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Regular Solution and Lattice Miura Transformation of Bigraded Toda Hierarchy*

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Abstract The authors give finite dimensional exponential solutions of the bigraded Toda hierarchy (BTH). As a specific example of exponential solutions of the BTH, the authors consider a regular solution for the $(1, 2)$-BTH with a 3×3-sized Lax matrix, and discuss some geometric structures of the solution from which the difference between the $(1, 2)$-BTH and the original Toda hierarchy is shown. After this, the authors construct another kind of Lax representation of $(N, 1)$-BTH which does not use the fractional operator of Lax operator. Then the authors introduce the lattice Miura transformation of $(N, 1)$-BTH which leads to equations depending on one field, and meanwhile some specific examples which contain the Volterra lattice equation (a useful ecological competition model) are given.

Keywords Regular solution, Lattice Miura transformation, Bigraded Toda hierarchy, Moment polytope, Volterra lattice
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1 Introduction

The Toda lattice hierarchy introduced by Toda [1–2] is a completely integrable system and has important applications in many different fields, such as the classical quantum fields theory. It is well-known that the Toda lattice equation (see [3]) can be reduced from the two-dimensional Toda hierarchy. Adding additional logarithm flows to the Toda lattice hierarchy, it becomes the extended Toda hierarchy (see [4]) which governs the Gromov-Witten invariant of CP^1. The bigraded Toda hierarchy (BTH for short) of the (N, M)-type (or simply the (N, M)-BTH) is the generalized Toda lattice hierarchy whose infinite Lax matrix has N upper and M lower non-zero diagonals. The BTH can be seen as a natural extension of the original Toda lattice hierarchy which is just of the $(1, 1)$-type. The BTH can also be treated as a general reduction of the two-dimensional Toda lattice hierarchy. The extended bigraded Toda hierarchy (EBTH for short) is the extension of the BTH which includes additional logarithm flows (see [5]). The dispersionless version of the extended bigraded Toda hierarchy was firstly introduced

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by Aoyama and Kodama in [6]. Later the dispersive extended bigraded Toda hierarchy was introduced by Gudio Carlet [5] who hoped that the EBTH might also be relevant to the theory of Gromov-Witten invariants.

We generalized the Sato theory to the EBTH and gave the HBEs of EBTH in [7]. The close relation of the BTH and the two-dimensional Toda hierarchy becomes a great motivation for us to consider the solutional structure of the BTH. In paper [8], we proved that the BTH has an equivalent relation between the (N, M)-BTH (whose infinite Lax matrix has N upper and M lower non-zero diagonals) and the (M, N)-BTH. In [9–10], the BTH is proved to have a natural Block-type Lie algebraic symmetry and so is the dispersionless BTH. As we know, the (N, M)-BTH is equivalent to the (N, M)-band bi-infinite matrix-formed Toda hierarchy, so we consider its reduction, i.e., the semi-finite and finite matrix form of the BTH. Then we give the solutions of the BTH using orthogonal polynomials in the matrix form. Some rational solutions of the BTH and the corresponding Young diagrams were also given in [8]. But there is one missing part about the regular exponential solutions of the BTH. Therefore from the general structure of the solutions of the BTH, the regular exponential solution which depends only on the primary time variables will be introduced in this article. Also we will tell the difference between the $(1, 2)$-BTH and the original Toda hierarchy from the orbit of flows in the graph of diagonal elements. Because its structure is of the Flag manifold similar to the original Toda hierarchy, a geometric description uses the moment polytope (see [11]) whose vertices correspond to the solutions of the BTH.

Comparing the equations for the primary flows of the $(N, 1)$-BTH with the equations constructed in [15], we find that they have a very close relation which will be shown in detail in the following sections.

This paper is organized as follows. In Section 2, the definitions of the BTH and its tau function are given. In Section 3, the exponential solutions of the BTH will be given where we also consider the finite dimensional exponential solutions and further survey the $(1, M)$-BTH. To see the geometry of the (N, M)-BTH, we consider the regular solution for the $(1, 2)$-BTH with a 3×3 Lax matrix, and discuss the geometric structure of the solution, i.e., the moment polytope for the $(1, 2)$-BTH in Section 4. In Section 5, some primary flows of the $(N, 1)$-BTH will be introduced. In Section 6, we construct another kind of Lax representation of the BTH. In Section 7, the lattice Miura transformation of the BTH will be given, and meanwhile, some concrete examples will be shown in detail. After that, the last section is devoted to related conclusions and discussions.

2 The Bigraded Toda Hierarchy

Firstly, the interpolated bigraded Toda hierarchy will be introduced. The Lax operator of the BTH is given by the Laurent polynomial of shift matrix Λ (see [5])

$$\mathcal{L} := \Lambda^N + u_{N-1}\Lambda^{N-1} + \cdots + u_0 + \cdots + u_{-M}\Lambda^{-M}, \tag{2.1}$$

where $N, M \geq 1$, Λ represents the shift operator with $\Lambda := \mathrm{e}^{\epsilon\partial_x}$ and "ϵ" is called the string coupling constant, i.e., for any function $f(x)$,

$$\Lambda f(x) = f(x+\epsilon).$$

The $\mathcal{L}$ can be written in two different ways by dressing the shift operator

$$\mathcal{L} = \mathcal{P}_L \Lambda^N \mathcal{P}_L^{-1} = \mathcal{P}_R \Lambda^{-M} \mathcal{P}_R^{-1}, \tag{2.2}$$

where the dressing operators have the form

$$\mathcal{P}_L = 1 + w_1\Lambda^{-1} + w_2\Lambda^{-2} + \cdots, \tag{2.3}$$

$$\mathcal{P}_R = \widetilde{w_0} + \widetilde{w_1}\Lambda + \widetilde{w_2}\Lambda^2 + \cdots. \tag{2.4}$$

Note that $\mathcal{P}_L$ is a lower triangular matrix, and $\mathcal{P}_R$ is an upper triangular matrix.

(2.2) is quite important because it gives the reduction condition for the two-dimensional Toda lattice hierarchy. The pair is unique up to multiplying $\mathcal{P}_L$ and $\mathcal{P}_R$ from the right by operators in the form $1 + a_1\Lambda^{-1} + a_2\Lambda^{-2} + \cdots$ and $\widetilde{a}_0 + \widetilde{a}_1\Lambda + \widetilde{a}_2\Lambda^2 + \cdots$, respectively, with coefficients independent of x. Given any difference operator $A = \sum\limits_k A_k\Lambda^k$, the positive and negative projections are defined by $A_+ = \sum\limits_{k\geq 0} A_k\Lambda^k$ and $A_- = \sum\limits_{k<0} A_k\Lambda^k$.

To write out explicitly the Lax equations of the BTH, fractional powers $\mathcal{L}^{\frac{1}{N}}$ and $\mathcal{L}^{\frac{1}{M}}$ are defined by

$$\mathcal{L}^{\frac{1}{N}} = \Lambda + \sum_{k\leq 0} a_k\Lambda^k, \quad \mathcal{L}^{\frac{1}{M}} = \sum_{k\geq -1} b_k\Lambda^k$$

with the relations

$$(\mathcal{L}^{\frac{1}{N}})^N = (\mathcal{L}^{\frac{1}{M}})^M = \mathcal{L}.$$

$\mathcal{L}^{\frac{1}{N}}$ and $\mathcal{L}^{\frac{1}{M}}$ are the lower Heisenberg triangular matrix and the upper triangular matrix, respectively.

Acting on free functions, these two fraction powers can be seen as two different locally expansions around zero and infinity respectively. It was stressed that $\mathcal{L}^{\frac{1}{N}}$ and $\mathcal{L}^{\frac{1}{M}}$ are two different operators even if $N = M$ $(N, M \geq 2)$ in [5] due to two different dressing operators. They can also be expressed as following

$$\mathcal{L}^{\frac{1}{N}} = \mathcal{P}_L\Lambda\mathcal{P}_L^{-1}, \quad \mathcal{L}^{\frac{1}{M}} = \mathcal{P}_R\Lambda^{-1}\mathcal{P}_R^{-1}.$$

Let us now define the following operators for the generators of the BTH flows

$$B_{\gamma,n} := \begin{cases} \mathcal{L}^{n+1-\frac{\alpha-1}{N}}, & \text{if } \gamma = \alpha = 1, 2, \cdots, N, \\ \mathcal{L}^{n+1+\frac{\beta}{M}}, & \text{if } \gamma = \beta = -M+1, \cdots, -1, 0. \end{cases} \tag{2.5}$$

Definition 2.1 *The bigraded Toda hierarchy in the Lax representation is given by the set of infinite number S of flows defined by* ([8])

$$\frac{\partial \mathcal{L}}{\partial t_{\gamma,n}} = \begin{cases} [(B_{\alpha,n})_+, \mathcal{L}], & \text{if } \gamma = \alpha = 1, 2, \cdots, N, \\ [-(B_{\beta,n})_-, \mathcal{L}], & \text{if } \gamma = \beta = -M+1, \cdots, -1, 0. \end{cases} \tag{2.6}$$

Sometimes we denote $\frac{\partial}{\partial t_{\gamma,n}}$ as $\partial_{\gamma,n}$ in this paper and denote $\{t_{\gamma,n}, \gamma = \alpha = 1, 2, \cdots, N, \}$, $\{t_{\gamma,n}, \gamma = \beta = -M+1, \cdots, -1, 0, \}$ as t_α, t_β respectively. We call the flows for $n = 0$ the primaries of the BTH and the time variables $t_{\gamma,n}$ $(n = 0)$ are primary time variables. The original tridiagonal Toda hierarchy corresponds to the case with $N = M = 1$.

The tau function and band structure According to [7], a function τ depending only on the dynamical variables t and ϵ is called the tau-function of BTH if it provides symbols related to wave operators as following

$$P_L := 1 + \frac{w_1}{\lambda} + \frac{w_2}{\lambda^2} + \cdots := \frac{\tau(x, t - [\lambda^{-1}]^N; \epsilon)}{\tau(x, t; \epsilon)}, \tag{2.7}$$

$$P_L^{-1} := 1 + \frac{w'_1}{\lambda} + \frac{w'_2}{\lambda^2} + \cdots := \frac{\tau(x+\epsilon, t + [\lambda^{-1}]^N; \epsilon)}{\tau(x+\epsilon, t; \epsilon)}, \tag{2.8}$$

$$P_R := \widetilde{w}_0 + \widetilde{w}_1\lambda + \widetilde{w}_2\lambda^2 + \cdots := \frac{\tau(x+\epsilon, t + [\lambda]^M; \epsilon)}{\tau(x, t; \epsilon)}, \tag{2.9}$$

$$P_R^{-1} := \widetilde{w}'_0 + \widetilde{w}'_1\lambda + \widetilde{w}'_2\lambda^2 + \cdots := \frac{\tau(x, t - [\lambda]^M; \epsilon)}{\tau(x+\epsilon, t; \epsilon)}, \tag{2.10}$$

where $[\lambda^{-1}]^N$ and $[\lambda]^M$ are defined by

$$[\lambda^{-1}]^N_{\gamma,n} := \begin{cases} \dfrac{\lambda^{-N(n+1-\frac{\gamma-1}{N})}}{N(n+1-\frac{\gamma-1}{N})}, & \gamma = N, N-1, \cdots 1, \\ 0, & \gamma = 0, -1 \cdots -(M-1), \end{cases}$$

$$[\lambda]^M_{\gamma,n} := \begin{cases} 0, & \gamma = N, N-1, \cdots 1, \\ \dfrac{\lambda^{M(n+1+\frac{\beta}{M})}}{M(n+1+\frac{\beta}{M})}, & \gamma = 0, -1, \cdots -(M-1). \end{cases}$$

Then we get

$$P_L := \sum_{n=0}^{\infty} \frac{P_n(-\widehat{\partial}_L)\tau(x, t; \epsilon)}{\tau(x, t; \epsilon)}\lambda^{-n}, \quad P_L^{-1} := \sum_{n=0}^{\infty} \frac{P_n(\widehat{\partial}_L)\tau(x+\epsilon, t; \epsilon)}{\tau(x+\epsilon, t; \epsilon)}\lambda^{-n}, \tag{2.11}$$

$$P_R := \sum_{n=0}^{\infty} \frac{P_n(\widehat{\partial}_R)\tau(x+\epsilon, t; \epsilon)}{\tau(x, t; \epsilon)}\lambda^{n}, \quad P_R^{-1} := \sum_{n=0}^{\infty} \frac{P_n(-\widehat{\partial}_R)\tau(x, t; \epsilon)}{\tau(x+\epsilon, t; \epsilon)}\lambda^{n}, \tag{2.12}$$

where the Schur polynomial $P_k(\widehat{\partial})$ is defined by

$$\exp\Big(\sum_{\mathrm{k}=1}^{\infty} \frac{1}{\mathrm{k}}\partial_{\mathrm{k}} \mathrm{z}^{\mathrm{k}}\Big) = \sum_{\mathrm{k}=0}^{\infty} \mathrm{P}_{\mathrm{k}}(\widehat{\partial})\mathrm{z}^{\mathrm{k}}, \quad \widehat{\partial} = \Big(\partial_1, \frac{1}{2}\partial_2, \frac{1}{3}\partial_3, \frac{1}{4}\partial_4, \cdots\Big). \tag{2.13}$$

Here the operators $\widehat{\partial}_L$ and $\widehat{\partial}_R$ are defined by

$$\widehat{\partial}_L = \Big\{\frac{1}{N(n+1-\frac{\alpha-1}{N})}\partial_{t_{\alpha,n}} : 1 \le \alpha \le N\Big\},$$

$$\widehat{\partial}_R = \Big\{\frac{1}{M(n+1+\frac{\beta}{M})}\partial_{t_{\beta,n}} : -M+1 \le \beta \le 0\Big\}.$$

The dressing operators $\mathcal{P}_L$ and $\mathcal{P}_R$ can be expressed by the function $\tau(x,t;\epsilon)$:

$$\mathcal{P}_L = \sum_{n=0}^{\infty} \frac{P_n(-\widehat{\partial}_L)\tau(x,t;\epsilon)}{\tau(x,t;\epsilon)}\Lambda^{-n}, \quad \mathcal{P}_L^{-1} = \sum_{n=0}^{\infty} \Lambda^{-n}\frac{P_n(\widehat{\partial}_L)\tau(x+\epsilon,t;\epsilon)}{\tau(x+\epsilon,t;\epsilon)}, \tag{2.14}$$

$$\mathcal{P}_R = \sum_{n=0}^{\infty} \frac{P_n(\widehat{\partial}_R)\tau(x+\epsilon,t;\epsilon)}{\tau(x,t;\epsilon)}\Lambda^{n}, \quad \mathcal{P}_R^{-1} = \sum_{n=0}^{\infty} \Lambda^{n}\frac{P_n(-\widehat{\partial}_R)\tau(x,t;\epsilon)}{\tau(x+\epsilon,t;\epsilon)}. \tag{2.15}$$

One can then find the explicit form of the coefficients $u_i(x,t)$ of the operator $\mathcal{L}$ in terms of the τ-function using (2.2) as that in [12–13],

$$\begin{aligned} u_i(x,t) &= \frac{P_{N-i}(\widehat{D}_L)\tau(x+(i+1)\epsilon,t;\epsilon)\circ\tau(x,t;\epsilon)}{\tau(x,t;\epsilon)\,\tau(x+(i+1)\epsilon,t;\epsilon)} \\ &= \frac{P_{M+i}(\widehat{D}_R)\tau(x+\epsilon,t;\epsilon)\circ\tau(x+i\epsilon,t;\epsilon)}{\tau(x,t;\epsilon)\,\tau(x+(i+1)\epsilon,t;\epsilon)}, \end{aligned} \tag{2.16}$$

where $\widehat{D}_L$ and $\widehat{D}_R$ are just the Hirota derivatives corresponding to $\widehat{\partial}_L$ and $\widehat{\partial}_R$ respectively.

As we all know, the interpolated BTH is equivalent to the bi-infinite or semi-infinite matrix-formed BTH. Because what we will consider next is the matrix-formed bigraded Toda hierarchy, the following equivalent definitions in the matrix form are introduced:

$$\Lambda := (E_{i,i+1})_{i\in\mathbb{Z}_+}, \quad u_i := \mathrm{diag}(u_{i,1}, u_{i,2}, u_{i,3}, \cdots).$$

After the following transformation $u_i(x) := u_{i,j} := a_{j,j+i}$, the matrix representation of $\mathcal{L}$ can be expressed by $(a_{i,j})_{i,j\ge 1}$ with

$$a_{i,j}(t) = \frac{P_{i-j+N}(\widehat{D}_L)\tau_j\circ\tau_{i-1}}{\tau_{i-1}\tau_j} = \frac{P_{j-i+M}(\widehat{D}_R)\tau_i\circ\tau_{j-1}}{\tau_{i-1}\tau_j}. \tag{2.17}$$

This immediately implies

$$a_{i,j} = 0, \quad \text{if } j < -M+i \text{ or } j > N+i.$$

That shows that the Lax matrix $\mathcal{L}$ has the (N,M)-band structure.

3 Exponential Solutions of the BTH

In [8], the rational solutions of the BTH were introduced already. In this section, we will introduce the exact (regular) solutions of the BTH, i.e., the non-negative exponential solutions.

The tau functions of the two-dimensional Toda lattice hierarchy (see [13]) can be expressed by

$$\tau_i = \begin{vmatrix} \overline{C}_{0,0} & \overline{C}_{0,1} & \cdots & \overline{C}_{0,i-1} \\ \overline{C}_{1,0} & \overline{C}_{1,1} & \cdots & \overline{C}_{1,i-1} \\ \vdots & \vdots & & \vdots \\ \overline{C}_{i-1,0} & \overline{C}_{i-1,1} & \cdots & \overline{C}_{i-1,i-1} \end{vmatrix}, \tag{3.1}$$

where $\overline{C}_{i,j}$ can be in the form of following inner product using the arbitrary density function $\rho(\lambda, \mu)$:

$$\overline{C}_{i,j} = \langle \lambda^i, \mu^j \rho(\lambda, \mu) \rangle.$$

Here the inner product can be chosen as the following integral representation:

$$\begin{aligned} \overline{C}_{i,j} &= \iint \rho(\lambda, \mu) \lambda^i \mu^j \exp\Big(\sum_{n=0}^{\infty} x_n \lambda^n + \sum_{n=0}^{\infty} y_n \mu^n \Big) \mathrm{d}\lambda \mathrm{d}\mu \\ &= \sum_{k,l=0}^{\infty} \overline{c}_{i,j,k,l} P_k(x) P_l(y), \end{aligned}$$

where $P_k(x)$ and $P_l(y)$ are Schur functions introduced in (2.13).

We should note here that the coefficients $\overline{c}_{i,j,k,l}$ are totally independent.

As the original tridiagonal Toda lattice is the $(1,1)$-reduction of the two-dimensional Toda lattice hierarchy, to get the solution of the tridiagonal Toda lattice hierarchy, we need to add the factor $\delta(\lambda - \mu)$ to the integral in the definition of $\overline{C}_{i,j}$, i.e., the element $\overline{C}_{i,j}$ in the tau function of the tridiagonal Toda lattice hierarchy (see [14]) becomes

$$\iint \rho(\lambda, \mu) \delta(\lambda - \mu) \lambda^i \mu^j \exp\Big(\sum_{n=0}^{\infty} x_n \lambda^n + \sum_{n=0}^{\infty} y_n \mu^n \Big) \mathrm{d}\lambda \mathrm{d}\mu, \tag{3.2}$$

which can further leads to

$$\int \rho(\lambda, \lambda) \lambda^{i+j} \exp\Big(\sum_{n=0}^{\infty} (x_n + y_n) \lambda^n \Big) \mathrm{d}\lambda. \tag{3.3}$$

After changing the time variables x, y to t_α, t_β in the BTH, respectively, (3.3) becomes a new function

$$\int \rho(\lambda, \lambda) \lambda^{i+j} \mathrm{e}^{\xi_L(\lambda, t_\alpha) + \xi_R(\lambda, t_\beta)} \mathrm{d}\lambda,$$

which corresponds to the $(1,1)$-BTH.

Denote ω_N and ω_M as the N-th root and M-th root of the unit, respectively. For the (N,M)-BTH which is a generalization of the tridiagonal Toda lattice hierarchy, the new function $C_{i,j}$ (the new form of $\overline{C}_{i,j}$) has the following form

$$\begin{aligned} C_{i,j} &= \iint \rho(\lambda, \mu) \delta(\lambda^N - \mu^M) \lambda^i \mu^j \mathrm{e}^{\xi_L(\lambda, t_\alpha) + \xi_R(\mu, t_\beta)} \mathrm{d}\lambda \mathrm{d}\mu \\ &= \sum_{p=0}^{N-1} \sum_{q=0}^{M-1} \int \rho(\omega_N^p \lambda^{\frac{1}{N}}, \omega_M^q \lambda^{\frac{1}{M}}) (\omega_N^p \lambda^{\frac{1}{N}})^i (\omega_M^q \lambda^{\frac{1}{M}})^j \mathrm{e}^{\xi_L(\omega_N^p \lambda^{\frac{1}{N}}, t_\alpha) + \xi_R(\omega_M^q \lambda^{\frac{1}{M}}, t_\beta)} \mathrm{d}\lambda. \end{aligned}$$

Therefore, the tau functions of the BTH can be explicitly written in the form (see [8])

$$\tau_i = \begin{vmatrix} C_{0,0} & C_{0,1} & \cdots & C_{0,i-1} \\ C_{1,0} & C_{1,1} & \cdots & C_{1,i-1} \\ \vdots & \vdots & & \vdots \\ C_{i-1,0} & C_{i-1,1} & \cdots & C_{i-1,i-1} \end{vmatrix}. \tag{3.4}$$

If we consider the case in a finite dimension (the dimension is n), i.e.,

$$\rho(\omega_N^p \lambda^{\frac{1}{N}}, \omega_M^q \lambda^{\frac{1}{M}}) = \sum_{k=1}^{n} \rho_0(\omega_N^p \lambda^{\frac{1}{N}}, \omega_M^q \lambda^{\frac{1}{M}}) \delta(\lambda - \lambda_k), \tag{3.5}$$

then

$$C_{i,j} = \sum_{p=0}^{N-1} \sum_{q=0}^{M-1} \sum_{k=1}^{n} \rho_0(\omega_N^p \lambda_k^{\frac{1}{N}}, \omega_M^q \lambda_k^{\frac{1}{M}})(\omega_N^p \lambda_k^{\frac{1}{N}})^i (\omega_M^q \lambda_k^{\frac{1}{M}})^j \mathrm{e}^{\xi_L(\omega_N^p \lambda^{\frac{1}{N}}, t_\alpha) + \xi_R(\omega_M^q \lambda^{\frac{1}{M}}, t_\beta)}. \tag{3.6}$$

After denoting $C_{0,0}$ as τ_1, we can rewrite the tau functions in the following bi-directional Wronskian form

$$\tau_i = \begin{vmatrix} \tau_1 & \partial_{t_{-M+1,0}} \tau_1 & \cdots & \partial_{t_{-M+1,0}}^{i-1} \tau_1 \\ \partial_{t_{N,0}} \tau_1 & \partial_{t_{N,0}} \partial_{t_{-M+1,0}} \tau_1 & \cdots & \partial_{t_{N,0}} \partial_{t_{-M+1,0}}^{i-1} \tau_1 \\ \vdots & \vdots & & \vdots \\ \partial_{t_{N,0}}^{i-1} \tau_1 & \partial_{t_{N,0}}^{i-1} \partial_{t_{-M+1,0}} \tau_1 & \cdots & \partial_{t_{N,0}}^{i-1} \partial_{t_{-M+1,0}}^{i-1} \tau_1 \end{vmatrix}. \tag{3.7}$$

Then τ_1 has the following form

$$\tau_1 = \sum_{p=0}^{N-1} \sum_{q=0}^{M-1} \sum_{k=1}^{n} \rho_0(\omega_N^p \lambda_k^{\frac{1}{N}}, \omega_M^q \lambda_k^{\frac{1}{M}}) \mathrm{e}^{\xi_L(\omega_N^p \lambda^{\frac{1}{N}}, t_\alpha) + \xi_R(\omega_M^q \lambda^{\frac{1}{M}}, t_\beta)}. \tag{3.8}$$

For the (N, M) case, for each $m \le n$,

$$\begin{aligned} \tau_m = & \sum_{1 \le i_1 \le i_2 \le \cdots \le i_m \le n} \sum_{i'_j, i'_k = 1}^{N} \sum_{\bar{i}_j, \bar{i}_k = 1}^{M} \prod_{1 \le k < j \le m} (\omega_N^{i'_j} \omega_{i_j}^{\frac{1}{N}} - \omega_N^{i'_k} \omega_{i_k}^{\frac{1}{N}})(\omega_M^{\bar{i}_j} \omega_{i_j}^{\frac{1}{M}} - \omega_M^{\bar{i}_k} \omega_{i_k}^{\frac{1}{M}}) \\ & \cdot \prod_{j,k,l=1}^{m} \rho_0(\omega_N^{i'_k} \lambda_{i_j}^{\frac{1}{N}}, \omega_M^{\bar{i}_l} \lambda_{i_j}^{\frac{1}{M}}) E_{i_j, i'_k, \bar{i}_l}, \end{aligned}$$

where

$$E_{i_j, i'_k, \bar{i}_l} = \mathrm{e}^{\xi_L(\omega_N^{i'_k} \lambda_{i_j}^{\frac{1}{N}}, t_\alpha) + \xi_R(\omega_M^{\bar{i}_l} \lambda_{i_j}^{\frac{1}{M}}, t_\beta)}. \tag{3.9}$$

We can find that there are $\binom{n}{m}(NM)^m$ terms in τ_m.

The theory above is about the (N, M)-BTH. To see it clearly, we will further consider a specific kind of example, i.e., the $(1, M)$-BTH, clearly in the following.

Before that, firstly we will consider the finite-sized Lax matrix of the $(1, M)$-BTH.

If the Lax matrix is supposed to have n different eigenvalues, i.e., there exists a matrix Φ, such that

$$\Phi L \Phi^{-1} = \mathrm{dig}(\lambda_1, \lambda_2, \cdots, \lambda_n),$$

then from the theory on the (N, M)-BTH, we can get the first tau function of the $(1, M)$-BTH as following

$$\tau_1 = \sum_{q=0}^{M-1} \sum_{k=1}^{n} \rho_0(\lambda_k, \omega_M^q \lambda_k^{\frac{1}{M}}) \mathrm{e}^{\xi_R(\omega_M^q \lambda_k^{\frac{1}{M}}, t_\beta)}. \tag{3.10}$$

Similarly for every integer $m \le n$, the tau function τ_m has the following form

$$\begin{aligned}\tau_m = & \sum_{1\le i_1 \le i_2 \le \cdots \le i_m \le n} \sum_{i'_j, i'_k = 1}^{M} \prod_{1 \le k < j \le m} (\omega_{i_j} - \omega_{i_k}) \\ & \cdot \prod_{1\le k<j\le m} (\omega_M^{i'_j} \omega_{i_j}^{\frac{1}{M}} - \omega_M^{i'_k} \omega_{i_k}^{\frac{1}{M}}) \prod_{j=1}^{m} \rho_0(\lambda_{i_j}, \omega_M^{i'_j} \lambda_{i_j}^{\frac{1}{M}}) E_{i_j, i'_j},\end{aligned}$$

where

$$E_{i_j, i'_j} = \mathrm{e}^{\xi_R(\omega_M^{i'_j} \lambda_{i_j}^{\frac{1}{M}}, t_\beta)}.$$

Considering the primary dependence, τ_1 can be written as

$$\tau_1 = \sum_{q=0}^{M-1} \sum_{k=1}^{n} \rho_0(\lambda_k, \omega_M^q \lambda_k^{\frac{1}{M}}) \exp\Big(\sum_{s=1}^{M} (\omega_M^q \lambda_k^{\frac{1}{M}})^s t_{s-M,0} \Big). \tag{3.11}$$

To see it more clearly, we will further consider a specific example in the following section, i.e., the exponential solution of the $(1, 2)$-BTH.

4 The Regular Solution of the $(1, 2)$-BTH

One can obtain the general finite dimensional solution for the (N, M)-BTH with exponential functions. However, most of the solutions are complex and have singular points. In this section, we will consider the exponential solution, particularly the regular solution of the $(1, 2)$-BTH. By this regular solution, we see the difference between $(1, 2)$-BTH and original Toda hierarchy from a geometric viewpoint.

For $(1, 2)$ case, the solution can be expressed as

$$\tau_1 = \sum_{k=1}^{n} \rho_0(\lambda_k, \lambda_k^{\frac{1}{2}}) \mathrm{e}^{\xi_R(\lambda_k^{\frac{1}{2}}, t_\beta)} + \rho_0(\lambda_k, \omega_2 \lambda_k^{\frac{1}{2}}) \mathrm{e}^{\xi_R(\omega_M \lambda_k^{\frac{1}{2}}, t_\beta)}, \tag{4.1}$$

where $\omega_2 = -1$.

If we only consider the primary dependence which means that we let the tau function depend only on the primary time variables, then the first solution τ_1 in the form of a one-by-one matrix has the following form

$$\tau_1 = \sum_{k=1}^{n} \rho_0(\lambda_k, \lambda_k^{\frac{1}{2}}) \mathrm{e}^{\omega_k^{\frac{1}{2}} \lambda_k^{\frac{1}{2}} t_{-1,0} + \omega_k \lambda_k t_{0,0}} + \rho_0(\lambda_k, -\lambda_k^{\frac{1}{2}}) \mathrm{e}^{-\omega_k^{\frac{1}{2}} \lambda_k^{\frac{1}{2}} t_{-1,0} + \omega_k \lambda_k t_{0,0}}. \tag{4.2}$$

Let us assume that the Lax matrix is semi-simple and has distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$. If the $\lambda_k, k = 1, 2, 3$ are negative, the real solution can be written in the following form:

$$\begin{aligned}\tau_1 &= \sum_{k=1}^{n} \rho_0(\lambda_k, \lambda_k^{\frac{1}{2}}) \mathrm{e}^{i|\lambda_k|^{\frac{1}{2}} t_{-1,0} + \lambda_k t_{0,0}} + \rho_0(\lambda_k, -\lambda_k^{\frac{1}{2}}) \mathrm{e}^{-i|\lambda_k|^{\frac{1}{2}} t_{-1,0} + \lambda_k t_{0,0}} \\ &= \sum_{k=1}^{n} A'_k \cos(|\lambda_k|^{\frac{1}{2}} t_{-1,0} + \theta_k) \mathrm{e}^{\lambda_k (t_{0,0} + t_{1,0})},\end{aligned} \tag{4.3}$$

where

$$A'_k := \sqrt{\rho_0(\lambda_k, \lambda_k^{\frac{1}{2}})^2 + \rho_0(\lambda_k, -\lambda_k^{\frac{1}{2}})^2}$$

and θ_k depend on $\{\rho_0(\lambda_k, \lambda_k^{\frac{1}{2}}), \rho_0(\lambda_k, -\lambda_k^{\frac{1}{2}}), |\lambda_k|^{\frac{1}{2}} t_{-1,0}\}$. This is in fact a periodic solution about the $t_{-1,0}$ time variable and it has singular points.

If we set $0 < \lambda_1 < \lambda_2 < \lambda_3$, $\rho_0 \geq 0$, it will lead to regular solutions. In this case as a simple but interesting example, we will consider a regular solution for the $(1, 2)$-BTH with a 3×3-sized Lax matrix, and discuss some geometric structures of the solution in the following part. Then the regular function τ_1 is given by

$$\tau_1 = \sum_{k=1}^{3} A_k \cosh(\lambda_k^{\frac{1}{2}} t_{-1,0} + \theta_k) \mathrm{e}^{\lambda_k (t_{0,0} + t_{1,0})} = \sum_{k=1}^{3} C_k E_k,$$

where A_k and θ_k are arbitrary constants, $C_k := A_k \cosh(\lambda_k^{\frac{1}{2}} t_{-1,0} + \theta_k)$ and $E_k := \mathrm{e}^{\lambda_k (t_{0,0} + t_{1,0})}$. We also write $S_k := A_k \sinh(\lambda_k^{\frac{1}{2}} t_{-1,0} + \theta_k)$ in the following part.

For τ_1 being positive definite, $A_i > 0$ is supposed to hold. Also we set all $\theta_k = 0$ (this is necessary for τ_k being sign definite). Then the second tau function τ_2 and the third tau function τ_3 in (3.7) are given by

$$\begin{aligned}\tau_2 &= \begin{vmatrix} \tau_1 & \partial_{1,0}\tau_1 \\ \partial_{-1,0}\tau_1 & \partial_{-1,0}\partial_{1,0}\tau_1 \end{vmatrix} = \left| \begin{pmatrix} C_1E_1 & C_2E_2 & C_3E_3 \\ \lambda_1^{\frac{1}{2}} S_1E_1 & \lambda_2^{\frac{1}{2}} S_2E_2 & \lambda_3^{\frac{1}{2}} S_3E_3 \end{pmatrix} \begin{pmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \\ 1 & \lambda_3 \end{pmatrix} \right| \\ &= \sum_{1 \leq i < j \leq 3} (C_iS_j\lambda_j^{\frac{1}{2}} - C_jS_i\lambda_i^{\frac{1}{2}})(\lambda_j - \lambda_i)E_iE_j, \\ \tau_3 &= \begin{vmatrix} \tau_1 & \partial_{1,0}\tau_1 & \partial_{1,0}^2\tau_1 \\ \partial_{-1,0}\tau_1 & \partial_{-1,0}\partial_{1,0}\tau_1 & \partial_{-1,0}\partial_{1,0}^2\tau_1 \\ \partial_{-1,0}^2\tau_1 & \partial_{-1,0}^2\partial_{1,0}\tau_1 & \partial_{-1,0}^2\partial_{1,0}^2\tau_1 \end{vmatrix} \\ &= \left| \begin{pmatrix} C_1E_1 & C_2E_2 & C_3E_3 \\ \lambda_1^{\frac{1}{2}} S_1E_1 & \lambda_2^{\frac{1}{2}} S_2E_2 & \lambda_3^{\frac{1}{2}} S_3E_3 \\ \lambda_1C_1E_1 & \lambda_2C_2E_2 & \lambda_3C_3E_3 \end{pmatrix} \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{pmatrix} \right| \\ &= \left(\sum_{i \to j \to k} \lambda_i^{\frac{1}{2}} |\lambda_j - \lambda_k| S_iC_jC_k \right) \prod_{i<j} (\lambda_j - \lambda_i) E_1E_2E_3,\end{aligned}$$

respectively, where $\sum\limits_{i \to j \to k}$ implies the sum over the cyclic permutation on $\{1, 2, 3\}$. Therefore, τ_1 is always positive.

For τ_2, taking derivatives can imply

$$\begin{aligned}
&\partial_{t_{-1,0}}(C_iS_j\lambda_j^{\frac{1}{2}}-C_jS_i\lambda_i^{\frac{1}{2}})\\
&=S_iS_j(\lambda_i\lambda_j)^{\frac{1}{2}}+C_iC_j\lambda_j-S_jS_i(\lambda_i\lambda_j)^{\frac{1}{2}}-C_jC_i\lambda_i\\
&=C_iC_j(\lambda_j-\lambda_i)>0, \qquad (4.4)
\end{aligned}$$

which means that τ_2 always increases with the variable $t_{-1,0}$. Because when $t_{-1,0}=0$,

$$C_iS_j\lambda_j^{\frac{1}{2}}-C_jS_i\lambda_i^{\frac{1}{2}}=0,$$

and when $t_{-1,0}>0$, τ_2 is always positive. Because

$$\begin{aligned}
\lim_{t_{-1,0}\to 0}\frac{\partial_{1,0}\tau_2}{\tau_2}&=\lim_{t_{-1,0}\to 0}\frac{\sum\limits_{1\le i<j\le 3}(C_iS_j\lambda_j^{\frac{1}{2}}-C_jS_i\lambda_i^{\frac{1}{2}})(\lambda_j^2-\lambda_i^2)E_iE_j}{\sum\limits_{1\le i<j\le 3}(C_iS_j\lambda_j^{\frac{1}{2}}-C_jS_i\lambda_i^{\frac{1}{2}})(\lambda_j-\lambda_i)E_iE_j}\\
&=\lim_{t_{-1,0}\to 0}\frac{\sum\limits_{1\le i<j\le 3}C_iC_j(\lambda_j^2-\lambda_i^2)E_iE_j}{\sum\limits_{1\le i<j\le 3}C_iC_j(\lambda_j-\lambda_i)E_iE_j}\\
&=\frac{\sum\limits_{1\le i<j\le 3}(\lambda_j^2-\lambda_i^2)E_iE_j}{\sum\limits_{1\le i<j\le 3}(\lambda_j-\lambda_i)E_iE_j},
\end{aligned}$$

$t_{-1,0}=0$ is a removable singular point. Using the formula (2.17) for $a_{i,j}$ of the Lax matrix, we have

$$\begin{aligned}
a_{1,1}&=\partial_{1,0}\ln\tau_1=\frac{\lambda_1C_1E_1+\lambda_2C_2E_2+\lambda_3C_3E_3}{C_1E_1+C_2E_2+C_3E_3},\\
a_{2,2}&=\partial_{1,0}\ln\frac{\tau_2}{\tau_1}=\frac{\sum\limits_{1\le i<j\le 3}(C_iS_j\lambda_j^{\frac{1}{2}}-C_jS_i\lambda_i^{\frac{1}{2}})(\lambda_j^2-\lambda_i^2)E_iE_j}{\sum\limits_{1\le i<j\le 3}(C_iS_j\lambda_j^{\frac{1}{2}}-C_jS_i\lambda_i^{\frac{1}{2}})(\lambda_j-\lambda_i)E_iE_j}-a_{11},\\
a_{3,3}&=\partial_{1,0}\ln\frac{\tau_3}{\tau_2}=\sum_{i=1}^{3}\lambda_i-(a_{1,1}+a_{2,2}).
\end{aligned}$$

Assuming the ordering for the eigenvalues as

$$\lambda_1<\lambda_2<\lambda_3,$$

one can obtain the following asymptotic sorting property of the Lax matrix:

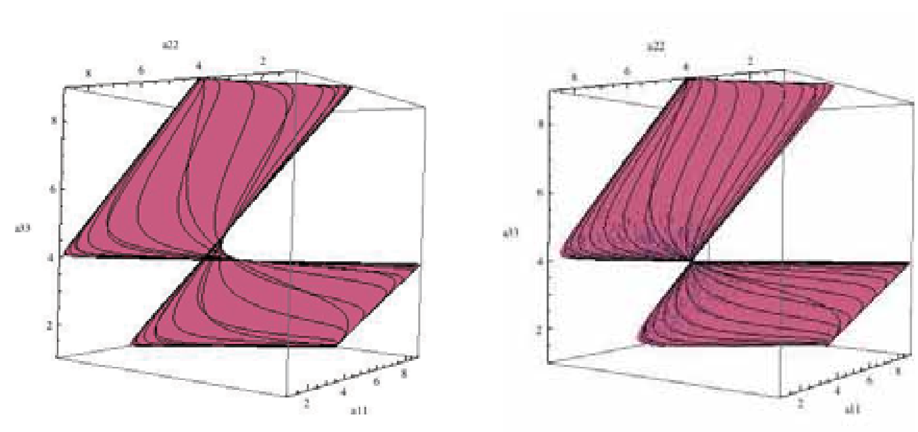
$$\mathcal{L}\longrightarrow\begin{pmatrix}\lambda_3&1&0\\0&\lambda_2&1\\0&0&\lambda_1\end{pmatrix}\quad\text{for } t_{-1,0}\to\infty,$$

$$\mathcal{L} \longrightarrow \begin{cases} \begin{pmatrix} \lambda_3 & 1 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix} & \text{for } t_{1,0} \to \infty, \\ \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_3 \end{pmatrix} & \text{for } t_{1,0} \to -\infty. \end{cases}$$

To see the orbit generated by the solution, we consider the projection π of the Lax matrix on the diagonal part, i.e.,

$$\pi: \quad \mathcal{L} = \begin{pmatrix} a_{1,1} & 1 & 0 \\ a_{2,1} & a_{2,2} & 1 \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \longmapsto \operatorname{diag}(\mathcal{L}) \equiv (a_{1,1}, a_{2,2}, a_{3,3}).$$

Figure 1 illustrates the image of the map π: The left panel (a) shows the image in the case of the $(1,2)$-BTH for $-5 \le t_{0,0} \le 5$ and $-5 \le t_{0,1} \le 5$. The right panel (b) of the figure shows the case of the original Toda lattice, that is, the corresponding Lax matrix is a 3×3-sized tridiagonal matrix. That example gives the following proposition.



(a) (b)

Figure 1 The image $\pi(\mathcal{L}) = (a_{1,1}, a_{2,2}, a_{3,3})$: (a) For the $(1,2)$-BTH; and (b) for the tridiagonal Toda hierarchy. Those orbits are obtained by changing $t_{0,0}$ and $t_{0,1}$ with fixed $t_{-1,0} = 1$. The eigenvalues are given by $\lambda_1 = 1, \lambda_2 = 4$ and $\lambda_3 = 9$.

Proposition 4.1 *In the case of the original Toda lattice, one can show that all the orbits have to cross the center point* $(\lambda_2, \sigma_1 - 2\lambda_2, \lambda_2)$ *with* $\sigma_1 = \sum_{j=1}^{3} \lambda_j$. *However, the orbits for the* $(1,2)$-*BTH have no such restriction, but those still go through the points close to the center.*

Proof Firstly because the Lax matrix has eigenvalues $\lambda_1, \lambda_2, \lambda_3$ with $\lambda_1 < \lambda_2 < \lambda_3$. We set the diagonal elements $(a_{1,1}, a_{2,2}, a_{3,3})$ to take values $(\Delta_1, \sigma_1 - 2\Delta_1, \Delta_1)$. Then considering that the matrix

$$\begin{pmatrix} \Delta_1 & 1 & 0 \\ a_{2,1} & \sigma_1 - 2\Delta_1 & 1 \\ a_{3,1} & a_{3,2} & \Delta_1 \end{pmatrix}$$

has eigenvalues $\lambda_1, \lambda_2, \lambda_3$ with $\lambda_1 < \lambda_2 < \lambda_3$, we get

$$(\Delta_1 - \lambda)[(\sigma_1 - 2\Delta_1 - \lambda)(\Delta_1 - \lambda) - a_{3,2}] - a_{2,1}(\Delta_1 - \lambda) + a_{3,1} = -\lambda^3 + \sigma_1\lambda^2 - \sigma_2\lambda + \sigma_3,$$

where $\sigma_1, \sigma_2, \sigma_3$ are the first three fundamental symmetric polynomials. The equation above implies

$$a_{3,2} + a_{2,1} = 2\sigma_1\Delta_1 - 3\Delta_1^2 - \sigma_2, \tag{4.5}$$

$$\Delta_1^2\sigma_1 - 2\Delta_1^3 - (a_{3,2} + a_{2,1})\Delta_1 + a_{3,1} = \sigma_3, \tag{4.6}$$

which further leads to

$$\Delta_1^3 - \sigma_1\Delta_1^2 + \sigma_2\Delta_1 - \sigma_3 + a_{3,1} = 0. \tag{4.7}$$

By (2.17), we get

$$a_{3,2}(t) = \frac{P_2(\widehat{D}_L)\tau_2 \circ \tau_2}{\tau_2\tau_2} = \frac{P_1(\widehat{D}_R)\tau_3 \circ \tau_1}{\tau_2\tau_2}. \tag{4.8}$$

Because we choose all eigenvalues to be positive,

$$\begin{aligned} D_{1,0}^2\tau_2 \circ \tau_2 = \sum_{1\le i,j,k\le 3} &(C_iS_j\lambda_j^{\frac{1}{2}} - C_jS_i\lambda_i^{\frac{1}{2}})(C_jS_k\lambda_k^{\frac{1}{2}} - C_kS_j\lambda_j^{\frac{1}{2}}) \\ &\cdot (\lambda_k - \lambda_j)(\lambda_j - \lambda_i)(\lambda_k + \lambda_i + 2\lambda_j) > 0, \end{aligned} \tag{4.9}$$

and therefore $a_{3,2}(t) > 0$. Similarly, by (2.17), we get

$$a_{2,1}(t) = \frac{P_2(\widehat{D}_L)\tau_1 \circ \tau_1}{\tau_1\tau_1} = \frac{P_1(\widehat{D}_R)\tau_2 \circ \tau_0}{\tau_1\tau_1} = \frac{\partial_{-1,0}\tau_2}{\tau_1\tau_1}. \tag{4.10}$$

Because of (4.4), we get

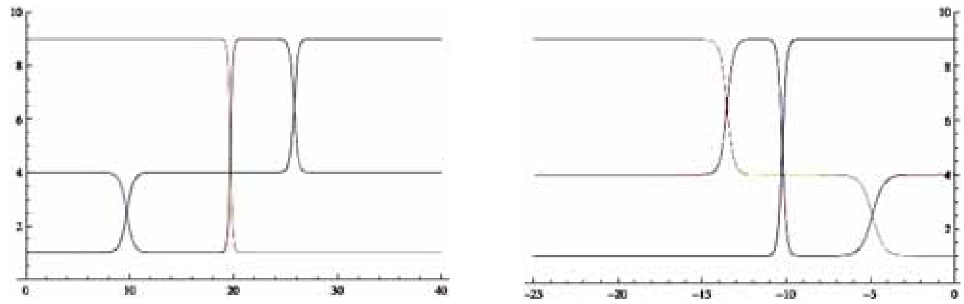
$$a_{2,1}(t) > 0. \tag{4.11}$$

By (4.5), we can get the following identity that must be correct:

$$f(\Delta_1) := 2\sigma_1\Delta_1 - 3\Delta_1^2 - \sigma_2 > 0. \tag{4.12}$$

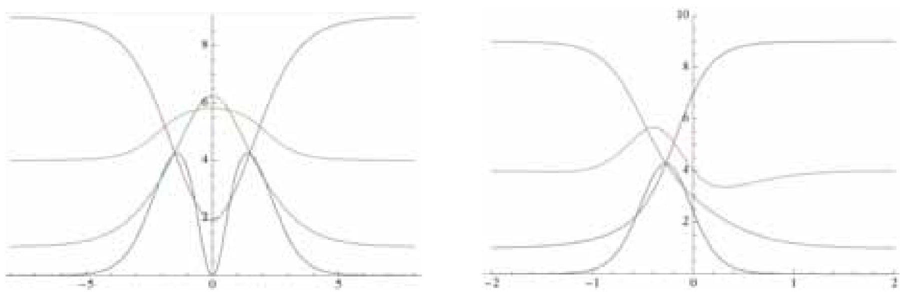
For the triangonal Toda hierarchy, $a_{3,1} = 0$, and (4.7) has three roots $\lambda_1, \lambda_2, \lambda_3$ with the order $\lambda_1 < \lambda_2 < \lambda_3$. Let $\Delta_1 = \lambda_3$, and we find

$$\begin{aligned} f(\lambda_3) &= 2\sigma_1\lambda_3 - 3\lambda_3^2 - \sigma_2 = 2(\lambda_1 + \lambda_2 + \lambda_3)\lambda_3 - 3\lambda_3^2 - (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) \\ &= (\lambda_1 - \lambda_3)(\lambda_3 - \lambda_2) < 0, \end{aligned} \tag{4.13}$$



(a) (b)

Figure 2 Graphs for $(a_{1,1}, a_{2,2}, a_{3,3})$ of the (1,2)-BTH: (a) Depending on the parameters $t_{0,0}$ with $t_{-1,0} = 20$ and $t_{0,1} = -2$; (b) depending on the parameter $t_{0,0}$ with $t_{-1,0} = 20$ and $t_{0,1} = 1$. The eigenvalues are $\lambda_1 = 1,\ \lambda_2 = 4,\ \lambda_3 = 9$ and $A_1 = A_2 = A_3 = 1$.



(a) (b)

Figure 3 Graphs for $(a_{3,1}, a_{1,1}, a_{2,2}, a_{3,3})$ for (1,2)-BTH: (a) Depending on the parameters $t_{-1,0}$ with $t_{0,1} = 0$ and $t_{0,0} = -0.4$; (b) depending on the parameters $t_{0,0}$ with $t_{0,1} = 0$ and $t_{-1,0} = 1$, where $\lambda_1 = 1,\ \lambda_2 = 4,\ \lambda_3 = 9$ and $A_1 = A_2 = A_3 = 1$.

which is in contradiction with (4.12). Similarly, we can also find that $\Delta_1 = \lambda_1$ is also in contradiction with (4.12). Therefore, the only choice is $\Delta_1 = \lambda_2$. That is why for the original Toda lattice, all the orbits have to cross the center point $(\lambda_2, \sigma_1 - 2\lambda_2, \lambda_2)$ with $\sigma_1 = \sum\limits_{j=1}^{3} \lambda_j$ as shown in Figure 1(b). By (2.17), we get

$$a_{3,1}(t) = \frac{\tau_3 \tau_0}{\tau_2 \tau_1}. \tag{4.14}$$

So for the $(1,2)$-BTH, $a_{3,1}(t)$ is always positive when $t_{-1,0} > 0$, because of the positivity of the tau functions. From that and considering (4.7), we can get that there will be one more part which is close to the crossing point $(\lambda_2, \sigma_1 - 2\lambda_2, \lambda_2)$ just as Figure 1(a).

The boundaries of Figure 1(a) are characterized by Figure 2(a) and Figure 2(b). Fixing another time variable $t_{1,1}$, their two-dimensional graphs are as Figure 3(a) and Figure 3(b).

After these geometric pictures, the moment map (see [14]) related to the $(1,2)$-BTH will be given in the following.

Moment polytope for the $(1,2)$-BTH From the last section, τ_1 and τ_2 have the following form:

$$\tau_1 = \sum_{k=1}^{3} C_k E_k, \quad \tau_2 = \sum_{1\le i<j\le 3} (C_i S_j \lambda_j^{\frac{1}{2}} - C_j S_i \lambda_i^{\frac{1}{2}})(\lambda_j - \lambda_i) E_i E_j.$$

We can treat (τ_1, τ_2) as one point of the Flag manifold G/B, where $G := Sl(3, R)$ and B is a Borel subgroup (an upper triangular subgroup) containing the Cartan Lie subgroup of G (a diagonal torus). Here we describe the moment polytope defined by the map (see [11]),

$$\begin{aligned} \mu :\quad & G/B \longrightarrow \mathcal{H}^*, \\ & (\tau_1, \tau_2) \longmapsto M_{\tau_1} + M_{\tau_2}, \end{aligned}$$

where $\mathcal{H}^*$ is the dual of the Cartan Lie subgroup of G,

$$M_{\tau_1} = \frac{C_1E_1L_1 + C_2E_2L_2 + C_3E_3L_3}{C_1E_1 + C_2E_2 + C_3E_3},$$

$$M_{\tau_2} = \frac{\displaystyle\sum_{1\le i<j\le 3} (C_i S_j \lambda_j^{\frac{1}{2}} - C_j S_i \lambda_i^{\frac{1}{2}})(\lambda_j - \lambda_i) E_i E_j (L_i + L_j)}{\displaystyle\sum_{1\le i<j\le 3} (C_i S_j \lambda_j^{\frac{1}{2}} - C_j S_i \lambda_i^{\frac{1}{2}})(\lambda_j - \lambda_i) E_i E_j}.$$

Here the weight vectors L_i's are defined by

$$L_1 := (1, 0), \quad L_2 := \frac{1}{2}(-1, \sqrt{3}), \quad L_3 := \frac{1}{2}(-1, -\sqrt{3}).$$

Figure 4 illustrates the moment polytope (i.e., the graph of $M_{\tau_1} + M_{\tau_2}$) for our example.

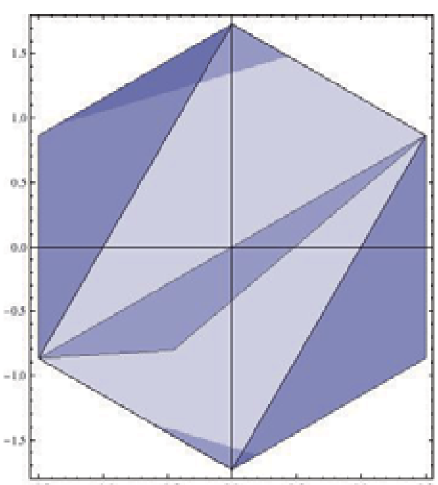


Figure 4 The moment polytope (i.e., $M_{\tau_1} + M_{\tau_2}$ in $\mathcal{H}^*$) of the (1,2)-BTH: The orbits are obtained by changing $t_{0,0}$ and $t_{0,1}$ with fixed $t_{-1,0} = 1$. The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 4$ and $\lambda_3 = 9$.

In the Figure 4, the vertex $\frac{1}{2}(3, \sqrt{3})$ represents the highest weight $L_1 - L_3$ which is the starting point of this $(1, 2)$-BTH flow (the $t_{0,1}$ flow and the $t_{0,0}$ flow). The vertex $\frac{1}{2}(-3, -\sqrt{3})$

represents the lowest weight $-L_1 + L_3$ which is the destination of the flow. The boundaries of Figure 4 corresponding to the $\overline{G} = Sl(2, R)$ for the $(1, 2)$-BTH associated with two of $a_{2,1},\ a_{3,1}$ and $a_{3,2}$ are zeroes. The six vertices in Figure 4 are fixed points of the Cartan subgroup of G which can be generated from the highest weight by the action of the symmetric group S_3. These six vertices are in bijection with the elements of the Weyl group.

Till now, we have found that the moment polytope can tell how the flow of the time variables of the BTH goes well, and meanwhile the sorting property of the flows and the representation in Lie algebra describing the orbit are also seen from Figure 4. What about the polytope corresponding to the higher rank matrix-formed BTH? It is an interesting question.

5 The $(N, 1)$-Bigraded Toda Hierarchy

In the previous section, the $(1, M)$-BTH was introduced in geat detail. In this section, we concentrate on the $(N, 1)$-BTH. For convenience of calculation, we firstly use the interpolated form of the BTH.

In the following part, we will introduce some concrete primary flows of the $(N, 1)$-BTH.

(2,1)-BTH: The Lax operator of the $(2, 1)$-BTH is as following

$$\mathcal{L}_{2,1} = \Lambda^2 + u_1\Lambda + u_0 + u_{-1}\Lambda^{-1}. \tag{5.1}$$

(2.6) in this case are as following

$$\partial_{2,0}\mathcal{L}_{2,1} = [\Lambda + (1 + \Lambda)^{-1}u_1(x), \mathcal{L}_{2,1}], \tag{5.2}$$

which further leads to the following concrete equations

$$\begin{cases} \partial_{2,0}u_1(x) = u_1(x + \epsilon) - u_1(x) + u_1(x)(1 - \Lambda)(1 + \Lambda)^{-1}u_1(x), \\ \partial_{2,0}u_0(x) = u_{-1}(x + \epsilon) - u_{-1}(x), \\ \partial_{2,0}u_{-1}(x) = u_{-1}(x)(1 - \Lambda^{-1})(1 + \Lambda)^{-1}u_1(x), \end{cases} \tag{5.3}$$

which are equivalent to (40) in [15] and are also related to the system (10)–(12) proposed in [16].

(3,1)-BTH: The equations of the $(3, 1)$-BTH are as following

$$\begin{cases} \partial_{3,0}u_2(x) = u_1(x + \epsilon) - u_1(x) + u_2(x)(1 - \Lambda^2)(1 + \Lambda + \Lambda^2)^{-1}u_2(x), \\ \partial_{3,0}u_1(x) = u_0(x + \epsilon) - u_0(x) + u_1(x)(1 - \Lambda)(1 + \Lambda + \Lambda^2)^{-1}u_2(x), \\ \partial_{3,0}u_0(x) = u_{-1}(x + \epsilon) - u_{-1}(x), \\ \partial_{3,0}u_{-1}(x) = u_{-1}(x)(1 - \Lambda^{-1})(1 + \Lambda + \Lambda^2)^{-1}u_2(x), \end{cases} \tag{5.4}$$

which can be rewritten as

$$\begin{cases} \partial_{3,0}\Big(\sum_{i=0}^{2} v_2(x + i\epsilon)\Big) = v_1(x + \epsilon) - v_1(x) + \Big(\sum_{i=0}^{2} v_2(x + i\epsilon)\Big)(1 - \Lambda^2)v_2(x), \\ \partial_{3,0}v_1(x) = v_0(x + \epsilon) - v_0(x) + v_1(x)(1 - \Lambda)v_2(x), \\ \partial_{3,0}v_0(x) = v_{-1}(x + \epsilon) - v_{-1}(x), \\ \partial_{3,0}v_{-1}(x) = v_{-1}(x)(1 - \Lambda^{-1})v_2(x), \end{cases} \tag{5.5}$$

by the transformation $u_2 = (1+\Lambda+\Lambda^2)v_2;\ u_i = v_i,\ i = 1,0,-1$. This is the case when $n=1,\ m=-2,\ l=4$ for (10) in [15] after treating $v(x+i\epsilon)$ as $v(i)$, and

$$\begin{aligned}\sum_{s=1}^{3} v_2'(i+s-1) &= \sum_{s=1}^{3} v_2(i+s-1)\times(v_2(i)-v_2(i+2)) + v_1(i+1)-v_1(i),\\ v_k'(i) &= v_k(i)\Big(\sum_{s=1}^{2} v_0(i+s-1) - \sum_{s=1}^{2} v_0(i+s-1+k)\Big)\\ &\quad + v_{k-1}(i-1) - v_{k-1}(i), \quad k=1,0,-1.\end{aligned} \tag{5.6}$$

To generalize the results above, the $(N,1)$-BTH will be considered as following.

$(N,1)$-BTH: The concrete equations of the $(N,1)$-BTH are as following

$$\begin{cases}\partial_{N,0}\Big(\sum_{i=0}^{N-1} v_{N-1}(x+i\epsilon)\Big) = v_{N-2}(x+\epsilon) - v_{N-2}(x)\\ \quad +\Big(\sum_{i=0}^{N-1} v_{N-1}(x+i\epsilon)\Big)(1-\Lambda^{N-1})v_{N-1}(x),\\ \partial_{N,0}v_{N-2}(x) = v_{N-3}(x+\epsilon) - v_{N-3}(x) + v_{N-2}(x)(1-\Lambda^{N-2})v_{N-1}(x),\\ \qquad\qquad \cdots\cdots\cdots\cdots\cdots\cdots\\ \partial_{N,0}v_1(x) = v_0(x+\epsilon) - v_0(x) + v_1(x)(1-\Lambda)v_2(x),\\ \partial_{N,0}v_0(x) = v_{-1}(x+\epsilon) - v_{-1}(x),\\ \partial_{N,0}v_{-1}(x) = v_{-1}(x)(1-\Lambda^{-1})v_2(x),\end{cases}$$

where $u_{N-1} = (1+\Lambda+\cdots+\Lambda^{N-1})v_{N-1};\ u_i = v_i,\ i = N-2,\cdots,0,-1$.

Similarly we can rewrite the equations of $(N,1)$-BTH as following

$$\begin{cases}\sum_{s=1}^{N} \partial_{N,0}v_{N-1}(i+s-1) = \sum_{s=1}^{N} v_{N-1}(i+s-1)\times(v_{N-1}(i) - v_{N-1}(i+N-1))\\ \qquad\qquad +v_{N-2}(i+1) - v_{N-2}(i),\\ \partial_{N,0}v_k(i) = v_k(i)(v_{N-1}(i) - v_{N-1}(i+k)) + v_{k-1}(i+1) - v_{k-1}(i),\\ \qquad\qquad k = N-2, N-3, \cdots, -1,\end{cases} \tag{5.7}$$

where $v_j(i) := v_j(x+i\epsilon)$ and $v_{-2}=0$.

This is exactly the case when $n=1$, $m=-N+1$, $l=N+1$ for (10) in [15]. It has another Lax representation which will be discussed in the next section.

6 Another Lax Representation for Primary Flows of the $(N,1)$-BTH

In this section, we will introduce another Lax representation of the $\partial_{N,0}$ flow (i.e., the primary flow) of the $(N,1)$-BTH.

For an arbitrary pair of integers $n\in\mathbb{N}$ and $m\le n-1$, we define an infinite collection of the first-order differential operators

$$H_i = \partial_{N,0} - v_{N-1}(i, t_{N,0}), \quad i\in\mathbb{Z} \tag{6.1}$$

and

$$G_i = \partial_{N,0} + \sum_{k=1}^{N-1} v_{N-1}(i-k, t_{N,0}) + \sum_{k=1}^{N} v_{N-1-k}(i-N+1, t_{N,0}) H_{i-k}^{-1} \cdots H_{i-2}^{-1} H_{i-1}^{-1}.$$

Let us define the following auxiliary equations on the infinite collection of dressing operators $\{\widehat{w}_i,\ i \in \mathbb{Z}\}$:

$$G_i \widehat{w}_i = \widehat{w}_{i+1-N} \partial_{N,0}, \quad H_i \widehat{w}_i = \widehat{w}_{i+1} \partial_{N,0}, \tag{6.2}$$

which can be rewritten in terms of the Baker-Akhiezer (BA) function ψ_i as

$$G_i \psi_i = z\psi_{i+1-N}, \quad H_i \psi_i = z\psi_{i+1}. \tag{6.3}$$

We can show that the compatibility conditions of (6.3) are the well-determined system of equations for the fields $\{v_{-1}(i, t_{N,0}), v_0(i, t_{N,0}), \cdots, v_{N-1}(i, t_{N,0})\}$.

Formally, the consistency condition of (6.3) is given by

$$G_{i+1} H_i = H_{i-N+1} G_i.$$

The technical observation will lead to that (6.3) can be rewritten in terms of the $(L, A_{N,0})$-pair

$$L(\psi_i) = z\psi_i, \quad \partial_{N,0}\psi_i = A_{N,0}(\psi_i),$$

where L and $A_{N,0}$ are difference operators acting on the BA functions $\{\psi_i, i \in \mathbb{Z}\}$ as

$$\begin{aligned} L(\psi_i) &= z\psi_{i+N} + \Big(\sum_{s=1}^{N} v_{N-1}(i+s-1)\Big)\psi_{i+N-1} + \sum_{j=1}^{N} \frac{1}{z^j} v_{N-1-j}(i)\psi_{i+N-1-j}, \\ A_{N,0}(\psi_i) &= z\psi_{i+1} + v_{N-1}(i)\psi_i. \end{aligned} \tag{6.4}$$

That means

$$\begin{aligned} L &= G_{i+N}^{-1} H_1 \partial_{N,0} + \Big(\sum_{s=1}^{N} v_{N-1}(i+s-1)\Big) G_{i+N-1}^{-1} \\ &\quad + \sum_{j=1}^{N} \frac{1}{z} v_{N-1-j}(i) G_{i+N-1-j}^{-1} H_{i-j}^{-1} \cdots H_{i-2}^{-1} H_{i-1}^{-1}, \\ A_{N,0} &= H_1 + v_{N-1}(i). \end{aligned} \tag{6.5}$$

Then consistency conditions of (6.3) are expressed in the form of the Lax equation

$$\partial_{N,0} L = [A_{N,0}, L] = A_{N,0} L - L A_{N,0}, \tag{6.6}$$

which exactly leads to (5.7).

Till now we have given another Lax construction for the primary flows of the $(N, 1)$-BTH.

7 Lattice Miura Transformation

As we all know, many one-field lattice equations are very useful in a lot of branches of science such as biology, medical science, physics and so on. In this section, we will introduce a kind of Miura mapping which connects the one-field lattice equations with the $(N+1)$-field ones (i.e., the $(N,1)$-BTH).

Defining

$$F_i = G_{i+N}H_{i+N-1}\cdots H_{i+1}H_i, \quad i\in\mathbb{Z},$$

which is an N-order differential operator, we obtain

$$F_i\psi_i = z^{N+1}\psi_{i+1}, \quad H_i\psi_i = z\psi_{i+1}, \quad i\in\mathbb{Z}.$$

Because

$$G_i\psi_i = z\psi_{i+1-N}, \quad H_i\psi_i = z\psi_{i+1}, \quad i\in\mathbb{Z}, \tag{7.1}$$

we define

$$\overline{G}_i\overline{\psi}_i = z\overline{\psi}_{i+1}, \quad \overline{H}_i\overline{\psi}_i = z\overline{\psi}_{i+N+1}, \quad \psi_i = \overline{\psi}_{(N+1)i}, \tag{7.2}$$

where

$$\overline{H}_i = \partial_{N,0} - \sum_{k=1}^{N+1} r_{i+k-1}, \quad \overline{G}_i = \partial_{N,0} - r_i.$$

The compatibility of the system (7.2) leads to the following one-field lattice equations

$$\sum_{s=1}^{N}\partial_{N,0}r_{i+s-1} = \sum_{s=1}^{N} r_{i+s-1}\times(r_{i+N}-r_{i-1}), \quad i\in\mathbb{Z}. \tag{7.3}$$

Define

$$\overline{F}_i = \overline{G}_{i+N}\overline{G}_{i+N-1}\cdots\overline{G}_{i+1}\overline{G}_i,$$

which is an N-order differential operator. Now we consider a new system

$$\overline{F}_i\overline{\psi}_i = z^{N+1}\overline{\psi}_{i+N+1}, \quad \overline{H}_i\overline{\psi}_i = z\overline{\psi}_{i+N+1}.$$

Comparing the system (7.2) with (7.1) will lead to the following identification

$$F_i = \overline{F}_{(N+1)i}, \quad H_i = \overline{H}_{(N+1)i}, \tag{7.4}$$

which will tell us the Miura transformation in detail.

(7.4) can be rewritten in the following equivalent form

$$G_{i+N}H_{i+N-1}\cdots H_{i+1}H_i = \overline{G}_{(N+1)i+N}\overline{G}_{(N+1)i+N-1}\cdots\overline{G}_{(N+1)i+1}\overline{G}_{(N+1)i}.$$

In the following, we will give two specific examples of the original Toda hierarchy and the $(2,1)$-BTH including their corresponding lattice Miura transformations and one-field equations.

Example 7.1 For the $(1,1)$-BTH, i.e., the original Toda hierarchy, the case $N=1$, $l=2$, $n=1$, $m=0$, $\overline{n}=2$, $\overline{m}=1$ in [15] is as following

$$\begin{cases}\partial_{1,0}v_0(x)=v_{-1}(x+\epsilon)-v_{-1}(x),\\ \partial_{1,0}v_{-1}(x)=v_{-1}(x)(1-\Lambda^{-1})v_1(x),\end{cases} \tag{7.5}$$

which can be rewritten in the following discrete form

$$\begin{cases}\partial_{1,0}v_0(i)=v_{-1}(i+1)-v_{-1}(i),\\ \partial_{1,0}v_{-1}(i)=v_{-1}(i)(v_1(i)-v_1(i-1)),\end{cases} \tag{7.6}$$

by the transformation $v_j(i):=v_j(x+i\epsilon)$, $j=0,-1$. The lattice Miura transformation is as following

$$\begin{cases}v_1(i)=r_{2i}+r_{2i+1},\\ v_0(i)=r_{2i-1}r_{2i}.\end{cases} \tag{7.7}$$

After the lattice Miura transformation (7.7), the $t_{1,0}$ flow of the $(1,1)$-BTH can be transformed into the following one-field equation, i.e., the Volterra lattice which is a very useful ecological competition model in biology,

$$\partial_{1,0}r_i=r_i(r_{i+1}-r_{i-1}),\quad i\in\mathbb{Z}. \tag{7.8}$$

Example 7.2 For the $(2,1)$-BTH, $N=2$, $l=3$, $n=1$, $m=-1$, $\overline{n}=3$, $\overline{m}=1$, the lattice Miura transformation is as following

$$\begin{cases}v_1(i)=r_{3i+2}+r_{3i+1}+r_{3i},\\ v_0(i)=r_{3i-1}r_{3i}+r_{3i-1}r_{3i+1}+r_{3i}r_{3i+2}+r_{3i+1}r_{3i+3}+r_{3i+2}r_{3i+3}\\ \qquad\qquad +r_{3i+2}^2+r_{3i+1}^2+r_{3i}^2+2r_{3i+2}r_{3i+1}+2r_{3i}r_{3i+1},\\ v_{-1}(i)=-r_{3i-3}r_{3i-2}r_{3i-1}+r_{3i-2}r_{3i-1}r_{3i-1}\\ \qquad\qquad +(r_{3i-2}+r_{3i-1})(r_{3i-1}+r_{3i})(r_{3i+1}+r_{3i}).\end{cases} \tag{7.9}$$

After the lattice Miura transformation (7.9), the $t_{2,0}$ flow of the $(2,1)$-BTH can be transformed into the following one-field equation

$$\partial_{2,0}(r_i+r_{i+1})=(r_i+r_{i+1})(r_{i+2}-r_{i-1}),\quad i\in\mathbb{Z}. \tag{7.10}$$

The relation between the $(N,1)$-BTH and the one-field lattice equations gives us some hints on how to get the solutions of the one-field lattice equations from the solutions of the BTH (see [8]) by certain transformation.

8 Conclusions and Discussions

We give the finite dimensional exponential solutions of the bigraded Toda Hierarchy. As a specific example of the exponential solutions of the BTH, we consider a regular solution for the $(1,2)$-BTH with a 3×3 Lax matrix. The difference between the $(1,2)$-BTH and the original

Toda hierarchy is found from a geometric viewpoint by diagonal projection and the moment map. Our future work contains finding other regular solutions corresponding to other cases of the BTH and their geometric descriptions. After that, we construct another Lax representation of the bigraded Toda hierarchy (BTH) and introduce the lattice Miura transformation of the BTH. These Miura transformations give a good connection between the primary equation of the $(N, 1)$-BTH and the one-field lattice equations which include the Volterra lattice equation. What kinds of one-field lattice equations will correspond to the whole hierarchies in the BTH? It is an interesting question.

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