $r = 2$ or $r = 3$.*

(4) $\mathrm{Exp}(|S_{p+3}|,2) = \sum\limits_{i=1}^{\infty}\big[\frac{p+3}{2^i}\big]$. *In particular,* $\mathrm{Exp}(|S_{p+3}|,2) \le p+3$.

(5) $\mathrm{Exp}(|S_{p+3}|,r) = \sum\limits_{i=1}^{\infty}\big[\frac{p+3}{r^i}\big]$. *Furthermore,* $\mathrm{Exp}(|S_{p+3}|,r) < \frac{p-1}{2}$, *where* $5 \le r \in \pi(S_{p+3})$. *Particularly, if $r > [\frac{p+3}{2}]$, then* $\mathrm{Exp}(|S_{p+3}|,r) = 1$.

Lemma 3.6 (cf. [16]) *Let a be an arbitrary integer and m be a positive integer. If $(a, m) = 1$, then the equation $a^x \equiv 1 \,(\mathrm{mod}\ m)$ has solutions. Moreover, if the order of a modulo m is $h(a)$, then $h(a) \mid \varphi(m)$, where $\varphi(m)$ is the Euler's function of m.*

Lemma 3.7 *Let A_{p+3} be an alternating group of degree $p+3$, where $p+2$ is a composite number, $p+4$ is a prime and $97 < p \in \pi(1000!)$. Set $P \in \mathrm{Syl}_p(A_{p+3})$ and $Q \in \mathrm{Syl}_q(A_{p+3})$, where $5 \le q < p$. Then the following assertions hold*:

(i) *$q^{s(q)} \nmid |N_G(P)|$, where* $s(q) = \mathrm{Exp}(|A_{p+3}|,q)$.

(ii) *If* $p \in \{103, 109, 163, 193, 223, 229, 277, 349, 439, 463, 499, 613, 643, 739, 769, 823, 853, 877, 907, 967\}$, *then $p \nmid |N_G(Q)|$.*

(iii) *If* $p \in \{127, 307, 313, 379, 397, 457, 487, 673, 757, 859, 883, 937\}$, *then there exists at least a prime number, say r, such that the order of r modulo p is less than $p-1$, where $5 \le r < p$ and $r \in \pi(A_{p+3})$.*

Proof Clearly, the equation $q^x \equiv 1 \ (\mathrm{mod}\, p)$ has solutions by Lemma 3.6. Suppose that the order of q modulo p is $h(q)$. If $h(q) = p-1$, then q is a primitive root of modulo p. By hypothesis, it is easy to check that there are only 32 such groups satisfying the conditions that

$p+2$ is a composite number, $p+4$ is a prime number and $97 < p \in \pi(1000!)$. Using Maple 8.0, we can obtain $h(q)$. For convenience, we have tabulated p and $h(q)$ in Table 1 of this article.

Table 1 p and $h(q)$

p	$h(q)$	Condition	p	$h(q)$	Condition	p	$h(q)$	Condition
103	$2 \cdot 3 \cdot 17$	none	109	$2^2 \cdot 3^3$	none	163	$2 \cdot 3^4$	none
127	$2 \cdot 3^2 \cdot 7$	$q \neq 19$	127	3	$q = 19$	193	$2^6 \cdot 3$	none
223	$2 \cdot 3 \cdot 37$	none	229	$2^2 \cdot 3 \cdot 19$	none	277	$2^2 \cdot 3 \cdot 23$	none
307	$2 \cdot 3^2 \cdot 17$	$q \neq 17$	307	3	$q = 17$	349	$2^2 \cdot 3 \cdot 29$	none
313	$2^3 \cdot 3 \cdot 13$	$q \neq 5$	313	8	$q = 5$	439	$2 \cdot 3 \cdot 73$	none
379	$2 \cdot 3^3 \cdot 7$	$q \neq 5$	379	21	$q = 5$	397	$2^2 \cdot 3^2 \cdot 11$	$q \neq 31$
397	11	$q = 31$	457	$2^3 \cdot 3 \cdot 19$	$q \neq 109$	457	4	$q = 109$
463	$2 \cdot 3 \cdot 7 \cdot 11$	none	487	$2 \cdot 3^5$	$q \neq 5, 41$	487	54	$q = 5$
487	9	$q = 41$	499	$2 \cdot 3 \cdot 83$	none	613	$2^2 \cdot 3^2 \cdot 17$	none
643	$2 \cdot 3 \cdot 107$	none	673	$2^5 \cdot 3 \cdot 7$	$q \neq 23$	673	14	$q = 23$
739	$2 \cdot 3^2 \cdot 41$	none	757	$2^2 \cdot 3^3 \cdot 7$	$q \neq 59$	757	7	$q = 59$
769	$2^8 \cdot 3$	none	823	$2 \cdot 3 \cdot 137$	none	853	$2^2 \cdot 3 \cdot 71$	none
859	$2 \cdot 3 \cdot 11 \cdot 13$	$q \neq 13$	859	11	$q = 13$	877	$2 \cdot 3^2 \cdot 73$	none
883	$2 \cdot 3^2 \cdot 7^2$	$q \neq 71$	883	7	$q = 71$	907	$2 \cdot 3 \cdot 151$	none
937	$2^3 \cdot 3^2 \cdot 13$	$q \neq 13, 23$	937	18	$q = 13$	937	24	$q = 23$
967	$2 \cdot 3 \cdot 7 \cdot 23$	none						

Now, using the $n-c$ Theorem, the factor group $N_G(P)/C_G(P)$ is isomorphic to a subgroup of $\mathrm{Aut}(P) \cong \mathbb{Z}_{p-1}$. Hence, $|N_G(P)/C_G(P)| \mid (p-1)$. By Table 1, if there exists a prime number, say q, where $5 \leq q < p$ and $q \in \pi(A_{p+3})$, such that $q^{s(q)} \mid |N_G(P)|$, and then $q \mid |C_G(P)|$. Thus $\deg(p) \geq 2$, a contradiction to Lemma 3.4(3), and (i) is proved.

Next, assume that $p \in \{103, 109, 163, 193, 223, 229, 277, 349, 439, 463, 499, 613, 643, 739, 769, 823, 853, 877, 907, 967\}$. If $p \mid |N_G(Q)|$, by Table 1 and $\mathrm{Exp}(|A_{p+3}|, q) < p$, then $p \mid |C_G(Q)|$, which leads to a contradiction as above. Thus (ii) holds. The remaining parts of (iii) follow at once from Table 1. This completes the proof of Lemma 3.7.

Lemma 3.8 *Let M be a finite non-abelian simple group with an order having prime divisors at most* 997*. Then M is isomorphic to one of the following simple groups listed in Tables* 1–3 *in* [17]*. In particular, if $|\pi(\mathrm{Out}(M))| \neq 1$, then $\pi(\mathrm{Out}(M)) \subseteq \{2, 3, 5, 7\}$.*

Proof Let p be a prime and $\mathcal{F}_p$ denotes the set of non-abelian finite simple groups M such that $p \in \pi(G) \subseteq \{2, 3, 5, \cdots, p\}$. By [17], the members of $\mathcal{F}_p$ are ordered according to the size of their prime spectrum (listed in Tables 1–3). The number of groups in each set $\mathcal{F}_p$ is given after the symbol "♯". For each group, we also know the prime decomposition of the order. However, since the members of $\mathcal{F}_p$ are too many and the order decompositions occupy too much space, the detailed Tables 1–3 are omitted. In the latter case, i.e., if $|\pi(\mathrm{Out}(M))| \neq 1$, using [2], it is easy to check that the statement of the lemma is correct by checking each choice of p. Since the method is not very complicated by checking computations, the detailed process is omitted, too.

Note that, the full list of all non-abelian simple groups in $\mathcal{F}_{131}$ has been determined in [17]. In fact, there are 407 such groups, and for convenience we list them in Table 2 of this article.

Table 2 Finite non-abelian simple groups with $\pi(M) \subseteq \{2,\ 3,\ 5,\ 7, \cdots, 131\}$

S	$\lvert\mathrm{Out}(S)\rvert$	S	$\lvert\mathrm{Out}(S)\rvert$	$\lvert S\rvert$	$\lvert\mathrm{Out}(S)\rvert$	S	$\lvert\mathrm{Out}(S)\rvert$	S	$\lvert\mathrm{Out}(S)\rvert$
A_5	2	$U_3(5)$	6	$L_2(7)$	2	$S_4(7)$	2	A_6	4
$L_2(8)$	3	$O_8^+(2)$	6	$L_2(17)$	2	$L_2(11)$	3	$L_2(16)$	2
A_9	2	$U_5(2)$	2	$S_4(4)$	4	A_7	2	$U_6(2)$	6
A_{11}	2	$U_3(3)$	2	J_2	2	He	2	A_{10}	2
$O_8^-(2)$	2	A_{12}	2	$L_4(4)$	4	$U_4(3)$	8	A_8	2
M_{11}	1	$L_3(4)$	12	M_{12}	2	$U_4(2)$	2	$S_8(2)$	2
$L_2(49)$	4	M^cL	2	$L_2(13)$	2	$L_2(5^2)$	4	$S_6(2)$	1
$L_3(3^3)$	6	$L_2(2^6)$	6	$L_3(3)$	2	$L_3(3^2)$	4	$L_4(3)$	4
HS	2	M_{22}	2	$L_5(3)$	2	$L_6(3)$	4	$U_3(2^2)$	4
$U_4(5)$	4	A_{18}	2	$L_2(19)$	2	$S_4(5)$	2	$S_4(2^3)$	6
$S_6(3)$	2	$O_7(3)$	2	$O_8^+(3)$	24	$G_2(3)$	2	${}^3D_4(2)$	3
$G_2(2^2)$	2	A_{13}	2	A_{14}	2	A_{15}	2	A_{16}	2
$Sz(2^3)$	3	${}^2F_4(2)'$	2	Suz	2	F_{i22}	2	$L_2(13^2)$	4
$L_3(2^4)$	24	$U_3(17)$	6	$U_4(2^2)$	4	$S_4(13)$	2	$S_6(2^2)$	2
$O_7(2^2)$	2	$O_8^+(2^2)$	2	$O_{10}^-(2)$	2	$F_4(2)$	2	A_{17}	6
$L_3(7)$	6	$U_3(8)$	18	$U_3(19)$	2	$L_4(7)$	4	J_3	2
J_1	1	$L_3(11)$	2	HN	2	$U_4(8)$	4	A_{19}	2
A_{20}	2	A_{21}	2	A_{22}	2	${}^2E_6(2)$	6	$L_2(23)$	2
$U_3(23)$	4	M_{23}	1	Co_3	1	M_{24}	1	Co_2	1
Co_1	1	A_{23}	2	A_{24}	2	A_{25}	2	A_{26}	2
A_{27}	2	$L_2(27)$	6	A_{28}	2	$L_2(29)$	2	$L_2(17^2)$	4
$S_4(17)$	2	Ru	1	Fi'_{24}	8	A_{29}	2	A_{30}	2
$L_2(31)$	2	$L_3(5)$	2	$L_2(32)$	5	$L_2(5^3)$	12	$G_2(5)$	1
$L_5(2)$	2	$L_6(2)$	2	$L_4(5)$	8	$L_3(25)$	12	$O_7(5)$	2
$S_6(5)$	2	$O_8^+(5)$	24	$O_{10}^+(2)$	2	$U_3(31)$	2	$L_5(4)$	4
$S_{10}(2)$	1	$O_{12}^+(2)$	2	ON	2	Th	1	$O_{12}^-(2)$	2
$L_6(4)$	12	$S_{12}(2)$	1	$L_2(37)$	2	$A_{31}, \cdots, A_{36}$	2	$U_3(11)$	6
$L_2(31^2)$	4	$S_4(31)$	2	${}^2G_{27}$	3	$U_3(37)$	6	$L_2(11^3)$	6
$G_2(11)$	1	$U_4(31)$	4	$L_3(3^4)$	8	$A_{37}, \cdots, A_{40}$	2	$S_4(9)$	4
$Sz(32)$	5	$L_2(41)$	2	$O_8^-(3)$	4	$L_4(9)$	16	$S_8(3)$	2
$O_9(3)$	2	$L_2(41^2)$	4	$S_4(41)$	2	$L_2(2^{10})$	10	$S_4(32)$	10
$U_5(4)$	20	$O_{10}^+(3)$	4	$U_6(4)$	4	A_{41}	2	A_{42}	2
$U_3(7)$	2	$U_4(7)$	8	$L_2(43)$	2	$L_2(7^3)$	6	$G_2(7)$	2
$U_7(2)$	2	$L_3(49)$	12	$S_6(7)$	2	$O_7(7)$	2	$O_8^+(7)$	24
$U_3(37)$	2	$U_8(2)$	2	$L_2(43^2)$	4	$S_4(43)$	2	$U_9(2)$	6
$O_{14}^-(2)$	2	$U_{10}(2)$	2	J_4	1	$A_{43}, \cdots, A_{46}$	2	$L_2(47)$	2
$L_2(47^2)$	4	$S_4(47)$	2	B	1	$A_{47}, \cdots, A_{52}$	2	$L_2(53)$	2
$L_2(23^2)$	4	$S_4(23)$	2	$U_4(23)$	4	$A_{53}, \cdots, A_{59}$	2	$L_2(59)$	2
A_{60}	2	$L_2(3^5)$	10	$U_5(3)$	2	$L_2(11^2)$	4	$S_4(11)$	2
$L_2(61)$	2	$L_3(13)$	6	$U_6(3)$	4	$U_4(11)$	8	$L_3(47)$	2
$L_4(11)$	4	$L_4(13)$	8	$O_{10}^-(3)$	8	$L_5(9)$	4	$S_{10}(3)$	2
$O_{11}(3)$	2	$O_{12}^+(3)$	8	$L_3(11^2)$	12	$S_6(11)$	2	$O_7(11)$	2
$O_8^+(11)$	24	$L_4(47)$	4	$L_2(67)$	2	$A_{61}, \cdots, A_{66}$	2	$L_3(37)$	6
$L_3(29)$	2	$L_3(67)$	2	Ly	1	$L_2(37^3)$	6	$G_2(37)$	2
$L_2(71)$	2	$L_5(5)$	2	$L_6(5)$	4	$A_{67}, \cdots, A_{70}$	2	M	1
A_{71}, A_{72}	2	$U_3(9)$	4	$L_3(8)$	6	$L_2(73)$	24	$U_4(9)$	8
${}^3D_4(3)$	3	$L_2(2^9)$	9	$G_2(8)$	3	$L_2(3^6)$	12	$S_4(27)$	6
$G_2(9)$	2	$L_4(8)$	6	$L_3(64)$	36	$S_6(8)$	3	$O_8^+(8)$	18
$L_3(3^4)$	8	$S_6(9)$	4	$O_7(9)$	4	$F_4(3)$	1	$O_8^+(9)$	48
$L_4(23)$	4	$L_3(23^2)$	12	$S_6(23)$	2	$A_{79}, \cdots, A_{82}$	2	$O_7(23)$	2

Table 2 (Continued)

S	$\|\mathrm{Out}(S)\|$	S	$\|\mathrm{Out}(S)\|$	$\|S\|$	$\|\mathrm{Out}(S)\|$	S	$\|\mathrm{Out}(S)\|$	S	$\|\mathrm{Out}(S)\|$
$O_8^+(23)$	24	$L_2(83)$	2	$L_2(83^2)$	4	$A_{83}, \cdots, A_{88}$	2	$S_4(83)$	2
$L_2(89)$	2	$L_2(97)$	2	$L_3(61)$	6	$A_{89}, \cdots, A_{96}$	2	$L_2(101)$	2
A_{101}, A_{102}	2	$U_3(101)$	6	$U_5(17)$	2	$A_{97}, \cdots, A_{100}$	2	$L_2(103)$	2
$U_3(47)$	6	$U_3(103)$	6	$L_2(47^3)$	6	$A_{103}, \cdots, A_{106}$	2	$G_2(47)$	2
$L_3(47^2)$	12	$S_6(47)$	2	$O_7(47)$	2	$A_{103}, \cdots, A_{106}$	2	$O_8^+(47)$	24
$L_2(131)$	2	$L_2(107)$	2	$L_2(109)$	2	$A_{107}, \cdots, A_{112}$	2	$U_3(64)$	12
${}^3D_4(8)$	9	$Sz(2^9)$	9	${}^2F_4(8)$	3	$L_2(2^{18})$	18	$G_2(64)$	12
$S_4(2^9)$	9	$L_2(113)$	2	$U_7(4)$	2	$A_{113}, \cdots, A_{126}$	2	$L_2(127)$	2
$L_2(2^7)$	7	$L_3(19)$	6	$Sz(2^7)$	7	$L_2(19^3)$	6	$G_2(19)$	2
$L_7(2)$	2	$L_8(2)$	6	$L_2(2^{14})$	14	$S_4(2^7)$	7	$L_3(107)$	2
$L_9(2)$	2	$O_{14}^+(2)$	2	$L_{10}(2)$	2	$L_7(4)$	4	$S_{14}(2)$	1
$O_{16}^+(2)$	2	$L_{11}(2)$	2	$E_7(2)$	1	$A_{127}, \cdots, A_{136}$	2	$L_{12}(2)$	2

Lemma 3.9 (cf. [18]) *Let* $S = P_1 \times P_2 \times \cdots \times P_r$*, where* P_i*s are isomorphic non-abelian simple groups. Then* $\mathrm{Aut}(S) = (\mathrm{Aut}(P_1) \times \mathrm{Aut}(P_2) \times \cdots \times \mathrm{Aut}(P_r)) \rtimes \mathbf{S}_r$.

4 *OD*-Characterization of the Symmetric Groups S_{p+3}

In this section, we are going to give an affirmative answer to the open problem of this article for the symmetric groups S_{p+3} satisfying the conditions that $p+2$ is a composite number, $p+4$ is a prime and $97 < p \in \pi(1000!)$. In other words, we will prove Theorem 2.1.

Proof of Theorem 2.1 Let G be a finite group satisfying the conditions that (1) $|G| = |S_{p+3}|$ and (2) $D(G) = D(S_{p+3})$, where $p+2$ is a composite number, $p+4$ is a prime and $97 < p \in \pi(1000!)$. By these hypotheses, we obtain that $\{r\} \cup \{rs \mid r+s \le p+3\} \subseteq \pi_e(G)$ and $\{rs \mid r+s > p+3\} \cap \pi_e(G) = \varnothing$, where r, $s \in \pi(G)$. By Lemma 3.4, the prime graph of G is connected since $\deg(3) = d-1$, where $d = |\pi(G)|$. Moreover, by the structure of $D(G)$, it is easy to check that $\Gamma(G) = \Gamma(S_{p+3})$. In the following, we will write the proof in a number of separate lemmas.

Lemma 4.1 *Let K be the maximal normal solvable subgroup of G. Then K is a $\{2,3\}$-group. Particularly, G is nonsolvable.*

Proof We first show that K is a p'-group. If not, let p divide the order of K. Set $P \in \mathrm{Syl}_p(G)$. By Lemma 3.7(i), we have $q^{s(q)} \nmid |N_G(P)|$ for each prime $q \in \pi(G)$ and $5 \le q < p$. If $q \mid |N_G(P)|$, then either $q \mid |C_G(P)|$ or $q \in \pi(K)$. For the former, by Lemma 3.4(3), this leads to an obvious contradiction since $q \sim p$. In the latter case, i.e., $q \in \pi(K)$, by Table 1, it is easy to check that there necessarily exists such a prime r such that $r \nsim q$, where $5 \le r < p$ and $r \in \pi(K)$. In fact, by Lemma 3.2(1), it is sufficient to find such a prime r such that $r + q > p$, and then $r \nsim q$. Since K is solvable, it possesses a Hall $\{p,q,r\}$-subgroup T. It follows that T

is solvable. Since there exists no edge between p, q and r in $\Gamma(G)$, all elements in T are of the prime power order. Hence $|\pi(T)| \leq 2$ by Lemma 3.3, a contradiction. Thus K is a p'-group.

We shall argue next that K is a q'-group for each $q \in \pi(G)\backslash\{2,3,p\}$. Set $Q \in \mathrm{Syl}_q(K)$, where $q \in \pi(K)$. Suppose that the order of q modulo p is $h(q)$. By the Frattini argument, $G = KN_G(Q)$, and hence p divides the order of $N_G(Q)$. By Lemma 3.7(ii) and (iii), it is easy to see that p is equal to one of the following possible primes: 127, 307, 313, 379, 397, 457, 487, 673, 757, 859, 883 and 937. In this case, there necessarily exists at least a prime, say q, such that $h(q) < p-1$. We prove the lemma up to the choice of p one by one. The proof is written in 3 cases.

Case 1 To prove that the lemma follows if $p = 127$.

By Table 1, if there exists a prime q such that $p \mid |N_G(Q)|$, where $Q \in \mathrm{Syl}_q(G)$, then $q = 19$. Now, by the n-c theorem, the factor group $N_G(Q)/C_G(Q)$ is isomorphic to a subgroup of $\mathrm{Aut}(Q)$. By Lemma 3.5(5), we have $\mathrm{Exp}(|G|, 19) = 6$, and thus $|N_G(Q)/C_G(Q)| \mid \prod\limits_{i=1}^{6} 19^{15} \cdot (19^i - 1)$. It is easy to check that $113 \nmid \prod\limits_{i=1}^{6} 19^{15} \cdot (19^i - 1)$. If $113 \mid |N_G(Q)|$, then $113 \in \pi(C_G(Q))$. Thus $19 \sim 113$, a contradiction. Hence $113 \in \pi(K)$. Since K is solvable, it possesses a Hall $\{19, 113\}$-subgroup H of order $19^6 \cdot 113$. Obviously, H is abelian, so $19 \sim 113$, which leads to a contradiction as above.

Case 2 To prove that the lemma follows if $p = 307$.

It is easy to see that there exists a prime, say q, such that $p \mid |N_G(Q)|$, where $Q \in \mathrm{Syl}_q(G)$. Then $q = 17$ by Table 1. On the other hand, the factor group $N_G(Q)/C_G(Q)$ is isomorphic to a subgroup of $\mathrm{Aut}(Q)$ by the n-c theorem and $\mathrm{Exp}(|G|, 17) = 19$ by Lemma 3.4, so $|N_G(Q)/C_G(Q)| \mid \prod\limits_{i=1}^{19} 17^{171} \cdot (17^i - 1)$. It is easy to check that $31 \nmid \prod\limits_{i=1}^{19} 17^{171} \cdot (17^i - 1)$. If $31 \mid |N_G(Q)|$, then $31 \in \pi(C_G(Q))$. Set $N = N_G(Q)$, $C = C_G(Q)$ and $K_{31} \in \mathrm{Syl}_{31}(C_G(Q))$. By Lemma 3.5, we have $\mathrm{Exp}(|G|, 31) = 9$. Again, by the Frattini argument $N = CN_N(K_{31})$ and hence $p \nmid |N_N(K_{31})|$. Thus $p \mid |C_G(Q)|$, and so $\deg(p) \geq 3$, a contradiction. Therefore $31 \nmid |N_G(Q)|$ and $31 \in \pi(K)$. Set $P_{31} \in \mathrm{Syl}_{31}(K)$. Since $G = KN_G(P_{31})$, then $p \mid |N_G(P_{31})|$. It is easy to see that this is impossible by Table 1.

Case 3 Till now we have proved that K is a q'-group while $p = 127$ or 307. Assume that p is one of the remaining possible primes. Now, we have to discuss 10 cases. If K is a q-group for each $q \in \pi(G)\backslash\{2,3,p\}$, it is easy to show that this is impossible by checking each choice of p. Since the method used below is the same as in Case 2, the detailed processes are omitted. Therefore K is a $\{2,3\}$-group. Since $K \neq G$, it follows at once that G is a nonsolvable group. This completes the proof of Lemma 4.1.

Lemma 4.2 *The quotient group G/K is an almost simple group. In fact, $S \lesssim G/K \lesssim \mathrm{Aut}(S)$, where S is a non-abelian simple group.*

Proof Let $\overline{G} := G/K$ and $S := \mathrm{Soc}(\overline{G})$. Then $S = B_1 \times B_2 \times \cdots \times B_m$, where B_i ($i = 1, 2, \cdots, m$) are non-abelian simple groups and $S \lesssim \overline{G} \lesssim \mathrm{Aut}(S)$. We assert that $m = 1$.

Suppose that $m \geq 2$. We assert that p does not divide the order of S. Otherwise, there exists a prime, say r, such that $r \sim p$, where $5 \le r < p$ and $r \in \pi(G)$, which is impossible by Lemma 3.4(3). Hence, for every i we have $B_i \in \mathcal{F}_p$. On the other hand, by Lemma 3.7, we observe that $p \in \pi(\overline{G}) \subseteq \pi(\mathrm{Aut}(S))$. Thus p divides the order of $\mathrm{Out}(S)$. But

$$\mathrm{Out}(S) = \mathrm{Out}(S_1) \times \mathrm{Out}(S_2) \times \cdots \times \mathrm{Out}(S_r),$$

where the groups S_j are direct products of all isomorphic $B_i's$ such that

$$S = S_1 \times S_2 \times \cdots \times S_r.$$

Therefore for some j, p divides the order of an outer automorphism group of a direct product S_j of t isomorphic simple groups B_i for some $1 \le i \le m$. Since $B_i \in \mathcal{F}_p$, it follows that $|\mathrm{Out}(B_i)|$ is not divided by p by Lemma 3.8. Now, by Lemma 3.9, we obtain $|\mathrm{Aut}(S_j)| = |\mathrm{Aut}(B_i)|^t \cdot t!$. Therefore $t \ge p$ and so 2^{2p} must divide the order of G. However, $\mathrm{Exp}(|S_{p+3}|, 2) \le p+3 < 2p$ by Lemma 3.5(4), which is a contradiction. Thus $m = 1$ and $S = B_1$. This completes the proof of Lemma 4.2.

Lemma 4.3 *$S \cong A_{p+3}$ and G is isomorphic to one of the following groups: S_{p+3}, $\mathbb{Z}_2 \cdot A_{p+3}$ or $\mathbb{Z}_2 \times A_{p+3}$. In other words, S_{p+3} is 3-fold OD-characterizable.*

Proof By Lemmas 3.8 and 4.1, we may assume that $|S| = 2^{\alpha_1} \cdot 3^{\alpha_2} \cdot 5^{\alpha_3} \cdots p^{\alpha_s}$, where $2 \le \alpha_1 \le |G|_2 = \mathrm{Exp}(|S_{p+3}|, 2)$ and $1 \le \alpha_2 \le |G|_3 = \mathrm{Exp}(|S_{p+3}|, 3)$. Let $p_1, p_2, p_3, \cdots, p_s$ be distinct consecutive prime numbers and $2 = p_1 < 3 = p_2 < 5 = p_3 < \cdots < p = p_s$, and then $\alpha_j = |G|_{p_j} = \mathrm{Exp}(|S_{p+3}|, p_j)$ for each $j \ge 3$. Using Tables 1–3 in [17], we see that S can only be isomorphic to one of the simple groups: A_p, A_{p+1}, A_{p+2} and A_{p+3}.

If $S \cong A_p$, then K is a 2-group. In this case, it is easy to see that $3p \in \pi_e(\overline{G}) \setminus \pi_e(S_p)$, a contradiction.

Similarly, S cannot be isomorphic to the groups A_{p+1} and A_{p+2}. Therefore, $S \cong A_{p+3}$. According to Lemma 4.2, we have $A_{p+3} \lesssim G/K \lesssim \mathrm{Aut}(A_{p+3}) \cong S_{p+3}$.

If $G/K \cong S_{p+3}$, then by comparing orders we deduce that $G \cong S_{p+3}$.

If $G/K \cong A_{p+3}$, then $|K| = 2$. Therefore G is a central extension of $\mathbb{Z}_2$ by A_{p+3}. If G is a non-split extension of $\mathbb{Z}_2$ by A_{p+3}, then $G \cong \mathbb{Z}_2 \cdot A_{p+3}$. If G is a split extension of $\mathbb{Z}_2$ by A_{p+3}, then $G \cong \mathbb{Z}_2 \times A_{p+3}$. Moreover, whether G is isomorphic to $\mathbb{Z}_2 \cdot A_{p+3}$ or $\mathbb{Z}_2 \times A_{p+3}$, it is easy to see that the groups $\mathbb{Z}_2 \cdot A_{p+3}$ and $\mathbb{Z}_2 \times A_{p+3}$ satisfy the conditions (1) $|G| = |S_{p+3}|$ and (2) $D(G) = D(S_{p+3})$. Hence, S_{p+3} is 3-fold OD-characterizable. This completes the proof of Lemma 4.3 and also the proof of Theorem 2.1.

5 *OD*-Characterization of Alternating Groups A_{116} and A_{134}

We again recall that all the alternating groups A_p, A_{p+1} and A_{p+2} (p is a prime) are OD-characterizable (see Proposition 1.1(1)). Proposition 1.1(2) says that all the alternating groups A_{p+3}, where p is a prime and $7 \ne p \in \pi(100!)$, are OD-characterizable. On the other hand, in [13], we also proved that the alternating groups A_{p+3}, where $p+2$ is a composite number, $p+4$ is a prime and $7 \ne p \in \pi(1000!)$, are OD-characterizable. So far no alternating group, which is not OD-characterizable, has been found. Hence, the authors in [5] put forward the following conjecture.

Conjecture 5.1 All alternating groups A_{p+3} with $p \neq 7$ are OD-characterizable.

In this section, we continue this investigation in [5]. In particular, we are going to give an affirmative answer to the conjecture for another two alternating groups A_{116} and A_{134} and prove that the alternating groups A_{116} and A_{134} are OD-characterizable.

Theorem 5.1 *The alternating group A_{116} is OD-characterizable.*

Proof Let G be a finite group satisfying

(1) $|G| = |A_{116}| = 2^{111} \cdot 3^{55} \cdot 5^{27} \cdot 7^{18} \cdot 11^{10} \cdot 13^{8} \cdot 17^{6} \cdot 19^{6} \cdot 23^{5} \cdot 29^{4} \cdot 31^{3} \cdot 37^{3} \cdot 41^{2} \cdot 43^{2} \cdot 47^{2} \cdot 53^{2} \cdot 59 \cdot 61 \cdot 67 \cdot 71 \cdot 73 \cdot 79 \cdot 83 \cdot 89 \cdot 97 \cdot 101 \cdot 103 \cdot 107 \cdot 109 \cdot 113$;

(2) $D(G) = D(A_{116}) = (28, 29, 28, 28, 26, 26, 24, 24, 23, 22, 22, 21, 20, 20, 18, 17, 16, 16, 5, 14, 14, 12, 11, 9, 8, 6, 6, 4, 4, 1)$.

We have to show that $G \cong A_{116}$. By these hypotheses, we conclude that $\{2, p\} \cup \{pq \mid p+q \leq 116\} \cup \{2p \mid p+4 \leq 116\} \subseteq \pi_e(G)$ and $(\{2p \mid p+4 > 116\} \cup \{pq \mid p+q > 116\}) \cap \pi_e(G) = \varnothing$, where $2 \neq p, q \in \pi(G)$. Obviously, the prime graph of G is connected since $\deg(3) = 29$ and $|\pi(G)| = 30$. Moreover, it is easy to check that $\Gamma(G) = \Gamma(A_{116})$ by the structure of $D(G)$. For convenience, we break up the proof into a sequence of lemmas.

Lemma 5.1 *Let K be the maximal normal solvable subgroup of G. Then K is a $\{2, 3\}$-group. In particular, G is nonsolvable.*

Proof We first prove that K is a $113'$-group. Indeed, if not, then K would contain an element x of order 113. Set $C = C_G(x)$ and $N = N_G(\langle x \rangle)$. By the structure of $D(G)$, it follows that C is a $\{3, 113\}$-group. Using the n-c theorem, the factor group N/C is isomorphic to a subgroup of $\mathrm{Aut}(\langle x \rangle) \cong \mathbb{Z}_{16} \times \mathbb{Z}_7$. Hence, $N_G(\langle x \rangle)$ is a $\{2, 3, 7, 113\}$-group. By the Frattini argument, we have that $G = KN_G(\langle x \rangle)$. This implies that $r \in \pi(K)$ for each $r \in \pi(G) \backslash \{2, 3, 7, 113\}$, and for example, 107 divides the order of K. Since K is solvable, it possesses a Hall $\{107, 113\}$-subgroup H , which is a nilpotent subgroup of order $107 \cdot 113$. Hence $107 \sim 113$ and $\deg(113) \geq 2$, a contradiction.

Next, we prove that K is a q'-group for each $q \in \pi(G) \backslash \{2, 3, 113\}$. Let $q \in \pi(K)$, $Q \in \mathrm{Syl}_q(K)$ and $N = N_G(Q)$. Again, by the Frattini argument, $G = KN_G(Q)$, and hence 113 divides the order of N. Let T be a subgroup of N of order 113. Since T normalizes Q, by the n-c theorem, we have that $N_G(Q)/C_G(Q) \lesssim \mathrm{Aut}(Q)$. It is easy to check that 113 divides the order of $\mathrm{Aut}(Q)$ if and only if $q = 7$. Thus, if $113 \nmid |\mathrm{Aut}(Q)|$, then $T \leq C_G(Q)$. In this case, $113q \in \pi_e(G)$, so $\deg(113) \geq 2$, a contradiction. On the other hand, $q = 7$ and $113 \mid |\mathrm{Aut}(Q)|$, where $Q \in \mathrm{Syl}_7(K)$. Since $\mathrm{Exp}(|G|, 7) = 18$, hence $|N_G(Q)/C_G(Q)| \mid \prod\limits_{i=1}^{18} 7^{153} \cdot (7^i - 1)$. It is easy to check that $67 \nmid \prod\limits_{i=1}^{18} 7^{153} \cdot (7^i - 1)$. If $67 \mid |N_G(Q)|$, then $67 \in \pi(C_G(Q))$. Set $C = C_G(Q)$ and $K_{67} \in \mathrm{Syl}_{67}(C_G(Q))$. By hypothesis, we have $\mathrm{Exp}(|G|, 67) = 1$. Again, by the Frattini argument, $N = CN_N(K_{67})$. This implies that $p \nmid |N_N(K_{67})|$. Thus $113 \mid |C_G(Q)|$, and so

$\deg(113) \geq 2$, a contradiction. Therefore $67 \nmid |N_G(Q)|$ and $67 \in \pi(K)$. Set $P_{67} \in \mathrm{Syl}_{67}(K)$. Since $G = KN_G(P_{67})$, then $113 \mid |N_G(P_{67})|$, a contradiction. Hence, K is a $\{2,3\}$-group. Since $K \neq G$, it follows at once that G is nonsolvable. This completes the proof of Lemma 5.1.

Lemma 5.2 *The quotient group G/K is an almost simple group. In fact, $S \lesssim G/K \lesssim \mathrm{Aut}(S)$, where S is a non-abelian simple group.*

Proof Let $\overline{G} := G/K$ and $S := \mathrm{Soc}(\overline{G})$. Then $S = B_1 \times B_2 \times \cdots \times B_m$, where B_i ($1 \leq i \leq m$) are non-abelian simple groups and $S \lesssim \overline{G} \lesssim \mathrm{Aut}(S)$. In what follows, we will show that $m = 1$.

Suppose that $m \geq 2$. We assert that 113 does not divide the order of S. Otherwise $2 \sim 113$, which is impossible for $\Gamma(G) = \Gamma(A_{116})$. Hence, for every i we have $B_i \in \mathcal{F}_p$, where p is a prime and $p < 113$. On the other hand, by Lemma 3.8, we observe that $113 \in \pi(\overline{G}) \subseteq \pi(\mathrm{Aut}(S))$. Thus 113 divides the order of $\mathrm{Out}(S)$. But

$$\mathrm{Out}(S) = \mathrm{Out}(S_1) \times \mathrm{Out}(S_2) \times \cdots \times \mathrm{Out}(S_r),$$

where the groups S_j ($j = 1, 2, \cdots, r$) are direct products of all isomorphic $B_i's$ such that

$$S = S_1 \times S_2 \times \cdots \times S_r.$$

Therefore for some j, 113 divides the order of an outer automorphism group of a direct product S_j of t isomorphic simple groups B_i for some $1 \leq i \leq m$. Since $B_i \in \mathcal{F}_p$, it follows that $|\mathrm{Out}(B_i)|$ is not divided by 113 from Table 2. Now, by Lemma 3.9, we obtain that $|\mathrm{Aut}(S_j)| = |\mathrm{Aut}(B_i)|^t \cdot t!$. Therefore $t \geq 113$ and so 2^{226} divides the order of G. However, $\mathrm{Exp}(|A_{116}|, 2) = \mathrm{Exp}(|G|, 2) = 111 < 226$ by Lemma 3.4 (4), a contradiction. Thus $m = 1$ and $S = B_1$. This completes the proof of Lemma 5.2.

Lemma 5.3 *G is isomorphic to the alternating group A_{116}.*

Proof By Lemmas 3.7 and 5.1, we may assume that

$$|S| = 2^a \cdot 3^b \cdot 5^{27} \cdot 7^{18} \cdot 11^{10} \cdot 13^8 \cdot 17^6 \cdot 19^6 \cdot 23^5 \cdot 29^4 \cdot 31^3 \cdot 37^3 \cdot 41^2 \cdot 43^2 \cdot 47^2 \cdot 53^2 \cdot 59 \cdot 61 \cdot 67 \cdot 71 \cdot 73 \cdot 79 \cdot 83 \cdot 89 \cdot 97 \cdot 101 \cdot 103 \cdot 107 \cdot 109 \cdot 113,$$

where $2 \leq a \leq 111,\ 1 \leq b \leq 55$. Using Tables 1–3 in [17], S can only be isomorphic to one of the simple groups: A_{113}, A_{114}, A_{115}, A_{116}, A_{117}, A_{118}, A_{119}, A_{120}, A_{121}, A_{122}, A_{123}, A_{124}, A_{125} and A_{126}.

If $S \cong A_{113}$, then $A_{113} \lesssim G/K \lesssim \mathrm{Aut}(A_{113}) \cong S_{113}$, and so it follows that $G/K \cong S_{113}$ or A_{113}. In the case $G/K \cong S_{113}$, it is easy to see that $3 \cdot 113 \in \pi_e(\overline{G}) \backslash \pi_e(S_{113})$, a contradiction. In the latter case, $G/K \cong A_{113}$ by comparing orders, we deduce that $5 \mid |K|$, a contradiction to Lemma 5.1.

Similarly, we see that S can not be isomorphic to the alternating groups A_{114} and A_{115}. On the other hand, since $\mathrm{Exp}(|A_i|, 13) = 9$, where $i = 117, 118, \cdots, 126$, but $\mathrm{Exp}(|A_{116}|, 13) = 8$, S can not be isomorphic to one of the alternating groups: A_{117}, A_{118}, A_{119}, A_{120}, A_{121},

A_{122}, A_{123}, A_{124}, A_{125} and A_{126}. Therefore, $S \cong A_{116}$. According to Lemma 5.2, we have that $A_{116} \lesssim G/K \lesssim \mathrm{Aut}(A_{116}) \cong S_{116}$. By comparing orders we see that G/K can only be isomorphic to A_{116}. Hence, we obtain that $K = 1$ and $G \cong A_{116}$, This completes the proof of the lemma, which concludes the theorem.

Theorem 5.2 *The alternating group A_{134} is OD-characterizable.*

Proof Let G be a finite group satisfying

$$|G| = |A_{134}| = 2^{130} \cdot 3^{63} \cdot 5^{32} \cdot 7^{21} \cdot 11^{12} \cdot 13^{10} \cdot 17^{7} \cdot 19^{7} \cdot 23^{5} \cdot 29^{4} \cdot 31^{4} \cdot 37^{3} \cdot 41^{3} \cdot 43^{3} \cdot 47^{2} \cdot 53^{2} \cdot 59^{2} \cdot 61^{2} \cdot 67^{2} \cdot 71 \cdot 73 \cdot 79 \cdot 83 \cdot 89 \cdot 97 \cdot 101 \cdot 103 \cdot 107 \cdot 109 \cdot 113 \cdot 127 \cdot 131$$

and

$$D(G) = D(A_{134}) = (30, 31, 30, 30, 29, 29, 29, 29, 28, 26, 26, 24, 23, 23, 22, 21, 20, 20, 18, 18, 18, 16, 15, 14, 10, 10, 10, 9, 9, 8, 4, 1).$$

Clearly, the prime graph of G is connected since $\deg(3) = 28$ and $|\pi(G)| = 29$. Furthermore, it is easy to check that $\Gamma(G) = \Gamma(A_{134})$ by the structure of $D(G)$.

Let K denote the maximal normal solvable subgroup of G. For the same reason as in the proof of Theorem 5.1, K is a $\{2, 3\}$-group and $A_{134} \lesssim G/K \lesssim \mathrm{Aut}(A_{134}) \cong S_{134}$. Hence $G/K \cong A_{134}$ or S_{134}. In the case that $G/K \cong A_{134}$, by considering orders, we deduce that $K = 1$ and $G \cong A_{134}$, and the desired conclusion follows in this case. In the latter case, we see that $2^{131} \mid |G|$, a contradiction. We omit the detailed processes for A_{134}, since the method used is quite similar to that for A_{116}. Hence, A_{134} is OD-characterizable and the proof of the theorem and also the proof of Theorem 2.2 are complete.

In 1989, Shi [19] put forward the following conjecture.

Conjecture 5.2 (cf. [19]) Let G be a group and M a finite simple group. Then $G \cong M$ if and only if (1) $|G| = |M|$ and (2) $\pi_e(G) = \pi_e(M)$.

The above Conjecture 5.2 was proved by joint works of many mathematicians, and the last part of the proof was given by Mazurov etc. in [20]. That is, the following theorem holds.

Theorem 5.3 (cf. [20]) *Let G be a group and M a finite simple group. Then $G \cong M$ if and only if* (1) $|G| = |M|$ *and* (2) $\pi_e(G) = \pi_e(M)$.

About the relation of Conjecture 5.2 and OD-characterizable groups, we have the following facts: For two finite groups G and M, if $\pi_e(G) = \pi_e(M)$, then G and M must have the same prime graph. Hence they have the same degree pattern. Therefore, we can have the following Corollary 5.1 by Theorem 2.2.

Corollary 5.1 *If G is a finite group such that* (1) $|G| = |A_{p+3}|$ *and* (2) $\pi_e(G) = \pi_e(A_{p+3})$, *where $7 \neq p \in \pi(136!)$, then $G \cong A_{p+3}$.*

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