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Carleson-Type Maximal Operators with Variable Kernels*

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Abstract The author considers the L^p boundedness for two kinds of Carleson-type maximal operators with variable kernels $\frac{\Omega(x,y')}{|y|^n}$, where $\Omega(x,y') \in L^\infty(\mathbb{R}^n) \times W_2^s(\mathbf{S}^{n-1})$ for some $s > 0$.

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1 Introduction

For $f \in L^2([-\pi,\pi])$ and $x \in [-\pi,\pi]$, the Carleson operator $\mathcal{C}^*$ is defined by

$$\mathcal{C}^* f(x) = \sup_{\lambda \in \mathbb{R}} \Big| \int_{-\pi}^{\pi} \frac{\mathrm{e}^{-\mathrm{i}\lambda t} f(t)}{x-t} \mathrm{d}t \Big|.$$

In 1966, using the fact that $\mathcal{C}^*$ is of the weak type (2,2), Carleson [6] proved that the Fourier series of a function in $L^2([-\pi,\pi])$ converges pointwise almost everywhere. Later, Hunt [12] extended Carleson's theorem to the $L^p([-\pi,\pi])$ for $1 < p < \infty$.

In 1970, Sjölin [17] studied an analogue of the Carleson operator $\mathcal{C}^*$ on $\mathbb{R}^n$ defined by

$$\mathcal{S}^* f(x) = \sup_{\lambda \in \mathbb{R}^n} \Big| \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}\lambda \cdot y} K(x-y) f(y) \mathrm{d}y \Big|.$$

He showed that $\mathcal{S}^*$ is bounded on L^p for $1 < p < \infty$, where K is an appropriate Calderón-Zygmund kernel.

In 2001, Stein and Wainger [18] considered the following more general Carleson-type maximal operator,

$$\mathcal{T}^* f(x) = \sup_{\lambda} \Big| \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}P_\lambda(y)} K(y) f(x-y) \mathrm{d}y \Big|,$$

where $P_\lambda(x) = \sum\limits_{1 \le |\alpha| \le d} \lambda_\alpha x^\alpha$, $\lambda := (\lambda_\alpha)_{1 \le |\alpha| \le d}$. They got the result as following.

Theorem A (cf. [18]) *Suppose that $P_\lambda(x) = \sum\limits_{2 \le |\alpha| \le d} \lambda_\alpha x^\alpha$ and K satisfies the following conditions:*

(1) *K is a tempered distribution and agrees with a C^1 function $K(x)$ for $x \neq 0$;*

(2) *$\widehat{K} \in L^\infty$;*

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(3) $|\partial_x^\gamma K(x)| \le A|x|^{-n-|\gamma|}$ *for* $0 \le |\gamma| \le 1$.
Then $\|\mathcal{T}^* f\|_{L^p} \le C_p \|f\|_{L^p}$ *for* $1 < p < \infty$.

Recently, Ding and Liu [9] improved Theorem A and obtained weighted L^p boundedness for the Carleson-type maximal operator $\mathcal{T}^*$, that is the following theorem.

Theorem B (cf. [9]) *Suppose that* $P_\lambda(x) = \sum_{2\le|\alpha|\le d} \lambda_\alpha x^\alpha$, $K(x) = \Omega(x')|x|^{-n}$, *and* Ω *satisfies*

$$\int_{\mathbf{S}^{n-1}} \Omega(x')\mathrm{d}\sigma(x') = 0, \tag{1.1}$$

$$\Omega \in L^q(\mathbf{S}^{n-1}) \quad \textit{and} \quad \int_0^1 \frac{\omega_q(\delta)}{\delta}\mathrm{d}\delta < \infty \quad \textit{for some } 1 < q \le \infty,$$

where $\omega_q(\delta)$ *is the* L^q*-modulus of the continuity of* Ω. *Then for* $1 \le q' < p < \infty$, *there exists a constant* $C > 0$ *such that*

$$\|\mathcal{T}^* f\|_{L^p} \le C\Big(\|\Omega\|_{H^1(\mathbf{S}^{n-1})} + \int_0^1 \frac{\omega_q(\delta)}{\delta}\mathrm{d}\delta\Big)\|f\|_{L^p},$$

where $H^1(\mathbf{S}^{n-1})$ *is the Hardy space on* $\mathbf{S}^{n-1}$(*see* [8] *for the definition of* $H^1(\mathbf{S}^{n-1})$).

In 2009, the L^p bound for another Carleson-type maximal operator is considered, which is the following theorem.

Theorem C (cf. [10]) *Suppose that* $Q_\mu(t) = \sum_{2\le k\le d} \mu_k t^k$, $K(x) = \Omega(x')|x|^{-n}$, *and* $\Omega \in H^1(\mathbf{S}^{n-1})$ *and satisfies* (1.1). *Then the Carleson-type maximal operator*

$$\mathfrak{T}^* f(x) = \sup_{\mu\in\mathbb{R}^{d-1}} \Big| \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}Q_\mu(|y|)} \frac{\Omega(y')}{|y|^n} f(x-y)\mathrm{d}y \Big|$$

is bounded on L^p *for* $1 < p < \infty$. *Further,* $\|\mathfrak{T}^*\|_{L^p\to L^p} \le C\|\Omega\|_{H^1(\mathbf{S}^{n-1})}$.

If $\Omega(x, y')$ is integrable on $\mathbf{S}^{n-1}$ for every $x \in \mathbb{R}^n$ and satisfies

$$\int_{\mathbf{S}^{n-1}} \Omega(x, y')\mathrm{d}\sigma(y') = 0 \quad \text{for all } x \in \mathbb{R}^n, \tag{1.2}$$

then singular integrals with the variable kernel

$$\mathcal{K} f(x) = \mathrm{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x, y')}{|y|^n} f(x-y)\mathrm{d}y$$

exist. The continuity property of the singular integral $\mathcal{K}$ has been extensively studied, see [3–5] for more details.

Motivated by the above results about Carleson-type maximal operators and singular integrals with variable kernels, we will consider the L^p boundedness for the following two Carleson-type maximal operators with the variable kernel

$$\mathcal{K}^* f(x) = \sup_\lambda \Big| \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}P_\lambda(y)} \frac{\Omega(x, y')}{|y|^n} f(x-y)\mathrm{d}y \Big| \tag{1.3}$$

and

$$\mathfrak{K}^* f(x) = \sup_{\mu\in\mathbb{R}^{d-1}} \Big| \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}Q_\mu(|y|)} \frac{\Omega(x, y')}{|y|^n} f(x-y)\mathrm{d}y \Big|, \tag{1.4}$$

where $\Omega(x, y')$ is integrable on $\mathbf{S}^{n-1}$ for every $x \in \mathbb{R}^n$ and satisfies (1.2).

Our main results in this note are as follows.

Theorem 1.1 *Let $\mathcal{K}^*$ be given as in (1.3) and $P_\lambda(x) = \sum\limits_{2\le|\alpha|\le d} \lambda_\alpha x^\alpha$. If $\Omega(x,y') \in L^\infty(\mathbb{R}^n) \times W_2^s(\mathbf{S}^{n-1})$ for $s > \frac{3n-2}{4}$, then for $1 < p < \infty$, there exists a constant $C > 0$ such that*

$$\big\|\mathcal{K}^* f\big\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}.$$

Remark 1.1 The Sobolev imbedding theorem on $\mathbf{S}^{n-1}$ (see [14, § 7.1]) states that $W_2^s(\mathbf{S}^{n-1}) \subset C^1(\mathbf{S}^{n-1})$ for $s > \frac{n+1}{2}$. $\frac{3n-2}{4} < \frac{n+1}{2}$ when $2 \le n < 4$, so, Theorem 1.1 can be looked as an improvement of Theorem A in some sense. Moreover, in the same way, we can prove that $\mathcal{K}^*$ is also bounded on $L^p(w)$ for $w \in A_p$, $1 < p < \infty$.

Theorem 1.2 *Let $\mathfrak{K}^*$ be given as in (1.4) and $Q_\mu(t) = \sum\limits_{2\le k\le d} \mu_k t^k$. If $\Omega(x,y') \in L^\infty(\mathbb{R}^n) \times W_2^s(\mathbf{S}^{n-1})$ for $s > \frac{n-1}{2}$, then for $1 < p < \infty$, there exists a constant $C > 0$ such that*

$$\big\|\mathfrak{K}^* f\big\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}.$$

Remark 1.2 The study of singular integrals with an oscillating factor $\mathrm{e}^{\mathrm{i}Q_\mu(|y|)}$ has an important motivation. In fact, the singular integral in (1.4) is a generalization of the strongly singular convolution operator, which has been well studied by Fefferman (see [11]).

2 Notations and Lemmas

This section is devoted to the description of some basic facts and notations about spherical harmonics. An exposition of further details can be found in [2] or [16].

For $m \in \mathbb{N}$, denote the space of spherical harmonics of degree m on $\mathbf{S}^{n-1}$ by $\mathcal{H}_m$, and let h_m be the dimension of $\mathcal{H}_m$. If $\{Y_{m,j}\}_{j=1}^{h_m}$ denotes the normalized complete system in $\mathcal{H}_m$, then, $\{Y_{m,j} : m \in \mathbb{N}, j = 1, 2\cdots, h_m\}$ is an orthonormal basis for the Hilbert spaces $L^2(\mathbf{S}^{n-1})$ with the inner product

$$\langle f, g\rangle := \int_{\mathbf{S}^{n-1}} f(x')g(x')\mathrm{d}\sigma(x'), \quad f, g \in L^2(\mathbf{S}^{n-1}).$$

Thus, for any $f \in L^2(\mathbf{S}^{n-1})$, its Fourier-Laplace series converges to f in the L^2 norm, that is, $f(x') = \sum\limits_{m\in\mathbb{N}} \sum\limits_{j=1}^{h_m} a_{m,j} Y_{m,j}(x')$, where the coefficients $a_{m,j}$ are given by

$$a_{m,j} := \langle f, Y_{m,j}\rangle = \int_{\mathbf{S}^{n-1}} f(x')Y_{m,j}(x')\mathrm{d}\sigma(x'). \tag{2.1}$$

Definition 2.1 *For a secondly differentiable function f on $\mathbf{S}^{n-1}$, set F by*

$$F(z) = f\Big(\frac{z}{|z|}\Big), \quad z \in \mathbb{R}^n \setminus \{0\}.$$

The Laplace-Beltrami operator Λ on $\mathbf{S}^{n-1}$ is defined by

$$\Lambda f(x') := \Delta F(z)\Big|_{z=x'} = \sum_{k=1}^n \frac{\partial^2}{\partial z_k^2} F(z)\Big|_{z=x'}, \quad x' \in \mathbf{S}^{n-1}.$$

Essentially, the following lemma is Lemma 2.8 in [7], which can also be found in [2, §6].

Lemma 2.1 *For the above h_m, $Y_{m,j}$ and Λ, we have*

$$h_m = \binom{m+n-1}{n-1} - \binom{m+n-3}{n-1} \le C_n m^{n-2}, \quad m \in \mathbb{N}, \tag{2.2}$$

$$|Y_{m,j}(x')| \le C_n m^{\frac{n}{2}-1}, \quad \forall x' \in \mathbf{S}^{n-1},\ j = 1, 2, \cdots, h_m, \tag{2.3}$$

$$\Lambda Y_{m,j} = -m(m+n-2) Y_{m,j}, \quad m \in \mathbb{N},\ j = 1, 2, \cdots, h_m. \tag{2.4}$$

For $s \in \mathbb{N}$, Λ^s denotes the s-th order Laplace-Beltrami operator. According to (2.4), we have

$$(-\Lambda)^s Y_{m,j}(x') = m^s (m+n-2)^s Y_{m,j}(x'), \quad m \in \mathbb{N},\ j = 1, 2, \cdots, h_m.$$

For $s > 0$, we define the s-th order Laplace-Beltrami operator on $\mathbf{S}^{n-1}$ in a distributive sense by

$$H_m\big((-\Lambda)^s f\big)(x') = m^s (m+n-2)^s H_m(f)(x'), \quad m \in \mathbb{N},$$

where f is a distribution on $\mathbf{S}^{n-1}$, and $H_m(f)$ denotes the orthogonal projection of f onto $\mathcal{H}_m$. We call $f^{(s)} := (-\Lambda)^{\frac{s}{2}} f$ the s-th order derivative of the distribution f. Similar to the definition of Sobolev spaces on $\mathbb{R}^n$, the Sobolev spaces on $\mathbf{S}^{n-1}$ can be given as follows (see [13, §2.2] and [15, §1.7]).

Definition 2.2 *For $s \ge 0$, the Sobolev space $W_2^s(\mathbf{S}^{n-1})$ is defined by*

$$W_2^s(\mathbf{S}^{n-1}) := \Big\{ f(x') = \sum_{m\in\mathbb{N}} \sum_{j=1}^{h_m} a_{m,j} Y_{m,j}(x') : \sum_{m\in\mathbb{N}} m^s (m+n-2)^s \sum_{j=1}^{h_m} a_{m,j}^2 < \infty \Big\}$$

with the inner product $\langle f, g\rangle_s := \langle f^{(s)}, g^{(s)}\rangle$.

The "by parts integration" formula is a crucial property of the operator Λ, which can be found in [2, §6] and [7].

Lemma 2.2 *Let $f, g \in C^2(\mathbf{S}^{n-1})$, and then*

$$\int_{\mathbf{S}^{n-1}} f(x') \Lambda g(x') \mathrm{d}\sigma(x') = \int_{\mathbf{S}^{n-1}} \Lambda f(x') g(x') \mathrm{d}\sigma(x').$$

3 Proof of the Main Results

For $\Omega(x, y') \in L^\infty(\mathbb{R}^n) \times W_2^s(\mathbf{S}^{n-1})$, by a limit argument in [5, §5] or [3, §2], we may assume that

$$\Omega(x, y') = \sum_{m\in\mathbb{N}} \sum_{j=1}^{h_m} a_{m,j}(x) Y_{m,j}(y') \tag{3.1}$$

is a finite sum, where $a_{m,j}(x)$ are the Fourier-Laplace coefficients given by (2.1).

By (2.4) in Lemma 2.1, Lemma 2.2 and Hölder's inequality,

$$\begin{aligned} |a_{m,j}(x)| &= m^{-s}(m+n-2)^{-s} \Big| \int_{\mathbf{S}^{n-1}} \Omega(x, z') (-\Lambda)^s Y_{m,j}(z') \mathrm{d}\sigma(z') \Big| \\ &= m^{-s}(m+n-2)^{-s} \Big| \int_{\mathbf{S}^{n-1}} (-\Lambda)^s \Omega(x, z') Y_{m,j}(z') \mathrm{d}\sigma(z') \Big| \\ &\le m^{-s}(m+n-2)^{-s} \|(-\Lambda)^s \Omega(x, \cdot)\|_{L^2(\mathbf{S}^{n-1})} \|Y_{m,j}\|_{L^2(\mathbf{S}^{n-1})}. \end{aligned} \tag{3.2}$$

Applying the Plancherel theorem, we have

$$\|(-\Lambda)^s\Omega(x,\cdot)\|^2_{L^2(\mathbf{S}^{n-1})} = \sum_{m\in\mathbb{N}} m^s(m+n-2)^s \sum_{k=1}^{h_m} a^2_{m,j}(x) = \|\Omega(x,\cdot)\|^2_{W_2^s(\mathbf{S}^{n-1})} < \infty. \tag{3.3}$$

Note that $Y_{m,j}$ can be normalized such that $\|Y_{m,j}\|_{L^2(\mathbf{S}^{n-1})} = 1$. (3.2) and (3.3) imply that

$$|a_{m,j}(x)| \le C_n m^{-2s}. \tag{3.4}$$

3.1 Proof of Theorem 1.1

According to the representation of $\Omega(x,y')$ in (3.1), we get the following estimate:

$$\mathcal{K}^*f(x) \le \sum_{m\in\mathbb{N}}\sum_{j=1}^{h_m} |a_{m,j}(x)|\mathcal{K}^*_{m,j}f(x),$$

where

$$\mathcal{K}^*_{m,j}f(x) = \sup_\lambda \Big|\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}P_\lambda(y)}\frac{Y_{m,j}(y')}{|y|^n} f(x-y)\mathrm{d}y\Big|.$$

By Minkowski's inequality and (3.4), we have

$$\|\mathcal{K}^*f\|_{L^p} \le \sum_{m\in\mathbb{N}}\sum_{j=1}^{h_m}\|a_{m,j}\|_{L^\infty}\|\mathcal{K}^*_{m,j}f\|_{L^p} \le C_n \sum_{m\in\mathbb{N}} m^{-2s}\sum_{j=1}^{h_m}\|\mathcal{K}^*_{m,j}f\|_{L^p}. \tag{3.5}$$

Therefore, it suffices to consider the terms $\|\mathcal{K}^*_{m,j}f\|_{L^p}$. It is easy to see that, for $1<q<\infty$,

$$L^\infty(\mathbf{S}^{n-1}) \subset L^q(\mathbf{S}^{n-1}) \subset H^1(\mathbf{S}^{n-1}) \ \text{ and } \ \int_0^1 \frac{\omega_q(\delta)}{\delta}\mathrm{d}\delta \le \int_0^1 \frac{\omega_\infty(\delta)}{\delta}\mathrm{d}\delta.$$

Then, Theorem B implies that

$$\|\mathcal{K}^*_{m,j}f\|_{L^p} \le C(\|Y_{m,j}\|_{L^2(\mathbf{S}^{n-1})} + C(m,j))\|f\|_{L^p},$$

where $C(m,j) := \int_0^1 \frac{\omega_\infty^{m,j}(\delta)}{\delta}\mathrm{d}\delta$, and $\omega_\infty^{m,j}(\delta)$ denotes the continuous modulus of $Y_{m,j}$.

To estimate the constants $C(m,j)$, we need the following Markov inequality, which can be found in [13, §2] and [1].

Lemma 3.1 *For $Y_{m,j}\in\mathcal{H}_m$, $x',y'\in\mathbf{S}^{n-1}$, we have*

$$\big|Y_{m,j}(x') - Y_{m,j}(y')\big| \le m|x'-y'|\|Y_{m,j}\|_{L^\infty(\mathbf{S}^{n-1})}.$$

By Lemma 3.1 and (2.3), we have $C(m,j)\le Cm^{\frac{n}{2}}$ and $\|\mathcal{K}^*_{m,j}f\|_{L^p} \le C_n m^{\frac{n}{2}}\|f\|_{L^p}$. Then, (3.5) and (2.2) imply that

$$\|\mathcal{K}^*f\|_{L^p} \le C_n\sum_{m\in\mathbb{N}}\sum_{j=1}^{h_m} m^{-2s}m^{\frac{n}{2}}\|f\|_{L^p} \le C_n\sum_{m\in\mathbb{N}} m^{n-2}m^{-2s}m^{\frac{n}{2}}\|f\|_{L^p} \le C_n\|f\|_{L^p},$$

where we use the fact that $\frac{3n}{2}-2-2s<-1$, because $s>\frac{3n-2}{4}$.

3.2 Proof of Theorem 1.2

Maximal operators $\mathfrak{K}^*$ can also be treated as $\mathcal{K}^*$ in the previous subsection. Define

$$\mathfrak{K}^*_{m,j}f(x)=\sup_{\mu\in\mathbb{R}^{d-1}}\Big|\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}Q_\mu(|y|)}\frac{Y_{m,j}(y')}{|y|^n}f(x-y)\mathrm{d}y\Big|.$$

Theorem C shows that

$$\|\mathfrak{K}^*_{m,j}f\|_{L^p}\le C\|Y_{m,j}\|_{H^1(\mathbf{S}^{n-1})}\|f\|_{L^p}\le C\|Y_{m,j}\|_{L^2(\mathbf{S}^{n-1})}\|f\|_{L^p}.$$

Then, in a similar way as (3.5), for $s>\frac{n-1}{2}$, we have

$$\begin{aligned}\|\mathfrak{K}^*f\|_{L^p}&\le\sum_{m\in\mathbb{N}}\sum_{j=1}^{h_m}\|a_{m,j}\|_{L^\infty}\|\mathfrak{K}^*_{m,j}f\|_{L^p}\le C_n\sum_{m\in\mathbb{N}}\sum_{j=1}^{h_m}m^{-2s}\|f\|_{L^p}\\&\le C_n\sum_{m\in\mathbb{N}}m^{n-2}m^{-2s}\|f\|_{L^p}\le C_n\|f\|_{L^p}.\end{aligned}$$

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