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Some Subsystems of a Lie Triple System Closely Related to Its Frattini Subsystem*

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Abstract The main purpose of the present paper is to give some properties of the Jacobson radical, the Frattini subsystem and c-ideals of a Lie triple system. Some further results concerning the Frattini subsystems of nilpotent and solvable Lie triple systems are obtained. Moreover, we develop initially c-ideals for a Lie triple system and make use of them to give some characterizations of a solvable Lie triple system.

Keywords Frattini subsystem, Jacobson radical, c-ideals, Solvable, Nilpotent
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1 Introduction

The theory of Frattini subgroup of a group was well developed and is useful in the study of certain types of problems in the group theory. The corresponding concept for algebras has been widely recognized and is of independent interest (see [1–2, 4–7, 13, 15–19, 22]). It is well known that a Lie algebra can become a Lie triple system in a natural way whereas a Lie triple system can be imbedded into a Lie algebra. Since Lie triple systems are intimately connected with Lie algebras, it seems desirable to investigate the possibility of establishing a parallel theory for Lie triple systems. The Frattini subsystem of a Lie triple system was defined and some elementary properties were investigated (see [22]).

The main purpose of this present paper is to give some properties of the Jacobson radical, the Frattini subsystem and c-ideals of Lie triple systems. Basic definitions on Lie triple systems are collected in Section 1. Section 2 is devoted to studying the Jacobson radical of a Lie triple system T and showing that $J(T) = [R(T), T, T] \subseteq N(T)$, which generalizes some results in [22]. In Section 3, some further results concerning the Frattini subsystems of nilpotent and solvable Lie triple systems are obtained.

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A subsystem B of a Lie triple system T is called a c-ideal of T if there is an ideal C of T such that $T = B + C$ and $B \cap C \subseteq B_T$, where B_T is the largest ideal of T contained in B. This is analogous to the concept of a c-normal subgroup, which has been studied by a number of authors (see [11–12]). Towers gave some properties of c-ideals and made use of them to give some properties of solvable and super-solvable Lie algebras (see [20–21]). The purpose of this paper is to study the corresponding idea for Lie triple systems. In dealing with c-ideals of Lie triple systems, however, we can not always employ the methods used in Lie algebras. This is because the product in a Lie triple system is ternary but not binary. By c-ideals of a Lie triple system, we obtain some sufficient conditions for solvable Lie triple systems and some further results concerning c-ideals in Section 4.

Some notations used in this paper are as follows: T is the category of finite-dimensional Lie triple systems over a field $\mathbf{F}$. A proper subsystem M of T (with $\dim T > 1$) is called a maximal subsystem of T if the only subsystem properly containing M is T itself. The Frattini subsystem $F(T)$ of T is the intersection of all maximal subsystems of T. The maximal ideal of T contained in $F(T)$ is denoted by $\phi(T)$. The Jacobson radical $J(T)$ of T is the intersection of all the maximal ideals of T. Our notations and terminologies are standard as may be found in [8, 14, 22].

2 Basic Definitions

We begin by reviewing some definitions, notations and facts which can be found in [3, 14, 22–23].

Definition 2.1 *A Lie triple system* (*LTS*) T *is a vector space with a ternary product* $[\cdot,\cdot,\cdot]$ *satisfying the following identities:*

(1) $[x, y, z] = -[y, x, z]$;

(2) $[x, y, z] + [y, z, x] + [z, x, y] = 0$;

(3) $[u, v, [x, y, z]] = [[u, v, x], y, z] + [x, [u, v, y], z] + [x, y, [u, v, z]]$

for any $x, y, z, u, v \in T$.

Definition 2.2 *For* $a, b, x \in T$, *define a mapping* $L(a, b) : T \to T$ *by* $L(a, b)(x) = [a, b, x]$. *Then Definition* 2.1(3) *becomes, writing* $D = L(a, b)$,

$$D([x, y, z]) = [D(x), y, z] + [x, D(y), z] + [x, y, D(z)], \quad \forall x, y, z \in T.$$

Any linear endomorphism D *of* T *satisfying the above property will be called a derivation of* T. $\mathrm{Der}(T)$ *denotes the set of all derivations of* T. *We can prove that* $\mathrm{Der}(T)$ *is a Lie algebra. As a special case, the derivation* $L(a, b)$ *will be called an inner derivation.*

Define $\mathrm{InnDer}(T) = \{D \mid D = \Sigma L(x, y)\}$ for any $x, y \in T$, and then $\mathrm{InnDer}(T)$ is a subalgebra of $\mathrm{Der}(T)$. $\mathrm{InnDer}(T)$ is called the inner derivation algebra of T.

For a Lie triple system T, set $L_s(T)$ to be the vector space $L_s(T) = T \oplus \mathrm{InnDer}(T)$. The product in $L_s(T)$ is

$$[x_1 + D_1, x_2 + D_2] = L(x_1, x_2) + [D_1, D_2] + D_1(x_2) - D_2(x_1)$$

for any $x_1, x_2 \in T$, $D_1, D_2 \in \mathrm{InnDer}(T)$. It is easy to prove that $L_s(T)$ together with the above bracket is a Lie algebra which is called the standard imbedding Lie algebra of T.

Note that for the ternary composition in T and the binary bracket in $L_s(T)$, we have $[x, y, z] = [[x, y], z]$ for $x, y, z \in T$. For $x \in L(T)$, as usual, $\mathrm{Ad}x \in \mathrm{End}L(T)$ is defined by $\mathrm{Ad}x(y) := [x, y]$. Clearly $\mathrm{Ad}x \in \mathrm{Der}L_s(T)$, the derivation algebra of $L_s(T)$. For $x, y \in T$, we have $\mathrm{Ad}x(y) \in L(T, T)$, $(\mathrm{Ad}x)^2(y) \in T$.

Definition 2.3 *Let T be an LTS, $T^{(1)} = T$ and $T^{(k+1)} = [T, T^{(k)}, T^{(k)}]$. T is called solvable if there is some positive integer k such that $T^{(k)} = \{0\}$.*

Definition 2.4 *Let T be an LTS, $T^1 := T$ and $T^{n+1} = [T^n, T, T]$ for $n \geq 1$. T is called nilpotent if $T^n = 0$ for some n.*

3 The Jacobson Radical of a Lie Triple System

Lemma 3.1 (see [22]) *Let T be an LTS. Then the following statements hold:*

(1) *If B is a subsystem of T such that*

$$B + F(T) = T \ (or B + \phi(T) = T),$$

then $B = T$.

(2) *$F(T) \subseteq T^2$ and $J(T) \subseteq T^2$.*

(3) *Let T be solvable. Then $F(T)$ is an ideal of T.*

(4) *If B is an ideal of T, then there is a proper subsystem C of T such that $T = B + C$ if and only if $B \nsubseteq F(T)$.*

(5) *Let C be a subsystem of T and B an ideal of T. If $B \subseteq F(C)$ (respectively, $B \subseteq \phi(C)$), then $B \subseteq F(T)$ (respectively, $B \subseteq \phi(T)$).*

We introduce a class Θ: An LTS T belongs to the class Θ if $[T/N, T/N, T/N] \subseteq T/N$ and $[T/N, T/N, T/N] \neq T/N$ for every proper ideal N of T. Clearly, a solvable LTS belongs to the class Θ.

Lemma 3.2 *For an arbitrary ideal K of T belonging to the class Θ, $T^2 \cap K \subseteq J(T)$.*

Proof We may assume that $K \neq \{0\}$. Suppose that there exists a maximal ideal I of T such that $T^2 \cap K \nsubseteq I$. Then $T^2 \nsubseteq I$ and $K \nsubseteq I$. Thus $T = I + K$. We see that $T/I = (I + K)/I \cong K/K \cap I$. Since $K \in \Theta$, we have

$$[K/K \cap I, K/K \cap I, K/K \cap I] \subseteq K/K \cap I$$

and therefore $[T/I, T/I, T/I] \subseteq T/I$. Then $[T/I, T/I, T/I] = \{0\}$ since I is a maximal ideal of T. Hence we have $T^2 \subseteq I$, which is a contradiction. The result follows.

Theorem 3.1 *If $T \in \Theta$ or $T^2 \in \Theta$, then $J(T) = T^2$. In particular, if T is solvable, then $J(T) = T^2$.*

Proof It follows from Lemma 3.1(2) and Lemma 3.2.

Theorem 3.2 *Let T be an LTS over a field of characteristic zero. Then $J(T) = [R(T), T, T] \subseteq N(T)$.*

Proof By Theorem 2.21 in [14], T has the Levi decomposition $T = R(T) + S$, where $R(T)$ is the solvable radical of T and S is a semisimple subsystem of T. Then

$$\begin{aligned}T^2 = [T,T,T] &= [R(T)+S, R(T)+S, R(T)+S]\\ &= [S,S,S] + [R(T),T,T]\\ &= S + [R(T),T,T].\end{aligned}$$

By means of Theorem 4.4 in [22], we have $J(T) = T^2 \cap R(T)$. Since $[R(T),T,T] \subseteq R(T)$,

$$J(T) = T^2 \cap R(T) = R(T) \cap S + [R(T),T,T] = [R(T),T,T].$$

Combine Theorem 3.1 in [8] with Corollary 3.8 in [8], we obtain that $N(T) = N(L_s(T)) \cap T$ and $R(T) = R(L_s(T)) \cap T$, where $N(T)$ is the nil-radical of T and $N(L_s(T))$ is the nil-radical of $L_s(T)$. It follows from [9] that

$$L_s(T)^2 \cap R(L_s(T)) = [L_s(T), R(L_s(T))] \subseteq N(L_s(T)).$$

So

$$L_s(T)^2 \cap R(L_s(T)) \cap T = [L_s(T), R(L_s(T))] \cap T \subseteq N(L_s(T)) \cap T,$$

i.e.,

$$T^2 \cap R(T) \subseteq L_s(T)^2 \cap R(T) \subseteq N(T).$$

Hence

$$J(T) = [R(T),T,T] \subseteq N(T).$$

4 The Frattini Subsystem of a Lie Triple System

In this section, all Lie triple systems considered will be finite dimensional over a field $\mathbf{F}$ of characteristic zero.

Theorem 4.1 *Suppose that A is an ideal of T and B is an ideal of A such that $B \subseteq A \cap F(T)$. If A/B is nilpotent, then A is nilpotent.*

Proof Take any element a of A. Let $D = (\mathrm{Ad}a)^2$. Then, by Fitting's Lemma in [10, p. 113], $T = T_0 + T_1$ is the Fitting decomposition relative to D, where $D|_{T_0}$ is nilpotent and $D|_{T_1}$ is an isomorphism of T_1. So $T_1 \subseteq A$ since A is an ideal of T. Since A/B is nilpotent, there exists an integer n such that $T_1 = D^n(T_1) \subseteq B$. Then $T = T_0 + F(T)$ by Lemma 3.1(1) implies that $T = T_0$. Hence $D|_A$ is nilpotent. Therefore A is nilpotent by Engle's Theorem in [8].

Corollary 4.1 *If A is an ideal of T such that $A \subseteq F(T)$, then*

$$N(T/A) = N(T)/A,$$

where $N(T)$ is the nil-radical of T.

Proof It is a straightforward result of Theorem 4.1.

Theorem 4.2 *Let T be an LTS. The following statements are equivalent:*

(1) *T is nilpotent.*

(2) *If A is an ideal of T such that $A \subseteq \phi(T)$, then T/A is nilpotent.*

(3) *All maximal subsystems are ideals.*

(4) $F(T) = \phi(T) = J(T) = T^2$.

Proof (1) $\Leftrightarrow$ (2). It is clear by Theorem 4.1.

(1) $\Leftrightarrow$ (3) and (4) $\Rightarrow$ (1). They are clear by Theorem 3.8 in [22].

(1) $\Rightarrow$ (4). Let T be nilpotent. Combine Theorem 4.2 in [22] with Theorem 4.4 in [22], we obtain $F(T) = \phi(T) = J(T) = T^2$.

Theorem 4.3 *Let A be an ideal of T. If U/A is a maximal nilpotent subsystem of T/A, then $U = C + A$, where C is a maximal nilpotent subsystem of T.*

Proof If $A \subseteq \phi(U)$, then it follows that $U/\phi(U)$ is nilpotent, whence U is nilpotent by Theorem 4.2 and the result is clear. So suppose that $A \nsubseteq \phi(U)$. Then $U = A + M$ for some maximal subsystem M of U. If we choose B to be minimal with respect to $U = A + B$, then $A \cap B \subseteq \phi(B)$ by Lemma 5.1 in [22]. Also $U/A \cong B/(A \cap B)$ is nilpotent, which yields that B is nilpotent. If we now choose C to be the biggest nilpotent subsystem of U such that $U = A + C$, it is easy to see that C is a maximal nilpotent subsystem of T.

In the following, we will give some necessary and sufficient conditions for solvable Lie triple systems. First, we give a lemma.

Lemma 4.1 *Let T be an LTS. If $F(T)$ contains every proper ideal of T, and $T/\phi(T)$ is not abelian, then $\phi(T) = N(T)$.*

Proof If T is nilpotent, then $F(T) = \phi(T)$ by means of Lemma 3.1(3). Since $F(T)$ contains all ideals of T, $\phi(T)$ is the largest ideal of T. Then $T/\phi(T)$ has no proper ideals. If $T/\phi(T)$ is not abelian, then there exists $k \in \mathbb{N}$ such that $(T/\phi(T))^{(k)} \neq \{0\}$ is an abelian ideal of $T/\phi(T)$ since $T/\phi(T)$ is nilpotent. So we have a contradiction and T can not be nilpotent. Then $N(T)$ is a proper ideal of T. Hence $\phi(T) \supseteq N(T)$ since $\phi(T)$ is the largest ideal of T. The result follows by means of Theorem 4.1.

Theorem 4.4 *Let T be an LTS. If $\phi(S) \neq N(S)$ for any subsystem S of T, then the following statements are equivalent:*

(1) *T is solvable.*

(2) *$T/\phi(T)$ is solvable.*

(3) *If S is a nonzero subsystem of T and $F(S)$ contains all ideals of S, then $S/\phi(S)$ is abelian.*

Proof (1) $\Leftrightarrow$ (2). By means of Theorem 4.1, $\phi(T)$ is nilpotent. So T is solvable. The converse is clear.

(1) $\Rightarrow$ (3). Let S be a nonzero subsystem of T and let $F(S)$ contain all ideals of S. Suppose that $S/\phi(S)$ is not abelian, then $\phi(S) = N(S)$ by Lemma 4.1. This is a contradiction by the assumption. So $S/\phi(S)$ is abelian.

(3) $\Rightarrow$ (1). Suppose that T is not solvable. Let S be a minimal subsystem of T such that S is not solvable. So every proper subsystem of S is solvable. Assume that there exists a proper ideal A of S and A is not contained in $F(S)$. Clearly, A is solvable. Let T_1 be a maximal subsystem of S which does not contain A. T_1 exists since $A \nsubseteq F(S)$. T_1 is solvable and $S = A + T_1$. But

$$S/A = (A + T_1)/A \cong T_1/T_1 \cap A$$

is solvable. Hence S is solvable. This is a contradiction and every proper ideal of S must be contained in $F(S)$. Since $F(S)$ contains all ideals of S, $\phi(S)$ is the largest ideal of S. Then $S/\phi(S)$ has no proper ideals. If $S/\phi(S)$ is not abelian, then there exists $k \in \mathbb{N}$ such that $(S/\phi(S))^{(k)} \neq \{0\}$ is an abelian ideal of $S/\phi(S)$ since $S/\phi(S)$ is solvable, a contradiction. Hence we have that $S/\phi(S)$ is abelian. By means of Theorem 4.1, $\phi(S)$ is nilpotent, so S is solvable since $S/\phi(S)$ is abelian, again a contradiction. Therefore, T must be solvable.

5 The c-ideals of a Lie Triple System

Lemma 5.1 *Let T be an LTS over any field.*

(1) *If B is a c-ideal of T and $B \subseteq K \subseteq T$, then B is a c-ideal of K.*

(2) *If I is an ideal of T and $I \subseteq B$, then B is a c-ideal of T if and only if B/I is c-ideal of T/I.*

Proof Their proofs are similar to [21].

Theorem 5.1 *Let T be an LTS over a field of characteristic zero. Then all maximal subsystems of T are c-ideals of T if and only if T is solvable.*

Proof Let T be a non-solvable LTS of the smallest dimension in which maximal subsystems are c-ideals of T. Then all proper factor algebras of T are solvable by Lemma 5.1(2). Suppose first that T is simple. Let M be a maximal subsystem of T. Then M is a c-ideal, so there exists an ideal C of T such that $T = M + C$ and $M \cap C \subseteq M_T = \{0\}$ since T is simple. This yields that C is a non-trivial proper ideal of T, a contradiction. If T has two minimal ideals B_1 and B_2, then T/B_1 and T/B_2 are solvable. So there is k such that

$$(T/B_1)^{(k)} = (T/B_2)^{(k)} = \{0\},$$

i.e.,

$$T^{(k)} \subseteq B_1 \cap B_2 = \{0\},$$

and so T is solvable. Hence T has a unique minimal ideal B and T/B is solvable.

Suppose that there exists an element $b \in B$ such that $(\mathrm{Ad}b)^2$ is not nilpotent. Let $T = T_0 \oplus T_1$ be the Fitting decomposition in [10] relative to $(\mathrm{Ad}b)^2$, where $(\mathrm{Ad}b)^2|_{T_0}$ is nilpotent in T_0 and $(\mathrm{Ad}b)^2|_{T_1}$ is an isomorphism of T_1. Then $T \neq T_0$, so let M be a maximal subsystem of T containing T_0. As M is a c-ideal, there is an ideal C of T such that $T = M + C$ and $M \cap C \subseteq M_T$. Now $T_1 \subseteq B$, so $B \not\subseteq M_T$. It follows that $M_T = \{0\}$, whence $M = T_0$ and $B = C = T_1$. But $b \in M \cap B = \{0\}$. Hence $(\mathrm{Ad}b)^2$ is nilpotent for every $b \in B$, yielding that B is nilpotent by the Engle's Theorem in [8] and so T is solvable, a contradiction.

Now suppose that T is solvable and let M be a maximal subsystem of T. Then there is an integer $k \geq 2$ such that $T^{(k)} \subseteq M$, but $T^{(k-1)} \not\subseteq M$. We have that $T^{(k-1)}$ is an ideal of T, $T = M + T^{(k-1)}$ and $M \cap T^{(k-1)} \subseteq M_T$, so M is a c-ideal of T.

The following two theorems contain analogous results to the corresponding ones for Lie algebras (see [21]), and their proofs are similar.

Theorem 5.2 *Let T be an LTS over a field of characteristic zero. Then T has a solvable maximal subsystem that is a c-ideal of T if and only if T is solvable.*

Theorem 5.3 *Let T be an LTS over any field, such that all maximal nilpotent subsystems of T are c-ideals of T. Then T is solvable.*

Proposition 5.1 *Let T be an LTS over any field $\mathbf{F}$ and let B, C be subsystems of T with $B \subseteq \phi(C)$. If B is a c-ideal in T, then B is an ideal of T and $B \subseteq \phi(T)$.*

Proof Suppose that $T = B + K$ and $B \cap K \subseteq B_T$. Then

$$C = C \cap T = C \cap (B + K) = B + C \cap K \subseteq \phi(C) + C \cap K \subseteq C$$

since $B \subseteq \phi(C)$. So

$$C = \phi(C) + K \cap C.$$

It follows from Lemma 3.1(1) that $C = K \cap C$. Hence $B \subseteq C \subseteq K$, given $B = B \cap K \subseteq B_T$ and B is an ideal of T. So $B \subseteq \phi(T)$ by Lemma 3.1(5).

The Lie triple system T is called elementary if $\phi(B) = \{0\}$ for every subsystem B of T; T is called an E-LTS if $\phi(B) \subseteq \phi(T)$ for all subsystems B of T. Then we have the following theorem.

Theorem 5.4 *If every subsystem B of T is a c-ideal in T, then T is an E-LTS.*

Proof Simply put $B = \phi(C)$ in Proposition 5.1.

A subsystem B of T is c-supplemented in T if there exists a subsystem C of T with $T = B+C$ and $B \cap C \subseteq B_T$, where B_T is the largest ideal of T contained in B. We say that T is c-supplemented if every subsystem of T is c-supplemented in T. An LTS T is called completely factorisable if for every subsystem B of T there exists a subsystem C such that $T = B + C$ and $B \cap C = \{0\}$; T is called ϕ-free if $\phi(T) = \{0\}$; T is called elementary if $\phi(B) = \{0\}$ for every subsystem B of T; T is called completely factorisable if for every subsystem B of T there exists a subsystem C such that $T = B + C$ and $B \cap C = \{0\}$.

Theorem 5.5 *Let T be an LTS. Then the following are equivalent:*

(1) *Every subsystem B of T is a c-ideal in T.*

(2) *$T/\phi(T)$ is completely factorisable and every subsystem of $\phi(T)$ is an ideal of T.*

Proof (1) $\Rightarrow$ (2). Suppose first that T is ϕ-free and c-ideal, and let B be a subsystem of T. Then there exists a subsystem C of T such that $T = B + C$. Choose D to be a subsystem of T which is minimal with respect to $T = B + D$. It follows from Lemma 5.1 in [22] that $B \cap D \subseteq \phi(D)$. Since T is elementary by Theorem 5.2, $B \cap D = \{0\}$. Hence T is completely factorisable, and (2) follows from Lemma 5.1(2) and Proposition 5.1.

(2) $\Rightarrow$ (1). Suppose that (2) holds and let B be a subsystem of T. Then there exists a subsystem $C/\phi(T)$ of $T/\phi(T)$ such that

$$T/\phi(T) = ((B + \phi(T))/\phi(T)) + (C/\phi(T))$$

and

$$\{0\} = ((B + \phi(T))/\phi(T)) \cap (C/\phi(T)) = (B \cap C + \phi(T))/\phi(T).$$

Hence $T = B + C$ and $B \cap C \subseteq \phi(T)$. Thus $B \cap C$ is an ideal of T and $B \cap C \subseteq B_T$; that is, T is c-supplemented.

By Theorem 5.4, if T is c-ideal, then T is an E-LTS. Clearly, every elementary LTS is an E-LTS. In the following, we will give some properties of an E-LTS.

Lemma 5.2 *Let T, G be two LTSs over $\mathbf{F}$. If f is a surjective homomorphism from T to G, then $f(F(T)) \subseteq F(G)$.*

Proof Let N be a maximal subsystem of G. Then $M = f^{-1}(N)$ is a maximal subsystem of T, i.e., $f(M) = N$. If M is a subsystem of T, then N is a subsystem of G. Hence $f(x) \in f(M) = N$ for every $x \in F(T)$. Since this is valid for all maximal subsystems of G, we have $f(F(T)) \subseteq F(G)$.

Lemma 5.3 *Let A be an ideal of T and B a subsystem of T which is minimal with respect to $T = A + B$. Then*

$$\phi(T/A) \cong (A + \phi(B))/A.$$

Proof In the light of Lemma 5.1 in [22], we obtain $A \cap B \subseteq \phi(B)$ and $A \cap B \supseteq \phi(B) \cap A$. So $A \cap B = \phi(B) \cap A$. Clearly, $A \cap B$ is an ideal of B. Since $T/A = (A + B)/A \cong B/(A \cap B)$, we have

$$\phi(T/A) \cong \phi(B/(A \cap B)).$$

And since $A \cap B \subseteq \phi(B)$, it follows from Lemma 3.1(5) that

$$\phi(B/(A \cap B)) \cong \phi(B)/(A \cap B) = \phi(B)/(A \cap \phi(B)) \cong (A + \phi(B))/A.$$

Hence $\phi(T/A) \cong (A + \phi(B))/A$.

Theorem 5.6 *Let T be an LTS over $\mathbf{F}$. If T is a solvable E-LTS such that $f : T \to T/\mathrm{Ker}f$ is a surjective homomorphism, then $f(\phi(T)) = \phi(f(T))$.*

Proof It follows from Lemma 3.1(3) that $F(T) = \phi(T)$ since T is solvable. By virtue of Lemma 5.2, $f(\phi(T))$ is always contained in $\phi(f(T))$. It is clear that $\mathrm{Ker}f$ is an ideal of T. If $\mathrm{Ker}f \subseteq \phi(T)$, then

$$\phi(f(T)) = \phi(T/\mathrm{Ker}f) = \phi(T)/\mathrm{Ker}f = f(\phi(T)).$$

If $\mathrm{Ker}f \not\subseteq \phi(T)$, then there is a system K which is minimal with respect to $T = K + \mathrm{Ker}f$. It then follows from Lemma 5.3 that

$$\phi(f(T)) = \phi(T/\mathrm{Ker}f) \cong (\mathrm{Ker}f + \phi(K))/\mathrm{Ker}f = f(\mathrm{Ker}f + \phi(K)) = f(\phi(K)).$$

Since T is an E-LTS, we have $\phi(K) \subseteq \phi(T)$. Then $f(\phi(K)) \subseteq f(\phi(T))$. Thus

$$f(\phi(T)) = \phi(f(T)).$$

Theorem 5.7 *Let T be an LTS over $\mathbf{F}$. Then T is an E-LTS if and only if $T/\phi(T)$ is elementary.*

Proof ($\Rightarrow$) Suppose that T is an E-LTS, and let $S/\phi(T)$ be a subsystem of $T/\phi(T)$. Choose a subsystem U of T which is minimal with respect to $\phi(T) + U = S$ (U could be equal to S). Let N be an ideal of S such that $N/\phi(T) = \phi(S/\phi(T))$. If $N \subset \phi(T)$, then $\phi(S/\phi(T)) = \{0\}$. So $T/\phi(T)$ is elementary. If $N = \phi(T)$, then it is clear that $T/\phi(T)$ is elementary.

We claim that $N = \phi(T)$. Suppose that $N \supset \phi(T)$.

Since $N \supseteq \phi(T) + N \cap U$ and $\dim(\phi(T) + N \cap U) \geq \dim N$, we have $N = \phi(T) + N \cap U$. Then

$$N = N \cap S = N \cap (\phi(T) + U) = \phi(T) + N \cap U.$$

If $N \cap U \subseteq \phi(T)$, then

$$N = \phi(T) + N \cap U = \phi(T),$$

a contradiction. So $N \cap U \not\subseteq \phi(T)$. It follows that $N \cap U \not\subseteq \phi(U)$, since T is an E-LTS. But $N \cap U$ is an ideal of U, so $N \cap U \not\subseteq F(U)$. Hence there is a maximal subsystem M of U such that $N \cap U \not\subseteq M$ and $U = M + N \cap U$.

By the minimality of U we must have $\phi(T) + M \neq S$. We claim that $\phi(T) + M$ is a maximal subsystem of S. Suppose that $\phi(T) + M \subset J \subset S$. Then $M \subseteq J \cap U \subseteq U$ and so, by the maximality of M, either $J \cap U = M$ or $J \cap U = U$.

Since $J \supseteq \phi(T) + J \cap U$ and $\dim(\phi(T) + J \cap U) \geq \dim J$, we have $J = \phi(T) + J \cap U$.

Furthermore, since $J \supseteq J \cap (\phi(T) + U)$ and $\dim(J \cap (\phi(T) + U)) = \dim J$, we have $J = J \cap (\phi(T) + U)$. So

$$J = J \cap (\phi(T) + U) = \phi(T) + J \cap U.$$

Then $J \cap U = M$ implies that

$$\phi(T) + M = \phi(T) + J \cap U = J \cap (\phi(T) + U) = J \cap S = J,$$

a contradiction. $J \cap U = U$ gives $U \subseteq J$ and hence $J \supseteq \phi(T) + U = S$, also a contradiction. Hence $\phi(T) + M$ is a maximal subsystem of S. Thus

$$(\phi(T) + M)/\phi(T) \supseteq \phi(S/\phi(T)) = N/\phi(T),$$

and so $N \subseteq \phi(T) + M$. But now $N \cap U \subseteq N \subseteq \phi(T) + M$ and so

$$S = \phi(T) + U = \phi(T) + M + N \cap U = \phi(T) + M$$

since M is a maximal subsystem of U such that $U = M + N \cap U$, contradicting the minimality of U. We conclude that $N = \phi(T)$, and hence $\phi(S/\phi(T)) = \{0\}$ and $T/\phi(T)$ is an elementary LTS.

($\Leftarrow$) Suppose that $T/\phi(T)$ is an elementary LTS and let S be a subsystem of T. It follows from Lemma 3.1(5) that

$$(\phi(S) + \phi(T))/\phi(T) \subseteq \phi((S + \phi(T))/\phi(T)) = \{0\}.$$

So $\phi(S) \subseteq \phi(T)$ for any subsystem S of T, i.e., T is an E-LTS.

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