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Global Well-Posedness and Scattering for the Defocusing $\dot{H}^s$-Critical NLS*

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Abstract The authors consider the scattering phenomena of the defocusing $\dot{H}^s$-critical NLS. It is shown that if a solution of the defocusing NLS remains bounded in the critical homogeneous Sobolev norm on its maximal interval of existence, then the solution is global and scatters.

Keywords Nonlinear Schrödinger equation, Scattering, Global well-posedness
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1 Introduction

We consider the Cauchy problem for the semilinear defocusing Schrödinger equation in $\mathbb{R}^{1+d}$

$$\begin{cases} \mathrm{i}u_t + \Delta u = |u|^\alpha u, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where

$$\frac{4}{d} < \alpha < \begin{cases} \dfrac{4}{d-2}, & \text{if } d \geq 3, \\ \infty, & \text{if } d = 1, 2, \end{cases} \tag{1.2}$$

and $u(t,x)$ is a complex-valued field in spacetime $\mathbb{R} \times \mathbb{R}^d$.

A simple computation shows that the equation is invariant under the scaling

$$u_\lambda = \lambda^{\frac{2}{\alpha}} u(\lambda^2 t, \lambda x), \tag{1.3}$$

and the corresponding scale-invariant Sobolev norm is $\dot{H}^s(\mathbb{R}^d)$ under (1.3), where $s = \frac{d}{2} - \frac{2}{\alpha}$. We will use both notations, s and α, throughout this paper. We restrict the initial data to the $\dot{H}^s(\mathbb{R}^d)$ class. Thus the Cauchy problem (1.1) will be critical at the $\dot{H}^s$ level.

The Cauchy problem for (1.1) was intensively studied (see [2–3, 21]). It is known (see, e.g., [2–3] and also see Theorem 3.3 below) that if the initial data $u_0(x)$ have a finite norm, then the Cauchy problem is locally well-posed, in the sense that there exists a unique solution to (1.1)

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in $C(I,\, \dot{H}^s(\mathbb{R}^d)) \cap L^{\boldsymbol{q}}(I,\, L^{\boldsymbol{r}}(\mathbb{R}^d))$ for some time interval I, where $(\boldsymbol{q},\, \boldsymbol{r})$ denotes some special Lebesgue exponent pair which we will define later (see (1.8) below).

If the initial data are small, then the solution is known to exist globally in time, and scatters to a solution $u_\pm(t)$ to the free Schrödinger equation $(\mathrm{i}\partial_t + \Delta)u_\pm = 0$, in the sense that $\|u(t) - u_\pm(t)\|_{\dot{H}^s(\mathbb{R}^d)} \to 0$ as $t \to \pm\infty$. For (1.1) with general initial data, the arguments in [2–3] do not extend to yield global well-posedness, because the time of existence given by the local theory depends on the profile of the data as well as on the normal of the initial data. The situation is quite similar to the mass-critical and energy-critical Schrödinger equations. The first major step toward verifying global well-posedness is Bourgain's method of "induction on energy" (see [1]), and he obtained the global space-time bound for the defocusing energy-critical NLS in three and four dimensions with spherically symmetric data. Another important breakthrough to non-spherically symmetric initial data was made in [8]. The authors developed the argument of "induction on energy" and introduced the "minimal energy blowup solutions" which are localized in both space and frequency (comparing with the critical solution in our paper). And the remarkable paper [20] first adapted the concentration-compactness-rigidity method to simplify the progress in [8] for the radial energy-critical case. Then there are many papers focusing on these topics (see [9–11, 26, 28] for the mass-critical case, and [25, 30] for the energy-critical case).

Our main result is the following global well-posedness result for (1.1).

Theorem 1.1 *Suppose that the dimension d and the regularity exponent s satisfy*

$$d \geq 6 \tag{1.4}$$

and

$$2s^2 - (d+4)s + d < 0 \tag{1.5}$$

or

$$d = 4,\, 5, \quad \frac{2}{3} < s < 1. \tag{1.6}$$

Suppose that u is a solution to (1.1) *with the initial value $u_0 \in \dot{H}^s(\mathbb{R}^d)$ and the maximal lifespan I. Assume that $\sup\limits_{t\in I}\|u(t)\|_{\dot{H}^s} = A < +\infty$. Then $I = \mathbb{R}$ and u scatters forward and backward.*

Remark 1.1 In [21], the authors proved that if $\sup\limits_{t\in I}\|u(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} = A < +\infty$, then u is global and it scatters. The rigidity part of [21] depends on the following consequences of the Morawetz type identity. Let $u_0 \in H^1(\mathbb{R}^3) \cap \dot{\mathrm{H}}^{\frac{1}{2}}(\mathbb{R}^3)$. Then, for each $0 < T < T_+(u_0)$, we have

$$\int_0^T \int \frac{|u(x)|^4}{|x|}\mathrm{d}x\mathrm{d}t \leq C_0[\|u(T)\|^2_{\dot{\mathrm{H}}^{\frac{1}{2}}} + \|u(0)\|^2_{\dot{\mathrm{H}}^{\frac{1}{2}}}], \tag{1.7}$$

where u is the solution to (1.1), and C_0 is independent of T (see, for instance, [7, Proposition 2.1, Lemma 2.3] for the proof). (1.7) seems not true for $0 < s < 1$ except $s = \frac{1}{2}$.

Remark 1.2 It is easy to see that d, s, satisfying condition (1.6), also satisfy (1.5). These assumptions come out when we prove Theorem 4.1 (see Remark 4.3 below).

1.1 Outline of the proof

In this paper, we adopt the concentration-compactness-rigidity argument. First we set

$$\boldsymbol{q} = \frac{2\alpha(\alpha+2)}{4-(d-2)\alpha}, \quad \boldsymbol{b} = \frac{2\alpha(\alpha+2)}{d\alpha^2+(d-2)\alpha-4}, \quad \boldsymbol{r} = \alpha+2, \tag{1.8}$$

throughout this paper. Following the argument in [21, Section 3] closely, we can get a critical solution in the following sense.

Theorem 1.2 (Reduction to Critical Solutions) *Suppose that Theorem* 1.1 *fails. Then there exists a solution u with the maximal lifespan I, $\|u\|_{L^{\boldsymbol{q}}(I,L^{\boldsymbol{r}}(\mathbb{R}^d))} = \infty$ and $\sup\limits_{t\in I}\||\nabla|^s u(t)\|_{L^2} < \infty$, which is called a critical solution. Furthermore, there exists a frequency scale function $N(t): I \to \mathbb{R}^+$ and a spacial center function $x(t): I \to \mathbb{R}^d$, such that the set*

$$\{N(t)^{-\frac{d-2s}{2}} u(t, N(t)^{-1}(x-x(t)));\ t \in I\} \tag{1.9}$$

is precompact in $\dot{H}^s(\mathbb{R}^d)$.

Reduction to the critical solution is by now a standard technique in the analysis of Schröding -er equation at critical regularity. We postpone the proof of this theorem to the appendix following the argument in [21, Section 3] with some slight changes.

Remark 1.3 Due to the precompactness of the set (1.9), for any η, there exists a time-free constant $C(\eta)$, such that

$$\int_{|x+x(t)|\geq \frac{C(\eta)}{N(t)}} ||\nabla|^s u(t,x)|^2 \mathrm{d}x \leq \eta, \quad \int_{|\xi|\geq C(\eta)N(t)} ||\xi|^s \widehat{u}(t,\xi)|^2 \mathrm{d}\xi \leq \eta$$

for any $t \in I$.

Remark 1.4 By the precompactness of (1.9), for any $\eta > 0$, we can find a time-free constant $c(\eta) > 0$, such that

$$\int_{|x+x(t)|\leq \frac{c(\eta)}{N(t)}} ||\nabla|^s u(t,x)|^2 \mathrm{d}x \leq \eta, \quad \int_{|\xi|\leq c(\eta)N(t)} ||\xi|^s \widehat{u}(t,\xi)|^2 \mathrm{d}\xi \leq \eta$$

for any $t \in I$.

Concerning the behavior of critical solutions at the endpoints of their maximal lifespan, we can get some reduced Duhamel formulae as in the mass and energy cases. For the proof, see [24, Proposition 5.23].

Theorem 1.3 (Duhamel's Formula) *Let u be the solution as in the above theorem with its maximal-lifespan I. Then, for all $t \in I$,*

$$\begin{aligned} u(t) &= \lim_{T\nearrow \sup I} \mathrm{i}\int_t^T \mathrm{e}^{\mathrm{i}(t-t')\Delta} F(u(t'))\mathrm{d}t' \\ &= -\lim_{T\searrow \inf I} \mathrm{i}\int_T^t \mathrm{e}^{\mathrm{i}(t-t')\Delta} F(u(t'))\mathrm{d}t', \end{aligned} \tag{1.10}$$

as weak limits in $\dot{H}^s$.

To get more information about the critical solution, we classify the frequency scale function $N(t)$ as in [25, Section 4] for the energy case (see also [24, Theorem 5.24] for the mass case).

Theorem 1.4 (Three Special Scenarios for Critical Solutions) *Suppose that Theorem* 1.1 *fails. We can ensure that there exists a critical solution u as in Theorem* 1.2 *with the maximal lifespan I and the frequency scale function $N : I \to \mathbb{R}^d$ matching one of the following three scenarios:*

(I) (*Finite-Time Blowup*) *We have that either $|\inf I| < \infty$ or $\sup I < \infty$.*

(II) (*Soliton-Like Solution*) *We have $I = \mathbb{R}$ and*

$$N(t) = 1 \quad \textit{for all } t \in \mathbb{R}.$$

(III) (*Low-to-High Frequency Cascade*) *We have $I = \mathbb{R}$,*

$$\inf_{t\in\mathbb{R}} N(t) \geq 1 \quad \textit{and} \quad \limsup_{t\to+\infty} N(t) = \infty.$$

For the rigidity part, we utilize different techniques to exclude these three scenarios. The main idea is to gain a negative regularity for the critical solutions as in the energy-critical case. The reduced Duhamel formula (1.10) is important in the proceedings. And to exclude the soliton-like solution, we use a localized interaction Morawetz identity (see [9, Section 6]).

The remainder of this paper is organized as follows. In Section 3, we review some classical results and some useful lemmas which will be used throughout this paper. In Section 4, we get a negative regularity for both the soliton-like solution and the low-to-high frequency cascade. In Section 5, the soliton-like solution scenario is excluded by a long-time Strichartz's estimate and a localized interaction Morawetz identity. Sections 6 and 7 are devoted to the other scenarios according to their frequency scale functions.

2 Basic Tools

We need some tools from the Littlewood-Paley theory. Let $\varphi(\xi) \in \mathcal{S}(\mathbb{R}^d)$ be a radial function supported in the ball $\{\xi \in \mathbb{R}^d : |\xi| \leq 2\}$ and equal to 1 on the ball $\{\xi \in \mathbb{R}^d : |\xi| \leq 1\}$. For each dyadic number $N > 0$, we define the Fourier multipliers as follows:

$$\begin{aligned}
\widehat{P_{\leq N} f}(\xi) &\equiv \varphi\Big(\frac{\xi}{N}\Big)\widehat{f}(\xi),\\
\widehat{P_{>N} f}(\xi) &\equiv \Big(1 - \varphi\Big(\frac{\xi}{N}\Big)\Big)\widehat{f}(\xi),\\
\widehat{P_N f}(\xi) &\equiv \Big(\varphi\Big(\frac{\xi}{N}\Big) - \varphi\Big(\frac{2\xi}{N}\Big)\Big)\widehat{f}(\xi),\\
P_{M<\cdot\leq N} &\equiv P_{\leq N} - P_{\leq M}.
\end{aligned}$$

Sometimes we use $f_{\leq M}$ instead of $P_{\leq N} f$ for short and similarly for the others. By the definition, we have

$$P_{\leq N} f = \sum_{M\leq N} P_M f; \quad P_{>N} f = \sum_{M>N} P_M f; \quad f = \sum_M P_M f.$$

With these notations, we obtain the extremely useful Bernstein ineqality.

Lemma 2.1 (Bernstein's Ineqality) *Letting $s \geq 0$ and $1 \leq p \leq q \leq \infty$, the following inequalities hold:*

$$\begin{aligned}
\|P_{\geq N} f\|_{L^p} &\lesssim_{p,s,d} N^{-s} \||\nabla|^s P_{\geq N} f\|_{L^p}, \\
\|P_{\leq N} |\nabla|^s f\|_{L^p} &\lesssim_{p,s,d} N^{s} \||\nabla|^s P_{\leq N} f\|_{L^p}, \\
\|P_N |\nabla|^{\pm s} f\|_{L^p} &\sim_{p,s,d} N^{\pm s} \||\nabla|^s P_N f\|_{L^p}, \\
\|P_{\leq N} f\|_{L^q} &\lesssim_{p,q,d} N^{\frac{d}{p}-\frac{d}{q}} \|P_{\leq N} f\|_{L^p}, \\
\|P_N f\|_{L^q} &\lesssim_{p,q,d} N^{\frac{d}{p}-\frac{d}{q}} \|P_N f\|_{L^p}.
\end{aligned}$$

Lemma 2.2 *Given $\gamma > 0$, $0 < \eta < \frac{1}{2}(1 - 2^{-\gamma})$, and $\{b_k\} \in \ell^\infty(\mathbb{Z}^+)$, let $x_k \in \ell^\infty(\mathbb{Z}^+)$ be a non-negative sequence obeying*

$$x_k \leq b_k + \eta \sum_{l=0}^{\infty} 2^{-\gamma|k-l|} x_l \quad \textit{for all } k \geq 0. \tag{2.1}$$

Then

$$x_k \lesssim \sum_{l=0}^{k} r^{|k-l|} b_l \quad \textit{for all } k \geq 0 \tag{2.2}$$

for some $r = r(\eta) \in (2^{-\gamma}, 1)$. Moreover, $r \downarrow 2^{-\gamma}$ as $\eta \downarrow 0$.

For the proof, see [25, Lemma 2.14].

Lemma 2.3 (Fractional Chain Rule) *Suppose $G \in C^1(\mathbb{C})$, $s \in (0, 1]$ and $1 < q, q_1, q_2 < \infty$, such that $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Then,*

$$\||\nabla|^s G(u)\|_{L^q} \lesssim \|G'(u)\|_{L^{q_1}} \||\nabla|^s u\|_{L^{q_2}}.$$

The readers can find a proof in [6].

3 The Cauchy Problem

It is well-known that, given any $\phi \in \mathcal{S}'(\mathbb{R}^d)$, the solution of linear Schrödinger equation is

$$u(t) = \mathrm{e}^{\mathrm{i}t\Delta}\phi = K_t \star \phi, \tag{3.1}$$

where the kernel K_t is given by

$$K_t(x) = (4\pi \mathrm{i}t)^{-\frac{N}{2}} \mathrm{e}^{\frac{\mathrm{i}|x|^2}{4t}}. \tag{3.2}$$

It follows from (3.1)–(3.2) that if $\phi \in L^1(\mathbb{R}^N)$, then $|t|^{\frac{N}{2}} \|u(t)\|_{L^\infty} \leq \|\phi\|_{L^1}$. Since $\mathrm{e}^{\mathrm{i}t\Delta}$ is an isometry of $L^2(\mathbb{R}^N)$, the previous estimate (together with the Riesz-Thorin interpolation theorem) shows that if $2 \leq q \leq \infty$ and $\phi \in L^{q'}(\mathbb{R}^N)$, then

$$|t|^{\frac{N(q-2)}{2q}} \|u(t)\|_{L^q} \leq \|\phi\|_{L^{q'}}. \tag{3.3}$$

The well-known Strichartz's estimate is another way to express the dispersive effect of the operator $\mathrm{e}^{\mathrm{i}t\Delta}$. To state the estimates, we first need the following definition.

Definition 3.1 (Admissible Pair) *For $d \geq 1$, we say that a Lebesgue exponents pair (q, r) is admissible if*

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \quad 2 \leq q, r \leq \infty, \quad (d, q, r) \neq (2, 2, \infty).$$

We are now ready to state the Strichartz's estimate.

Theorem 3.1 (Strichartz's Estimate) *The following properties hold:*
(i) *For any $\varphi \in L^2(\mathbb{R}^d)$, the function $t \mapsto \mathrm{e}^{\mathrm{i}t\Delta}\varphi$ belongs to*

$$L^q(\mathbb{R}, L^r(\mathbb{R}^d)) \cup C(\mathbb{R}, L^2(\mathbb{R}^d))$$

for every admissible pair (q, r), and there exists a constant C, such that

$$\|\mathrm{e}^{\mathrm{i}\cdot\Delta}\varphi\|_{L^q(\mathbb{R}, L^r)} \leq C\|\varphi\|_{L^2} \quad \text{for every } \varphi \in L^2(\mathbb{R}^d).$$

(ii) *Let I be an interval of $\mathbb{R}$, $J = \overline{I}$, and $t_0 \in J$. If (γ, ρ) is an admissible pair and $f \in L^{\gamma'}(I, L^{\rho'})$, then for every admissible pair (q, r), the function*

$$t \mapsto \int_{t_0}^{t} \mathrm{e}^{\mathrm{i}(t-s)\Delta} f(s)\mathrm{d}s \quad \text{for } t \in I$$

belongs to $L^q(I, L^r(\mathbb{R}^d)) \cup C(I, L^2(\mathbb{R}^d))$, and there exists a constant C independent of I, such that

$$\Big\| \int_{t_0}^{t} \mathrm{e}^{\mathrm{i}(t-s)\Delta} f(s)\mathrm{d}s \Big\|_{L^q(I, L^r)} \leq C\|f\|_{L^{\gamma'}(I, L^{\rho'})}.$$

For a proof of the non-endpoint case, see [2, Theorem 2.3.3] and [19] for the endpoint case. We also use an inhomogeneous Strichartz's estimate in the sequel. As above, we need a new definition.

Definition 3.2 ($\dot{H}^s$-Admissible Pair) *For $d \geq 1$ and $s \in (-1, 1)$, we say that a Lebesgue exponents pair (q, r) is $\dot{H}^s$-admissible if*

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2} - s, \quad \frac{2d}{d-2s} < r < \frac{2d}{d-2}.$$

Now we state the inhomogeneous estimate.

Theorem 3.2 *Let $s > 0$, (q_1, r_1) be an $\dot{H}^s$-admissible pair and (q_2', r_2') be an $\dot{H}^{-s}$-admissible pair. Then*

$$\Big\| \int_{t_0}^{t} \mathrm{e}^{\mathrm{i}(t-s)\Delta} f(s)\mathrm{d}s \Big\|_{L^{q_1}(I, L^{r_1})} \leq C\|f\|_{L^{q_2'}(I, L^{r_2'})}.$$

For the proof, see [4, Lemma 2.1] and [18].

Theorem 3.3 (see [4–5]) *Assume $u_0 \in \dot{H}^s(\mathbb{R}^d)$, $t_0 \in I$ and $\|u_0\|_{\dot{H}^s(\mathbb{R}^d)} \leq A$. Then there exists a $\delta = \delta(A)$, such that if $\|\mathrm{e}^{\mathrm{i}(t-t_0)\Delta}u_0\|_{L^q(I, L^r)} < \delta$, we can find a unique solution u to*

(1.1) in $\mathbb{R}^d \times I$ with $u \in C(I;\dot{H}^s(\mathbb{R}^d))$. Furthermore, we can also find a constant C, such that for any admissible pair (q,r), we have

$$\||\nabla|^s u\|_{L^q(I,L^r)} + \sup_{t\in I}\|u(t)\|_{\dot{H}^s} \le CA, \quad \|u\|_{L^q(I,L^r)} \le 2\delta.$$

Moreover, if $u_{0,k} \to u_0$ in $\dot{H}^s(\mathbb{R}^d)$, then the corresponding solutions $u_k \to u$ in $C(I;\dot{H}^s(\mathbb{R}^d))$.

The following bilinear estimates will also be used in this paper.

Lemma 3.1 *Suppose that $\widehat{v}(t,\xi)$ is supported on $|\xi| \le M$ and that $\widehat{u}(t,\xi)$ is supported on $|\xi| > N$, $M \ll N$. Then, for the interval $I = [a,b]$, $d \ge 1$,*

$$\|uv\|_{L^2(I,L^2)} \lesssim \frac{M^{\frac{d-1}{2}}}{N^{\frac{1}{2}}}\|u\|_{S^0_*(I)}\|v\|_{S^0_*(I)}, \tag{3.4}$$

where

$$\|u\|_{S^0_*(I)} \equiv \|u(a)\|_{L^2} + \sup_{\substack{(q,r) \text{ admissible pair} \\ q>2}} \|(\mathrm{i}\partial_t + \Delta)u\|_{L^q(I,L^r)}. \tag{3.5}$$

For a proof, we refer readers to [31, Lemma 2.5] for example.

Definition 3.3 *Let $v_0 \in \dot{H}^s$, $v(t) = \mathrm{e}^{\mathrm{i}t\Delta}v_0$ and let $\{t_n\}$ be a sequence with $\lim_{n\to\infty} t_n = \overline{t} \in [-\infty,+\infty]$. We say that $u(x,t)$ is a nonlinear profile associated with $(v_0,\{t_n\})$ if there exists an interval I with $\overline{t} \in I$ (if $\overline{t} = \pm\infty$, $I = [a,+\infty)$ or $I = (-\infty,a]$), such that u is a solution to (1.1) in I and*

$$\lim_{n\to\infty}\|u(t_n) - v(t_n)\|_{\dot{H}^s} = 0.$$

Remark 3.1 There always exists a unique nonlinear profile associated to $(v_0,\{t_n\})$ (for the proof, see the analogous one in [20, Remark 2.13]). We can hence define a maximal interval I of the existence for the nonlinear profile associated to $(v_0,\{t_n\})$.

4 Negative Regularity

The main result of this section is the following theorem, and a similar proof can be found in [25, Section 6].

Theorem 4.1 (Negative Regularity) *Let u be a critical solution with the maximal lifespan I. If $I = \mathbb{R}$,*

$$\||\nabla|^s u\|_{L^\infty(\mathbb{R},L^2)} < \infty \tag{4.1}$$

and

$$\inf_{\mathbb{R}} N(t) \ge 1, \tag{4.2}$$

then there exists an $\varepsilon > 0$ such that $\||\nabla|^{-\varepsilon}u\|_{L^\infty(\mathbb{R},L^2(\mathbb{R}^d))} < \infty$.

Remark 4.1 Let u be the solution satisfying the assumption of this theorem, and interpolation easily implies that $u \in L^\infty(\mathbb{R},\ L^2(\mathbb{R}^d))$, hence the solution shares the mass conservation.

Remark 4.2 It is easy to see that the soliton-like solutions and the low-to-high frequency cascades satisfy the condition of this theorem, and hence the mass conservation.

Remark 4.3 The conditions in (1.4)–(1.6) come from the proof of this theorem. Practically, (1.4)–(1.5) refer to the following case of $\frac{4}{d-2s} < 1$, while (1.6) refers to $1 \le \frac{4}{d-2s} \le 2$.

By the precompactness of (1.9) and the assumption on the frequency function, for any $\eta > 0$ (chosen later), we can find a uniform $N_0 = N_0(\eta) > 0$, such that

$$\int_{|\xi|\le N_0} |\xi|^{2s}|\widehat{u}(t)|^2 \mathrm{d}\xi < \eta \tag{4.3}$$

for any $t \in I$. We now turn to the proof. To this end, we set

$$A(N) = N^{-\mu} \sup_{t\in\mathbb{R}} \|u_N(t)\|_{L^\rho} \quad \text{for } N \le 10N_0, \tag{4.4}$$

where $\frac{d}{2} - \frac{d}{\rho} - \mu = s$. We pick $\mu = \frac{d}{d-2s} - s$ if $\frac{4}{d-2s} < 1$, while we will choose ρ later if $1 \le \frac{4}{d-2s} \le 2$.

By Bernstein's inequality, it is easy to see that $A(N) \le \sup\limits_{t\in\mathbb{R}} N^s \|u_N\|_{L^2} < \infty$. Without loss of generality, we calculate $N^{-\mu}\|u_N(0)\|_{L^\rho}$ for convenience.

First we consider the case of $\frac{4}{d-2s} < 1$. By (1.10), we have

$$\begin{aligned} N^{-\mu}\|u_N(0)\|_{L^\rho} &\le N^{-\mu}\Big\| \int_0^\infty \mathrm{e}^{-\mathrm{i}t\Delta} P_N F(u(t))\mathrm{d}t\Big\|_{L^\rho} \\ &\le N^{-\mu}\Big\| \int_0^{N^{-2}} \mathrm{e}^{-\mathrm{i}t\Delta} P_N F(u(t))\mathrm{d}t\Big\|_{L^\rho} \\ &\quad + N^{-\mu}\Big\| \int_{N^{-2}}^\infty \mathrm{e}^{-\mathrm{i}t\Delta} P_N F(u(t))\mathrm{d}t\Big\|_{L^\rho}. \end{aligned} \tag{4.5}$$

Then (3.3) and Bernstein's inequality imply that

$$\begin{aligned} N^{-\mu}\|u_N(0)\|_{L^\rho} &\lesssim N^{-\mu+\frac{d}{2}-\frac{d}{\rho}}\Big\| \int_0^{N^{-2}} \mathrm{e}^{-\mathrm{i}t\Delta} P_N F(u(t))\mathrm{d}t\Big\|_{L^2} \\ &\quad + N^{-\mu}\int_{N^{-2}}^\infty t^{-(\frac{d}{2}-\frac{d}{\rho})}\mathrm{d}t \sup_{t\in\mathbb{R}} \|P_N F(u(t))\|_{L^{\rho'}} \\ &\lesssim N^{-\mu+\frac{d}{2}-\frac{d}{\rho}-2+\frac{d}{\rho'}-\frac{d}{2}}\|P_N F(u(\cdot))\|_{L^\infty(\mathbb{R},L^{\rho'})} \\ &\quad + N^{-\mu-2(1-\frac{d}{2}+\frac{d}{\rho})}\|P_N F(u(\cdot))\|_{L^\infty(\mathbb{R},L^{\rho'})} \\ &= N^{s-\frac{d-4s}{d-2s}}\|P_N F(u(\cdot))\|_{L^\infty(\mathbb{R},L^{\rho'})}. \end{aligned} \tag{4.6}$$

Notice that the power $s - \frac{d-4s}{d-2s}$ is positive by (1.5). For the nonlinearity of $F(u)$, we use the fundamental theorem of calculus, and decompose it as

$$\begin{aligned} F(u) &= F(u_{\frac{N}{10}\le\cdot\le N_0}) + u_{<\frac{N}{10}}\int_0^1 F_z(u_{\frac{N}{10}\le\cdot\le N_0}+\theta u_{<\frac{N}{10}})\mathrm{d}\theta \\ &\quad + \overline{u_{<\frac{N}{10}}}\int_0^1 F_{\overline{z}}(u_{\frac{N}{10}\le\cdot\le N_0}+\theta u_{<\frac{N}{10}})\mathrm{d}\theta \\ &\quad + O(|u_{>N_0}||u_{\le N_0}|^{\alpha}) + O(|u_{>N_0}|^{\alpha+1}). \end{aligned} \tag{4.7}$$

Using (4.1), Hölder's and Bernstein's inequalities, we estimate the last two terms of $F(u)$ as

$$\begin{aligned} &N^{s-\frac{d-4s}{d-2s}}\|P_N(O(|u_{>N_0}||u|^{\alpha}))\|_{L^\infty(\mathbb{R},L^{\rho'})} \\ &\le N^{s-\frac{d-4s}{d-2s}}\|u_{>N_0}\|_{L^\infty(\mathbb{R},L^{\frac{2d(d-2s)}{d^2-2sd-2d+8s}})}\||u|^{\frac{4}{d-2s}}\|_{L^\infty(\mathbb{R},L^{\frac{d}{2}})} \\ &\lesssim N^{s-\frac{d-4s}{d-2s}}\||\nabla|^{\frac{d-4s}{d-2s}}u_{>N_0}\|_{L^\infty(\mathbb{R},L^2)} \\ &\lesssim N^{s-\frac{d-4s}{d-2s}}N_0^{\frac{d-4s}{d-2s}-s} = \Big(\frac{N}{N_0}\Big)^{s-\frac{d-4s}{d-2s}}. \end{aligned} \tag{4.8}$$

Next we turn to the contribution to the right-hand side of (4.5) coming from the second and third terms in (4.7). Without loss of generality, it suffices to estimate the first term of them. An intrinsic equivalent norm for Besov spaces shows that $F_z(u)\in L^\infty\big(\mathbb{R},\dot B^{\frac{4s}{d-2s}}_{\frac{d-2s}{2},\infty}\big)$ (see [29, Theorem 4.4.1]). Indeed,

$$\begin{aligned} \|F_z(u)\|_{B^{\frac{4s}{d-2s}}_{\frac{d-2s}{2},\infty}} &= \sup_{y\in\mathbb{R}^d}\frac{1}{|y|^{\frac{4s}{d-2s}}}\||u(\cdot-y)|^{\frac{4}{d-2s}}-|u(\cdot)|^{\frac{4}{d-2s}}\|_{L^{\frac{d-2s}{2}}} \\ &\le \sup_{y\in\mathbb{R}^d}\frac{1}{|y|^{\frac{4s}{d-2s}}}\||u(\cdot-y)-u(\cdot)|^{\frac{4}{d-2s}}\|_{L^{\frac{d-2s}{2}}} \\ &= \Big(\sup_{y\in\mathbb{R}^d}\frac{1}{|y|^s}\|u(\cdot-y)-u(\cdot)\|_{L^2}\Big)^{\frac{4}{d-2s}} \\ &= \|u\|_{\dot B^s_{2,\infty}}^{\frac{4}{d-2s}} \le \|u\|_{\dot B^s_{2,2}}^{\frac{4}{d-2s}} = \|u\|_{\dot H^s}^{\frac{4}{d-2s}}. \end{aligned}$$

Thus by the dyadic decomposition, we have

$$\begin{aligned} \|P_{>\frac{N}{10}}F_z(u)\|_{L^{\frac{d-2s}{2}}} &\sim \Big\|\Big(\sum_{M>\frac{N}{10}}|P_M(|u|^{\frac{4}{d-2s}})|^2\Big)^{\frac12}\Big\|_{L^{\frac{d-2s}{2}}} \\ &\lesssim \Big(\sum_{M>\frac{N}{10}}\|P_M(|u|^{\frac{4}{d-2s}})\|^2_{L^{\frac{d-2s}{2}}}\Big)^{\frac12} \\ &\lesssim \Big(\sum_{M>\frac{N}{10}}\|u\|_{\dot H^s}^{\frac{8}{d-2s}}M^{-\frac{8s}{d-2s}}\Big)^{\frac12} \\ &\lesssim N^{-\frac{4s}{d-2s}}\|u\|_{\dot H^s}^{\frac{4}{d-2s}}. \end{aligned}$$

Together with (4.3), Hölder's and Bernstein's inequalities, we obtain

$$N^{s-\frac{d-4s}{d-2s}}\Big\|u_{<\frac{N}{10}}\int_0^1 F_z(u_{\frac{N}{10}\le\cdot\le N_0}+\theta u_{<\frac{N}{10}})\mathrm{d}\theta\Big\|_{L^\infty(\mathbb{R},L^{\rho'})}$$

$$
\begin{aligned}
&\le N^{s-\frac{d-4s}{d-2s}}\|u_{<\frac{N}{10}}\|_{L^\infty(\mathbb{R},L^\rho)}\|P_{>\frac{N}{10}}F_z(u_{\le N_0})\|_{L^\infty(\mathbb{R},L^{\frac{d-2s}{2}})}\\
&\lesssim N^{s-\frac{d-4s}{d-2s}}\|u_{<\frac{N}{10}}\|_{L^\infty(\mathbb{R},L^\rho)}N^{-\frac{4s}{d-2s}}\|u_{<N_0}\|^{\frac{4}{d-2s}}_{L^\infty(\mathbb{R},\dot H^s)}\\
&\lesssim \eta^{\frac{4}{d-2s}}\sum_{N_1<\frac{N}{10}}N_1^\mu N^{s-\frac{d-4s}{d-2s}-\frac{4s}{d-2s}}A(N_1)\\
&\lesssim \eta^{\frac{4}{d-2s}}\sum_{N_1<\frac{N}{10}}\Big(\frac{N_1}{N}\Big)^\mu A(N_1).
\end{aligned}\tag{4.9}
$$

We are left to estimate the contribution of $F(u_{\frac{N}{10}\le\cdot\le N_0})$ to the right-hand side of (4.5). We estimate

$$
\begin{aligned}
&\|F(u_{\frac{N}{10}\le\cdot\le N_0})\|_{L^\infty(\mathbb{R},L^{\rho'})}\\
\lesssim&\sum_{\frac{N}{10}\le N_1,N_2\le N_0}\|u_{N_1}|u_{N_2}|^{\frac{4}{d-2s}}\|_{L^\infty(\mathbb{R},L^{\rho'})}\\
\lesssim&\sum_{\frac{N}{10}\le N_2\le N_1\le N_0}\||u_{N_1}|^{\frac{4}{d-2s}}\|_{L^\infty(\mathbb{R},L^{\rho_1})}\||u_{N_1}|^{1-\frac{4}{d-2s}}\|_{L^\infty(\mathbb{R},L^{\rho_2})}\||u_{N_2}|^{\frac{4}{d-2s}}\|_{L^\infty(\mathbb{R},L^{\rho_3})}\\
&+\sum_{\frac{N}{10}\le N_1\le N_2\le N_0}\|u_{N_1}\|_{L^\infty(\mathbb{R},L^\rho)}\||u_{N_2}|^{\frac{4}{d-2s}}\|_{L^\infty(\mathbb{R},L^{\frac{d-2s}{2}})}=\mathrm{I}+\mathrm{II},
\end{aligned}
$$

where $\rho_1=\frac{d-2s}{2}$, $\rho_2=\frac{2(d-2s)^2}{(d-2s-2)(d-2s-4)}$ and $\rho_3=\frac{(d-2s)^2}{2(d-2s-2)}$. For the second term, we employ (4.3) and Bernstein's inequality, such that

$$
\begin{aligned}
\mathrm{II}&\le\sum_{\frac{N}{10}\le N_1\le N_2\le N_0}\|u_{N_1}\|_{L^\infty(\mathbb{R},L^\rho)}\|u_{N_2}\|^{\frac{4}{d-2s}}_{L^\infty(\mathbb{R},L^2)}\\
&\lesssim\sum_{\frac{N}{10}\le N_1\le N_2\le N_0}\|u_{N_1}\|_{L^\infty(\mathbb{R},L^\rho)}(N_2^{-s})^{\frac{4}{d-2s}}\eta^{\frac{4}{d-2s}}.
\end{aligned}
$$

Taking the sum of N_2, we obtain

$$
\mathrm{II}\lesssim\eta^{\frac{4}{d-2s}}\sum_{\frac{N}{10}\le N_1\le N_0}A(N_1)N_1^\mu(N_1^{-s})^{\frac{4}{d-2s}}=\eta^{\frac{4}{d-2s}}\sum_{\frac{N}{10}\le N_1\le N_0}A(N_1)N_1^{\mu-\frac{4s}{d-2s}}.
$$

Finally, we turn to the first term. According to Berstein's inequality and the definition of $A(N)$, we have

$$
\begin{aligned}
\mathrm{I}&\le\sum_{\frac{N}{10}\le N_2\le N_1\le N_0}\|u_{N_1}\|^{\frac{4}{d-2s}}_{L^\infty(\mathbb{R},L^2)}\|u_{N_1}\|^{1-\frac{4}{d-2s}}_{L^\infty(\mathbb{R},L^{\frac{2(d-2s)}{d-2s-2}})}\|u_{N_2}\|^{\frac{4}{d-2s}}_{L^\infty(\mathbb{R},L^{\frac{2(d-2s)}{d-2s-2}})}\\
&\le\sum_{\frac{N}{10}\le N_2\le N_1\le N_0}(N_1^{-s}\eta)^{\frac{4}{d-2s}}(N_1^\mu A(N_1))^{1-\frac{4}{d-2s}}(N_2^\mu A(N_2))^{\frac{4}{d-2s}}.
\end{aligned}
$$

Now we set $-\delta=\mu-\frac{4s}{d-2s}<0$, $B(N)=N^{-\delta}A(N)$ and $C(N)=N^{\epsilon-\delta}A(N)$, for some small $\epsilon>0$, such that $s-\frac{d-4s}{d-2s}-\epsilon>0$ (see (1.5)). With these notations, we rewrite the above inequality as

$$
\begin{aligned}
\mathrm{I} &\lesssim \eta^{\frac{4}{d-2s}} \sum_{\frac{N}{10}\le N_2\le N_1\le N_0} N_1^{-\frac{4s}{d-2s}} (N_1^{\frac{4s}{d-2s}} B(N_1))^{1-\frac{4}{d-2s}} (N_2^{\frac{4s}{d-2s}} B(N_2))^{\frac{4}{d-2s}} \\
&= \eta^{\frac{4}{d-2s}} \sum_{\frac{N}{10}\le N_2\le N_1\le N_0} \Big(\frac{N_2}{N_1}\Big)^{\frac{16s}{(d-2s)^2}} B(N_1)^{1-\frac{4}{d-2s}} B(N_2)^{\frac{4}{d-2s}} \\
&= \eta^{\frac{4}{d-2s}} \sum_{\frac{N}{10}\le N_2\le N_1\le N_0} \Big(\frac{N_2}{N_1}\Big)^{\frac{16s}{(d-2s)^2}} (N_1^{-\epsilon} C(N_1))^{1-\frac{4}{d-2s}} (N_2^{-\epsilon} C(N_2))^{\frac{4}{d-2s}}.
\end{aligned}
$$

Taking the sum over N_2 first, we have

$$
\begin{aligned}
\mathrm{I} &= \eta^{\frac{4}{d-2s}} \sum_{\frac{N}{10}\le N_1\le N_0} N_1^{-\frac{16s}{(d-2s)^2}-\epsilon(1-\frac{4}{d-2s})} C(N_1)^{1-\frac{4}{d-2s}} \\
&\quad\times \Big(\sum_{\frac{N}{10}\le N_2\le N_1} N_2^{\frac{16s}{(d-2s)^2}-\frac{4\epsilon}{d-2s}} C(N_2)^{\frac{4}{d-2s}} \Big) \\
&\lesssim \eta^{\frac{4}{d-2s}} \sum_{\frac{N}{10}\le N_1\le N_0} N_1^{-\frac{16s}{(d-2s)^2}-\epsilon(1-\frac{4}{d-2s})} C(N_1)^{1-\frac{4}{d-2s}} N_1^{\frac{16s}{(d-2s)^2}-\frac{4\epsilon}{d-2s}} \\
&\quad\times \Big(\sum_{\frac{N}{10}\le N_2\le N_0} C(N_2) \Big)^{\frac{4}{d-2s}} \\
&= \eta^{\frac{4}{d-2s}} \sum_{\frac{N}{10}\le N_1\le N_0} N_1^{-\epsilon} C(N_1)^{1-\frac{4}{d-2s}} \Big(\sum_{\frac{N}{10}\le N_2\le N_0} C(N_2) \Big)^{\frac{4}{d-2s}} \\
&\lesssim \eta^{\frac{4}{d-2s}} N^{-\epsilon} \sum_{\frac{N}{10}\le N_1\le N_0} C(N_1).
\end{aligned}
$$

Putting the above estimates together, we obtain

$$
\begin{aligned}
&N^{s-\frac{d-4s}{d-2s}} \|P_N F(u_{\frac{N}{10}\le\cdot\le N_0})\|_{L^\infty(\mathbb{R},L^{\rho'})} \\
&\lesssim \eta^{\frac{4}{d-2s}} N^{s-\frac{d-4s}{d-2s}} \Big(\sum_{\frac{N}{10}\le N_1\le N_0} A(N_1) N_1^{\mu-\frac{4s}{d-2s}} + N^{-\epsilon} \sum_{\frac{N}{10}\le N_1\le N_0} C(N_1) \Big).
\end{aligned} \tag{4.10}
$$

Collecting (4.5) and (4.7)–(4.10), we estimate

$$
\begin{aligned}
A(N) &\lesssim \Big(\frac{N}{N_0}\Big)^{s-\frac{d-4s}{d-2s}} + \eta^{\frac{4}{d-2s}} \sum_{N_1<\frac{N}{10}} \Big(\frac{N_1}{N}\Big)^{\mu} A(N_1) \\
&\quad + \eta^{\frac{4}{d-2s}} \Big(\sum_{\frac{N}{10}\le N_1\le N_0} A(N_1) \Big(\frac{N}{N_1}\Big)^{s-\frac{d-4s}{d-2s}} + \sum_{\frac{N}{10}\le N_1\le N_0} \Big(\frac{N}{N_1}\Big)^{s-\frac{d-4s}{d-2s}-\epsilon} A(N_1) \Big) \\
&\lesssim \Big(\frac{N}{N_0}\Big)^{s-\frac{d-4s}{d-2s}} + \eta^{\frac{4}{d-2s}} \Big(\sum_{N_1<\frac{N}{10}} \Big(\frac{N_1}{N}\Big)^{\mu} A(N_1) + \sum_{\frac{N}{10}\le N_1\le N_0} \Big(\frac{N}{N_1}\Big)^{s-\frac{d-4s}{d-2s}-\epsilon} A(N_1) \Big).
\end{aligned}
$$

With the above estimate, applying the discrete Gronwall's inequality (2.2), we have

$$
\sup_{t\in\mathbb{R}} \|u_N(t)\|_{L^\rho} \lesssim N^{\mu+\omega-}, \tag{4.11}
$$

where $\omega=\min\{\mu,\ s-\frac{d-4s}{d-2s}-\epsilon\}>0$.

Now we turn to the case that $1\le\frac{4}{d-2s}\le 2$. We pick ρ satisfying

$$\frac{1}{2}-\frac{1}{d}-\frac{s}{2d}\le\frac{1}{\rho}<\frac{1}{2}-\frac{2}{d}+\frac{s}{d}, \tag{4.12}$$

which gives $s>\frac{2}{3}$ as in (1.6). As the similar estimates in (4.5)–(4.7), we first calculate

$$\begin{aligned}
&N^{-\mu-2(1-\frac{d}{2}+\frac{d}{\rho})}\|P_N(|u_{>N_0}||u|^{\alpha})\|_{L^\infty(\mathbb{R},\,L^{\rho'})}\\
&\le N^{-\mu-2(1-\frac{d}{2}+\frac{d}{\rho})}\|u_{>N_0}\|_{L^\infty(\mathbb{R},\,L^{p_1})}\||u|^{\frac{4}{d-2s}}\|_{L^\infty(\mathbb{R},\,L^{\frac{d}{2}})}\\
&\le N^{-\mu-2(1-\frac{d}{2}+\frac{d}{\rho})}N_0^{-\beta}\||\nabla|^{\beta}u_{>N_0}\|_{L^\infty(\mathbb{R},\,L^{p_1})}\|u\|^{\frac{4}{d-2s}}_{L^\infty(\mathbb{R},\,L^{\frac{2d}{d-2s}})}\\
&\lesssim N^{-\mu-2(1-\frac{d}{2}+\frac{d}{\rho})}N_0^{-\beta}\||\nabla|^{s}u_{>N_0}\|_{L^\infty(\mathbb{R},\,L^{2})}\\
&=\Big(\frac{N}{N_0}\Big)^{-\mu-2(1-\frac{d}{2}+\frac{d}{\rho})},
\end{aligned} \tag{4.13}$$

where $0<\beta=\frac{d}{2}+s-2-\frac{d}{\rho}\le\frac{3}{2}s-1$ (comparing with (4.12)) and $p_1=\frac{2d}{d-2(s-\beta)}$. Next, we consider the integrals in the decomposition (4.7). By the proposition of support, we have

$$\begin{aligned}
&N^{-\mu-2(1-\frac{d}{2}+\frac{d}{\rho})}\Big\|P_N\Big(u_{<\frac{N}{10}}\int_0^1F_z(u_{\frac{N}{10}\le\cdot\le N_0}+\theta u_{<\frac{N}{10}})\mathrm{d}\theta\Big)\Big\|_{L^\infty(\mathbb{R},\,L^{\rho'})}\\
&\le N^{-\mu-2(1-\frac{d}{2}+\frac{d}{\rho})}\|u_{<\frac{d}{10}}\|_{L^\infty(\mathbb{R},\,L^{\rho})}\Big\|P_{>\frac{N}{10}}\Big(\int_0^1F_z(u_{\frac{N}{10}\le\cdot\le N_0}+\theta u_{<\frac{N}{10}})\mathrm{d}\theta\Big)\Big\|_{L^\infty(\mathbb{R},\,L^{\frac{\rho}{\rho-2}})}.
\end{aligned}$$

We use (2.3) and (4.3) to give a bound to the second factor on the right-hand side by

$$N^{-\gamma}\||\nabla|^{\gamma}u_{\le N_0}\|_{L^\infty(\mathbb{R},\,L^{q_1})}\||u_{\le N_0}|^{\frac{4}{d-2s}-1}\|_{L^\infty(\mathbb{R},\,L^{q_2})}\lesssim N^{-\gamma}\eta^{\frac{4}{d-2s}},$$

where $q_1=\frac{2d}{d-2(s-\gamma)}$ for some $0<\gamma=d-\frac{2d}{\rho}-2\le s$ (compareing with (1.6)) and $q_2=\frac{2d}{4-d+2s}$. Then we continue to consider the integrals in the above inequality as follows:

$$\begin{aligned}
&N^{-\mu-2(1-\frac{d}{2}+\frac{d}{\rho})}\Big\|P_N\Big(u_{<\frac{N}{10}}\int_0^1F_z(u_{\frac{N}{10}\le\cdot\le N_0}+\theta u_{<\frac{N}{10}})\mathrm{d}\theta\Big)\Big\|_{L^\infty(\mathbb{R},\,L^{\rho'})}\\
&\lesssim\eta^{\frac{4}{d-2s}}N^{-\mu-2(1-\frac{d}{2}+\frac{d}{\rho})-\gamma}\|u_{<\frac{N}{10}}\|_{L^\infty(\mathbb{R},\,L^{\rho})}\\
&\lesssim\eta^{\frac{4}{d-2s}}N^{-\mu-2(1-\frac{d}{2}+\frac{d}{\rho})-\gamma}\sum_{N_1<\frac{N}{10}}\|u_{>N_1}\|_{L^\infty(\mathbb{R},\,L^{\rho})}\\
&\lesssim\eta^{\frac{4}{d-2s}}\sum_{N_1<\frac{N}{10}}A(N_1)\Big(\frac{N_1}{N}\Big)^{\mu}.
\end{aligned} \tag{4.14}$$

Finally, we consider the first term of (4.7). According to the Littlewood-Paley decomposition, we deduce

$$\begin{aligned}
&N^{-\mu-2(1-\frac{d}{2}+\frac{d}{\rho})}\|P_NF(u_{\frac{N}{10}\le\cdot\le N_0})\|_{L^\infty(\mathbb{R},\,L^{\rho'})}\\
&\lesssim N^{-\mu-2(1-\frac{d}{2}+\frac{d}{\rho})}\sum_{\frac{N}{10}\le N_1,\,N_2,\,N_3\le N_0}\|u_{N_1}u_{N_2}|u_{N_3}|^{\frac{4}{d-2s}-1}\|_{L^\infty(\mathbb{R},\,L^{\rho'})}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim N^{-\mu-2(1-\frac{d}{2}+\frac{d}{\rho})}\sum_{\frac{N}{10}\le N_1\le N_2,\, N_3\le N_0}\|u_{N_1}u_{N_2}|u_{N_3}|^{\frac{4}{d-2s}-1}\|_{L^\infty(\mathbb{R},\,L^{\rho'})}\\
&\quad + N^{-\mu-2(1-\frac{d}{2}+\frac{d}{\rho})}\sum_{\frac{N}{10}\le N_3\le N_1\le N_2\le N_0}\|u_{N_1}u_{N_2}|u_{N_3}|^{\frac{4}{d-2s}-1}\|_{L^\infty(\mathbb{R},\,L^{\rho'})}\equiv \mathrm{I}+\mathrm{II}.
\end{aligned}
$$

Applying Hölder's and Bernstein's inequalities to the terms of I, we estimate them by

$$
\begin{aligned}
&\|u_{N_1}\|_{L^\infty(\mathbb{R},\,L^{p_1})}\|u_{N_2}\|_{L^\infty(\mathbb{R},\,L^{p_2})}\||u_{N_3}|^{\frac{4}{d-2s}-1}\|_{L^\infty(\mathbb{R},\,L^{p_3})}\\
&\lesssim \|u_{N_1}\|_{L^\rho}\||\nabla|^s u_{N_2}\|_{L^\infty(\mathbb{R},\,L^2)}N_2^{\frac{d-2s}{2}-\frac{d}{p_2}}\big(\||\nabla|^s u_{N_3}\|_{L^\infty(\mathbb{R},\,L^2)}N_3^{\frac{d-2s}{2}-\frac{d}{p_3(\frac{4}{d-2s}-1)}}\big)^{\frac{4}{d-2s}-1}\\
&\lesssim \eta^{\frac{4}{d-2s}}A(N_1)N_1^{\mu}N_2^{\frac{d-2s}{2}-\frac{d}{p_2}}N_3^{2-\frac{d-2s}{2}-\frac{d}{p_3}},
\end{aligned}
$$

where $\frac{1}{\rho'}=\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}$, with $p_1=\rho$, $2\le p_2<\frac{2d}{d-2s}$ and $2\le p_3(\frac{4}{d-2s}-1)<\frac{2d}{d-2s}$ (due to the definitions of ρ and μ, such p_2 and p_3 exist). Taking the sum of N_2 and N_3, we obtain

$$
\begin{aligned}
I&\lesssim \eta^{\frac{4}{d-2s}}N^{-\mu-2(1-\frac{d}{2}+\frac{d}{\rho})}\sum_{\frac{N}{10}\le N_1\le N_0}A(N_1)N_1^{\mu+2-d(\frac{1}{p_2}+\frac{1}{p_3})}\\
&=\eta^{\frac{4}{d-2s}}\sum_{\frac{N}{10}\le N_1\le N_0}A(N_1)\Big(\frac{N}{N_1}\Big)^{-\mu-2(1-\frac{d}{2}+\frac{d}{\rho})},
\end{aligned}
\tag{4.15}
$$

where $-\mu-2(1-\frac{\mathrm{d}}{2}+\frac{\mathrm{d}}{\rho})>0$. Similarly, applying Hölder's inequality to the terms of II in space, we obtain

$$
\begin{aligned}
&\|u_{N_1}u_{N_2}|u_{N_3}|^{\frac{4}{d-2s}-1}\|_{L^\infty(\mathbb{R},\,L^{\rho'})}\\
&\le \||u_{N_1}|^\theta\|_{L^{q_1}}\||u_{N_1}|^{1-\theta}\|_{L^{q_2}}\|u_{N_2}\|_{L^{q_3}}\||u_{N_3}|^{\frac{4}{d-2s}-1}\|_{L^{q_4}}\\
&\le \|u_{N_1}\|_{L^{\theta q_1}}^\theta\|u_{N_1}\|_{L^{(1-\theta)q_2}}^{1-\theta}\|u_{N_2}\|_{L^{q_3}}\|u_{N_3}\|_{L^{(\frac{4}{d-2s}-1)q_4}}^{\frac{4}{d-2s}-1}\\
&\lesssim \|u_{N_1}\|_{L^\rho}^\theta\|u_{N_1}\|_{L^{(1-\theta)q_2}}^{1-\theta}\|u_{N_2}\|_{L^{q_3}}\|u_{N_3}\|_{L^\rho}^{1-\theta}\|u_{N_3}\|_{L^\rho}^{\frac{4}{d-2s}-2+\theta},
\end{aligned}
\tag{4.16}
$$

where $\frac{1}{\rho'}=\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{q_3}+\frac{1}{q_4}$ with $\theta q_1=\rho$, $(\frac{4}{d-2s}-1)q_4=\rho$, $(1-\theta)q_2>2$, $2\le q_3<\frac{2d}{d-2s}$ and $2-\frac{4}{d-2s}\le\theta\le\min\big(1,\,\frac{8}{(d-2s)(\rho-2)}\big)$ (Remark 1.2 ensures the existence of these exponents). By Hölder's and Bernstein's inequalities, we have

$$
\begin{aligned}
&\|u_{N_1}u_{N_2}|u_{N_3}|^{\frac{4}{d-2s}-1}\|_{L^\infty(\mathbb{R},\,L^{\rho'})}\\
&\lesssim \|u_{N_1}\|_{L^\rho}^\theta\big(\||\nabla|^s u_{N_1}\|_{L^2}N_1^{\frac{d-2s}{2}-\frac{d}{(1-\theta)q_2}}\big)^{1-\theta}\||\nabla|^s u_{N_2}\|_{L^2}N_2^{\frac{d-2s}{2}-\frac{d}{q_3}}\\
&\quad\times\|u_{N_3}\|_{L^\rho}^{1-\theta}\big(\||\nabla|^s u_{N_3}\|_{L^2}N_3^{\frac{d-2s}{2}-\frac{d}{\rho}}\big)^{\frac{4}{d-2s}-2+\theta}\\
&\lesssim \eta^{\frac{4}{d-2s}}\|u_{N_1}\|_{L^\rho}^\theta\|u_{N_3}\|_{L^\rho}^{1-\theta}N_1^{(\frac{d-2s}{2}-\frac{d}{(1-\theta)q_2})(1-\theta)}N_2^{\frac{d-2s}{2}-\frac{d}{q_3}}N_3^{(\frac{d-2s}{2}-\frac{d}{\rho})(\frac{4}{d-2s}-2+\theta)}.
\end{aligned}
\tag{4.17}
$$

Taking the sum, we have

$$
\begin{aligned}
\mathrm{II}&\lesssim \eta^{\frac{4}{d-2s}}N^{-\mu-2(1-\frac{d}{2}+\frac{d}{\rho})}\sum_{\frac{N}{10}\le N_3\le N_0}\|u_{N_3}\|_{L^\rho}^{1-\theta}N_3^{(\frac{d-2s}{2}-\frac{d}{\rho})(\frac{4}{d-2s}-2+\theta)}\\
&\quad\times\sum_{N_3\le N_1\le N_0}\|u_{N_1}\|_{L^\rho}^\theta N_1^{\frac{d-2s}{2}(2-\theta)-\frac{d}{q_3}-\frac{d}{q_2}}
\end{aligned}
$$

$$\begin{aligned}
&\lesssim \eta^{\frac{4}{d-2s}} N^{-\mu-2(1-\frac{d}{2}+\frac{d}{\rho})} \\
&\quad \times \sum_{\frac{N}{10}\le N_3\le N_0} (N_3^{\mu+2(1-\frac{d}{2}+\frac{d}{\rho})} A(N_3))^{1-\theta} N_3^{-2(1-\theta)(1-\frac{d}{2}+\frac{d}{\rho})+(\frac{d-2s}{2}-\frac{d}{\rho})(\frac{4}{d-2s}-2+\theta)} \\
&\quad \times \sum_{N_3\le N_1\le N_0} (N_1^{\mu+2(1-\frac{d}{2}+\frac{d}{\rho})} A(N_1))^{\theta} N_1^{\frac{d-2s}{2}(2-\theta)-\frac{d}{q_3}-\frac{d}{q_2}-2\theta(1-\frac{d}{2}+\frac{d}{\rho})} \\
&\lesssim \eta^{\frac{4}{d-2s}} \sum_{\frac{N}{10}\le N_3\le N_0} (\frac{N}{N_3})^{-\mu-2(1-\frac{d}{2}+\frac{d}{\rho})-\epsilon} A(N_3)
\end{aligned} \tag{4.18}$$

for some small enough $\epsilon > 0$, such that $-\mu - 2\big(1 - \frac{d}{2} + \frac{d}{\rho}\big) - \epsilon > 0$ (comparing with (4.12)), and we note that the power of N_1 in the second inequality is negative due to the definitions of θ and ρ. Thus combining (4.13)–(4.15), (4.18) and the discrete Gronwall's inequality (2.2), we have

$$\sup_{t\in\mathbb{R}} \|u_N(t)\|_{L^\rho} \lesssim N^{\mu+\omega-}, \tag{4.19}$$

where $\omega = \min\big\{\mu,\ -\mu - 2\big(1-\frac{d}{2}+\frac{d}{\rho}\big) - \epsilon\big\} > 0$.

Summing all the dyadic frequency-localized parts leads to the following property.

Lemma 4.1 *Let u be as Theorem 4.1. Then*

$$u \in L^\infty(\mathbb{R}, L^a(\mathbb{R}^d)) \quad \textit{for } a \in \Big(\frac{2\rho(s+\mu+\omega-)}{s+(\rho-1)(\mu+\omega-)}, \frac{2d}{d-2s}\Big), \tag{4.20}$$

which furthermore implies

$$|\nabla|^s F(u) \in L^\infty(\mathbb{R}, L^b(\mathbb{R}^d)) \quad \textit{for } \frac{1}{b} = \frac{1}{2} + \frac{4}{a(d-2s)}. \tag{4.21}$$

Proof By interpolation and (4.11), we have

$$\begin{aligned}
\|u_N\|_{L^\infty(\mathbb{R},L^a)} &\le \|u_N\|^{1-\theta}_{L^\infty(\mathbb{R},L^\rho)} \|u_N\|^{\theta}_{L^\infty(\mathbb{R},L^2)} \\
&\le (N^{\mu+\omega-})^{1-\theta} N^{-s\theta} \\
&= N^{(\mu+\omega-)-\theta(s+\mu+\omega-)},
\end{aligned}$$

where $\frac{1}{a} = \frac{1-\theta}{\rho} + \frac{\theta}{2}$, and

$$\frac{\frac{1}{2}-\frac{s}{d}-\frac{1}{\rho}}{\frac{1}{2}-\frac{1}{\rho}} < \theta < \frac{\mu+\omega-}{s+\mu+\omega-}, \tag{4.22}$$

which is not empty. From this and Bernstein's inequality, we obtain

$$\begin{aligned}
\|u\|_{L^\infty(\mathbb{R},L^a)} &\le \|u_{\le N_0}\|_{L^\infty(\mathbb{R},L^a)} + \|u_{>N_0}\|_{L^\infty(\mathbb{R},L^a)} \\
&\le \sum_{N\le N_0} N^{(\mu+\omega-)-\theta(s+\mu+\omega-)} + \sum_{N>N_0} \||\nabla|^s u_N\|_{L^\infty(\mathbb{R},L^2)} N^{\frac{d}{2}-\frac{d}{a}-s} < \infty.
\end{aligned}$$

With the fractional chain rule Lemma 2.3, we deduce from these that

$$\||\nabla|^s F(u)\|_{L^\infty(\mathbb{R},L^b)} \le \||\nabla|^s u\|_{L^\infty(\mathbb{R},L^2)} \|u^{\frac{4}{d-2s}}\|_{L^\infty(\mathbb{R},L^{\frac{(d-2s)a}{4}})} < \infty.$$

To conclude Theorem 4.1, we first obtain an inductive lemma.

Lemma 4.2 *If $|\nabla|^\tau F(u) \in L^\infty(\mathbb{R}, L^b(\mathbb{R}^d))$, where $0 \le \tau \le s$ and b is as in the above lemma, then there exists an $s_0 = s_0(b,d) > 0$, such that $u \in L^\infty(\mathbb{R}, \dot{H}^{\tau - s_0+})$.*

Proof We first claim that

$$\||\nabla|^\tau u_N\|_{L^\infty(\mathbb{R},L^2)} \le N^{s_0} \quad \text{with } s_0 = \frac{d}{b} - \frac{d+4}{2} > 0. \tag{4.23}$$

With this claim, we can conclude the result by

$$\begin{aligned}\||\nabla|^{\tau-s_0+}u\|_{L^\infty(\mathbb{R},L^2)} &\le \||\nabla|^{\tau-s_0+}u_{\le 1}\|_{L^\infty(\mathbb{R},L^2)} + \||\nabla|^{\tau-s_0+}u_{>1}\|_{L^\infty(\mathbb{R},L^2)} \\ &\le \sum_{N\le 1} N^{0+} + \sum_{N>1} N^{\tau-s-s_0+} < \infty.\end{aligned}$$

Now we turn to (4.23). Without loss of generality, we use (1.10) to deduce the claim at time $t=0$,

$$\begin{aligned}&\||\nabla|^\tau u_N(0)\|_{L^2}^2 \\ &= \lim_{\substack{T\to\infty \\ T'\to\infty}} \Big\langle \mathrm{i}\int_0^{\mathrm{T}} \mathrm{e}^{-\mathrm{i}t\Delta} P_N |\nabla|^\tau F(u(t))\mathrm{d}t, -\mathrm{i}\int_{T'}^0 \mathrm{e}^{-\mathrm{i}\bar{t}\Delta} P_N|\nabla|^\tau F(u(\bar{t}))\mathrm{d}\bar{t}\Big\rangle \\ &\le \int_0^\infty \int_{-\infty}^0 |\langle P_N|\nabla|^\tau F(u(t)), \mathrm{e}^{-\mathrm{i}(t-\bar{t})\Delta} P_N|\nabla|^\tau F(u(\bar{t}))\rangle| \mathrm{d}t\mathrm{d}\bar{t} \\ &\equiv \int_0^\infty \int_{-\infty}^0 A(N,t,\bar{t})\mathrm{d}t\mathrm{d}\bar{t}.\end{aligned}$$

By the dispersion of Schrödinger operator and Bernstein's inequality, we estimate the integrand

$$\begin{aligned}A(N,t,\bar{t}) &\le \|P_N|\nabla|^\tau F(u(t))\|_{L^b} \|\mathrm{e}^{-\mathrm{i}(t-\bar{t})\Delta} P_N|\nabla|^\tau F(u(\bar{t}))\|_{L^{b'}} \\ &\le |t-\bar{t}|^{\frac{d}{2}-\frac{d}{b}} \||\nabla|^\tau F(u(t))\|_{L^b} \||\nabla|^\tau F(u(\bar{t}))\|_{L^b}, \\ A(N,t,\bar{t}) &\le \|P_N|\nabla|^\tau F(u(t))\|_{L^2} \|\mathrm{e}^{-\mathrm{i}(t-\bar{t})\Delta} P_N|\nabla|^\tau F(u(\bar{t}))\|_{L^2} \\ &\le N^{2(\frac{d}{b}-\frac{d}{2})} \||\nabla|^\tau F(u(t))\|_{L^b} \||\nabla|^\tau F(u(\bar{t}))\|_{L^b}.\end{aligned}$$

Hence, we calculate elementarily

$$\begin{aligned}&\||\nabla|^\tau u_N(0)\|_{L^2}^2 \\ &\le \||\nabla|^\tau F(u)\|^2_{L^\infty(\mathbb{R},L^b)} \int_0^\infty \int_{-\infty}^0 \min\{|t-\bar{t}|^{-1}, N^2\}^{\frac{d}{b}-\frac{d}{2}} \mathrm{d}t\mathrm{d}\bar{t} \\ &= \||\nabla|^\tau F(u)\|^2_{L^\infty(\mathbb{R},L^b)} \\ &\quad \times \Big(\int_{\frac{1}{N^2}}^\infty \int_{-\infty}^0 + \int_0^{\frac{1}{N^2}} \int_{-\infty}^{t-\frac{1}{N^2}} + \int_0^{\frac{1}{N^2}} \int_{t-\frac{1}{N^2}}^0 \min\{|t-\bar{t}|^{-1}, N^2\}^{\frac{d}{b}-\frac{d}{2}} \mathrm{d}t\mathrm{d}\bar{t}\Big) \\ &= N^{2(\frac{d}{b}-\frac{d+4}{2})} \||\nabla|^\tau F(u)\|^2_{L^\infty(\mathbb{R},L^b)}.\end{aligned}$$

Now we show how to use Lemma 4.2 to prove Theorem 4.1. By Lemma 4.1, we can apply Lemma 4.2 to $\tau = s$, and we conclude $u \in L^\infty(\mathbb{R}, \dot{H}^{s-s_0+})$. Thus as the proof of Lemma 4.1, (4.20) and Lemma 2.3 imply $|\nabla|^{s-s_0+}F(u) \in L^\infty(\mathbb{R}, L^b(\mathbb{R}^d))$ for some b as in (4.21). We apply Lemma 4.2 to $\tau = s - s_0+$ and obtain $u \in L^\infty(\mathbb{R}, \dot{H}^{s-2s_0+})$. If b is suitably chosen, we can iterate this procedure finitely many times, and we derive $u \in L^\infty(\mathbb{R}, \dot{H}^{-\varepsilon})$ for some $0 < \varepsilon < s_0$, which completes the proof.

5 The Soliton-Like Solution

Pay attention that the soliton-like solution satisfies the condition of Theorem 4.1, and thus we have the mass conservation by interpolation and the classical local theory. In this case, we first prove a relationship between the length of the time interval and the Strichartz norms on it.

Proposition 5.1 *If u is a soliton-like solution as in Theorem* 1.4, *and J is a compact time interval, then*

$$|J|^{\frac{1}{q}} \lesssim \|u\|_{L^q(J,L^r)} \lesssim 1 + |J|^{\frac{1}{q}}. \tag{5.1}$$

Proof On one hand, by Remark 1.4, for any $\eta > 0$, we have

$$\int_{|x-x(t)|\geq c(\eta)} |u(t)|^{\frac{2d}{d-2s}} \mathrm{d}x \leq \Big(\int_{|x-x(t)|\geq c(\eta)} ||\nabla|^s u|^2 \mathrm{d}x \Big)^{\frac{d}{d-2s}} \leq \eta \quad \text{for all } t \in \mathbb{R}. \tag{5.2}$$

On the other hand, we claim that

$$\int_{|x-x(t)|\leq c(\eta)} |u|^{\frac{2d}{d-2s}} \mathrm{d}x \gtrsim 1 \quad \text{for all } t \in \mathbb{R}. \tag{5.3}$$

Indeed, if (5.3) fails, there exists a time sequence $t_n \underset{n\to\infty}{\longrightarrow} +\infty$, such that

$$\int_{|x-x(t_n)|\leq c(\eta)} |u(t_n)|^{\frac{2d}{d-2s}} \mathrm{d}x \underset{n\to\infty}{\longrightarrow} 0,$$

which together with (5.2) shows that $u(t_n, x - x(t_n)) \underset{n\to\infty}{\longrightarrow} 0$ in $L^{\frac{2d}{d-2s}}$. However, the precompactness and the blowup property of the critical solution show that $u(t_n, x - x(t_n)) \underset{n\to\infty}{\longrightarrow} w$ in $\dot{H}^s$ for some $w \neq 0$, which is a contradiction. We conclude (5.3). A simple application of Hölder's inequality yields

$$\Big[\int_{|x-x(t_n)|\leq c(\eta)} |u(t_n)|^{\frac{2d}{d-2s}} \mathrm{d}x \Big]^{\frac{d-2s}{2d}} \leq [c(\eta)]^{\frac{(d-2s)(1-s)}{d+2(1-s)}} \Big[\int_{\mathbb{R}^d} |u(t_n)|^r \mathrm{d}x \Big]^{\frac{1}{r}}.$$

Thus, using (5.3) and integrating over J, we conclude the first inequality.

Now we turn to the second inequality. Let $\eta > 0$ be a small parameter chosen soon. Partition J into subintervals $\bigcup\limits_k J_k$, such that $|J_k| \leq \eta$, and let $J_k = [t_k, t_{k+1}]$. It requires at most $\eta^{-1}|J| + 1$ intervals. On every subinterval, we use Strichartz's, Hölder's and Bernstein's inequalities

$$\begin{aligned}
&\|u\|_{L^q(J_k,L^r)} \\
&\lesssim \|\mathrm{e}^{\mathrm{i}t-t_k\Delta} u(t_k)\|_{L^q(J_k,L^r)} + \|u\|^{\alpha+1}_{L^q(J_k,L^r)} \\
&\leq \|\mathrm{e}^{\mathrm{i}t-t_k\Delta} u_{\geq N_0}(t_k)\|_{L^q(J_k,L^r)} + \|\mathrm{e}^{\mathrm{i}t-t_k\Delta} u_{\leq N_0}(t_k)\|_{L^q(J_k,L^r)} + \|u\|^{\alpha+1}_{L^q(J_k,L^r)} \\
&\leq \||\nabla|^s u_{\geq N_0}\|_{L^\infty(\mathbb{R},L^2)} + |J_k|^{\frac{1}{q}} N_0^{\frac{d}{2}-\frac{d}{r}-s} \|u_{\leq N_0}(t_k)\|_{L^\infty(\mathbb{R},L^2)} + \|u\|^{\alpha+1}_{L^q(J_k,L^r)}.
\end{aligned} \tag{5.4}$$

By Remark 1.4 and the definition of the soliton-like sulotion, we can choose N_0 large enough so that the first term is small. Then we pick η small enough, depending on N_0 to make the second term small. A simple application of the bootstrap argument deduces

$$\|u\|_{L^q(J_k,L^r)} \le \epsilon \tag{5.5}$$

for some small $\epsilon > 0$. Collecting all the subintervals and the control of the amount of them give the second inequality of (5.1).

In order to defeat the soliton-like solution, we obtain a high-frequency Strichartz's estimate over any compact time interval as in [9, Theorem 5.1], which is used for a frequency localized interaction Morawetz estimate shortly.

Theorem 5.1 *Suppose that u is a soliton-like solution, and J is a compact time interval with $|J| = K$. Then we have*

$$\|P_{\ge N}u\|_{L^2(J,L^{\frac{2d}{d-2}})} \lesssim o\Big(\frac{K^{\frac12}}{N^{\frac12}}\Big) \tag{5.6}$$

for all $N \le K$.

Proof By the Duhamel principle and Strichartz's estimate, we have

$$\begin{aligned}\|P_{\ge N}u\|_{L^2(J,L^{\frac{2d}{d-2}})} &\lesssim \|P_{\ge N}\mathrm{e}^{\mathrm{i}t\Delta}u_0\|_{L^2(J,L^{\frac{2d}{d-2}})} + \|P_{\ge N}(|u|^{\frac{4}{d-2s}}u)\|_{L^2(J,L^{\frac{2d}{d+2}})}\\ &\lesssim \|u_{\ge N}(0)\|_{L^2} + \|P_{\ge N}(|u|^{\frac{4}{d-2s}}u)\|_{L^2(J,L^{\frac{2d}{d+2}})}.\end{aligned} \tag{5.7}$$

For the nonlinearity, we estimate it as

$$\begin{aligned}&\|P_{\ge N}(|u|^{\frac{4}{d-2s}}u)\|_{L^2(J,L^{\frac{2d}{d+2}})}\\ \lesssim{}& \|P_{\ge N}(|u_{\le \eta N}|^{\frac{4}{d-2s}}u_{\le \eta N})\|_{L^2(J,L^{\frac{2d}{d+2}})}\\ &+ \|u_{\ge \eta N}|u_{\ge C_0}|^{\frac{4}{d-2s}}\|_{L^2(J,L^{\frac{2d}{d+2}})} + \|(P_{\ge \eta N}u)|(1-\chi)u_{\le C_0}|^{\frac{4}{d-2s}}\|_{L^2(J,L^{\frac{2d}{d+2}})}\\ &+ \|u_{\ge \eta N}|\chi u_{\le C_0}|^{\frac{4}{d-2s}}\|_{L^2(J,L^{\frac{2d}{d+2}})},\end{aligned} \tag{5.8}$$

where C_0 is a fixed constant chosen later, and $\chi(t,x) \in C_c^\infty(\mathbb{R}^d)$ for every time t with value 1 if $|x - x(t)| \le C_0$, vanishing if $|x - x(t)| \ge 2C_0$. For the first term, by Bernstein's and Hölder's inequalities, the bounded $\dot{H}^s$ norm and the sobolev embedding, for any $\frac12 < \sigma < \frac{4}{d-2s} + 1$,

$$\begin{aligned}&\|P_{\ge N}(|u_{\le \eta N}|^{\frac{4}{d-2s}}u_{\le \eta N})\|_{L^2(J,L^{\frac{2d}{d+2}})}\\ \le{}& \frac{1}{N^\sigma}\||\nabla|^\sigma(|u_{\le \eta N}|^{\frac{4}{d-2s}}u_{\le \eta N})\|_{L^2(J,L^{\frac{2d}{d+2}})}\\ \lesssim{}& \frac{1}{N^\sigma}\||\nabla|^\sigma u_{\le \eta N}\|_{L^2(J,L^{\frac{2d}{d-2}})} + \|u_{\le \eta N}\|^{\frac{4}{d-2s}}_{L^\infty(\mathbb{R},\frac{2d}{d-2s})}\\ \lesssim{}& \sum_{M \le \eta N}\Big(\frac{M}{N}\Big)^\sigma \|u_M\|_{L^2(J,L^{\frac{2d}{d-2}})}\\ \le{}& \sum_{M \le \eta N}\Big(\frac{M}{N}\Big)^\sigma \|u_{\ge M}\|_{L^2(J,L^{\frac{2d}{d-2}})}.\end{aligned}$$

For the second and the third terms,

$$\begin{aligned}
&\|u_{\geq\eta N}|u_{\geq C_0}|^{\frac{4}{d-2s}}\|_{L^2(J,L^{\frac{2d}{d+2}})}+\|(P_{\geq\eta N}u)|(1-\chi)u_{\leq C_0}|^{\frac{4}{d-2s}}\|_{L^2(J,L^{\frac{2d}{d+2}})}\\
&\leq\|u_{\geq\eta N}\|_{L^2(J,L^{\frac{2d}{d-2}})}(\|(1-\chi)u_{\leq C_0}\|_{L^\infty(\mathbb{R},L^{\frac{2d}{d-2s}})}+\|u_{>C_0}\|_{L^\infty(\mathbb{R},L^{\frac{2d}{d-2s}})})^{\frac{4}{d-2s}}\\
&\lesssim\|u_{\geq\eta N}\|_{L^2(J,L^{\frac{2d}{d-2}})}(\|(1-\chi)u\|_{L^\infty(\mathbb{R},L^{\frac{2d}{d-2s}})}+\|u_{>C_0}\|_{L^\infty(\mathbb{R},L^{\frac{2d}{d-2s}})})^{\frac{4}{d-2s}}\\
&\lesssim\delta(C_0)\|u_{\geq\eta N}\|_{L^2(J,L^{\frac{2d}{d-2}})}
\end{aligned}$$

with $\delta(C_0)\to 0$, as $C_0\to\infty$ (see Remark 1.4). Finally, we partition J into $\cup J_k$ as in the proof of Proposition 5.1, and if $4<d-2s$,

$$\begin{aligned}
&\|u_{\geq\eta N}|\chi u_{\leq C_0}|^{\frac{4}{d-2s}}\|_{L^2(J,L^{\frac{2d}{d+2}})}\\
&\leq\||u_{\geq\eta N}u_{\leq C_0}|^{\frac{4}{d-2s}}\chi^{\frac{4}{d-2s}}\|_{L^{\frac{d-2s}{2}}(J,L^{\frac{d(d-2s)}{4(d-s-1)}})}\|u_{\geq\eta N}\|^{1-\frac{4}{d-2s}}_{L^2(J,L^{\frac{2d}{d-2}})}\\
&\lesssim\||u_{\geq\eta N}u_{\leq C_0}|^{\frac{4}{d-2s}}\|_{L^{\frac{d-2s}{2}}(J,L^{\frac{d-2s}{2}})}\|\chi^{\frac{4}{d-2s}}\|_{L^\infty(J,L^{\frac{d(d-2s)}{2(d-2s-2)}})}\|u_{\geq\eta N}\|^{1-\frac{4}{d-2s}}_{L^2(J,L^{\frac{2d}{d-2}})}\\
&\lesssim\Big(\sum_k\|u_{\geq\eta N}u_{\leq C_0}\|^2_{L^2(J_k,L^2)}\Big)^{\frac{2}{d-2s}}\|u_{\geq\eta N}\|^{1-\frac{4}{d-2s}}_{L^2(J,L^{\frac{2d}{d-2}})}.
\end{aligned}$$

Applying the bilinear Strichartz's estimate (3.4), we have

$$\begin{aligned}
&\|u_{\geq\eta N}|\chi u_{\leq C_0}|^{\frac{4}{d-2s}}\|_{L^2(J,L^{\frac{2d}{d+2}})}\\
&\lesssim\Big[\sum_k\Big(\frac{C_0^{\frac{d-1}{2}}}{(\eta N)^{\frac12}}\Big)^2\|u_{\geq\eta N}\|^2_{S^0_*(J_k)}\|u_{\leq C_0}\|^2_{S^0_*(J_k)}\Big]^{\frac{2}{d-2s}}\|u_{\geq\eta N}\|^{1-\frac{4}{d-2s}}_{L^2(J,L^{\frac{2d}{d-2}})}\\
&\lesssim\Big(\sum_k 1\Big)^{\frac{2}{d-2s}}\|u_{\geq\eta N}\|^{1-\frac{4}{d-2s}}_{L^2(J,L^{\frac{2d}{d-2}})}(\eta N)^{-\frac{2}{d-2s}}\sup_k\|u_{\geq\eta N}\|^{\frac{4}{d-2s}}_{S^0_*(J_k)}\\
&\lesssim\Big(\frac{K}{\eta N}\Big)^{\frac{2}{d-2s}}\|u_{\geq\eta N}\|^{1-\frac{4}{d-2s}}_{L^2(J,L^{\frac{2d}{d-2}})}\sup_k\|u_{\geq\eta N}\|^{\frac{4}{d-2s}}_{S^0_*(J_k)}.
\end{aligned}$$

Here we use a claim that $\sharp J_k\sim|J|$. Now we turn to the proof of the claim. By Proposition 5.1 and the definition of the subintervals, we have $|J_k|\lesssim\epsilon^{\boldsymbol{q}}$, and hence $|J|=\sum\limits_k|J_k|\lesssim\sharp J_k$. Choose η as in Remark 1.4 to be small depending on ϵ, by Strichartz estimates and the interpolation, we can show that

$$\|u_{>C(\eta)}\|_{L^{\boldsymbol{q}}(J_k,L^{\boldsymbol{r}})}\lesssim\epsilon.$$

On the other hand, by Bernsteins inequality,

$$\|u_{\leq C(\eta)}(t)\|_{L^{\boldsymbol{r}}}\leq\|u_{\leq C(\eta)}(t)\|_{L^{\frac{2d}{d-2s}}}(C(\eta))^{\frac{d-2s}{2}-\frac{d}{\boldsymbol{r}}}\lesssim 1.$$

Integrating this on J_k, we have $\epsilon^{\boldsymbol{q}}\lesssim|J_k|$, and thus $\sharp J_k\lesssim|J|$, which concludes the claim. If $2\leq d-2s\leq 4$, similarly, we have

$$\begin{aligned}
&\|u_{\geq\eta N}|\chi u_{\leq C_0}|^{\frac{4}{d-2s}}\|_{L^2(J,L^{\frac{2d}{d+2}})}\\
&\leq\|u_{\geq\eta N}|u_{\leq C_0}|\|_{L^2(J,\,L^2)}\|\chi\|_{L^\infty(J,\,L^{r_1})}\||u|^{\frac{4}{d-2s}-1}\|_{L^\infty(J,\,L^{r_2})},
\end{aligned}$$

where $\frac{d+2}{2d}=\frac{1}{2}+\frac{1}{r_1}+\frac{1}{r_2}$ with $2\le r_2\big(\frac{4}{d-2s}-1\big)\le\frac{2d}{d-2s}$. The assumption and (1.6) guarantee the existence of r_1 and r_2. We continue as

$$\begin{aligned}&\big\|u_{\ge\eta N}|\chi u_{\le C_0}|^{\frac{4}{d-2s}}\big\|_{L^2(J,L^{\frac{2d}{d+2}})}\\ &\lesssim(\eta N)^{-\frac12}\Big(\sum_k 1\Big)^{\frac12}\sup_k\|u_{\ge\eta N}\|_{S^0_*(J_k)}\\ &=\Big(\frac{K}{\eta N}\Big)^{\frac12}\sup_k\|u_{\ge\eta N}\|_{S^0_*(J_k)}.\end{aligned}$$

Collecting these estimates, we obtain for $d-2s>4$,

$$\begin{aligned}&\|P_{\ge N}u\|_{L^2(J,L^{\frac{2d}{d-2}})}\\ &\lesssim\|u_{\ge N}(0)\|_{L^2}+\delta(C_0)\|u_{\ge\eta N}\|_{L^2(J,L^{\frac{2d}{d-2}})}\\ &\quad+\sum_{M\le\eta N}\Big(\frac{M}{N}\Big)^{\sigma}\|u_{\ge M}\|_{L^2(J,L^{\frac{2d}{d-2}})}+\Big(\frac{K}{\eta N}\Big)^{\frac{2}{d-2s}}\|u_{\ge\eta N}\|^{1-\frac{4}{d-2s}}_{L^2(J,L^{\frac{2d}{d-2}})}\sup_k\|u_{\ge\eta N}\|^{\frac{4}{d-2s}}_{S^0_*(J_k)},\end{aligned}\tag{5.9}$$

and for $2\le d-2s\le4$,

$$\begin{aligned}&\|P_{\ge N}u\|_{L^2(J,L^{\frac{2d}{d-2}})}\\ &\lesssim\|u_{\ge N}(0)\|_{L^2}+\delta(C_0)\|u_{\ge\eta N}\|_{L^2(J,L^{\frac{2d}{d-2}})}\\ &\quad+\sum_{M\le\eta N}\Big(\frac{M}{N}\Big)^{\sigma}\|u_{\ge M}\|_{L^2(J,L^{\frac{2d}{d-2}})}+\Big(\frac{K}{\eta N}\Big)^{\frac12}\sup_k\|u_{\ge\eta N}\|_{S^0_*(J_k)}.\end{aligned}\tag{5.10}$$

With (5.9), we follow the induction argument as in the proof of [9, Theorem 5.1]. We start with the basic case.

Lemma 5.1 *Let J be a compact interval, and $\|u\|_{L^q(J,L^r)}=C<\infty$. Then Theorem 5.1 holds if $N\le\frac{K}{C}$.*

Proof Partition J into $O(C)$ subintervals $\cup J_k$ with $\|u\|_{L^q(J_k,L^r)}=\epsilon$, where ϵ is small. By Duhamel's principle and Strichartz's estimate, we have $\|u\|_{S^0_*(J_k)}\lesssim1$. Summing these up, we obtain

$$\|u\|^2_{L^2(J,L^{\frac{2d}{d-2}})}\le\sum_k\|u\|^2_{L^2(J_k,L^{\frac{2d}{d-2}})}\lesssim\sum_k1\lesssim C\le\frac{K}{N},$$

which includes the lemma.

We need a last lemma to complete the proof of Theorem 5.1.

Lemma 5.2 *There exists a function $\rho(N)\le C$ and $\lim\limits_{N\to\infty}\rho(N)=0$ with*

$$\|u_{\ge N}\|_{L^2}+\|u_{\ge\eta N}\|^{\frac{4}{d-2s}}_{S^0_*(J_k)}\le\rho(N).$$

Proof It is easy to see that $\lim\limits_{N\to\infty}\|u_{\ge N}\|_{L^2}=0$. For the second term, we estimate by using (5.5). For any $\dot{H}^s$-admissible pair (q_1,r_1) with $\frac{2d}{d-2s}<q_1<\frac{2d}{d-2}$ and $\frac{2}{q_1}+\frac{d}{r_1}=\frac{d}{2}-s$, the inhomogeneous Strichartz's estimate implies

$$\|u\|_{L^{q_1}(J_k,L^{r_1})} \lesssim \|e^{it\Delta}u_0\|^{\theta}_{L^\infty(J_k,\ \dot H^s)}\|e^{it\Delta}|\nabla|^s u_0\|^{1-\theta}_{L^2(J_k,\ L^{\frac{2d}{d-2}})} + \|u\|^{\alpha+1}_{L^{\mathbf{q}}(J_k,L^{\mathbf{r}})} \lesssim \epsilon.$$

Thus for any admissible pair (q,r), we estimate

$$\begin{aligned}\|u\|_{L^q(J_k,L^r)} &\lesssim \|u_0\|_{L^2} + \||u|^{\frac{4}{d-2s}}u\|_{L^{q_0}(J_k,\ L^{r_0})}\\ &\lesssim \|u_0\|_{L^2} + \||u|^{\frac{4}{d-2s}}\|_{L^{\frac{q_1}{\frac{4}{d-2s}}}(J_k,\ L^{\frac{r_1}{\frac{4}{d-2s}}})}\|u\|_{L^q(J_k,L^r)}\end{aligned}$$

for some suitable admissible pair (q_0,r_0) and some suitable $\dot H^s$-admissible pair (q_1,t_1) (the definitions of these two pairs ensure the existence of such two pairs). Now we turn to$\|u_{\geq\eta N}\|_{S^0_*(J_k)}$. By Strichartz's estimate and the element decomposition, we have

$$\begin{aligned}\|u_{\geq\eta N}\|_{S^0_*(J_k)} &\leq \|u_{\geq\eta N}\|_{L^\infty(L^2)} + \|P_{\geq\eta N}(|u|^{\frac{4}{d-2s}}u)\|_{L^{\widetilde q_0'}(J_k,\ L^{\widetilde r_0'})}\\ &\lesssim \|u_{\geq\eta N}\|_{L^\infty(L^2)} + \|P_{\geq\eta N}(|u_{<N^{\frac12}}|^{\frac{4}{d-2s}}u_{<N^{\frac12}})\|_{L^{\widetilde q_0'}(J_k,\ L^{\widetilde r_0'})}\\ &\quad + \||u_{\geq N^{\frac12}}|^{\frac{4}{d-2s}}u\|_{L^{\widetilde q_0'}(J_k,\ L^{\widetilde r_0'})} + +\||u|^{\frac{4}{d-2s}}u_{\geq N^{\frac12}}\|_{L^{\widetilde q_0'}(J_k,\ L^{\widetilde r_0'})}.\end{aligned}$$

Employing Bernstein's and Hölder's inequalities, we deduce

$$\begin{aligned}&\|u_{\geq\eta N}\|_{S^0_*(J_k)}\\ &\lesssim \|u_{\geq\eta N}\|_{L^\infty(L^2)}\\ &\quad + N^{-s}\||\nabla|^s u_{<N^{\frac12}}\|^{\theta_1}_{L^\infty(\mathbb{R},\ L^2)}\||\nabla|^s u_{<N^{\frac12}}\|^{1-\theta_1}_{L^{\widetilde q_1}(J_k,\ L^{\widetilde r_1})}\|u\|^{\theta_2}_{L^\infty(J_k,\ L^{\frac{2d}{d-2s}})}\|u\|^{\alpha-\theta_2}_{L^{\widetilde q_2}(J_k,\ L^{\widetilde r_2})}\\ &\quad + \|u_{\geq N^{\frac12}}\|^{\theta_1}_{L^\infty(\mathbb{R},\ L^2)}\|u\|^{1-\theta_1}_{L^{\widetilde q_1}(J_k,\ L^{\widetilde r_1})}\|u\|^{\theta_2}_{L^\infty(J_k,\ L^{\frac{2d}{d-2s}})}\|u\|^{\alpha-\theta_2}_{L^{\widetilde q_2}(J_k,\ L^{\widetilde r_2})}\\ &\quad + \|u\|^{\theta_1}_{L^\infty(\mathbb{R},\ L^2)}\|u\|^{1-\theta_1}_{L^{\widetilde q_1}(J_k,\ L^{\widetilde r_1})}\|u_{\geq N^{\frac12}}\|^{\theta_2}_{L^\infty(J_k,\ L^{\frac{2d}{d-2s}})}\|u\|^{\alpha-\theta_2}_{L^{\widetilde q_2}(J_k,\ L^{\widetilde r_2})}\\ &\lesssim \|u_{\geq\eta N}\|_{L^\infty(\mathbb{R},\ L^2)} + N^{-\frac{s}{2}}\|u\|^{\theta_1}_{L^\infty(\mathbb{R},\ L^2)} + \|u_{\geq N^{\frac12}}\|^{\theta_1}_{L^\infty(\mathbb{R},\ L^2)} + \|u_{\geq N^{\frac12}}\|^{\theta_2}_{L^\infty(\mathbb{R},\ L^{\frac{2d}{d-2s}})},\end{aligned}$$

where we choose suitable admissible pairs $(\widetilde q_0,\widetilde r_0)$, $(\widetilde q_1,\widetilde r_1)$, an $\dot H^s$-admissible pair $(\widetilde q_2,\widetilde r_2)$ and parameters $\theta_1\in(0,1)$ and $\theta_2\in(0,\alpha)$ (according to the definition of an admissible pair, we have such exponents). Letting $N\to\infty$, the RHS of the above inequality goes to 0 (wherein we use the $\dot H^s(\mathbb{R}^d)$- and $L^2(\mathbb{R}^d)$- compactness of the orbit of $\{u(t,\cdot-x(t)) : t\in\mathbb{R}\}$).

By taking $\rho(N) = \|u_{\geq N}\|_{L^\infty(L^2)} + \|u_{\geq\eta N}\|^{\frac{4}{d-2s}}_{S^0_*(J_k)}$, we complete the proof.

Applying this lemma to (5.9) with the induction assumption, we get

$$\begin{aligned}\|P_{\geq N}u\|_{L^2(J,L^{\frac{2d}{d-2}})} &\lesssim \rho(N) + \delta(C_0)o(1)\\ &\quad + \Big(\frac{K}{\eta N}\Big)^{\frac12}\sum_{M\leq\eta N}\Big(\frac{M}{N}\Big)^{\sigma-\frac12}o(1) + \Big(\frac{K}{\eta N}\Big)^{\frac{2}{d-2s}}\Big(\frac{K}{\eta N}\Big)^{\frac12(1-\frac{4}{d-2s})}\rho(\eta N)\\ &\lesssim \Big(\frac{K}{N}\Big)^{\frac12}(\rho(\eta N)+o(1)) \lesssim o\Big(\frac{K}{N}\Big)^{\frac12}.\end{aligned}$$

We can handle the case $2\leq d-2s\leq 4$ similarly and omit the proof. These complete the proof of Theorem 5.1.

We prove a frequency local interaction Morawetz estimate as [9, Theorem 6.1] to defeat the soliton-like solution scenario. First we define a frequency cut-off operator I. Let C be a large fixed constant chosen later. Define $I: \ L^2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)$ as an operator given by the Fourier multiplier $m(\xi) \in C_c^\infty(\mathbb{R}^d)$ as $\widehat{Iv}(\xi) = m(\xi)\widehat{v}(\xi)$, where

$$m(\xi) = \begin{cases} 1, & \text{if } |\xi| \le CK, \\ 0, & \text{if } |\xi| \ge 2CK, \end{cases}$$

and K is large and chosen later. We state the frequency local interaction Morawetz estimate as following.

Theorem 5.2 *Let u be a soliton-like solution. Given any $K > 0$, let J be a compact time interval with $|J| = K$. Then*

$$\int_J \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x-y|^3} |Iu(t,x)|^2 |Iu(t,y)|^2 \mathrm{d}x\mathrm{d}y\mathrm{d}t \lesssim o(K). \tag{5.11}$$

We postpone the proof of Theorem 5.2. Assuming this theorem, we defeat the soliton-like solutions. Choose K large enough. Fixing $\eta > 0$ small enough, we have

$$\int_{|\xi| > \frac{C(\eta)K}{2}} |\widehat{u}(t,\xi)|^2 \mathrm{d}\xi \le \frac{1}{(C(\eta)K)^{2s}} \int_{|\xi| > \frac{C(\eta)K}{2}} |\xi|^{2s} |\widehat{u}|^2 \mathrm{d}\xi \le \eta.$$

Hence

$$\int_{|x-x(t)| \le C(\eta)} |(1-I)u(t)|^2 \mathrm{d}x \le \int |(1-I)u(t)|^2 \mathrm{d}x \le \eta. \tag{5.12}$$

Since $\{u(t, \dot{-}x(t)); t \in \mathbb{R}\}$ is precompact in $\dot H^s$ and $\{u(t, \dot{-}x(t)); t \in \mathbb{R}\}$ is bounded in $\dot H^{-\epsilon}$ for some $\epsilon > 0$, as shown in Theorem 4.1, it is easy to see that $\{u(t, \dot{-}x(t)); t \in \mathbb{R}\}$ is precompact in L^2. By Remark 4.2, u has a conserved mass m_0, and with the help of the proposition of the precompactness in L^2, we have

$$\int_{|x-x(t)| \le C(\eta)} |u(t)|^2 \mathrm{d}x \ge m_0^2 - \eta. \tag{5.13}$$

(5.12)–(5.13) imply that

$$\int_{|x-x(t)| \le C(\eta)} |Iu(t)|^2 \mathrm{d}x \ge \frac{m_0^2}{2} \sim 1. \tag{5.14}$$

Taking a square of the above inequality, we obtain

$$\begin{aligned} 1 &\lesssim \int_{|x-x(t)| \le C(\eta)} |Iu(t,x)|^2 \mathrm{d}x \int_{|y-x(t)| \le C(\eta)} |Iu(t,y)|^2 \mathrm{d}y \\ &\lesssim \int_{|x-y| \le 2C(\eta)} |Iu(t,x)|^2 |Iu(t,y)|^2 \mathrm{d}x\mathrm{d}y \\ &\lesssim \int_{\mathbb{R}^{2d}} \frac{1}{|x-y|^3} |Iu(t,x)|^2 |Iu(t,y)|^2 \mathrm{d}x\mathrm{d}y. \end{aligned} \tag{5.15}$$

Integrating (5.15) on a compact time interval J with $|J| = K$ forms

$$K = |J| \lesssim \int_J \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{1}{|x-y|^3} |Iu(t,x)|^2 |Iu(t,y)|^2 \mathrm{d}x\mathrm{d}y\mathrm{d}t. \tag{5.16}$$

On the other hand, we have (5.11), which implies $K \lesssim o(K)$. This causes a contradiction if K is large enough. Hence the soliton-like solution does not exist. The remainder of this section is devoted to the proof of Theorem 5.2.

Proof of Theorem 5.2 We define the interaction Morawetz quantity $M_a(t)$ of the solution u at time t by

$$M_a = 2\sum_{j=1}^{2d} \int_{\mathbb{R}^{2d}} a_j(x,y)\mathrm{Im}[Iu(x)Iu(y)\partial_j(Iu(t,x)Iu(t,y))]\mathrm{d}x\mathrm{d}y,$$

where $a(x,y) = |x-y|$, and a_j is short for $\frac{\partial a}{\partial x_j}$ if $1 \le j \le d$ and $\frac{\partial a}{\partial y_j}$ if $d+1 \le j \le 2d$. We set $z = (x,y)$ and $w(z) = Iu(x)Iu(y)$. According to (1.1) and the definition of the operator I, w satisfies

$$\begin{aligned} &\mathrm{i}\partial_t w(t,z) + \Delta_z w(t,z) \\ &= Iu(t,x)I(|u|^\alpha u)(t,y) + I(|u|^\alpha u)(t,x)Iu(t,y) \\ &\equiv F_1(t,z) + F_2(t,z) \equiv F(t,z). \end{aligned} \tag{5.17}$$

The fundmental theorem of calculus shows that

$$M_a(T) - M_a(0) = \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} M_a(t)\mathrm{d}t = 2\sum_{j=1}^{2d} \int_0^{\mathrm{T}} \int_{\mathbb{R}^{2d}} a_j(x,y)\partial_t \mathrm{Im}(\overline{w}\partial_j w)\mathrm{d}x\mathrm{d}y\mathrm{d}t.$$

Inserting (5.17) into this identity, we deduce

$$\begin{aligned} M_a(T) - M_a(0) = &\int_0^T \int a_j \partial_{jkk}(|w|^2)\mathrm{d}z\mathrm{d}t - 4\int_0^T \int a_j \mathrm{Re}\partial_k(\partial_j \overline{w}\partial_k w)\mathrm{d}z\mathrm{d}t \\ &+ \int_0^T \int a_j F(t,z)\partial_j \overline{w}\mathrm{d}z\mathrm{d}t + \int_0^T \int a_j \overline{F}(t,z)\partial_j w\mathrm{d}z\mathrm{d}t \\ &- \int_0^T \int a_j \partial_j F(t,z)\overline{w}\mathrm{d}z\mathrm{d}t - \int_0^T \int a_j \partial_j \overline{F}(t,z) w\mathrm{d}z\mathrm{d}t, \end{aligned}$$

where we use the Einstein summation convention and sum from 1 to $2d$ for every subindex. Inserting the nonlinearity explicitly and integrating by parts, we have

$$\begin{aligned} M_a(T) - M_a(0) = &\int_0^T \int_{\mathbb{R}^{2d}} (-\Delta\Delta a(x,y))|Iu(t,x)|^2|Iu(t,y)|^2\mathrm{d}x\mathrm{d}y\mathrm{d}t \\ &+ 4\int_0^{\mathrm{T}} \int_{\mathbb{R}^{2d}} a_{jk}(x,y)\mathrm{Re}(\partial_j\overline{w}\partial_k w)\mathrm{d}x\mathrm{d}y\mathrm{d}t \\ &+ 2\int_0^{\mathrm{T}} \int_{\mathbb{R}^{2d}} a_j(x,y)I(|u|^\alpha u)(t,x)\partial_j I\overline{u}(t,x)|Iu(t,y)|^2\mathrm{d}x\mathrm{d}y\mathrm{d}t \\ &+ 2\int_0^{\mathrm{T}} \int_{\mathbb{R}^{2d}} a_j(x,y)I(|u|^\alpha \overline{u})(t,x)\partial_j Iu(t,x)|Iu(t,y)|^2\mathrm{d}x\mathrm{d}y\mathrm{d}t \end{aligned}$$

$$+\int_0^{\mathrm{T}}\int_{\mathbb{R}^{2d}} a_{jj}(x,y)I(|u|^\alpha u)(t,x)I\overline{u}(t,x)|Iu(t,y)|^2\mathrm{d}x\mathrm{d}y\mathrm{d}t$$
$$+\int_0^{\mathrm{T}}\int_{\mathbb{R}^{2d}} a_{jj}(x,y)I(|u|^\alpha \overline{u})(t,x)Iu(t,x)|Iu(t,y)|^2\mathrm{d}x\mathrm{d}y\mathrm{d}t$$
$$+\int_0^{\mathrm{T}}\int_{\mathbb{R}^{2d}} a_{j}(x,y)(I(|u|^\alpha u)(t,y)I\overline{u}(t,y)-I(|u|^\alpha \overline{u})(t,y)Iu(t,y))$$
$$\times Iu(t,x)\partial_j I\overline{u}(t,x)\mathrm{d}x\mathrm{d}y\mathrm{d}t$$
$$+\int_0^{\mathrm{T}}\int_{\mathbb{R}^{2d}} a_{j}(x,y)(I(|u|^\alpha \overline{u})(t,y)Iu(t,y)-I(|u|^\alpha u)(t,y)I\overline{u}(t,y))$$
$$\times \partial_j Iu(t,x)I\overline{u}(t,x)\mathrm{d}x\mathrm{d}y\mathrm{d}t$$
$$+\,8 \text{ similar terms as the last above with } x,\ y \text{ exchanged.}$$

For further estimation, we transform this identity into

$$M_a(T)-M_a(0)=\int_0^{T}\int_{\mathbb{R}^{2d}}(-\Delta\Delta a(x,y))|Iu(t,x)|^2|Iu(t,y)|^2\mathrm{d}x\mathrm{d}y\mathrm{d}t \tag{5.18}$$
$$+\,4\sum_{j,k=1}^{2d}\int_0^{\mathrm{T}}\int_{\mathbb{R}^{2d}} a_{jk}(x,y)\mathrm{Re}(\partial_j\overline{w}\partial_k w)\mathrm{d}x\mathrm{d}y\mathrm{d}t \tag{5.19}$$
$$+\,\frac{4}{d+2-2s}\sum_{j=1}^{d}\int_0^{\mathrm{T}}\int_{\mathbb{R}^{2d}} a_{jj}(x,y)|Iu(t,x)|^{\alpha+2}|Iu(t,y)|^2\mathrm{d}x\mathrm{d}y\mathrm{d}t \tag{5.20}$$
$$+\sum_{j=1}^{d}\int_0^{\mathrm{T}}\int_{\mathbb{R}^{2d}} a_{j}(x,y)[I(|u|^\alpha u)(t,y)I\overline{u}(t,y)-I(|u|^\alpha \overline{u})(t,y)Iu(t,y)]$$
$$\times Iu(t,x)\partial_j I\overline{u}(t,x)\mathrm{d}x\mathrm{d}y\mathrm{d}t \tag{5.21}$$
$$+\sum_{j=1}^{d}\int_0^{\mathrm{T}}\int_{\mathbb{R}^{2d}} a_{j}(x,y)[I(|u|^\alpha \overline{u})(t,y)Iu(t,y)-I(|u|^\alpha u)(t,y)I\overline{u}(t,y)]$$
$$\times \partial_j Iu(t,x)I\overline{u}(t,x)\mathrm{d}x\mathrm{d}y\mathrm{d}t \tag{5.22}$$
$$+\,2\sum_{j=1}^{d}\int_0^{\mathrm{T}}\int_{\mathbb{R}^{2d}} a_{j}(x,y)[I(|u|^\alpha u)(t,x)-|Iu|^\alpha(t,x)Iu(t,x)]$$
$$\times |Iu(t,y)|^2\partial_j I\overline{u}(t,x)\mathrm{d}x\mathrm{d}y\mathrm{d}t \tag{5.23}$$
$$+\,2\sum_{j=1}^{d}\int_0^{\mathrm{T}}\int_{\mathbb{R}^{2d}} a_{j}(x,y)[I(|u|^\alpha \overline{u})(t,x)-|Iu|^\alpha(t,x)I\overline{u}(t,x)]$$
$$\times |Iu(t,y)|^2\partial_j Iu(t,x)\mathrm{d}x\mathrm{d}y\mathrm{d}t \tag{5.24}$$
$$+\sum_{j=1}^{d}\int_0^{\mathrm{T}}\int_{\mathbb{R}^{2d}} a_{jj}(x,y)[I(|u|^\alpha u)(t,x)-|Iu|^\alpha(t,x)Iu(t,x)]$$
$$\times |Iu(t,y)|^2 I\overline{u}(t,x)\mathrm{d}x\mathrm{d}y\mathrm{d}t \tag{5.25}$$
$$+\sum_{j=1}^{d}\int_0^{\mathrm{T}}\int_{\mathbb{R}^{2d}} a_{jj}(x,y)[I(|u|^\alpha \overline{u})(t,x)-|Iu|^\alpha(t,x)I\overline{u}(t,x)]$$
$$\times |Iu(t,y)|^2 Iu(t,x)\mathrm{d}x\mathrm{d}y\mathrm{d}t \tag{5.26}$$

$$+ (5.20) + \cdots + (5.26) \text{ with } x,\ y \text{ exchanged.} \tag{5.27}$$

A simple calculus shows that $-\Delta\Delta a = \frac{4(d-1)(d-3)}{|x-y|^3}$. Comparing this with (5.11), we need to control the other terms in (5.18)–(5.27).

Estimates for (5.19)–(5.20) Since the matrix with elements a_{jk} is positively defined, $(5.19) \geq 0$. Direct computer shows $\sum\limits_{j}^{2d} a_{jj} = \frac{2(d-1)}{|x-y|}$, and therefore $(5.20) \geq 0$.

Estimates for (5.23)–(5.24) Since the two terms are similar, we only control (5.23). By the triangle inequality, we have

$$\begin{aligned}
&\|I(|u|^\alpha u) - |Iu|^\alpha Iu\|_{L^2(J, L^{\frac{2d}{d+2}})} \\
&\leq \|I(|Iu|^\alpha u) - |Iu|^\alpha Iu\|_{L^2(J, L^{\frac{2d}{d+2}})} + \|I(|u|^\alpha u - |Iu|^\alpha Iu)\|_{L^2(J, L^{\frac{2d}{d+2}})} \\
&\lesssim \|P_{>CK}(|Iu|^\alpha Iu)\|_{L^2(J, L^{\frac{2d}{d+2}})} + \||u|^\alpha u - |Iu|^\alpha Iu\|_{L^2(J, L^{\frac{2d}{d+2}})},
\end{aligned}$$

where the last inequality uses the definition of the operator I. Applying Bernstein's and Hölder's inequalities,

$$\begin{aligned}
\text{LHS} &\lesssim \frac{1}{CK}\|\nabla(|Iu|^\alpha Iu)\|_{L^2(J, L^{\frac{2d}{d+2}})} + \|u_{>CK}\|_{L^2(J, L^{\frac{2d}{d-2}})}\|u\|^{\frac{4}{d-2s}}_{L^\infty(\mathbb{R}, L^{\frac{2d}{d-2s}})} \\
&\lesssim \frac{1}{CK}\|\nabla Iu\|_{L^2(J, L^{\frac{2d}{d+2}})}\|Iu\|^{\frac{4}{d-2s}}_{L^\infty(\mathbb{R}, L^{\frac{2d}{d-2s}})} + \|u_{>CK}\|_{L^2(J, L^{\frac{2d}{d-2}})}\|u\|^{\frac{4}{d-2s}}_{L^\infty(\mathbb{R}, L^{\frac{2d}{d-2s}})}.
\end{aligned}$$

Since the $\dot{H}^s$-norm of the solution is bounded, by Sobolev embedding, Bernstein's inequality and (5.6), we continue

$$\begin{aligned}
\text{LHS} &\lesssim \frac{1}{CK}\sum_{M \leq CK}\|\nabla u_M\|_{L^2(J, L^{\frac{2d}{d+2}})} + \|u_{>CK}\|_{L^2(J, L^{\frac{2d}{d-2}})} \\
&\lesssim \frac{1}{CK}\sum_{M \leq CK} M o\Big(\Big(\frac{K}{M}\Big)^{\frac12}\Big) + o\Big(\Big(\frac{K}{K}\Big)^{\frac12}\Big) = o(1),
\end{aligned}$$

which is acceptable for later use. Similarly,

$$\begin{aligned}
\|\partial_j I\overline{u}\|_{L^2(J, L^{\frac{2d}{d-2}})} &\lesssim \sum_{M \leq 2K} M\|u_{>M}\|_{L^2(J, L^{\frac{2d}{d-2}})} \\
&\lesssim \sum_{M \leq 2K} M o\Big(\Big(\frac{K}{M}\Big)^{\frac12}\Big) \leq o(K).
\end{aligned}$$

Now we turn to (5.23). By Hölder's inequality,

$$(5.23) \leq \|I(|u|^\alpha u) - |Iu|^\alpha Iu\|_{L^2(J, L^{\frac{2d}{d+2}})}\|\partial_j I\overline{u}\|_{L^2(J, L^{\frac{2d}{d-2}})}\|Iu\|_{L^\infty(\mathbb{R}, L^2)} \lesssim o(K),$$

which is acceptable for Theorem 5.2.

Estimates for (5.21)–(5.22) As above, we only prove (5.21). The following lemma will imply acceptable estimates immediately.

Lemma 5.3

$$\begin{aligned}
&\Big|\int_0^T\int_{\mathbb{R}^{2d}} a_j[I(|u|^\alpha u)(t,y)I\overline{u}(t,y) - I(|u|^\alpha \overline{u})(t,y)Iu(t,y)]|Iu(t,x)|^2\mathrm{d}x\mathrm{d}y\mathrm{d}t\Big|\\
&= \Big|2\mathrm{i}\int_0^{\mathrm{T}}\int_{\mathbb{R}^{2d}} a_j\mathrm{Im}[I(|u|^\alpha u)(t,y)I\overline{u}(t,y)]|Iu(t,x)|^2\mathrm{d}x\mathrm{d}y\mathrm{d}t\Big|\\
&\lesssim o(1).
\end{aligned}$$

Proof We split $|u|^{\alpha+2}$ as

$$\begin{aligned}
|u|^{\alpha+2} = {}& I(|u|^\alpha u)I\overline{u} + I(|u|^\alpha u)(1-I)\overline{u}\\
&+ (1-I)(|u|^\alpha u)I\overline{u} + (1-I)(|u|^\alpha u)(1-I)\overline{u}.
\end{aligned}$$

We insert this into the equation in the lemma,

$$\begin{aligned}
&2\mathrm{i}\int_0^T\int_{\mathbb{R}^{2d}} a_j\mathrm{Im}I(|u|^\alpha u)(t,y)|I\overline{u}(t,y)|Iu(t,x)|^2\mathrm{d}x\mathrm{d}y\mathrm{d}t\\
&= -2\mathrm{i}\int_0^T\int_{\mathbb{R}^{2d}} a_j\mathrm{Im}(1-I)(|u|^\alpha u)(t,y)I\overline{u}(t,y)|Iu(t,x)|^2\mathrm{d}x\mathrm{d}y\mathrm{d}t \qquad (5.28)\\
&\quad - 2\mathrm{i}\int_0^T\int_{\mathbb{R}^{2d}} a_j\mathrm{Im}I(|u|^\alpha u)(t,y)(1-I)\overline{u}(t,y)|Iu(t,x)|^2\mathrm{d}x\mathrm{d}y\mathrm{d}t \qquad (5.29)\\
&\quad - 2\mathrm{i}\int_0^T\int_{\mathbb{R}^{2d}} a_j\mathrm{Im}(1-I)(|u|^\alpha u)(t,y)(1-I)\overline{u}(t,y)|Iu(t,x)|^2\mathrm{d}x\mathrm{d}y\mathrm{d}t. \qquad (5.30)
\end{aligned}$$

Consider (5.30) first by Hölder's inequality, (5.6) and the mass conservation

$$\begin{aligned}
(5.30) &\leq \|P_{>CK}(|u|^\alpha u)\|_{L^2(J,L^{\frac{2d}{d+2}})}\|u_{>CK}\|_{L^2(J,L^{\frac{2d}{d-2}})}\|Iu\|^2_{L^\infty(\mathbb{R},L^2)}\\
&\lesssim o(1)(\|P_{>CK}(|u_{\leq CK}|^\alpha u_{\leq CK})\|_{L^2(J,L^{\frac{2d}{d+2}})}\\
&\qquad + \|u_{>CK}\|_{L^2(J,L^{\frac{2d}{d-2}})}\|u\|^{\frac{4}{d-2s}}_{L^\infty(\mathbb{R},L^{\frac{2d}{d-2s}})})\\
&\lesssim o(1)\Big(\frac{1}{CK}\|\nabla u_{\leq CK}\|_{L^2(J,L^{\frac{2d}{d-2}})}\|u\|^{\frac{4}{d-2s}}_{L^\infty(\mathbb{R},L^{\frac{2d}{d-2s}})} + o(1)\Big)\\
&\lesssim o(1)\sum_{M\leq CK}\frac{M}{CK}\|u_{>M}\|_{L^2(J,L^{\frac{2d}{d-2}})} + o(1)\\
&\lesssim o(1)\sum_{M\leq CK}\frac{M}{CK}o\Big(\Big(\frac{K}{M}\Big)^{\frac{1}{2}}\Big) + o(1) \lesssim o(1).
\end{aligned}$$

This is acceptable. For (5.28),

$$\begin{aligned}
(5.28) &= -2\mathrm{i}\mathrm{Im}\int_0^T\int_{\mathbb{R}^{2d}} a_j\Big[\frac{\nabla_y\cdot\nabla_y}{\nabla_y\cdot\nabla_y}(1-I)(|u|^\alpha u)(t,y)\Big]I\overline{u}(t,y)|Iu(t,x)|^2\mathrm{d}x\mathrm{d}y\mathrm{d}t\\
&= 2\mathrm{i}\mathrm{Im}\int_0^{\mathrm{T}}\int_{\mathbb{R}^{2d}} \partial_\ell(a_jI\overline{u}(t,y))\Big[\frac{\partial_{y_\ell}}{\Delta_y}(1-I)(|u|^\alpha u)(t,y)\Big]|Iu(t,x)|^2\mathrm{d}x\mathrm{d}y\mathrm{d}t.
\end{aligned}$$

By Hölder's and Bernstein's inequalities,

$$
\begin{aligned}
(5.28) &\lesssim \Big\|\frac{\partial_\ell}{\Delta}(1-I)(|u|^\alpha u)\Big\|_{L^2(J,L^{\frac{2d}{d+2}})}\|\partial_\ell Iu\|_{L^2(J,L^{\frac{2d}{d-2}})}\|Iu\|^2_{L^\infty(J,L^2)}\\
&\quad+\int_0^T\int_{\mathbb{R}^{2d}}\frac{1}{|x-y|}\Big|\frac{\partial_\ell}{\Delta}(1-I)(|u|^\alpha u)(t,y)\Big||Iu(t,x)|^2|Iu(t,y)|\mathrm{d}x\mathrm{d}y\mathrm{d}t\\
&\lesssim o(k^{-1})o(K)+\int_0^T\int_{\mathbb{R}^{2d}}\frac{1}{|x-y|}\Big|\frac{\partial_\ell}{\Delta}(1-I)(|u|^\alpha u)(t,y)\Big||Iu(t,x)|^2|Iu(t,y)|\mathrm{d}x\mathrm{d}y\mathrm{d}t\\
&\equiv o(1)+\mathrm{A},
\end{aligned}
$$

and by Hölder's equality for the second term,

$$
\begin{aligned}
\mathrm{A}&\le\Big\|\int_{\mathbb{R}^d}\frac{|Iu(t,x)|^2}{|x-y|}\mathrm{d}x\Big\|_{L^4(J,L^{6d})}\|Iu\|_{L^4(J,L^{\frac{2d}{d-\frac{7}{3}}})}\Big\|\frac{\partial_\ell}{\Delta}(1-I)(|u|^\alpha u)\Big\|_{L^2(J,L^{\frac{2d}{d+2}})}\\
&\lesssim o(K^{-1})\Big\|\int_{\mathbb{R}^d}\frac{|Iu(t,x)|^2}{|x-y|}\mathrm{d}x\Big\|_{L^4(J,L^{6d})}\|Iu\|_{L^4(J,L^{\frac{2d}{d-\frac{7}{3}}})}.
\end{aligned}
$$

Interpolating

$$\|u_{>M}\|_{L^2(J,L^{\frac{2d}{d-2}})}\lesssim o\Big(\frac{K^{\frac12}}{M^{\frac12}}\Big)$$

with

$$\|u_{>M}\|_{L^\infty(\mathbb{R},L^2)}\lesssim 1,$$

we have

$$\|u_{>M}\|_{L^4(J,L^{\frac{2d}{d-1}})}\lesssim o\Big(\frac{K^{\frac14}}{M^{\frac14}}\Big).$$

With this and Bernstein's inequality, we have

$$
\begin{aligned}
\|Iu\|_{L^4(J,L^{\frac{2d}{d-\frac{7}{3}}})}&\lesssim\sum_{M\le CK}\|u_{>M}\|_{L^4(J,L^{\frac{2d}{d-\frac{7}{3}}})}\\
&\lesssim\sum_{M\le CK}M^{\frac{d-1}{2}-\frac{d-\frac{7}{3}}{2}}\|u_{>M}\|_{L^4(J,L^{\frac{2d}{d-1}})}\\
&\lesssim\sum_{M\le CK}M^{\frac{d-1}{2}-\frac{d-\frac{7}{3}}{2}}o\Big(\frac{K^{\frac14}}{M^{\frac14}}\Big)=o(K^{\frac23}), \qquad (5.31)\\
\|Iu\|_{L^4(J,L^{\frac{2d}{d-\frac{5}{3}}})}&\lesssim\sum_{M\le CK}M^{\frac13}\|u_{>M}\|_{L^4(J,L^{\frac{2d}{d-1}})}\lesssim o(K^{\frac13}). \qquad (5.32)
\end{aligned}
$$

Thus together with Hardy's inequality, we have

$$
\begin{aligned}
&\Big\|\int_{\mathbb{R}^d}\frac{|Iu(t,x)|^2}{|x-y|}\mathrm{d}x\Big\|_{L^4(J,L^{6d})}\\
&\lesssim\||Iu|^2\|_{L^4(J,L^{\frac{6d}{6d-5}})}\lesssim\|Iu\|_{L^\infty(\mathbb{R},L^2)}\|Iu\|_{L^4(J,L^{\frac{2d}{d-\frac{5}{3}}})}\\
&\lesssim o(K^{\frac13}).
\end{aligned}
$$

Collecting these estimates, we have

$$(5.28) \lesssim o(K^{-1})o(K^{\frac{2}{3}})o(K^{\frac{1}{3}}) \lesssim o(1).$$

Finally we turn to (5.29) and as an estimate for (5.28), integrating by parts

$$\begin{aligned}(5.29) &= -2\mathrm{i}\int_0^T\int_{\mathbb{R}^{2d}} a_j \mathrm{Im} I(|u|^\alpha u)(t,y)\frac{\Delta}{\Delta}(1-I)\overline{u}(t,y)|Iu(t,x)|^2\mathrm{d}x\mathrm{d}y\mathrm{d}t\\ &\lesssim \sum_\ell \int_0^{\mathrm{T}}\int_{\mathbb{R}^{2d}} \frac{1}{|x-y|}|I(|u|^\alpha u)(t,y)||\frac{\partial_\ell}{\Delta}(1-I)\overline{u}(t,y)||Iu(t,x)|^2\mathrm{d}x\mathrm{d}y\mathrm{d}t\\ &\quad+\sum_\ell\int_0^T\int_{\mathbb{R}^{2d}}|a_j\partial_\ell I(|u|^\alpha u)(t,y)\frac{\partial_\ell}{\Delta}(1-I)(1-I)\overline{u}(t,y)||Iu(t,x)|^2\mathrm{d}x\mathrm{d}y\mathrm{d}t.\end{aligned}$$

By Hölder's, Hardy's and Bernstein's inequalities and mass conservation, we continue as follows:

$$\begin{aligned}(5.29) &\lesssim \sum_\ell \Big\|\int\frac{|Iu(t,x)|^2}{|x-y|}\mathrm{d}x\Big\|_{L^\infty(J,L^{\frac{3d}{2}})}\|I(|u|^\alpha)u\|_{L^2(J,L^{\frac{2d}{d+\frac{2}{3}}})}\Big\|\frac{\ell}{\Delta}(1-I)u\Big\|_{L^2(J,L^{\frac{2d}{d-2}})}\\ &\quad+\sum_\ell\|\partial_\ell I(|u|^\alpha u)\|_{L^2(J,L^{\frac{2d}{d+2}})}\Big\|\frac{\partial_\ell}{\Delta}(1-I)u\Big\|_{L^2(J,L^{\frac{2d}{d-2}})}\|Iu\|^2_{L^\infty(J,L^2)}\\ &\lesssim K^{-1}\||Iu|^2\|_{L^\infty(J,L^{\frac{3d}{3d-1}})}\|I(|u|^\alpha u)\|_{L^2(J,L^{\frac{2d}{d+\frac{2}{3}}})}\|(1-I)u\|_{L^2(J,L^{\frac{2d}{d-2}})}\\ &\quad+\sum_\ell K^{-1}\|\partial_\ell I(|u|^\alpha u)\|_{L^2(J,L^{\frac{2d}{d+2}})}\|(1-I)u\|_{L^2(J,L^{\frac{2d}{d-2}})}.\end{aligned}$$

Hölder's and Sobolev inequalities give

$$\begin{aligned}\||Iu|^2\|_{L^\infty(J,L^{\frac{3d}{3d-1}})} &\le \|Iu\|_{L^\infty(J,L^2)}\|Iu\|_{L^\infty(J,L^{\frac{3d}{3d-1}})}\lesssim \||\nabla|^{\frac{1}{3}}Iu\|_{L^\infty(J,L^2)},\\ \|I(|u|^\alpha u)\|_{L^2(J,L^{\frac{2d}{d+\frac{2}{3}}})} &\lesssim \||\nabla|^{\frac{2}{3}}I(|u|^\alpha u)\|_{L^2(J,L^{\frac{2d}{d+2}})}.\end{aligned}$$

By the triangle inequality, we have

$$\begin{aligned}\|I(|u|^\alpha u)\|_{L^2(J,L^{\frac{2d}{d+\frac{2}{3}}})} &\lesssim \||\nabla|^{\frac{2}{3}}I(|u|^\alpha u-|Iu|^\alpha Iu)\|_{L^2(J,L^{\frac{2d}{d+2}})}\\ &\quad+\||\nabla|^{\frac{2}{3}}I(|Iu|^\alpha Iu)\|_{L^2(J,L^{\frac{2d}{d+2}})}\\ &\lesssim K^{\frac{2}{3}}\|u\|^\alpha_{L^\infty(J,L^{\frac{2d}{d-2s}})}\|(1-I)u\|_{L^2(J,L^{\frac{2d}{d-2}})}\\ &\quad+\|Iu\|^\alpha_{L^\infty(J,L^{\frac{2d}{d-2s}})}\||\nabla|^{\frac{2}{3}}Iu\|_{L^2(J,L^{\frac{2d}{d-2}})}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|\partial_\ell I(|u|^\alpha u)\|_{L^2(J,L^{\frac{2d}{d+2}})} &\le \|\partial_\ell I(|u|^\alpha u-|Iu|^\alpha Iu)\|_{L^2(J,L^{\frac{2d}{d+2}})}+\|\partial_\ell I(|Iu|^\alpha Iu)\|_{L^2(J,L^{\frac{2d}{d+2}})}\\ &\lesssim K\|u\|^\alpha_{L^\infty(J,L^{\frac{2d}{d-2s}})}\|(1-I)u\|_{L^2(J,L^{\frac{2d}{d-2}})}\\ &\quad+\|Iu\|^\alpha_{L^\infty(J,L^{\frac{2d}{d-2s}})}\|\partial_\ell Iu\|_{L^2(J,L^{\frac{2d}{d-2}})}.\end{aligned}$$

By these together with (5.6), we have

$$(5.29) \lesssim K^{-1}K^{\frac{1}{3}}K^{\frac{2}{3}}o(1) + K^{-1}K^{\frac{1}{3}}o(1)\sum_{M\leq CK} M^{\frac{2}{3}}o\Big(\Big(\frac{K}{M}\Big)^{\frac{1}{2}}\Big)$$
$$+ K^{-1}Ko(1) + K^{-1}\sum_{M\leq CK} Mo\Big(\Big(\frac{K}{M}\Big)^{\frac{1}{2}}\Big) \sim o(1),$$

which completes the proof.

Now (5.21) follows from this lemma, if we add one order partial differential operator on $Iu(t,x)$ in the integral which produces a K.

Estimates for (5.25)–(5.26) Similarly, by symmetry, we only show that for (5.25). Noting that $\sum\limits_{j=1}^{d} a_{jj}(x,y) = \frac{d-1}{|x-y|}$, by Hölder's and Hardy's inequalities,

$$\begin{aligned}
(5.25) \leq &\Big\|\int \frac{|Iu(t,x)|^2}{|x-y|}\mathrm{d}x\Big\|_{L^4(J,L^{6d})}\|Iu\|_{L^4(J,L^{\frac{2d}{d-\frac{7}{3}}})}\\
&\times \|I(|u|^{\alpha}u) - |Iu|^{\alpha}Iu\|_{L^2(J,L^{\frac{2d}{d+2}})}\\
\lesssim &\,\||Iu|^2\|_{L^4(J,L^{\frac{6d}{6d-5}})}\|Iu\|_{L^4(J,L^{\frac{2d}{d-\frac{7}{3}}})}\|I(|u|^{\alpha}u) - |Iu|^{\alpha}Iu\|_{L^2(J,L^{\frac{2d}{d+2}})}\\
\lesssim &\,\|Iu\|_{L^\infty(J,L^2)}\|Iu\|_{L^4(J,L^{\frac{2d}{d-\frac{5}{3}}})}\|Iu\|_{L^4(J,L^{\frac{2d}{d-\frac{7}{3}}})}\\
&\times \|I(|u|^{\alpha}u) - |Iu|^{\alpha}Iu\|_{L^2(J,L^{\frac{2d}{d+2}})}.
\end{aligned}$$

(5.31)–(5.32), (5.6) and mass conservation show the estimate for (5.25).

Noting (5.18)–(5.26), we need to control $M_a(t)$ for any $t \in [0,T]$. By the definition and Hölder's and Bernstein's inequalities, we obtain

$$\begin{aligned}
|M_a(t)| &\leq \|\partial_j Iu\|_{L^2}\|Iu\|_{L^2}^3\\
&\lesssim \sum_{M\leq CK} M\|u_M\|_{L^2}\\
&\leq \sum_{M\leq K^{1-}} M\|u_M\|_{L^2} + \sum_{K^{1-}<M\leq CK} M\|u_{>M}\|_{L^2}\\
&\leq K^{1-} + \|u_{>k^{1-}}\|_{L^2}\sum_{K^{1-}<M\leq CK} M\\
&\lesssim K^{1-} + o(1)\sum_{K^{1-}<M\leq CK} M \sim o(K).
\end{aligned}$$

Now we conclude Theorem 5.2.

6 The Low-to-High Frequency Cascade

In this section, we defeat the low-to-high frequency cascade solutions as in Theorem 1.4. As Remark 4.2, we have

$$\sup_{t\in\mathbb{R}}\int |\xi|^{-2\varepsilon}|\widehat{u}(t,\xi)|^2\mathrm{d}\xi \lesssim 1. \tag{6.1}$$

Remark 1.4 deduce that, for any $\eta > 0$, there exists a constant $c(\eta)$ independent of t, such that

$$\int_{|\xi|\le c(\eta)N(t)} |\xi|^{2s}|\widehat{u}(t,\xi)|^2\mathrm{d}\xi \le \eta. \tag{6.2}$$

Interpolating (6.1)–(6.2), we have

$$\int_{|\xi|\le c(\eta)N(t)} |\widehat{u}(t,\xi)|^2\mathrm{d}\xi \lesssim \eta^{\frac{\varepsilon}{\varepsilon+s}}. \tag{6.3}$$

On the other hand, by the definition of the low-to-high frequency cascade solutions, we can pick up a time sequence $t_n \underset{n\to\infty}{\longrightarrow} \infty$, such that $N(t_n) \underset{n\to\infty}{\longrightarrow} \infty$. Thus

$$\int_{|\xi|\ge c(\eta)N(t)} |\widehat{u}(t_n,\xi)|^2\mathrm{d}\xi \lesssim \frac{1}{(c(\eta)N(t_n))^{2s}}\int |\xi|^{2s}|\widehat{u}(t_n,\xi)|^2\mathrm{d}\xi \underset{n\to\infty}{\longrightarrow} 0. \tag{6.4}$$

Then (6.3)–(6.4) show $\|u(t)\|_{L^2} \equiv 0$ by the mass conservation, which is a contradiction to the definition of u, i.e., low-to-high frequency cascade solutions can not exist.

7 The Finite-Time Blowup

Finally, we deal with the finite-time blowup scenario. Without loss of generality, let $T > 0$ be the finite endpoint of the life interval of such a solution u. Choosing $\epsilon = \frac{3}{2} - s$, $r_0 = \frac{2d}{d-1}$ and (q_0, r_0) as the corresponding admissible pair, by (1.10), we have

$$\begin{aligned}
\||\nabla|^{-\epsilon}u(t)\|_{L^2} &= \Big\|\int_t^T \mathrm{e}^{\mathrm{i}(t-s)\Delta}|\nabla|^{-\epsilon}(|u|^\alpha u)(s)\mathrm{d}s\Big\|_{L^2} \\
&\le \||\nabla|^{-\epsilon}(|u|^\alpha u)\|_{L^{q_0'}((t,T),L^{r_0'})} \lesssim (T-t)^{\frac{1}{q_0'}}\||\nabla|^{-\epsilon}(|u|^\alpha u)\|_{L^\infty((t,T),L^{r_0'})} \\
&\lesssim (T-t)^{\frac{1}{q_0'}}\||u|^{\alpha+1}\|_{L^\infty((t,T),L^{\frac{r'd}{d+\epsilon r'}})} \sim (T-t)^{\frac{1}{q_0'}}\|u\|^{\alpha+1}_{L^\infty((t,T),L^{\frac{2d}{d-2s}})} \underset{t\to T}{\longrightarrow} 0.
\end{aligned}$$

Interpolating this with $\sup\limits_{t\in I}\||\nabla|^s u(t)\|_{L^2}$, we have

$$\|u(t)\|_{L^2} \underset{t\to T}{\longrightarrow} 0,$$

which implies $\|u(t)\|_{L^2} \equiv 0$ and that there does not exist any finite-time blowup critical solution.

The main results of the last three sections show that the assumption in Theorem 1.2 causes a contradiction, so we prove Theorem 1.1.

8 Appendix

In this section, we prove the existence of the critical solution. Let $A > 0$, and set

$$\begin{aligned}
B(A) = \Big\{&u_0 \in \dot{H}^s; \text{if } u \text{ is the solution to (1.1) with initial data } u_0, \\
&\text{and lifespan } I(u_0), \sup_{t\in I(u_0)}\||\nabla|^s u(t)\|_{L^2} \le A\Big\}.
\end{aligned} \tag{8.1}$$

Definition 8.1 *We say that $SC(A)$ holds if for every $u_0 \in B(A)$, $I(u_0) = \mathbb{R}$ and u is the solution with initial data u_0, $\|u\|_{L^q(\mathbb{R},L^r)} < \infty$. We say that $SC(A; u_0)$ holds if $u_0 \in B(A)$, $I(u_0) = \mathbb{R}$ and u is the solution with initial data u_0, $\|u\|_{L^q(\mathbb{R},L^r)} < \infty$.*

By Theorem 3.3, for $A_0 > 0$ small enough, we have that $SC(A_0)$ holds. Our main result, Theorem 1.1, is equivalent to that $SC(A)$ holds for all $A > 0$. Thus if Theorem 1.1 fails, there exists a critical $A_C > A_0 > 0$ with the property that for any $A < A_C$, $SC(A)$ holds but for any $A > A_C$, $SC(A)$ fails. The key tool in the proof of Theorem 1.2 is the following profile decomposition.

Lemma 8.1 *Given a bounded sequence $(v_{0,n})_{j=1}^{\infty} \subset \dot{H}^s(\mathbb{R}^d)$, there exists a sequence $(V_j)_{j=1}^{\infty} \subset \dot{H}^s(\mathbb{R}^d)$, a subsequence of $(v_{0,n})_{j=1}^{\infty}$, and a sequence of triples vectors $(\lambda_{j,n}, x_{j,n}, t_{j,n})_{j=1}^{\infty} \subset \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}$, which are orthogonal in the following sense:*

$$\frac{\lambda_{j,n}}{\lambda_{j',n}} + \frac{\lambda_{j',n}}{\lambda_{j,n}} + \frac{|t_{j,n} - t_{j',n}|}{\lambda_{j,n}^2} + \frac{|x_{j,n} - x_{j',n}|}{\lambda_{j,n}} \underset{n\to\infty}{\longrightarrow} \infty \tag{8.2}$$

for $j \neq j'$, such that for each $J \geq 1$, we have

(1)

$$v_{0,n} = \sum_{j=1}^{J} \Big(\frac{1}{\lambda_{j,n}}\Big)^{\frac{d-2s}{2}} V_j^l\Big(\frac{\cdot - t_{j,n}}{\lambda_{j,n}^2}, \frac{\cdot - x_{j,n}}{\lambda_{j,n}}\Big) + w_n^J, \tag{8.3}$$

where $V_j^l(t) = \mathrm{e}^{\mathrm{i}t\Delta} V_j$;

(2)

$$\limsup_{n\to\infty} \|\mathrm{e}^{\mathrm{i}t\Delta} w_n^J\|_{L^q(\mathbb{R},L^r)} \underset{J\to\infty}{\longrightarrow} 0; \tag{8.4}$$

(3) *for any $J > 0$, we have a Pythagorean-like property,*

$$\|v_{0,n}\|_{\dot{H}^s}^2 = \sum_{j=1}^{J} \|V_j\|_{\dot{H}^s}^2 + \|w_n^J\|_{\dot{H}^s}^2 + o^J(1), \tag{8.5}$$

where $o^J(1) \underset{n\to\infty}{\longrightarrow} 0$.

The proof is analogous to the one in [22], and we omit it.

Finally, we need a perturbation result. It is analogous to [21, Theorem 2.14] (for the energy-critical equation).

Proposition 8.1 (Perturbation) *Given any $A \geq 0$, there exist $\varepsilon(A) > 0$ and $C(A) > 0$ with the following property. If $u \in C([0,\infty), H^s(\mathbb{R}^d))$ is a solution to* (1.1), *if $\widetilde{u} \in C([0,\infty), H^s(\mathbb{R}^d))$ and $e \in L^{b'}([0,\infty), L^{r'}(\mathbb{R}^d))$ satisfy*

$$\mathrm{i}\widetilde{u}_t + \Delta\widetilde{u} = |\widetilde{u}|^{\alpha}\widetilde{u} + e$$

for a.e. $t > 0$, and if

$$\begin{aligned} &\|\widetilde{u}\|_{L^q([0,\infty),L^r)} \leq A, \quad \|e\|_{L^{b'}([0,\infty),L^{r'})} \leq \varepsilon \leq \varepsilon(A), \\ &\|\mathrm{e}^{\mathrm{i}\Delta}\Big(u(0) - \widetilde{u}(0)\Big)\|_{L^q([0,\infty),L^r)} \leq \varepsilon \leq \varepsilon(A), \end{aligned} \tag{8.6}$$

then $u \in L^q((0,\infty), L^r(\mathbb{R}^d))$ and $\|u - \widetilde{u}\|_{L^q([0,\infty),L^r)} \leq C\varepsilon$.

Proof (see [13]) By the definition of A_C, there exists a sequence $A_n \to A_C$ as $n \to \infty$, such that there exists a sequence $(u_{0,n})_{n=1}^{\infty} \subset \dot{H}^s$ with $u_{0,n} \in SC(A_n)$ and the corresponding solution u_n is the blowup in its lifespan

$$\|u_n\|_{L^q(I_n,L^r)} = \infty.$$

Using the time-translation symmetry of (1.1), we replace $u_{0,n}$ by $u_n(s_n)$ (still denoted by $u_{0,n}$), such that

$$\lim_{n\to\infty} \|u_n\|_{L^q((\inf I_n,s_n],L^r)} = \lim_{n\to\infty} \|u_n\|_{L^q([s_n,\sup I_n),L^r)} = \infty. \tag{8.7}$$

Appling Lemma 8.1 to the sequence $(u_{0,n})_{n=1}^{\infty}$, we have

$$u_{0,n} = \sum_{j=1}^{J} \Big(\frac{1}{\lambda_{j,n}}\Big)^{\frac{d-2s}{2}} V_j^l\Big(\frac{\cdot - t_{j,n}}{\lambda_{j,n}^2}, \frac{\cdot - x_{j,n}}{\lambda_{j,n}}\Big) + w_n^J. \tag{8.8}$$

Let $s_{j,n} = -\frac{t_{j,n}}{\lambda_{j,n}^2}$, and let U_j be the non-linear profile associated with $(V_j, (s_{j,n})_{n=1}^{\infty})$ and lifespan $\widetilde{I}_j$. We denote

$$\widetilde{U}_{j,n}(t,x) = \Big(\frac{1}{\lambda_{j,n}}\Big)^{\frac{d-2s}{2}} U_j\Big(\frac{t}{\lambda_{j,n}^2} + s_{j,n}, \frac{x - x_{j,n}}{\lambda_{j,n}}\Big).$$

We will prove that there is only one non-trivial V_j in the decomposition (8.8). We proceed by contradiction.

Step 1 There exist J_0 and a constant C, such that for any $j > J_0$, $\widetilde{I}_j = \mathbb{R}$, the admissible pair (q,r) satisfies

$$\sup_{\mathbb{R}} \|U_j(t)\|_{\dot{H}^s} + \|U_j\|_{L^q(\mathbb{R},L^r)} + \||\nabla|^s U_j\|_{L^q(\mathbb{R},L^r)} \le C\|V_j\|_{\dot{H}^s},$$

due to small data and the Strichartz's estimate. Thus by (8.5), we have

$$\sum_{j>J_0} \sup_{\mathbb{R}} \|U_j(t)\|_{\dot{H}^s}^2 + \|U_j\|_{L^q(\mathbb{R},L^r)}^2 + \||\nabla|^s U_j\|_{L^q(\mathbb{R},L^r)}^2 \le C \sum_{j>J_0} \|V_j\|_{\dot{H}^s}^2.$$

Step 2 It can not happen that for all $1 \le j \le J_0$ and n large enough, we have

$$\|U_j\|_{L^q([s_{j,n},\sup \widetilde{I}_j),L^r)} < \infty.$$

If it is not true, $\sup \widetilde{I}_j = \infty$ for all $j \ge 1$ and

$$\sum_{j=1}^{\infty} \sup_{[s_{j,n},+\infty)} \|U_j(t)\|_{\dot{H}^s}^2 + \|U_j\|_{L^q([s_{j,n},\sup \widetilde{I}_j),L^r)}^2 < \infty.$$

For any $\epsilon_0 > 0$, by (8.4), there exist $J(\epsilon_0)$ and n large enough, such that

$$\|e^{it\Delta} w_n^{J(\epsilon_0)}\|_{L^q(\mathbb{R},L^r)} \le \epsilon_0. \tag{8.9}$$

Let

$$H_{n,\epsilon_0}(t,x) = \sum_{j=1}^{J(\epsilon_0)} \widetilde{U}_{j,n}(t,x) = \sum_{j=1}^{J(\epsilon_0)} \Big(\frac{1}{\lambda_{j,n}}\Big)^{\frac{d-2s}{2}} U_j\Big(\frac{t}{\lambda_{j,n}^2} + s_{j,n}, \frac{x - x_{j,n}}{\lambda_{j,n}}\Big).$$

We claim that, for some $C_0 > 0$,

$$\|H_{n,\epsilon_0}\|_{L^{\boldsymbol{q}}([0,+\infty),L^{\boldsymbol{r}})} \le C_0. \tag{8.10}$$

We use Sobolev embedding and interpolation, such that

$$\begin{aligned}\|H_{n,\epsilon_0}\|_{L^{\boldsymbol{q}}([0,+\infty),L^{\boldsymbol{r}})} &\le \||\nabla|^{\widetilde{s}} H_{n,\epsilon_0}\|_{L^{\boldsymbol{q}}([0,+\infty),L^{\tilde{\boldsymbol{r}}})} \\ &\lesssim \||\nabla|^{\widetilde{s}} H_{n,\epsilon_0}\|^{\theta}_{L^{\boldsymbol{\rho}}([0,+\infty),L^{\boldsymbol{\rho}})} \||\nabla|^{\widetilde{s}} H_{n,\epsilon_0}\|^{1-\theta}_{L^{\infty}([0,+\infty),\dot{H}^{s-\widetilde{s}})}\end{aligned} \tag{8.11}$$

for some suitable $\widetilde{s} \in (0,s)$, $\widetilde{\boldsymbol{r}}$, $\boldsymbol{\rho}$ and θ. Using the argument as in [20, p. 663] to the two terms in (8.11), we can give a bound for $\|H_{n,\epsilon_0}\|_{L^{\boldsymbol{q}}([0,+\infty),L^{\boldsymbol{r}})}$. We omit the details. Now, we consider the equation

$$\begin{cases} \mathrm{i}\partial_t H_{n,\epsilon_0} + \Delta H_{n,\epsilon_0} = \displaystyle\sum_{j=1}^{J(\epsilon_0)} |\widetilde{U}_{j,n}|^{\alpha} \widetilde{U}_{j,n}, \\ H_{n,\epsilon_0}(0) = \displaystyle\sum_{j=1}^{J(\epsilon_0)} \widetilde{U}_{j,n}(0). \end{cases} \tag{8.12}$$

We can rewrite the nonlinearity as

$$\begin{aligned}\sum_{j=1}^{J(\epsilon_0)} |\widetilde{U}_{j,n}|^{\alpha} \widetilde{U}_{j,n} &= |H_{n,\epsilon_0}|^{\alpha} H_{n,\epsilon_0} + \sum_{j=1}^{J(\epsilon_0)} |\widetilde{U}_{j,n}|^{\alpha} \widetilde{U}_{j,n} - |H_{n,\epsilon_0}|^{\alpha} H_{n,\epsilon_0} \\ &\equiv |H_{n,\epsilon_0}|^{\alpha} H_{n,\epsilon_0} + e_{n,\epsilon_0}.\end{aligned}$$

By (8.8)–(8.9), we have

$$\Big\|\mathrm{e}^{\mathrm{i}t\Delta}\Big(u_{0,n} - \sum_{j=1}^{J(\epsilon_0)} \widetilde{U}_{j,n}\Big)\Big\|_{L^{\boldsymbol{q}}([0,+\infty),L^{\boldsymbol{r}})} = \|\mathrm{e}^{\mathrm{i}t\Delta} w_n^{J(\epsilon_0)}\|_{L^{\boldsymbol{q}}([0,+\infty),L^{\boldsymbol{r}})} \le \epsilon_0. \tag{8.13}$$

To get a contradiction, we use Proposition 8.1. Hence, we need to estimate e_{n,ϵ_0}. We recall that for every $P > 1$ and $\ell \ge 2$, there exists a constant $C_{P,\ell}$, such that

$$\Big|\Big|\sum_{j=1}^{\ell} z_j\Big|^P - \sum_{j=1}^{\ell} |z_j|^P\Big| \le C_{P,\ell} \sum_{j\ne k} |z_j|\,|z_k|^{P-1} \tag{8.14}$$

for all $(z_j)_{1\le j\le \ell} \subset \mathbb{C}^{\ell}$ (see [14, (1.10)]). This implies

$$\begin{aligned}\|e_{n,\epsilon_0}\|_{L^{\boldsymbol{b}'}([0,+\infty),L^{\boldsymbol{r}'})} &\lesssim \sum_{j\ne j'} \||\widetilde{U}_{j,n}|^{\alpha} \widetilde{U}_{j',n}\|_{L^{\boldsymbol{b}'}([0,+\infty),L^{\boldsymbol{r}'})} \\ &\sim \sum_{j\ne j'} \Bigg\|\Bigg|\frac{U_j\big(\frac{\cdot-t_{j,n}}{\lambda_{j,n}^2}, \frac{\cdot-x_{j,n}}{\lambda_{j,n}}\big)}{\lambda_{j,n}^{\frac{d-2s}{2}}}\Bigg|^{\alpha} \frac{U_{j'}\big(\frac{\cdot-t_{j',n}}{\lambda_{j',n}^2}, \frac{\cdot-x_{j',n}}{\lambda_{j',n}}\big)}{\lambda_{j',n}^{\frac{d-2s}{2}}}\Bigg\|_{L^{\boldsymbol{b}'}([0,+\infty),L^{\boldsymbol{r}'})}.\end{aligned}$$

We will prove that

$$\|e_{n,\epsilon_0}\|_{L^{\boldsymbol{b}'}([0,+\infty),L^{\boldsymbol{r}'})} \underset{n\to\infty}{\longrightarrow} 0. \tag{8.15}$$

For every j, we can find $-\infty \leq a_j < +\infty$ such that

$$\|U_j\|_{L^q((a_j,\sup \widetilde{I}_j),L^r)} \leq C\|U_j\|_{L^q([s_{j,n},\sup \widetilde{I}_j),L^r)} < \infty$$

for n large and $s_{j,n} \in (a_j, +\infty)$. Picking any $j \neq j'$ in the above sum, by a change of variables, we have

$$\begin{aligned}
&\Big\|\Big|\frac{U_j\big(\frac{\cdot - t_{j,n}}{\lambda_{j,n}^2}, \frac{\cdot - x_{j,n}}{\lambda_{j,n}}\big)}{\lambda_{j,n}^{\frac{d-2s}{2}}}\Big|^\alpha \frac{U_{j'}\big(\frac{\cdot - tj',n}{\lambda_{j',n}^2}, \frac{\cdot - x_{j',n}}{\lambda_{j',n}}\big)}{\lambda_{j',n}^{\frac{d-2s}{2}}}\Big\|_{L^{b'}([0,+\infty),L^{r'})} \\
&= \Big(\frac{\lambda_{j',n}}{\lambda_{j,n}}\Big)^{\frac{d-2s}{2}} \Big\|\Big|U_j\Big(\frac{\lambda_{j',n}^2 \cdot + t_{j,n} - t_{j',n}}{\lambda_{j,n}^2}, \frac{\lambda_{j',n} \cdot + x_{j',n} - x_{j,n}}{\lambda_{j,n}}\Big)\Big|^\alpha U_{j'}(\cdot,\cdot)\Big\|_{L^{b'}([s_{j',n},+\infty),L^{r'})}. \quad (8.16)
\end{aligned}$$

For every $s \in [s_{j',n}, +\infty)$ (though we use the same notation as the one in regularity, whose meaning is clear and will not cause a delusion),

$$\frac{\lambda_{j',n}^2}{\lambda_{j,n}^2} s + \frac{t_{j,n} - t_{j',n}}{\lambda_{j,n}^2} \in [s_{j,n}, \infty) \subset (a_j, +\infty).$$

By (8.2), if $\frac{\lambda_{j,n}}{\lambda_{j',n}} \underset{n\to\infty}{\longrightarrow} \infty$, then (8.16) tends to 0 as $n \to \infty$. If $\lambda_{j,n} = \lambda_{j',n}$, but $\frac{|t_{j',n} - t_{j,n}|}{\lambda_{j,n}^2} \underset{n\to\infty}{\longrightarrow} \infty$, or $\lambda_{j,n} = \lambda_{j',n}$ $\frac{|t_{j',n} - t_{j,n}|}{\lambda_{j,n}^2} \leq C$ but $\frac{|x_{j',n} - x_{j,n}|}{\lambda_{j,n}} \underset{n\to\infty}{\longrightarrow} \infty$, the same result holds. By Proposition 8.1, this together with (8.10) and (8.13) implies that u_n is bounded in $L^q([0,+\infty), L^r(\mathbb{R}^d))$ for n large enough, which contradicts (8.7).

According to steps 1 and 2, we can arrange the order of V_j and find J_1, such that $1 \leq J_1 \leq J_0$ and for all $1 \leq j \leq J_1$ with

$$\|U_j\|_{L^q([s_{j,n},\sup \widetilde{I}_j),L^r)} = \infty,$$

while $j > J_1$, we have

$$\|U_j\|_{L^q([s_{j,n},\sup \widetilde{I}_j),L^r)} < \infty.$$

As a consequence of steps 1 and 2, we have

$$\sum_{j>J_1} \|U_j\|^2_{L^q([s_{j,n},+\infty),L^r)} < \infty$$

for n large enough. Now for $k \in \mathbb{N}$, $1 \leq j \leq J_1$, we set

$$T_{j,k}^+ = \begin{cases} \sup \widetilde{I}_j - \dfrac{1}{k}, & \text{if } \sup \widetilde{I}_j < \infty, \\ k, & \text{if } \sup \widetilde{I}_j = \infty, \end{cases}$$

and set $t_{j,k}^n$ by $s_{j,n} + \frac{t_{j,k}^n}{\lambda_{j,n}^2} = T_{j,k}^+$ and $t_k^n = \min\limits_{1\leq j\leq J_1} t_{j,k}^n$. With these definitions, $\widetilde{U}_{j,n}$ is defined on $[0, t_k^n]$ for all j, and there exists a C_k, such that for n large enough,

$$\sum_{j=1}^{\infty} \|\widetilde{U}_j\|^2_{L^q([0,t_k^n],L^r)} + \sup_{[0,t_k^n]} \|\widetilde{U}_{j,n}\|^2_{\dot{H}^s} \leq C_k. \quad (8.17)$$

Step 3 We claim as follows. Fixed $J \geq 1$, there exists an $n(J,\epsilon)$, such that for any $1 \leq J_2 \leq J$, we have

$$\Big|\Big\| \sum_{j=J_2}^{J} \frac{V_j^l(s_{j,n}, \frac{\cdot - x_{j,n}}{\lambda_{j,n}})}{\lambda_{j,n}^{\frac{d-2s}{2}}} \Big\|_{\dot{H}^s}^2 - \sum_{j=J_2}^{J} \Big\| \frac{V_j^l(s_{j,n}, \frac{\cdot - x_{j,n}}{\lambda_{j,n}})}{\lambda_{j,n}^{\frac{d-2s}{2}}} \Big\|_{\dot{H}^s}^2 \Big| \leq \epsilon. \tag{8.18}$$

In order to prove (8.18), we need to show that, for any $J_2 \leq j \neq j' \leq J$,

$$\lim_{n\to\infty} \frac{\Big\langle \Big((|\nabla|^s V_j^l\Big)\Big(s_{j,n}, \frac{\cdot - x_{j,n}}{\lambda_{j,n}}\Big), (|\nabla|^s V_{j'}^l)\Big(s_{j',n}, \frac{\cdot - x_{j',n}}{\lambda_{j',n}}\Big)\Big\rangle}{\lambda_{j,n}^{\frac{d}{2}} \lambda_{j',n}^{\frac{d}{2}}} = 0. \tag{8.19}$$

We will use the following formula frequently:

$$(\mathrm{e}^{\mathrm{i}t_0\Delta} v)\Big(\frac{x - x_0}{\lambda_0}\Big) = \Big(\mathrm{e}^{\mathrm{i}\lambda_0^2 t_0 \Delta}\Big(v\Big(\frac{\cdot - x_0}{\lambda_0}\Big)\Big)(x). \tag{8.20}$$

By the definition of V_j^l, (8.20) and a change of variance, we have

$$\begin{aligned} &\frac{\Big\langle (|\nabla|^s V_j^l)\Big(s_{j,n}, \frac{x - x_{j,n}}{\lambda_{j,n}}\Big),\ (|\nabla|^s V_{j'}^l)\Big(s_{j',n}, \frac{x - x_{j',n}}{\lambda_{j',n}}\Big)\Big\rangle_{L_x^2}}{\lambda_{j,n}^{\frac{d}{2}} \lambda_{j',n}^{\frac{d}{2}}} \\ &= \frac{\Big\langle \mathrm{e}^{\mathrm{i}(t_{j',n} - t_{j,n})\Delta}\Big((|\nabla|^s V_j)\Big(\frac{\cdot - x_{j,n}}{\lambda_{j,n}}\Big)\Big)(x),\ (|\nabla|^s V_{j'})\Big(\frac{x - x_{j',n}}{\lambda_{j',n}}\Big)\Big\rangle_{L_x^2}}{\lambda_{j,n}^{\frac{d}{2}} \lambda_{j',n}^{\frac{d}{2}}}. \end{aligned} \tag{8.21}$$

We consider different cases inspired by (8.2).

Case (i) $\frac{\lambda_{j',n}}{\lambda_{j,n}} \underset{n\to\infty}{\longrightarrow} 0$. We make a change of the variable, $y = \frac{x - x_{j',n}}{\lambda_{j',n}}$, and then (8.21) equals

$$\Big(\frac{\lambda_{j',n}}{\lambda_{j,n}}\Big)^{\frac{d}{2}} \Big\langle \mathrm{e}^{\mathrm{i}\frac{t_{j',n} - t_{j,n}}{\lambda_{j',n}^2}\Delta}\Big((|\nabla|^s V_j)\Big(\frac{\cdot + \frac{x_{j',n} - x_{j,n}}{\lambda_{j',n}}}{\frac{\lambda_{j,n}}{\lambda_{j',n}}}\Big)\Big)(y),\ |\nabla|^s V_{j'}(y)\Big\rangle_{L_y^2}. \tag{8.22}$$

Subcase (ia) $\frac{|t_{j'n} - t_{j,n}|}{\lambda_{j',n}^2} \leq C$. In this subcase, we can find a subsequence of n (still denoted by n) such that $\frac{t_{j'n} - t_{j,n}}{\lambda_{j',n}^2} \underset{n\to\infty}{\longrightarrow} t_{j,j'}$ for some $t_{j,j'} \in \mathbb{R}$, and therefore we only need to consider

$$\Big(\frac{\lambda_{j',n}}{\lambda_{j,n}}\Big)^{\frac{d}{2}} \Big\langle (|\nabla|^s V_j)\Big(\frac{\cdot + \frac{x_{j',n} - x_{j,n}}{\lambda_{j',n}}}{\frac{\lambda_{j,n}}{\lambda_{j',n}}}\Big)(y),\ \mathrm{e}^{-\mathrm{i}t_{j,j'}\Delta}(|\nabla|^s V_{j'})(y)\Big\rangle_{L_y^2} \underset{n\to\infty}{\longrightarrow} 0,$$

which implies (8.19) in the subcase.

subcase (ib) $\frac{|t_{j',n} - t_{j,n}|}{\lambda_{j',n}^2} \underset{n\to\infty}{\longrightarrow} \infty$. Without loss of generality, denote $t_{j,j',n} = \frac{t_{j,n} - t_{j',n}}{\lambda_{j',n}^2} \underset{n\to\infty}{\longrightarrow} +\infty$. We need a lemma in this subcase and in the sequel.

Lemma 8.2 *Assume that $\|h_n\|_{\dot{H}^s} \leq A$ and that $||\mathrm{e}^{\mathrm{i}t\Delta} h_n||_{L^q([0,+\infty), L^r)} \underset{n\to\infty}{\longrightarrow} 0$. Then $|\nabla|^s h_n \rightharpoonup 0$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$.*

For the proof, see [21, Lemma 3.6].

Take

$$h_n(x)=\Big(\frac{\lambda_{j,n}}{\lambda_{j',n}}\Big)^{\frac{d}{2}}\mathrm{e}^{\mathrm{i}\frac{t_{j,n}-t_{j',n}}{\lambda_{j,n}^2}\Delta}\Big((|\nabla|^sV_{j'})\Big(\frac{\cdot+\frac{x_{j,n}-x_{j',n}}{\lambda_{j,n}}}{\frac{\lambda_{j',n}}{\lambda_{j,n}}}\Big)\Big)(x).$$

As in the lemma, we act it by $\mathrm{e}^{\mathrm{i}t\Delta}$ and calculus as (8.20), so we obtain

$$\begin{aligned}\mathrm{e}^{\mathrm{i}t\Delta}h_n(x)&=\Big(\frac{\lambda_{j,n}}{\lambda_{j',n}}\Big)^{\frac{d}{2}}\mathrm{e}^{\mathrm{i}(t+\frac{t_{j,n}-t_{j',n}}{\lambda_{j,n}^2})\Delta}\Big((|\nabla|^sV_{j'})\Big(\frac{\cdot+\frac{x_{j,n}-x_{j',n}}{\lambda_{j,n}}}{\frac{\lambda_{j',n}}{\lambda_{j,n}}}\Big)\Big)(x)\\&=\Big(\frac{\lambda_{j,n}}{\lambda_{j',n}}\Big)^{\frac{d}{2}}\Big(\mathrm{e}^{\mathrm{i}\frac{t+\frac{t_{j,n}-t_{j',n}}{\lambda_{j,n}^2}}{(\frac{\lambda_{j',n}}{\lambda_{j,n}})^2}\Delta}(|\nabla|^sV_{j'})\Big)\Big(\frac{x+\frac{x_{j,n}-x_{j',n}}{\lambda_{j,n}}}{\frac{\lambda_{j',n}}{\lambda_{j,n}}}\Big).\end{aligned}$$

We check the assumption in the lemma,

$$\begin{aligned}\|\mathrm{e}^{\mathrm{i}t\Delta}h_n\|_{L^q([0,+\infty),L^r)}&=\Big\|\mathrm{e}^{\mathrm{i}\frac{t+\frac{t_{j,n}-t_{j',n}}{\lambda_{j,n}^2}}{(\frac{\lambda_{j',n}}{\lambda_{j,n}})^2}\Delta}(|\nabla|^sV_j)\Big\|_{L^q([0,+\infty),L^r)}\\&=\|\mathrm{e}^{\mathrm{i}\tau\Delta}(|\nabla|^sV_j)\|_{L^q([t_{j,j',n},+\infty),L^r)}\to 0,\end{aligned}$$

since $t_{j,j',n}\underset{n\to\infty}{\longrightarrow}+\infty$. Thus Lemma 8.2 includes the result in this subcase.

Case (ii) $\lambda_{j,n}=\lambda_{j',n}$ and $\frac{|t_{j',n}-t_{j,n}|}{\lambda_{j',n}^2}\underset{n\to\infty}{\longrightarrow}\infty$. In this case, (8.22) takes the form

$$\Big\langle\mathrm{e}^{\mathrm{i}\frac{t_{j',n}-t_{j,n}}{\lambda_{j',n}^2}\Delta}\Big((|\nabla|^sV_j)\Big(\cdot+\frac{x_{j',n}-x_{j,n}}{\lambda_{j',n}}\Big)\Big)(y),\ |\nabla|^sV_{j'}(y)\Big\rangle_{L_y^2}.$$

This case is similar to the subcase (ib) and we omit the proof.

Case (iii) $\lambda_{j,n}=\lambda_{j',n}$, $\frac{|t_{j',n}-t_{j,n}|}{\lambda_{j',n}^2}\le C$ and $\frac{|x_{j',n}-x_{j,n}|}{\lambda_{j',n}}\underset{n\to\infty}{\longrightarrow}\infty$. we can find a subsequence of n (still denoted by n) such that $\frac{t_{j'n}-t_{j,n}}{\lambda_{j',n}^2}\underset{n\to\infty}{\longrightarrow}t_{j,j'}$ for some $t_{j,j'}\in\mathbb{R}$ and $\frac{x_{j',n}-x_{j,n}}{\lambda_{j',n}}\underset{n\to\infty}{\longrightarrow}\infty$, and therefore we only need to consider

$$\Big\langle(|\nabla|^sV_j)\Big(y+\frac{x_{j',n}-x_{j,n}}{\lambda_{j',n}}\Big),\ \mathrm{e}^{\mathrm{i}t_{j,j'}\Delta}(|\nabla|^sV_{j'})(y)\Big\rangle_{L_y^2}.$$

Since $\frac{x_{j',n}-x_{j,n}}{\lambda_{j',n}}\underset{n\to\infty}{\longrightarrow}\infty$, it is easy to conclude the result.

Step 4 For fixed J, we set

$$\begin{aligned}e_{J,n}^{(1)}&=f(\widetilde{U}_{J,n})-\sum_{j=1}^{J}f(\widetilde{U}_{j,n}),\\e_{J,n}^{(2)}&=f(\widetilde{U}_{J,n}+w_n^{l,J})-f(\widetilde{U}_{J,n}),\end{aligned}$$

where $\widetilde{U}_{J,n}=\sum\limits_{j=1}^{J}\widetilde{U}_{j,n}$, $f(z)=|z|^{\alpha}z$ and $w_n^{l,J}(t)=\mathrm{e}^{\mathrm{i}t\Delta}w_n^J$. Then

(1) For any $J\ge 1$, $k\in\mathbb{N}$, $\epsilon>0$, there exists an $n(J,k,\epsilon)$, such that

$$\|e_{J,n}^{(1)}\|_{L^{b'}([0,t_k^n),L^{r'})}\le\epsilon \tag{8.23}$$

for all $n \geq n(J,k,\epsilon)$;

(2) For all $k \in \mathbb{N}$, $\epsilon > 0$, there exists a $J(k,\epsilon)$, so that for any $J \geq J(k,\epsilon)$, there exists an $n(J,k,\epsilon)$, such that

$$\|e^{(2)}_{J,n}\|_{L^{b'}([0,t^n_k),L^{r'})} \leq \epsilon. \tag{8.24}$$

We complete the proof.

Fixed $1 \leq j \leq J_1$, if we set $s_{j,n} \to s_j$ along some subsequence, then $s_j < +\infty$. Indeed, if $s_j = +\infty$, by the definition of the non-linear profile, we have

$$\|U_j(s_{j,n}) - \mathrm{e}^{\mathrm{i}s_{j,n}\Delta}V_j\|_{\dot{H}^s} \underset{n\to\infty}{\longrightarrow} 0.$$

By the triangle inequality and the linear Strichartz's estimate, we have

$$\begin{aligned}
&\|\mathrm{e}^{\mathrm{i}(t-s_{j,n})\Delta}U_j(s_{j,n})\|_{L^q([s_{j,n},+\infty),L^r)}\\
&\leq \|\mathrm{e}^{\mathrm{i}(t-s_{j,n})\Delta}U_j(s_{j,n}) - \mathrm{e}^{\mathrm{i}t\Delta}V_j\|_{L^q([s_{j,n},+\infty),L^r)} + \|\mathrm{e}^{\mathrm{i}t\Delta}V_j\|_{L^q([s_{j,n},+\infty),L^r)}\\
&\lesssim \|\mathrm{e}^{-\mathrm{i}s_{j,n}\Delta}U_j(s_{j,n}) - V_j\|_{\dot{H}^s} + \|\mathrm{e}^{\mathrm{i}t\Delta}V_j\|_{L^q([s_{j,n},+\infty),L^r)},
\end{aligned}$$

which is small for n large enough, and by Theorem 3.3, $\|U_j\|_{L^q([s_{j,n},+\infty),L^r)} < \infty$, which contradicts the result in step 2. Thus for fixed $k \in \mathbb{N}$ and $1 \leq j \leq J_1$, there exists $-\infty \leq a_j < +\infty$, such that $(s_{j,n,T^+_{j,k}}) \subset (a_j, T^+_{j,k})$ and

$$\|U_j\|_{L^q((a_j,T^+_{j,k}),L^r)} \leq C\|U_j\|_{L^q((s_{j,n,T^+_{j,k}}),L^r)} < \infty.$$

Then by the argument as in step 2, we conclude (8.23). To get (8.24), by Hölder's inequality,

$$\begin{aligned}
\|e^{(2)}_{J,n}\|_{L^{b'}([0,t^n_k),L^{r'})} &\lesssim \|w^{l,J}_n\|_{L^q([0,t^n_k],L^r)}\\
&\quad\times (\|\widetilde{U}_{J,n}\|^{\alpha}_{L^q([0,t^n_k],L^r)} + \|w^{l,J}_n\|^{\alpha}_{L^q([0,t^n_k],L^r)}) \underset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

This follows from (8.4). We complete the proof of step 4.

According to step 4 and (8.4), for any fixed $k, m \in \mathbb{N}$, we can find a $J(m,k)$, so that for any $J \geq J(m,k)$, there exists an $n_1(J,m,k)$, such that for any $n \geq n_1(J,m,k)$, we have

$$\|\mathrm{e}^{\mathrm{i}\Delta}w^J_n\|_{L^q(\mathbb{R},L^r)} \leq \frac{1}{m}, \quad \|e^{(2)}_{J,n}\|_{L^{b'}([0,t^n_k),L^{r'})} \leq \frac{1}{2m}. \tag{8.25}$$

Now we choose $J = J(m,k)$ for $m,\ k$. For this J (and corresponding $m,\ k$), we can find an $n(m,k) \geq n_1(J,m,k)$, such that

(1)

$$o^{J(m,k)}(1) \leq \frac{1}{m} \tag{8.26}$$

as in (8.5) for any $n \geq n(m,k)$;

(2)

$$\|e^{(1)}_{J(m,k),n}\|_{L^{b'}([0,t^n_k),L^{r'})} \leq \frac{1}{2m} \tag{8.27}$$

as in (8.23) for any $n \geq n(m,k)$;

(3)

$$\Big|\Big\| \sum_{j=J_2}^{J(m,k)} \frac{V_j^l(s_{j,n}, \frac{\cdot - x_{j,n}}{\lambda_{j,n}})}{\lambda_{j,n}^{\frac{d-2s}{2}}} \Big\|_{\dot H^s}^2 - \sum_{j=J_2}^{J(m,k)} \Big\| \frac{V_j^l(s_{j,n}, \frac{\cdot - x_{j,n}}{\lambda_{j,n}})}{\lambda_{j,n}^{\frac{d-2s}{2}}} \Big\|_{\dot H^s}^2 \Big| \leq \frac{1}{(2m)^2}, \tag{8.28}$$

as in (8.18) for any $n \geq n(m,k)$ and any $1 \leq J_2 \leq J(m,k)$;

(4)

$$\sum_{j=1}^{J(m,k)} \Big\| \widetilde{U}_{j,n}(x,0) - \frac{1}{\lambda_{j,n}^{\frac{d-2s}{2}}} V_j^l\Big(s_{j,n},\ \frac{x - x_{j,n}}{\lambda_{j,n}}\Big) \Big\|_{\dot H^s} \leq \frac{1}{m} \tag{8.29}$$

for any $n \geq n(m,k)$, which is a simple result by the definition of the nonlinear profile.

We can also assume that $J(m,k) < J(m+1,k)$ and $n(m,k) < n(m+1,k)$ without influencing any result listed above.

Step 5 For $0 \leq t \leq t_k^{n(m,k)}$ and m large enough, we have $t_k^{n(m,k)} \leq \sup I_{n(m,k)}$ and

$$u_{n(m,k)}(t) = \widetilde{U}_{J(m,k),n(m,k)}(t) + w_{n(m,k)}^{l,J(m,k)}(t) + r_{m,k}(t)$$

with

$$\|r_{m,k}\|_{L^q([0,t_k^{n(m,k)}),L^r)} \underset{m\to\infty}{\longrightarrow} 0.$$

We set $\widetilde{U}_{m,k}(t) = \widetilde{U}_{J(m,k),n(m,k)}(t) + w_{n(m,k)}^{l,J(m,k)}(t)$, and thus $\widetilde{U}_{m,k}$ will satisfy the following equation on $[0, t_k^{n(m,k)}]$:

$$\begin{cases} \mathrm{i}\partial_t \widetilde{U}_{m,k} + \Delta \widetilde{U}_{m,k} = \displaystyle\sum_{j=1}^{J(m,k)} |\widetilde{U}_{j,n(m,k)}|^\alpha \widetilde{U}_{j,n(m,k)}, \\ \widetilde{U}_{m,k}(0,x) = \widetilde{U}_{J(m,k),n(m,k)}(0) + w_{n(m,k)}^{l,J(m,k)}(0). \end{cases} \tag{8.30}$$

With the notation $f(z) = |z|^\alpha z$, we can rewrite the nonlinearity of (8.30) into

$$\begin{aligned} \sum_{j=1}^{J(m,k)} f(\widetilde{U}_{j,n(m,k)}) &= f(\widetilde{U}_{m,k}) + \sum_{j=1}^{J(m,k)} f(\widetilde{U}_{j,n(m,k)}) - f(\widetilde{U}_{m,k}) \\ &= f(\widetilde{U}_{m,k}) - e^{(1)}_{J(m,k),n(m,k)} - e^{(2)}_{J(m,k),n(m,k)} \\ &\equiv f(\widetilde{U}_{m,k}) + e_{m,k}. \end{aligned}$$

We note that, by (8.29),

$$\begin{aligned} &\|\mathrm{e}^{\mathrm{i}\Delta}(u_{n(m,k)}(0) - \widetilde{U}_{m,k})\|_{L^q([0,t_k^{n(m,k)}),L^r)} \\ \leq &\sum_{j=1}^{J(m,k)} \Big\| \frac{1}{\lambda_{j,n(m,k)}^{\frac{d-2s}{2}}} V_j^l\Big(s_{j,n(m,k)}, \frac{x - x_{j,n(m,k)}}{\lambda_{j,n(m,k)}}\Big) - \widetilde{U}_{j,n(m,k)}(x,0) \Big\|_{\dot H^s} \leq \frac{1}{m}. \end{aligned}$$

By this together with (8.25) and (8.27), the assumption of Proposition 8.1 is satisfied, so we conclude the result of step 5.

Step 6 There exists a j_0, $1 \le j_0 \le J_1$ and a subsequence $\{k_\iota\}$, $k_\iota \underset{\iota\to\infty}{\longrightarrow} \infty$, so that for each fixed k_ι, we can find a subsequence $m_\nu(k_\iota) \underset{\nu\to+\infty}{\longrightarrow} +\infty$, such that $n(m_\nu(k_\iota), k_\iota) \underset{\nu\to+\infty}{\longrightarrow} +\infty$ with

$$t_{j_0,k_\iota}^{n(m_\nu(k_\iota),k_\iota)} = t_{k_\iota}^{n(m_\nu(k_\iota),k_\iota)}$$

for each ι, ν.

The proof is a simple application of the pigeonhole principle as in [21], and we omit it.

Recall that for fixed $k \in \mathbb{N}$ and all large m, $\|U_{j_0}\|_{L^q([s_{j_0,n(m,k)}, \sup \widetilde{I}_{j_0}), L^r)} = \infty$ as in step 2, and $s_{j_0} = \lim\limits_{n\to\infty} s_{j_0,n} < +\infty$ as in the beginning of step 4. Then we can find a $-\infty < b_{j_0} < \sup \widetilde{I}_{j_0}$, such that $s_{j_0,n(m,k)} \le b_{j_0}$ for all large m and $\|U_{j_0}\|_{L^q([b_{j_0}, \sup \widetilde{I}_{j_0}), L^r)} = \infty$. By the definition of A_C, we have

$$A^2 = \sup_{[b_{j_0}, \sup \widetilde{I}_{j_0})} \|U_{j_0}(t)\|_{\dot{H}^s}^2 \ge A_C^2. \tag{8.31}$$

We also set $A_k^2 = \sup\limits_{[b_{j_0}, T_{j_0,k}^+]} \|U_{j_0}(t)\|_{\dot{H}^s}^2$, and thus $\lim\limits_{k\to\infty} A_k = A$.

Now, we set $T_{j_0,k} \in [b_{j_0}, T_{j_0,k}^+]$ by $A_k^2 = \|U_{j_0}(T_{j_0,k})\|_{\dot{H}^s}^2$. Define $\tau_{j_0,k}^{n(m,k)}$ by the formula

$$s_{j_0,n(m,k)} + \frac{\tau_{j_0,k}^{n(m,k)}}{\lambda_{j_0,n(m,k)}^2} = T_{j_0,k}.$$

Note that for fixed k and large m, $\tau_{j_0,k}^{n(m,k)} \ge 0$, since $T_{j_0,k} \le T_{j,k}^+$, we obtain that $\tau_{j_0,k}^{n(m,k)} \le t_{j_0,k}^{n(m,k)}$. Since $t_{j_0,k_\iota}^{n(m_\nu(k_\iota),k_\iota)} = t_{k_\iota}^{n(m_\nu(k_\iota),k_\iota)}$, for all ι, ν, we note that $\widetilde{U}_{j,n(m_\nu(k_\iota),k_\iota)}(\tau_{j_0,k_\iota}^{n(m_\nu(k_\iota),k_\iota)})$ is well-defined for all $1 \le j \le J_1$ by the definition of $t_{k_\iota}^{n(m_\nu(k_\iota),k_\iota)}$, and $j \ge J_1$ by the definition of J_1.

Step 7 For fixed k_ι, and ν large enough, we have

$$\begin{aligned} \|u_{n(m_\nu,k_\iota)}(\tau_{j_0,k_\iota}^{n(m_\nu,k_\iota)})\|_{\dot{H}^s}^2 = & \sum_{j=1}^{J(m_\nu(k_\iota),k_\iota)} \|\widetilde{U}_{j,\ n(m_\nu,k_\iota)}(\tau_{j_0,k_\iota}^{n(m_\nu,k_\iota)})\|_{\dot{H}^s}^2 \\ & + \|w_{n(m_\nu,k_\iota)}^{l,J(m_\nu(k_\iota),k_\iota)}(\tau_{j_0,k_\iota}^{n(m_\nu,k_\iota)})\|_{\dot{H}^s}^2 + \epsilon_{k_\iota}(\nu), \end{aligned} \tag{8.32}$$

where $\epsilon_{k_\iota}(\nu) \underset{\nu\to\infty}{\longrightarrow} 0$.

In order to simplify the notation, in this step, we set $J = J(m_\nu(k_\iota), k_\iota)$, $n = n(m_\nu, k_\iota)$ and $\tau_{j_0,k}^n = \tau_{j_0,k_\iota}^{n(m_\nu,k_\iota)}$ for short.

We first claim that, given any $\epsilon > 0$, we can find $J_2 = J_2(\epsilon)$ and a $\nu(\epsilon)$, such that for any $\nu \ge \nu(\epsilon)$, we have

$$\sup_{[0,t_{k_\iota}^n]} \Big\| \sum_{j=J_2}^{J} \widetilde{U}_{j,n}(t) \Big\|_{\dot{H}^s} \le \epsilon. \tag{8.33}$$

To prove this claim, for any $\epsilon_1 > 0$, by (8.5), we can find $J_2 = J_2(\epsilon_1)$, such that

$$\sum_{j=J_2}^{\infty} \|V_j\|_{\dot{H}^s}^2 \le \epsilon_1^2. \tag{8.34}$$

Thus by step 1, for any admissible pair (q,r), we have

$$\sum_{j=J_2}^{\infty}\sup_{\mathbb{R}}\|U_j(t)\|^2_{\dot{H}^s}+\|U_j\|^2_{L^{\mathbf{q}}(\mathbb{R},L^{\mathbf{r}})}+\||\nabla|^sU_j\|^2_{L^q(\mathbb{R},L^r)}\le C\epsilon_1^2.$$

For $0\le t<+\infty$, we have

$$\sum_{j=J_2}^{J}\widetilde{U}_{j,n}(t)=\mathrm{e}^{\mathrm{i}t\Delta}\Big(\sum_{j=J_2}^{J}\widetilde{U}_{j,n}(0)\Big)+\sum_{j=J_2}^{J}\int_0^t\mathrm{e}^{\mathrm{i}(t-t')\Delta}f(\widetilde{U}_{j,n})(t')\mathrm{d}t'.$$

By Strichartz's estimates, for some admissible (q_1,r_1) we obtain

$$\sup_{[0,t^n_{k_\iota}]}\Big\|\sum_{j=J_2}^{J}\widetilde{U}_{j,n}(t)\Big\|_{\dot{H}^s}\lesssim\Big\|\sum_{j=J_2}^{J}\widetilde{U}_{j,n}(0)\Big\|_{\dot{H}^s}+\sum_{j=J_2}^{J}\|\widetilde{U}_{j,n}\|^{\alpha}_{L^{\mathbf{q}}(\mathbb{R},L^{\mathbf{r}})}\||\nabla|^s\widetilde{U}_{j,n}\|_{L^{q_1}(\mathbb{R},L^{r_1})}.$$

By (8.29), Strichartz's estimates (8.28) and (8.34), we have

$$\begin{aligned}\sup_{[0,t^n_{k_\iota}]}\Big\|\sum_{j=J_2}^{J}\widetilde{U}_{j,n}(t)\Big\|_{\dot{H}^s}&\le\Big\|\sum_{j=J_2}^{J}\frac{1}{\lambda_{j,n}^{\frac{d-2s}{2}}}V_j^l\Big(s_{j,n},\frac{x-x_{j,n}}{\lambda_{j,n}}\Big)\Big\|_{\dot{H}^s}+\frac{1}{m_\nu}+C\epsilon_1\\&\le\frac{1}{m_\nu}+C\epsilon_1\le\epsilon,\end{aligned}$$

by choosing ν large enough and ϵ_1 small enough, which gives the claim. By step 5, (8.33) and (8.17), in order to prove (8.32), it suffices to prove

$$\langle|\nabla|^s\widetilde{U}_{j,\ n(m_\nu,k_\iota)}(\tau^{n(m_\nu,k_\iota)}_{j_0,k_\iota}),|\nabla|^s\widetilde{U}_{j',\ n(m_\nu,k_\iota)}(\tau^{n(m_\nu,k_\iota)}_{j_0,k_\iota})\rangle\underset{\nu\to\infty}{\longrightarrow}0\quad\text{for }1\le j\ne j'\le J_2,\tag{8.35}$$

$$\langle|\nabla|^s\widetilde{U}_{j,\ n(m_\nu,k_\iota)}(\tau^{n(m_\nu,k_\iota)}_{j_0,k_\iota}),|\nabla|^s w^{l,J(m_\nu(k_\iota),k_\iota)}_{n(m_\nu,k_\iota)}(\tau^{n(m_\nu,k_\iota)}_{j_0,k_\iota})\rangle\underset{\nu\to\infty}{\longrightarrow}0\quad\text{for }1\le j\le J_2.\tag{8.36}$$

To prove (8.35), we set

$$\widetilde{t}_{j,n}=\frac{\tau^n_{j_0,k}}{\lambda^2_{j,n}}-\frac{t_{j,n}}{\lambda^2_{j,n}},\quad\widetilde{t}_{j',n}=\frac{\tau^n_{j_0,k}}{\lambda^2_{j',n}}-\frac{t_{j',n}}{\lambda^2_{j',n}}.$$

As before, we discuss various cases.

(i) $|\widetilde{t}_{j',n}|\le C_{j'}$.

We take a subsequence, so that $\widetilde{t}_{j',n}\underset{\nu\to\infty}{\longrightarrow}\widetilde{t}_{j'}$. Since $0\le\tau^n_{j_0,k}\le t^{n(m_\nu,k_\iota)}_{j_0,k_\iota}=t^n_{k_\iota}$, we have

$$\begin{aligned}s_{j',n}=-\frac{t_{j',n}}{\lambda^2_{j',n}}&\le\frac{\tau^n_{j_0,k}}{\lambda^2_{j',n}}-\frac{t_{j',n}}{\lambda^2_{j',n}}\\&\le\frac{t^n_{k_\iota}}{\lambda^2_{j',n}}-\frac{t_{j',n}}{\lambda^2_{j',n}}\quad\le\frac{t_{j',n}}{\lambda^2_{j',n}}-\frac{t_{j',n}}{\lambda^2_{j',n}}=T^+_{j',k_\iota}.\end{aligned}$$

Thus $U_{j'}(t)$ is continuous in $\dot{H}^s$ in a neighborhood of $\widetilde{t}_{j',n}$. By the definition of $\widetilde{U}_j$, we consider

$$\frac{\langle(|\nabla|^sU_j)(\widetilde{t}_{j,n},\frac{\cdot-x_{j,n}}{\lambda_{j,n}}),(|\nabla|^sU_{j'})(\widetilde{t}_{j'},\frac{\cdot-x_{j',n}}{\lambda_{j',n}})\rangle}{(\lambda_{j,n}\lambda_{j',n})^{\frac{d}{2}}}.$$

(i.1) $\frac{\lambda_{j,n}}{\lambda_{j',n}}\underset{\nu\to\infty}{\longrightarrow}\infty$.

We proceed in two subcases.

(i.1.1) $\widetilde{t}_{j,n} \le C_j$, for some $C_j > 0$.

Passing along a subsequence, we suppose $\widetilde{t}_{j,n} \underset{\nu\to\infty}{\longrightarrow} \widetilde{t}_j$. We make a change of the variable $y = \frac{x - x_{j',n}}{\lambda_{j',n}}$, and we conclude this subcase.

(i.1.2) $\widetilde{t}_{j,n}$ is not bounded.

Passing to a subsequence, $\widetilde{t}_{j,n} \underset{\nu\to\infty}{\longrightarrow} \pm\infty$. Since for $j \le J_1$, $\widetilde{t}_{j,n} \le T_{j,k_\iota} < +\infty$, so $j > J_1$. If $\widetilde{t}_{j,n} \underset{\nu\to\infty}{\longrightarrow} +\infty$, U_j scatters at $+\infty$. If $\widetilde{t}_{j,n} \underset{\nu\to\infty}{\longrightarrow} -\infty$, since $\widetilde{t}_{j,n} \ge s_{j,n}$ and $s_{j,n} \underset{\nu\to\infty}{\longrightarrow} -\infty$, U_j scatters at $-\infty$. In either case, there exists an $h_j \in \dot{H}^s(\mathbb{R}^d)$, such that

$$\|U_j(\widetilde{t}_{j,n}) - \mathrm{e}^{\mathrm{i}\widetilde{t}_{j,n}\Delta} h_j\|_{\dot{H}^s} \underset{\nu\to\infty}{\longrightarrow} 0.$$

Thus we consider

$$\frac{\Big\langle |\nabla|^s \mathrm{e}^{\mathrm{i}\widetilde{t}_{j,n}\Delta} h_j\Big(\frac{\cdot - x_{j,n}}{\lambda_{j,n}}\Big), (|\nabla|^s U_{j'})\Big(\widetilde{t}_{j'}, \frac{\cdot - x_{j',n}}{\lambda_{j',n}}\Big)\Big\rangle}{(\lambda_{j,n}\lambda_{j',n})^{\frac{d}{2}}}.$$

We make a change of the variable $y = \frac{x - x_{j',n}}{\lambda_{j',n}}$ again, and this turns into

$$\Bigg\langle \frac{\mathrm{e}^{\mathrm{i}\frac{\widetilde{t}_{j,n}}{\frac{\lambda^2_{j',n}}{\lambda^2_{j,n}}}\Delta}\Bigg((|\nabla|^s h_j)\Bigg(\frac{\cdot - \frac{x_{j,n} - x_{j',n}}{\lambda_{j',n}}}{\frac{\lambda_{j,n}}{\lambda_{j',n}}}\Bigg)\Bigg)(y)}{\Big(\frac{\lambda_{j,n}}{\lambda_{j',n}}\Big)^{\frac{d}{2}}}, |\nabla|^s U_{j'}(\widetilde{t}_{j'}, y)\Bigg\rangle.$$

Since $\frac{\lambda_{j,n}}{\lambda_{j',n}} \underset{\nu\to\infty}{\longrightarrow} \infty$, which is an application of Lemma 8.2, we conclude this subcase.

(i.2) $\lambda_{j,n} = \lambda_{j',n}$ and $\frac{|t_{j,n} - t_{j',n}|}{\lambda^2_{j,n}} \underset{\nu\to\infty}{\longrightarrow} +\infty$.

Since $\widetilde{t}_{j,n} - \widetilde{t}_{j',n} = \frac{t_{j',n} - t_{j,n}}{\lambda^2_{j,n}}$ and $\widetilde{t}_{j',n}$ is bounded, we have $\widetilde{t}_{j,n}$, and thus with the argument as above applied again, we complete this case.

(i.3) $\lambda_{j,n} = \lambda_{j',n}$, $\frac{|t_{j,n} - t_{j',n}|}{\lambda^2_{j,n}} \le C_{j,j'}$ for some $C_{j,j'} > 0$, and $\frac{|x_{j,n} - x_{j',n}|}{\lambda_{j,n}} \underset{\nu\to\infty}{\longrightarrow} +\infty$.

For the same reason, we note that $\widetilde{t}_{j,n}$ is bounded. We make a change of the variable $y = \frac{x - x_{j',n}}{\lambda_{j',n}}$ and we obtain the result of this subcase easily.

By the symmetry, we reduce to the following case.

(ii) $\widetilde{t}_{j',n} \underset{\nu\to\infty}{\longrightarrow} +\infty$ and $\widetilde{t}_{j,n} \underset{\nu\to\infty}{\longrightarrow} +\infty$. Thus U_j and $U_{j'}$ scatter at $+\infty$. Thus we only consider

$$\begin{aligned}
&\frac{\Big\langle |\nabla|^s \mathrm{e}^{\mathrm{i}\widetilde{t}_{j,n}\Delta} h_j\Big(\frac{\cdot - x_{j,n}}{\lambda_{j,n}}\Big), |\nabla|^s \mathrm{e}^{\mathrm{i}\widetilde{t}_{j',n}\Delta} h_{j'}\Big(\frac{\cdot - x_{j',n}}{\lambda_{j',n}}\Big)\Big\rangle}{(\lambda_{j,n}\lambda_{j',n})^{\frac{d}{2}}}\\
&= \frac{\Big\langle \mathrm{e}^{\mathrm{i}\widetilde{t}_{j,n}\lambda^2_{j,n}\Delta}\Big((|\nabla|^s h_j)\Big(\frac{\cdot - x_{j,n}}{\lambda_{j,n}}\Big)\Big)(x), \mathrm{e}^{\mathrm{i}\widetilde{t}_{j',n}\lambda^2_{j',n}\Delta}\Big((|\nabla|^s h_{j'})\Big(\frac{\cdot - x_{j',n}}{\lambda_{j',n}}\Big)\Big)(x)\Big\rangle}{(\lambda_{j,n}\lambda_{j',n})^{\frac{d}{2}}}\\
&= \frac{\Big\langle (|\nabla|^s h_j)\Big(\frac{x - x_{j,n}}{\lambda_{j,n}}\Big), \mathrm{e}^{\mathrm{i}t_{j',n} - t_{j',n}\Delta}\Big((|\nabla|^s h_{j'})\Big(\frac{\cdot - x_{j',n}}{\lambda_{j',n}}\Big)\Big)(x)\Big\rangle}{(\lambda_{j,n}\lambda_{j',n})^{\frac{d}{2}}},
\end{aligned}$$

which is similar to (8.21), and we omit the details.

Now we turn to (8.36). Consider the case that $\widetilde{t}_{j,n}$ is bounded, and (8.36) follows from Lemma 8.2 and (8.4). If $\widetilde{t}_{j,n} \underset{\nu\to\infty}{\longrightarrow} +\infty$ passes to a subsequence, then U_j scatters at $+\infty$. Thus the proof is analogous, using Lemma 8.2 and (8.4). These complete the proof of step 7.

Now we arrive at the point to complete the proof of Theorem 1.2. By step 7, we have

$$A^2(n) \geq A^2_{k_\iota} + \epsilon_{k_\iota}(\nu).$$

Letting $\nu \to +\infty$, we note that $A_C^2 \geq A_{k_\iota^2}$. Then, letting $\iota \to +\infty$, we have $A_C^2 \geq A_{k_\iota^2}$. With (8.31), we have that $A_C = A$ and U_j is the required solution as in Theorem 1.2. The compactness is proved as in [20, Proposition 4.2] or [13].

Remark 8.1 As [21, Remark 3.8] shows, j_0 is the only one, such that V_j is non-trivial and $w_n^J \underset{n\to+\infty}{\longrightarrow} 0$ in $\dot{H}^s(\mathbb{R}^d)$. That means $V_j = 0$ if $j \neq j_0$.

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