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Simultaneous Identification of Two Parameters on the Reaction Diffusion System from Discrete Measurement Data*

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Abstract This article deals with an inverse problem of reconstructing two time independent coefficients in the reaction diffusion system from the final time space discretized measurement using the optimization method with the help of the smooth interpolation technique. The main objective of the article is to analyse the asymptotic behavior of the solution of the inverse problem for the linearly coupled reaction diffusion system with respect to the homogeneous Dirichlet boundary condition.

Keywords Inverse problem, Optimal control, Reaction diffusion systems, Asymptotic convergence
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1 Introduction

The parameter identification in the partial differential equation (PDE), intensively investigated during the last few decades, is a fertile and growing area of research with many applications, such as population dynamics, synaptic transmission at a neuromuscular junction, color negative film development, chemotaxis, epidemiology, and brain tumor growth. In ecological problems, different species interact with each other and in chemical reactions, different substances react and produce new substances. Systems of differential equations are used to model these events. For example, the reaction diffusion systems can be derived to model the spatial-temporal phenomena. Suppose that $u(x,t)$ and $v(x,t)$ are population density functions of two species or concentration of two chemicals, and then the reaction diffusion system with zero Dirichlet boundary conditions can be written as

$$\begin{array}{ll} u_t - u_{xx} + a(x)u + b(x)v = 0, & (x,t) \in \Omega_T = I \times (0,T], \\ v_t - v_{xx} + c(x)v + d(x)u = 0, & (x,t) \in \Omega_T, \\ u(x,0) = \phi(x), \quad v(x,0) = \varphi(x), & x \in I, \\ u(0,t) = u(1,t) = v(0,t) = v(1,t) = 0, & t \in (0,T], \end{array} \tag{1.1}$$

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where the interval $I = (0, 1)$ and $T > 0$ is an arbitrary but fixed moment of time. The initial conditions $\phi(x)$ and $\varphi(x)$, only depending on x, are sufficiently regular and the unknown coefficients $a(x), c(x)$ and the coefficients $b(x), d(x)$ are assumed to be sufficiently smooth and are kept independent of time t. We assume that there is a possibility to give the additional temperature for inverse heat problems, and for instance, the additional data $u(x,t),\ v(x,t)$ are given at some final time $t = T$ for a finite number of points, that is,

$$u(x_i, T) = m(x_i), \quad v(x_i, T) = n(x_i), \quad x_i \in I,\ i = 1, 2, \cdots, N, \tag{1.2}$$

where the given functions $m(x)$ and $n(x)$ satisfy the homogeneous Dirichlet boundary conditions. The additional temperature measurements given continuously throughout the interval in the following form:

$$u(x, T) = m(x), \quad v(x, T) = n(x), \quad x \in I \tag{1.3}$$

have been studied in [15]. Apart from the system of parabolic equations, a single parabolic equation from the continuous final time overspecified data has been studied by many researchers. For instance, the inverse problem of recovering the implied volatility coefficient in the Block-Scholes type equation has been studied by Jiang and Tao [12]. Chen and Liu [1] investigated the numerical reconstruction of the coefficient q in the parabolic equation $u_t - \Delta u + q(x)u = 0$ from the final measurement by using the optimization method combined with the finite element method. After these contributions to the study of inverse problems via the optimal control framework, there has been a lot of papers appearing in the literature. For instance, Deng et al. [3] studied an inverse problem of identifying the coefficient of a first-order term in a Cauchy problem of the second-order parabolic equation, and an inverse problem of recovering the nonlinear coefficient of heat conduction equations from the final time overspecified data was studied by Deng et al. in [6]. Moreover, Deng et al. [2] established an evolutional type inverse problem of recovering the radiative coefficient of a heat conduction equation. Yang et al. [18] investigated the inverse problem of reconstructing a space-dependent coefficient in the heat equation with homogeneous Neumann data using the final measurement data, and Deng et al. [4] studied an inverse problem of the determination of the implied volatility coefficient when all possible maturities from the current time to a chosen future time are known.

During the past few decades, various methods have been employed to study the inverse problems for partial differential equations. For example, the fixed point technique has been studied for identification of the diffusion coefficient and the reaction rate in a one-dimensional reaction-diffusion model by Friedman [8]. Hoffman and Jiang [10] investigated an inverse problem of reconstructing a source term in a phase field model for solidification, and Hasanov [9] established the simultaneous determination of source terms in a linear parabolic problem from the final overdetermination by adopting the methods of weak solutions and quasi-solutions. The inverse problem for the parabolic equations from the final overdetermination was investigated by Isakov [11].

It is interesting to note that the reconstruction from the continuous additional temperature measurement of the form (1.3) is very good for the theoretical aspect, but in practice, it is not very suitable because of the complexity or the cost for the continuous measurement. In order to overcome the difficulty, we only assume that the final observations are known at limited points x_i and then (1.2) is an appropriate form. For the sake of simplicity, in this work, we assume that the measurement points x_i are equidistant, that is,

$$0 < x_1 < x_2 < \cdots < x_{_N} < 1, \quad x_{i+1} - x_i = h, \quad i = 2, \cdots, N, \tag{1.4}$$

where the mesh parameter $h = \frac{1}{N+1}$. Even though the problem (1.1)–(1.2) is well defined, there is a lack of uniqueness and stability on the solution of the inverse problem, that is, the inverse problem (1.1)–(1.2) is improperly posed in the sense of Hadamard. In fact, the parameter identification problem (1.1)–(1.2) is underdetermined in mathematics; namely, from the given extra condition (1.2), one may not identify the unknown coefficients $a(x), c(x)$ uniquely and stably. In [15], we proved the uniqueness and stability of the identification of the coefficients $a(x), c(x)$ provided that the over-specified data are given in the form (1.3). On the basis of the idea of [15], we find a way to reconstruct $a(x), c(x)$ approximately by following the technique of Deng [5].

Initially, using the linear interpolation, we obtain new continuous functions $\alpha_{_N}(x)$ and $\beta_{_N}(x)$ from (1.2), that is,

$$\alpha_{_N}(x) = \begin{cases} m(x_1), & 0 \le x \le x_1, \\ \dfrac{x_{i+1} - x}{h} m(x_i) + \dfrac{x - x_i}{h} m(x_{i-1}), & x_1 \le x \le x_{i+1}, \quad i = 1, 2, \cdots, N-1, \\ m(x_{_N}), & x_{_N} \le x \le 1, \end{cases} \tag{1.5}$$

$$\beta_{_N}(x) = \begin{cases} n(x_1), & 0 \le x \le x_1, \\ \dfrac{x_{i+1} - x}{h} n(x_i) + \dfrac{x - x_i}{h} n(x_{i-1}), & x_1 \le x \le x_{i+1}, \quad i = 1, 2, \cdots, N-1. \\ n(x_{_N}), & x_{_N} \le x \le 1, \end{cases} \tag{1.6}$$

Then we consider the following over specified final time measurements:

$$u(x,T) = \alpha_{_N}(x), \quad v(x,T) = \beta_{_N}(x), \quad x \in I. \tag{1.7}$$

Approximation is a very useful technique when incomplete information prevents the use of exact measurements. Many problems in the real world are too complex to be solved analytically. Even when the exact measurement (1.3) is known, an approximation may yield a sufficiently accurate solution while reducing the complexity (it is impossible to obtain all the temperatures $u(x,T), v(x,T)$) of the problem significantly. From the result obtained in [15], one can easily prove the existence and uniqueness of the solution of the inverse problem (1.1) with the final time over specified data of the form (1.7). The main goal of this article is to deliberate the asymptotic behavior of the solution of the inverse problem, as $h \to 0$.

The rest of this paper is organized as follows: In Section 2, we study the existence of the optimal control and the optimality condition for the discrete measurement data by following the optimization technique (see [15]) and establish the energy estimates. In Section 3, we analyze the asymptotic behavior of the solution of the optimal control problem.

2 Optimal Control Problem

In order to analyze the inverse problem for the differential equations, the knowledge of the direct problem is essential. Using the well-known Schauder theory and the monotone method for parabolic equations, one can easily obtain the following existence result (see [7, 13–14, 17]).

Theorem 2.1 *Let $0 < \alpha < 1$ and the coefficients $a(x), b(x), c(x), d(x) \in C^{\alpha}(\overline{I})$. Then the system* (1.1) *has a unique solution $u(x,t), v(x,t) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}_T)$.*

As defined in [15], we consider the following optimal control problem: Find $(\overline{a}(x), \overline{c}(x)) \in \mathcal{M}$ satisfying

$$\mathcal{J}(\overline{a}, \overline{c}) = \min_{a,c \in \mathcal{M}} \mathcal{J}(a,c), \tag{2.1}$$

where

$$\mathcal{J}(a,c) = \frac{1}{2}\int_I (|u(x,T;a) - m(x)|^2 + |v(x,T;c) - n(x)|^2)\mathrm{d}x + \frac{N}{2}\int_I (|\nabla a|^2 + |\nabla c|^2)\mathrm{d}x, \tag{2.2}$$

$$\mathcal{M} = \{a(x), c(x) : 0 < a_0 \le a \le a_1,\ 0 < c_0 \le c \le c_1,\ \nabla a, \nabla c \in L^2(I)\}, \tag{2.3}$$

and (u,v) is the solution of the system (1.1) for the given coefficients $a(x), c(x) \in \mathcal{M}$. The constants a_0, a_1 and c_0, c_1 are given and N is the regularization parameter.

Suppose that (p,q) is the solution of the adjoint system associated with (1.1) of the form

$$\begin{array}{ll} -p_t - p_{xx} + ap + dq = 0, & (x,t) \in \Omega_T, \\ -q_t - q_{xx} + cq + bp = 0, & (x,t) \in \Omega_T, \\ p(x,T) = u(x,T) - m(x), & x \in I, \\ q(x,T) = v(x,T) - n(x), & x \in I, \\ p(0,t) = p(1,t) = q(0,t) = q(1,t) = 0, & t \in [0,T), \end{array} \tag{2.4}$$

where $m,\ n$ are the values of the solutions of the system (1.1) as defined in (1.3). Then we have the necessary optimality condition which has to be satisfied by each optimal control (a,c).

Theorem 2.2 *Let (a,c) be the solution of the optimal control problem* (2.1). *Then there exists a pair of functions $(u,v,p,q;a,c)$ satisfying*

$$\int_{\Omega_T} (pu(a-k) + qv(c-l))\mathrm{d}t\mathrm{d}x + N\int_I [\nabla a \cdot \nabla(k-a) + \nabla c \cdot \nabla(l-c)]\mathrm{d}x \ge 0 \tag{2.5}$$

for any $k, l \in \mathcal{M}$.

Now we consider the following optimal control problem along with (1.7): Find $(a_{N}(x), c_{N}(x)) \in \mathcal{M}$ satisfying

$$\mathcal{J}(a_N, c_N) = \min_{a,c \in \mathcal{M}} \mathcal{J}(a,c), \tag{2.6}$$

where

$$\begin{aligned}\mathcal{J}(a,c) = \frac{1}{2}\int_I (|u(x,T;a) - \alpha_N(x)|^2 + |v(x,T;c) - \beta_N(x)|^2)\mathrm{d}x \\ + \frac{N}{2}\int_I (|\nabla a|^2 + |\nabla c|^2)\mathrm{d}x. \end{aligned} \tag{2.7}$$

Similarly one can establish the following necessary optimality condition.

Theorem 2.3 *Let* (a_N, c_N) *be the solution of the optimal control problem* (2.6). *Then there exists a set of functions* $(u_N, v_N, p_N, q_N; a_N, c_N)$ *satisfying*

$$\begin{aligned}&\int_{\Omega_T} (p_N u_N (a_N - k) + q_N v_N (c_N - l))\mathrm{d}t\mathrm{d}x \\ &+ N\int_I [\nabla a_N \cdot \nabla(k - a_N) + \nabla c_N \cdot \nabla(l - c_N)]\mathrm{d}x \geq 0 \end{aligned} \tag{2.8}$$

for any $k, l \in \mathcal{M}$. *Here the pair* (u_N, v_N) *satisfies the following system*:

$$\begin{cases} \dfrac{\partial u_N}{\partial t} - \dfrac{\partial^2 u_N}{\partial x^2} + a_N(x)u_N + b(x)v_N = 0, & (x,t) \in \Omega_T, \\ \dfrac{\partial v_N}{\partial t} - \dfrac{\partial^2 v_N}{\partial x^2} + c_N(x)v_N + d(x)u_N = 0, & (x,t) \in \Omega_T, \\ u_N(x,0) = \phi(x), \quad v_N(x,0) = \varphi(x), & x \in I, \\ u_N(0,t) = u_N(1,t) = v_N(0,t) = v_N(1,t) = 0, & t \in (0,T], \end{cases} \tag{2.9}$$

and (p_N, q_N) *is the solution of the following system*:

$$\begin{cases} -\dfrac{\partial p_N}{\partial t} - \dfrac{\partial^2 p_N}{\partial x^2} + a_N(x)p_N + d(x)q_N = 0, & (x,t) \in \Omega_T, \\ -\dfrac{\partial q_N}{\partial t} - \dfrac{\partial^2 q_N}{\partial x^2} + c_N(x)q_N + b(x)p_N = 0, & (x,t) \in \Omega_T, \\ p_N(x,T) = u_N(x,T) - \alpha_N(x), & x \in I, \\ q_N(x,T) = v_N(x,T) - \beta_N(x), & x \in I, \\ p_N(0,t) = p_N(1,t) = q_N(0,t) = q_N(1,t) = 0, & t \in [0,T). \end{cases} \tag{2.10}$$

Lemma 2.1 *For the system* (1.1) *and its adjoint system* (2.4), *we have the estimates*

$$\max_{0\leq t\leq T}\int_I (|u|^2 + |v|^2)\mathrm{d}x \leq \exp(MT)(\|\phi\|^2_{L^2(I)} + \|\varphi\|^2_{L^2(I)}), \tag{2.11}$$

$$\begin{aligned}&\max_{0\leq t\leq T}\int_I (|p|^2 + |q|^2)\mathrm{d}x \\ &\leq \exp(MT)(\|u(x,T) - m(x)\|^2_{L^2(I)} + \|v(x,T) - n(x)\|^2_{L^2(I)}), \end{aligned} \tag{2.12}$$

where the constant

$$M = \Big(2 + \max_{x\in I}|b|^2 + \max_{x\in I}|d|^2\Big).$$

Proof Proof of (2.11) is available in [15] and the proof of (2.12) is similar to that of (2.11).

One can obtain the following similar results for the systems (2.9)–(2.10).

Lemma 2.2 *For the system* (2.9) *and the corresponding adjoint system* (2.10), *we have the estimates*

$$\max_{0\le t\le T}\int_I(|u_{N}|^2+|v_{N}|^2)\mathrm{d}x\le\exp(MT)(\|\phi\|^2_{L^2(I)}+\|\varphi\|^2_{L^2(I)}),\tag{2.13}$$

$$\max_{0\le t\le T}\int_I(|p_{N}|^2+|q_{N}|^2)\mathrm{d}x$$
$$\le\exp(MT)(\|u_{N}(x,T)-\alpha_{N}(x)\|^2_{L^2(I)}+\|v_{N}(x,T)-\beta_{N}(x)\|^2_{L^2(I)}).\tag{2.14}$$

3 Convergence Results

In this section, we discuss the asymptotic behavior of the discrete reconstruction (a_{N}, c_{N}) as $h\to 0$, for the system (1.1). In order to analyze the asymptotic behavior of the reconstruction, we require that the discrete measurements $m(x), n(x)$ in (1.2) satisfy

$$m(x),n(x)\in C^1(0,1),\quad \max_{x\in I}|m'(x)|\le\widehat{M},\quad \max_{x\in I}|n'(x)|\le\widehat{M}.\tag{3.1}$$

Lemma 3.1 *For* α_{N},β_{N} *defined in* (1.5) *and* (1.6), *respectively, we have the following estimates:*

$$\begin{aligned}|\alpha_{N}(x)-m(x)|\le 2\widehat{M}h,\\ |\beta_{N}(x)-n(x)|\le 2\widehat{M}h,\end{aligned}\qquad x\in[0,1].\tag{3.2}$$

Proof Let $R_{N}(x)=\alpha_{N}(x)-m(x),\ x\in[x_1,x_{N}]$. By using Taylor's approximation, we have the following reminder:

$$R_{N}(x)=m'(\xi)(x-x_i),\qquad\text{where } x,\xi\in[x_i,x_{i+1}],\ i=1,2,\cdots,N-1.\tag{3.3}$$

Estimation of the reminder terms along with the assumption (3.1) gives

$$|R_{N}(x)|\le\max_{x\in(0,1)}|m'(x)|\max_{x\in(0,1)}|(x-x_i)|\le\widehat{M}h.\tag{3.4}$$

For $x\in[0,x_1]$, we have

$$\begin{aligned}\alpha_{N}(x)=\alpha_{N}(x_1)=m(x_1)&=m(0)+m'(\xi_1)h\\ &=m'(\xi_1)h,\quad \xi_1\in[0,x_1],\end{aligned}\tag{3.5}$$

$$m(x)=m(0)+xm'(\xi_2),\quad \xi_2\in[0,x]\subset[0,x_1].\tag{3.6}$$

From (3.5)–(3.6) along with the assumption (3.1), we have

$$|\alpha_{N}(x)-m(x)|=|hm'(\xi_1)-xm'(\xi_2)|\le 2\widehat{M}h,\quad x\in[0,x_1].\tag{3.7}$$

For $x\in[x_{N},1]$, we also have

$$|\alpha_{N}(x)-m(x)|\le 2\widehat{M}h.\tag{3.8}$$

Combining (3.7) and (3.8), one can conclude that

$$|\alpha_{_N}(x) - m(x)| \le 2\widehat{M}h.$$

Similarly one can prove

$$|\beta_{_N}(x) - n(x)| \le 2\widehat{M}h$$

by considering the reminder as

$$S_{_N}(x) = \beta_{_N}(x) - n(x).$$

Setting $U = u_{_N} - u,\ V = v_{_N} - v,\ \mathcal{A} = a_{_N} - a$ and $\mathcal{C} = c_{_N} - c$, the subtraction of (2.9) from (1.1) yields

$$\begin{array}{ll} U_t - U_{xx} + a_{_N}U + bV = -\mathcal{A}u, & (x,t) \in \Omega_T, \\ V_t - V_{xx} + c_{_N}V + dU = -\mathcal{C}v, & (x,t) \in \Omega_T, \\ U(x,0) = 0, \quad V(x,0) = 0, & x \in I, \\ U(0,t) = U(1,t) = V(0,t) = V(1,t) = 0, & t \in (0,T]. \end{array} \tag{3.9}$$

Lemma 3.2 *Let (U,V) be the solution of the system* (3.9). *Then we have the following estimate:*

$$\begin{aligned} &\max_{0\le t\le T}\int_I (|U|^2 + |V|^2)\mathrm{d}x \\ &\le \exp(MT)\Big(\max_{x\in I}|\mathcal{A}|^2\int_{\Omega_T}|u|^2\mathrm{d}t\mathrm{d}x + \max_{x\in I}|\mathcal{C}|^2\int_{\Omega_T}|v|^2\mathrm{d}t\mathrm{d}x\Big). \end{aligned} \tag{3.10}$$

Proof The proof of this lemma is similar to that in [15].

Now by setting $P = p_{_N} - p$ and $Q = q_{_N} - q$, the subtraction of the adjoint systems (2.4) from (2.10) yields

$$\begin{array}{ll} -P_t - P_{xx} + a_{_N}P + dQ = -\mathcal{A}p, & (x,t) \in \Omega_T, \\ -Q_t - Q_{xx} + c_{_N}Q + bP = -\mathcal{C}q, & (x,t) \in \Omega_T, \\ P(x,T) = U(x,T) - [\alpha_{_N}(x) - m(x)], & x \in I, \\ Q(x,T) = V(x,T) - [\beta_{_N}(x) - n(x)], & x \in I, \\ P(0,t) = P(1,t) = Q(0,t) = Q(1,t) = 0, & t \in [0,T). \end{array} \tag{3.11}$$

Lemma 3.3 *Let (P,Q) be the solution of the system* (3.11). *Then there exists a constant $C > 0$ independent of a_0, c_0 such that*

$$\begin{aligned} &\max_{0\le t\le T}\int_I (|P|^2 + |Q|^2)\mathrm{d}x \\ &\le C\exp(2MT)\Big(\max_{x\in I}|\mathcal{A}|^2\int_{\Omega_T}(|p|^2 + |u|^2)\mathrm{d}t\mathrm{d}x + \max_{x\in I}|\mathcal{C}|^2\int_{\Omega_T}(|q|^2 + |v|^2)\mathrm{d}t\mathrm{d}x \\ &\quad + \int_I(|\alpha_{_N}(x) - m(x)|^2 + |\beta_{_N}(x) - n(x)|^2)\mathrm{d}x\Big). \end{aligned} \tag{3.12}$$

Proof Multiply the first equation (3.11) by P and integrate over I to have

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|P\|^2_{L^2(I)} + \int_I |P_x|^2\mathrm{d}x + \int_I a_{_N}|P|^2\mathrm{d}x = -\int_I dPQ\mathrm{d}x - \int_I \mathcal{A}Pp\ \mathrm{d}x. \tag{3.13}$$

Using the assumption on $a_{_N}$ and applying Cauchy's inequality, we get

$$\begin{aligned}&-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|P\|^2_{L^2(I)} + \int_I |P_x|^2\mathrm{d}x + a_0\int_I |P|^2\mathrm{d}x\\ &\le \int_I |P|^2\mathrm{d}x + \frac{1}{2}\max_{x\in I}|\mathcal{A}|^2\int_I |p|^2\mathrm{d}x + \frac{1}{2}\max_{x\in I}|d|^2\int_I |Q|^2\mathrm{d}x.\end{aligned} \tag{3.14}$$

Similarly, from the second equation of (3.11), we have

$$\begin{aligned}&-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Q\|^2_{L^2(I)} + \int_I |Q_x|^2\mathrm{d}x + c_0\int_I |Q|^2\mathrm{d}x\\ &\le \int_I |Q|^2\mathrm{d}x + \frac{1}{2}\max_{x\in I}|\mathcal{C}|^2\int_I |q|^2\mathrm{d}x + \frac{1}{2}\max_{x\in I}|b|^2\int_I |P|^2\mathrm{d}x.\end{aligned} \tag{3.15}$$

Coupling the above two inequalities, we have

$$\begin{aligned}&-\frac{\mathrm{d}}{\mathrm{d}t}[\exp(Mt)(\|P\|^2_{L^2(I)} + \|Q\|^2_{L^2(I)})]\\ &\le \exp(Mt)\Big(\max_{x\in I}|\mathcal{A}|^2\int_I |p|^2\mathrm{d}x + \max_{x\in I}|\mathcal{C}|^2\int_I |q|^2\mathrm{d}x\Big),\end{aligned} \tag{3.16}$$

where M is the constant defined in Lemma 2.1. Thus, integrating over (t,T), we arrive at

$$\begin{aligned}&\|P\|^2_{L^2(I)} + \|Q\|^2_{L^2(I)}\\ &\le \exp(-Mt)\Big(\max_{x\in I}|\mathcal{A}|^2\int_I\int_t^T \exp(Ms)|p|^2\mathrm{d}s\mathrm{d}x + \max_{x\in I}|\mathcal{C}|^2\int_I\int_t^T \exp(Ms)|q|^2\mathrm{d}s\mathrm{d}x\Big)\\ &\quad + 2\exp(M(T-t))\Big(\int_I(|U(x,T)|^2 + |V(x,T)|^2)\mathrm{d}x\\ &\quad + \int_I(|\alpha_{_N}(x) - m(x)|^2 + |\beta_{_N}(x) - n(x)|^2)\mathrm{d}x\Big).\end{aligned}$$

It is not difficult to conclude the proof by applying Lemma 3.1.

Theorem 3.1 *Let (a,c) and $(a_{_N}, c_{_N})$ be the solutions of the optimal control problems* (2.1) *and* (2.6) *respectively. Suppose that there exists a point $x_0 \in I$ such that $a_{_N}(x_0) = a(x_0)$ and $c_{_N}(x_0) = c(x_0)$. Then there exists an instant of time T_0 such that, for $T \ge T_0$,*

$$\left.\begin{aligned}a_{_N}(x) \to a(x)\\ c_{_N}(x) \to c(x)\end{aligned}\right\} \quad in\ \ C(0,1), \quad as\ h \to 0. \tag{3.17}$$

Proof Let us start the proof by taking $k = a_{_N},\ l = c_{_N}$ in (2.5) to have

$$\begin{aligned}&\int_{\Omega_T} qv(c - c_{_N})\mathrm{d}t\mathrm{d}x + \int_{\Omega_T} pu(a - a_{_N})\ \mathrm{d}t\mathrm{d}x\\ &+ N\int_I [\nabla a \cdot \nabla(a_{_N} - a) + \nabla c \cdot \nabla(c_{_N} - c)]\mathrm{d}x \ge 0.\end{aligned} \tag{3.18}$$

And, by taking $k = a,\ l = c$ in (2.8), we also have

$$\int_{\Omega_T} q_{_N} v_{_N} (c_{_N} - c)\mathrm{d}t\mathrm{d}x + \int_{\Omega_T} p_{_N} u_{_N} (a_{_N} - a)\ \mathrm{d}t\mathrm{d}x \\ + N\int_I [\nabla a_{_N} \cdot \nabla(a - a_{_N}) + \nabla c_{_N} \cdot \nabla(c - c_{_N})]\mathrm{d}x \ge 0, \tag{3.19}$$

where $(u, v), (u_{_N}, v_{_N})$ are the solutions of the systems (1.1) and (2.9) respectively and $(p, q), (p_{_N}, q_{_N})$ are the solutions of the corresponding adjoint systems (3.1) and (2.10) respectively. Now from (3.18)–(3.19), we get

$$N\Big(\int_I |\nabla(a_{_N} - a)|^2\mathrm{d}x + \int_I |\nabla(c_{_N} - c)|^2\mathrm{d}x\Big) \\ \le \int_{\Omega_T} \mathcal{A}(p_{_N} u_{_N} - pu)\mathrm{d}t\mathrm{d}x + \int_{\Omega_T} \mathcal{C}(q_{_N} v_{_N} - qv)\mathrm{d}t\mathrm{d}x \\ = \int_{\Omega_T} \mathcal{A}(p_{_N} U + Pu)\mathrm{d}t\mathrm{d}x + \int_{\Omega_T} \mathcal{C}(q_{_N} V + Qv)\mathrm{d}t\mathrm{d}x. \tag{3.20}$$

Applying Cauchy's inequality to each of the integrals on the right-hand side, we obtain

$$N\int_I (|\nabla\mathcal{A}|^2 + |\nabla\mathcal{C}|^2)\mathrm{d}x \\ \le \frac{1}{2}\Big(\max_{x\in I} |\mathcal{A}|^2 \int_{\Omega_T} (|p_{_N}|^2 + |u|^2)\mathrm{d}t\mathrm{d}x + \max_{x\in I} |\mathcal{C}|^2 \int_{\Omega_T} (|q_{_N}|^2 + |v|^2)\mathrm{d}t\mathrm{d}x\Big) \\ + \frac{1}{2}\int_{\Omega_T} (|U|^2 + |V|^2 + |P|^2 + |Q|^2)\mathrm{d}t\mathrm{d}x. \tag{3.21}$$

From Lemmas 3.1–3.2, it clear that

$$\int_{\Omega_T} (|U|^2 + |V|^2 + |P|^2 + |Q|^2)\mathrm{d}t\mathrm{d}x \\ \le CT\exp(2MT)\Big(\max_{x\in I} |\mathcal{A}|^2 \int_{\Omega_T} (|p|^2 + |u|^2)\mathrm{d}t\mathrm{d}x + \max_{x\in I} |\mathcal{C}|^2 \int_{\Omega_T} (|q|^2 + |v|^2)\mathrm{d}t\mathrm{d}x \\ + \int_I (|\alpha_{_N}(x) - m(x)|^2 + |\beta_{_N}(x) - n(x)|^2)\mathrm{d}x\Big). \tag{3.22}$$

Besides, from Lemma 2.1 and an analogue of Lemma 2.2, there exists a constant $\Gamma > 0$ such that

$$\int_{\Omega_T} (|u|^2 + |v|^2)\mathrm{d}t\mathrm{d}x \le T\exp(MT)\Gamma, \quad \int_{\Omega_T} (|p|^2 + |q|^2)\mathrm{d}t\mathrm{d}x \le T\exp(2MT)\Gamma. \tag{3.23}$$

Moreover, taking $\mathcal{A}(x_0) = 0$ into account and applying Hölder's inequality, we get

$$|\mathcal{A}(x)| = \Big|\int_{x_0}^{x} (\mathcal{A}(y))'\mathrm{d}y\Big| \le \Big(\int_I |\nabla\mathcal{A}|^2\mathrm{d}y\Big)^{\frac{1}{2}}, \tag{3.24}$$

so that

$$\max_{x\in I} |\mathcal{A}| \le \|\nabla\mathcal{A}\|_{L^2(I)}, \quad \forall x \in I. \tag{3.25}$$

Combining the preceding estimates with (3.21), we arrive at

$$\begin{aligned}\max_{x\in I}|\mathcal{A}|^2+\max_{x\in I}|\mathcal{C}|^2 &\le C_T(\max_{x\in I}|\mathcal{A}|^2+\max_{x\in I}|\mathcal{C}|^2)\\ &\quad+\frac{CT}{2N}\exp(2MT)\int_I(|\alpha_{_N}(x)-m(x)|^2+|\beta_{_N}(x)-n(x)|^2)\mathrm{d}x, \quad (3.26)\end{aligned}$$

where the constant

$$C_T=\frac{T}{2N}\exp(4MT)\Gamma(1+CT).$$

Now choosing $T_0>0$ such that $C_{T_0}=\frac{1}{2}$, we have

$$\begin{aligned}\max_{x\in I}|\mathcal{A}|^2+\max_{x\in I}|\mathcal{C}|^2 &\le \frac{CT}{N}\exp(2MT)\int_I(|\alpha_{_N}(x)-m(x)|^2+|\beta_{_N}(x)-n(x)|^2)\mathrm{d}x\\ &\le \frac{8CT}{N}\exp(2MT)\widehat{M}^2h^2. \quad (3.27)\end{aligned}$$

Therefore, from the above inequality, we easily conclude that

$$\max_{x\in(0,1)}|a_{_N}(x)-a(x)|^2+\max_{x\in(0,1)}|c_{_N}(x)-c(x)|^2\to 0, \quad \text{as } h\to 0.$$

This concludes the proof of Theorem 3.1.

Remark 3.1 It is interesting to note that we can easily establish the existence and uniqueness of the inverse problem of reconstructing two time-independent coefficients in the reaction diffusion system with zero flux boundary conditions from the final time overspecified measurement (1.3) as in [15]. But, in order to analyze the asymptotic behavior of the reconstruction from the discrete measurement $m(x),n(x)$ as in (1.2), the assumption (3.1) should be replaced with

$$m(x),n(x)\in C^2(0,1), \quad \max_{x\in I}|m''(x)|\le\widetilde{M}, \quad \max_{x\in I}|n''(x)|\le\widetilde{M}. \quad (3.28)$$

Lemma 3.4 *For $\alpha_{_N}$ and $\beta_{_N}$ defined in* (1.5) *and* (1.6)*, respectively, we have the following estimates:*

$$\begin{aligned}|\alpha_{_N}(x)-m(x)|\le\widetilde{M}h^2,\\ |\beta_{_N}(x)-n(x)|\le\widetilde{M}h^2,\end{aligned} \quad x\in[0,1]. \quad (3.29)$$

Proof We assume $R_{_N}(x)=\alpha_{_N}(x)-m(x),\ x\in[x_1,x_{_N}]$. By using Taylor's approximation, we have the following reminder (see [16]):

$$R_{_N}(x)=\frac{1}{2}m''(\xi)(x-x_i)(x-x_{i+1}), \quad x,\xi\in[x_i,x_{i+1}],\ i=1,2,\cdots,N-1. \quad (3.30)$$

Estimation of the reminder terms, along with the assumption (3.28) gives

$$|R_{_N}(x)|\le\frac{1}{2}\max_{x\in(0,1)}|m''(x)|\cdot\max_{x\in(0,1)}|(x-x_i)(x-x_{i+1})|\le\frac{\widetilde{M}h^2}{8}. \quad (3.31)$$

For $x \in [0, x_1]$, we have

$$\alpha_{_N}(x) = \alpha_{_N}(x_1) = m(x_1) = m(0) + m'(0)h + m''(\xi_1)\frac{h^2}{2}$$
$$= m(0) + m''(\xi_1)\frac{h^2}{2}, \quad \xi_1 \in [0, x_1], \tag{3.32}$$
$$m(x) = m(0) + m''(\xi_2)\frac{x^2}{2}, \quad \xi_2 \in [0, x] \subset [0, x_1]. \tag{3.33}$$

From (3.32)–(3.33) along with the assumption (3.28), we have

$$|\alpha_{_N}(x) - m(x)| = \left| m''(\xi_1)\frac{h^2}{2} - m''(\xi_2)\frac{x^2}{2} \right| \le \widetilde{M}h^2, \quad x \in [0, x_1]. \tag{3.34}$$

For $x \in [x_{_N}, 1]$, we also have

$$|\alpha_{_N}(x) - m(x)| \le \widetilde{M}h^2, \quad x \in [x_{_N}, 1]. \tag{3.35}$$

By combining (3.34) and (3.35), one can conclude that $|\alpha_{_N}(x) - m(x)| \le \widetilde{M}h^2$. Similarly we can prove

$$|\beta_{_N}(x) - n(x)| \le \widetilde{M}h^2.$$

Remark 3.2 By using the upper bound for the reminders on Theorem 3.1, we get

$$\max_{x\in I} |\mathcal{A}|^2 + \max_{x\in I} |\mathcal{C}|^2 \le \frac{8CT}{N} \exp(2MT)\widetilde{M}^2 h^4. \tag{3.36}$$

This yields the required asymptotic convergence of the reconstruction of the coefficients from the linearly coupled reaction diffusion system with zero flux boundary condition through the final time discrete measurement data.

Remark 3.3 The main difference in the convergence for the system with Dirichlet boundary conditions and zero flux boundary condition is that the Dirichlet boundary conditions require more numbers of discrete measurements than the zero flux boundary condition. For the reconstruction of the coefficients on the linearly coupled reaction diffusion system with zero flux boundary condition the cost of the measurement is less.

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