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New Lower Bounds for the Least Common Multiples of Arithmetic Progressions*

Rongjun WU[1] Qianrong TAN[2] Shaofang HONG[3]

Abstract For relatively prime positive integers u_0 and r, and for $0 \le k \le n$, define $u_k := u_0 + kr$. Let $L_n := \operatorname{lcm}(u_0, u_1, \cdots, u_n)$ and let $a, l \ge 2$ be any integers. In this paper, the authors show that, for integers $\alpha \ge a$, $r \ge \max(a, l-1)$ and $n \ge l\alpha r$, the following inequality holds

$$L_n \ge u_0 r^{(l-1)\alpha+a-l}(r+1)^n.$$

Particularly, letting $l = 3$ yields an improvement on the best previous lower bound on L_n obtained by Hong and Kominers in 2010.

Keywords Arithmetic progression, Least common multiple, Lower bound
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1 Introduction

Hanson and Nair initiated the research for effective estimates for the least common multiple of the terms in a finite arithmetic progression; and, in [6] and [13] they managed to produce good upper and lower bounds for $\operatorname{lcm}(1, 2, \cdots, n)$. In particular, Nair [13] discovered a nice new proof for the following well-known nontrivial lower bound

$$\operatorname{lcm}(1, 2, \cdots, n) \ge 2^{n-1} \tag{1.1}$$

for any integer $n \ge 1$. In [4], Farhi provided an identity involving the least common multiple of binomial coefficients and then used it to give a simple proof of the estimate (1.1). Inspired by Hanson's and Nair's works, Bateman, Kalb, and Stenger [1] and Farhi [2] respectively sought asymptotics and nontrivial lower bounds for the least common multiples of arithmetic progressions. Recently, Hong, Qian and Tan [10] extended the Bateman-Kalb-Stenger theorem from the linear polynomial to the product of linear polynomials. On the other hand, Farhi [2] obtained several nontrivial bounds and posed a conjecture which was later confirmed by Hong and Feng [7]. Hong and Feng [7] also got an improved lower bound for sufficiently long arithmetic progressions; this result was later sharpened further by Hong and Yang [11]. We notice that

Manuscript received December 28, 2011. Revised November 16, 2012.
[1]Mathematical College, Sichuan University, Chengdu 610064, China. E-mail: eugen_woo@163.com
[2]School of Mathematics and Computer Science, Panzhihua University, Panzhihua 617000, Sichuan, China. E-mail: tqrmei6@126.com
[3]Corresponding author. Mathematical College, Sichuan University, Chengdu 610064, China; Yangtze Center of Mathematics, Sichuan University, Chengdu 610064, China. E-mail: sfhong@scu.edu.cn; s-f.hong@tom.com; hongsf02@yahoo.com
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Hong and Yang [12] and Farhi and Kane [5] obtained some related results regarding the least common multiple of a finite number of consecutive integers. The theorem of Farhi and Kane [5] was extended by Hong and Qian [9] from the set of positive integers to the general arithmetic progression case. Recently, Qian, Tan and Hong [14] obtained some results about the least common multiple of consecutive terms in a quadratic progression.

In this paper, we study finite arithmetic progressions $\{u_k := u_0+kr\}_{k=0}^n$ with $u_0, r \geq 1$ being integers satisfying $(u_0, r) = 1$. Throughout this paper, we define $L_n := \mathrm{lcm}(u_0, u_1, \cdots, u_n)$ to be the least common multiple of the sequence $\{u_k\}_{k=0}^n$. We begin with the following lower bound on L_n.

Theorem 1.1 (see [11]) *Let $\alpha \geq 1$ be an integer. If $n > r^\alpha$, then we have $L_n \geq u_0 r^\alpha (r+1)^n$.*

If $r = 1$, then Theorem 1.1 is the conjecture of Farhi [2] proven by Hong and Feng [7]. If $\alpha = 1$, then Theorem 1.1 becomes the improved lower bound of Hong and Feng [7]. In [8], Hong and Kominers sharpened the lower bound in Theorem 1.1 whenever $\alpha, r \geq 2$. In particular, they proved the following theorem which replaces the exponential condition $n > r^\alpha$ of Theorem 1.1 with a linear condition $n \geq 2\alpha r$.

Theorem 1.2 (see [8]) *Let $a \geq 2$ be any given integer. Then for any integers $\alpha, r \geq a$ and $n \geq 2\alpha r$, we have $L_n \geq u_0 r^{\alpha+a-2}(r+1)^n$.*

Letting $a = 2$, we see that Theorem 1.2 improves upon Theorem 1.1 for all but three choices of $\alpha, r \geq 2$. In the present paper, we provide a more general lower bound as follows.

Theorem 1.3 *Let $a, l \geq 2$ be any given integers. Then for any integers $\alpha \geq a$, $r \geq \max(a, l-1)$ and $n \geq l\alpha r$, we have $L_n \geq u_0 r^{(l-1)\alpha+a-l}(r+1)^n$.*

Picking $l = 2$, then Theorem 1.3 becomes Theorem 1.2. Letting $l = 3$ in Theorem 1.3 gives us the following new lower bound.

Theorem 1.4 *Let $a \geq 2$ be any given integer. Then for any integers $\alpha, r \geq a$ and $n \geq 3\alpha r$, we have $L_n \geq u_0 r^{2\alpha+a-3}(r+1)^n$.*

Since $\alpha \geq a \geq 2$, we have $2\alpha + a - 3 > \alpha + a - 2$. Therefore, the lower bound in Theorem 1.4 is better than that in Theorem 1.2 when n is large enough.

This paper is organized as follows. In Section 2, we first introduce relevant notations and the previous results. Finally, we prove Theorem 1.3.

2 Proof of Theorem 1.3

For any real numbers x and y, we say that y divides x if there exists an integer z such that $x = yz$. If y divides x, then we write $y \mid x$. As usual, we let $\lfloor x \rfloor$ denote the largest integer no more than x.

Following Hong and Yang [11], we denote, for each integer $0 \leq k \leq n$,

$$C_{n,k} := \frac{u_k \cdots u_n}{(n-k)!}, \quad L_{n,k} := \mathrm{lcm}(u_k, \cdots, u_n).$$

From the latter definition, we have that $L_n = L_{n,0}$.

The following lemma first appeared in [2] and was reproved in [3] and [7].

Lemma 2.1 (see [2–3, 7]) *For any integer $n \geq 1$, $C_{n,0} \mid L_n$.*

From Lemma 2.1, we see immediately that

$$L_{n,k} = A_{n,k} \frac{u_k \cdots u_n}{(n-k)!} = A_{n,k} \cdot C_{n,k} \tag{2.1}$$

for some integer $A_{n,k} \geq 1$.

Following Hong and Feng [7] and Hong and Yang [11], we define, for any $n \geq 1$,

$$k_n := \max\Big\{0, \Big\lfloor \frac{n-u_0}{r+1} \Big\rfloor + 1\Big\}. \tag{2.2}$$

Hong and Feng [7] proved the following result.

Lemma 2.2 (see [7]) *For all $n \geq 1$ and $0 \leq k \leq n$,*

$$L_n \geq L_{n,k_n} \geq C_{n,k_n} \geq u_0(r+1)^n.$$

Now we are in a position to prove a lemma whose proof closely follows the approach of Hong and Yang [11].

Lemma 2.3 *Let $a, l \geq 2$ be any given integers. Then for any integers $\alpha \geq a$, $r \geq \max(a, l-1)$ and $n \geq l\alpha r$, we have $n - k_n > ((l-1)\alpha + a - l)r$.*

Proof If $n \leq u_0$, then by the definition (2.2), $k_n \leq 1$. Since $\alpha, r \geq a \geq 2$ and $n \geq l\alpha r$, we derive that $n - k_n \geq n - 1 \geq l\alpha r - 1 > ((l-1)\alpha + a - l)r$.

Now we suppose that $n > u_0$. In this case, we have

$$k_n = \Big\lfloor \frac{n-u_0}{r+1} \Big\rfloor + 1.$$

So we have

$$k_n \leq \frac{n-u_0}{r+1} + 1 \leq \frac{n-1}{r+1} + 1 = \frac{n+r}{r+1}.$$

It then follows that

$$n - k_n \geq n - \frac{n+r}{r+1} = \frac{(n-1)r}{r+1} \geq \frac{(l\alpha r - 1)r}{r+1}. \tag{2.3}$$

Note that $r \geq l-1$ tells us that $r - l + 1 \geq 0$. Then from the assumption $\alpha, r \geq a$ it follows that

$$\begin{aligned}(l\alpha r - 1) - (r+1)((l-1)\alpha + a - l) &= (r-l+1)\alpha - 1 - (r+1)(a-l) \\ &\geq a(r-l+1) - 1 - (r+1)(a-l) \\ &= l(r-a) + l - 1 > 0. \end{aligned} \tag{2.4}$$

Therefore by (2.4), we infer that

$$\frac{l\alpha r - 1}{r+1} > (l-1)\alpha + a - l. \tag{2.5}$$

The desired result then follows immediately from (2.3) and (2.5).

Using the similar argument as that of Theorem 1.1, by Lemma 2.3 we can now prove Theorem 1.3 as the conclusion of this paper.

Proof of Theorem 1.3 By hypothesis, we have $\alpha, r \geq a \geq 2$, $l \geq 2$ and $n \geq l\alpha r$. It follows from Lemma 2.3 that $r^{(l-1)\alpha+a-l} \mid (n-k_n)!$. Thus, we may express $(n-k_n)!$ in the form $r^{(l-1)\alpha+a-l} \cdot B_n = (n-k_n)!$, with $B_n \geq 1$ being an integer. Letting $k = k_n$ in (2.1), we find that

$$r^{(l-1)\alpha+a-l} \cdot B_n \cdot L_{n,k_n} = A_{n,k_n} \cdot u_{k_n} \cdots u_n.$$

It then follows that $r^{(l-1)\alpha+a-l} \mid A_{n,k_n}$, since the requirement $(r, u_0) = 1$ implies that $(r, u_k) = 1$ for all $0 \leq k \leq n$. Then, we get from (2.1) and Lemma 2.2 that

$$L_{n,k_n} \geq r^{(l-1)\alpha+a-l} C_{n,k_n} \geq u_0 r^{(l-1)\alpha+a-l} (r+1)^n.$$

Therefore the statement of Theorem 1.3 follows immediately. The proof of Theorem 1.3 is complete.

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