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Weighted L^p Estimates for Maximal Commutators of Multilinear Singular Integrals*

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Abstract This paper is concerned with the pointwise estimates for the sharp function of two kinds of maximal commutators of multilinear singular integral operators $T^*_{\Sigma b}$ and $T^*_{\Pi b}$, which are generalized by a weighted BMO function b and a multilinear singular integral operator T, respectively. As applications, some commutator theorems are established.

Keywords Weighted BMO space, Maximal commutator, Multilinear singular integral, Sharp maximal function
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1 Introduction

The theory of multilinear Calderór-Zygmund singular integral operators, originated from the works of Coifman and Meyer's, plays an important role in harmonic analysis. Its study has been attracting a lot of attention in the last few decades. So far, some properties of the multilinear operators are parallel to those of the classical linear Caldrón-Zygmund operators but new interesting phenomena have also been observed. A systematic analysis of many basic properties of such multilinear operators can be found in the articles by Coifman and Meyer [1], Grafakos and Torres [2–4], and Lerner et al. [5]. So we first recall the definition and results of multilinear Calderón-Zygmund operators as well as the corresponding maximal multilinear operators.

Definition 1.1 (Multilinear Calderón-Zygmund Operators) *Let T be a multilinear operator initially defined on the m-fold product of Schwartz space and taking values into the space of tempered distributions*

$$T : \mathcal{S}(\mathbb{R}^n) \times \cdots \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n).$$

Following [2], we say that T is an m-linear Calderón-Zygmund operator if for some $1 \le q_j < \infty$, it extends to a bounded multilinear operator from $L^{q_1} \times \cdots \times L^{q_m}$ to L^q, where $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, and if there exists a function K, defined off the diagonal $x = y_1 = \cdots = y_m$

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in $(\mathbb{R}^n)^{m+1}$, satisfying

$$T(f_1,\cdots,f_m)=\int_{(\mathbb{R}^n)^m} K(x,y_1,\cdots,y_m)f_1(y_1)\cdots f_m(y_m)\mathrm{d}y_1\cdots\mathrm{d}y_m$$

for all $x\notin\bigcap_{j=1}^{m}\mathrm{supp}f_j$,

$$|K(y_0,y_1,\cdots,y_m)|\le\frac{A}{\Big(\sum_{k,l=0}^{m}|y_k-y_l|\Big)^{mn}} \tag{1.1}$$

and

$$|K(y_0,\cdots,y_j,\cdots,y_m)-K(y_0,\cdots,y_j',\cdots,y_m)|\le\frac{A|y_j-y_j'|^{\varepsilon}}{\Big(\sum_{k,l=0}^{m}|y_k-y_l|\Big)^{mn+\varepsilon}} \tag{1.2}$$

for some $\varepsilon>0$ and all $0\le j\le m$, where $|y_j-y_j'|\le\frac{1}{2}\max\limits_{0\le k\le m}|y_j-y_k|$.

The maximal multilinear singular integral operator is defined by

$$T^*(\vec{f})=\sup_{\delta>0}|T_\delta(f_1,\cdots,f_m)(x)|, \tag{1.3}$$

where T_δ is the smooth truncation of T given by

$$T_\delta(f_1,\cdots,f_m)(x)=\int_{|x-y_1|^2+\cdots+|x-y_m|^2>\delta^2} K(x,y_1,\cdots,y_m)f_1(y_1)\cdots f_m(y_m)\mathrm{d}y_1\cdots\mathrm{d}y_m.$$

As pointed out in [4], $T^*(\vec{f})$ is pointwise well-defined when $f_j\in L^{q_j}(\mathbb{R}^n)$ with $1\le q_j<\infty$.

The study for the multilinear singular integral operator and its maximal operators attracts many authors' attention. For the maximal multilinear operator T^*, one can see for example [4] for details. We list some results for T^* as follows.

Theorem A (see [4]) *Let $1\le q_j<\infty$, and q be such that $\frac{1}{q}=\frac{1}{q_1}+\cdots+\frac{1}{q_m}$ and $\omega\in A_{q_1}\cap\cdots\cap A_{q_m}$. Let T be an m-linear Calderón-Zygmund operator. Then there exists a constant $C_{n,q}<\infty$, such that all $\vec{f}=(f_1,\cdots,f_m)$ satisfy*

$$\|T^*(\vec{f})\|_{L^q(\omega)}\le C_{n,q}(A+W)\prod_{j=1}^{m}\|f_j\|_{L^{q_j}(\omega)},$$

where W is the norm of T in the mapping T: $L^1\times\cdots L^1\to L^{\frac{1}{m},\infty}$.

Theorem B (see [4]) *Let T be an m-linear Calderón-Zygmund operator. Then, for all exponents $p,p_1,\cdots,p_m$, satisfying $\frac{1}{p_1}+\cdots+\frac{1}{p_m}=\frac{1}{p}$, we have*

$$T^*:L^{p_1}\times\cdots\times L^{p_m}\to L^p,$$

when $1<p_1,\cdots,p_m\le\infty$, and we also have

$$T^*:L^{p_1}\times\cdots\times L^{p_m}\to L^{p,\infty},$$

when at least one p_j equals one. In either case the norm of T^* is controlled by a constant multiple of $A+W$.

Definition 1.2 (see [6]) (Commutators in the j-th Entry) *Given a collection of locally integrable function $\vec{b} = (b_1, \cdots, b_m)$, we define the commutators of the m-linear Calderón-Zygmund operator T to be*

$$[\vec{b}, T](\vec{f}) = T_{\Sigma b}(f_1, \cdots, f_m) = \sum_{j=1}^{m} T^j_{b_j}(\vec{f}),$$

where each term is the commutator of b_j and T in the j-th entry of T, that is

$$T^j_{b_j}(\vec{f}) = b_j T(f_1, \cdots, f_j, \cdots, f_m) - T(f_1, \cdots, b_j f_j, \cdots, f_m).$$

In [7], the following more general iterated commutators of multilinear Calderón-Zygmund operators and pointwise multiplication with functions in BMO were defined and studied in products of Lebesgue spaces, including strong type and weak end-point estimates with multiple $A_{\vec{p}}$ weights, that is

$$\begin{aligned} T_{\Pi b}(\vec{f})(x) &= [b_1, [b_2, \cdots, [b_{m-1}, [b_m, T]_m]_{m-1}, \cdots]_2]_1 \\ &= \int_{(\mathbb{R}^n)^m} K(x, y_1, \cdots, y_m) \prod_{j=1}^{m} (b_j(x) - b_j(y_j)) f_1(y_1) \cdots f_m(y_m) \mathrm{d}y_1 \cdots \mathrm{d}y_m. \end{aligned}$$

Clearly, when $m = 1$,

$$T_{\Pi b}(f)(x) = T_{\Sigma b}(f)(x) = T_b f(x) = bTf(x) - T(bf)(x),$$

which is the commutator of the Coifman-Rochberg-Weiss type. When T is the Calderón-Zygmund singular integral operator and $b \in \mathrm{BMO}(\mathbb{R}^n)$, Coifman, Rochberg and Weiss showed that T_b is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. In 1985, Bloom [8] proved that if $b \in \mathrm{BMO}(\omega)$ and T is the singular integral operator, then T_b is bounded from $L^p(\mu)$ to $L^p(\nu)$ with the assumption that $\mu, \nu \in A_p$ and $\omega(x) = (\mu(x)\nu(x)^{-1})^{\frac{1}{p}}$. The main purpose of this paper is to extend these two weighted results to the following two maximal commutators of multilinear singular integral operators.

Now we present the definitions of two classes of maximal commutators of multilinear singular integral operators. One is

$$\begin{aligned} T^*_{\Sigma b}(\vec{f})(x) = \sup_{\delta>0} \Big| \sum_{j=1}^{m} \int_{\sum_{i=1}^{m} |x-y_i|^2 > \delta^2} & K(x, y_1, \cdots, y_m) f_1(y_1) \\ & \cdots (b_j(x) - b_j(y_j)) f_j(y_j) \cdots f_m(y_m) \mathrm{d}\vec{y} \Big|, \end{aligned} \tag{1.4}$$

and the other is

$$\begin{aligned} T^*_{\Pi b}(\vec{f})(x) &= \sup_{\delta>0} |[b_1, [b_2, \cdots, [b_{m-1}, [b_m, T_\delta]_m]_{m-1}, \cdots]_2]_1(\vec{f})(x)| \\ &= \sup_{\delta>0} \Big| \int_{\sum_{i=1}^{m} |x-y_i|^2 > \delta^2} K(x, y_1, \cdots, y_m) \\ &\quad \cdot \prod_{j=1}^{m} (b_j(x) - b_j(y_j)) f_1(y_1) \cdots f_m(y_m) \mathrm{d}\vec{y} \Big|, \end{aligned} \tag{1.5}$$

where $d\vec{y} = dy_1 \cdots dy_m$. It is obvious to see that

$$T^*_{\Sigma b}(\vec{f})(x) \le \sum_{j=1}^{m} T^{*,j}_{b_j}(\vec{f})(x).$$

We can formulate our results as follows.

Theorem 1.1 *Let T be an m-linear Calderón-Zygmund operator with the kernel K satisfying* (1.1) *and* (1.2). *Suppose that $1 < q_1, \cdots, q_m, q < \infty$ are given numbers satisfying $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, and T maps $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$. Further assume that $\mu, \nu \in A_p$, $\omega = (\mu\nu^{-1})^{\frac{1}{p}}$ and that (μ, ν) satisfies the following condition: There exists a constant $C_0 > 0$ such that for any cube $B \subset \mathbb{R}^n$,*

$$\Big(\frac{1}{|B|}\int_B \mu(x)dx\Big)\Big(\frac{1}{|B|}\int_B \nu(x)^{-\frac{1}{p-1}}dx\Big)^{p-1} \ge C_0 > 0. \tag{1.6}$$

If $b_j \in \mathrm{BMO}(\omega)$, for $j = 1, \cdots, m$, then we have

$$\|T^{*,j}_{b_j}(\vec{f})\|_{L^p(\nu)} \le C\|b_j\|_{\mathrm{BMO}(\omega)} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\mu)} \quad for\ j = 1, \cdots, m.$$

Furthermore,

$$\|T^*_{\Sigma b}(\vec{f})\|_{L^p(\nu)} \le C\sum_{j=1}^{m} \|b_j\|_{\mathrm{BMO}(\omega)} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\mu)},$$

where $1 < p_j < \infty$, $1 < p < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$.

Obviously, choosing $\omega(x) = \mu(x) = \nu(x) = 1$, we can get the following strong type estimate for the maximal iterated commutator for a multilinear singular integral operator.

Theorem 1.2 *Let T be an m-linear Calderón-Zygmund operator with the kernel K satisfying* (1.1) *and* (1.2). *Suppose that $1 < q_1, \cdots, q_m, q < \infty$ are given numbers satisfying $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, and T maps $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$. If $b_j \in \mathrm{BMO}(\omega)$, for $j = 1, \cdots, m$, then we have*

$$\|T^*_{\Sigma b}(\vec{f})\|_{L^p} \le C\sum_{j=1}^{m} \|b_j\|_{\mathrm{BMO}} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}},$$

where $1 < p_j < \infty$, $1 < p < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$.

Theorem 1.3 *Let T be an m-linear Calderón-Zygmund operator with the kernel K satisfying* (1.1) *and* (1.2). *Suppose that $1 < q_1, \cdots, q_m, q < \infty$ are given numbers satisfying $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, and T maps $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$. Further assume that $\mu, \nu \in A_p$, $\omega = (\mu\nu^{-1})^{\frac{1}{p}}$ and that (μ, ν) satisfies* (1.6). *If $b_j \in \mathrm{BMO}(\omega)$, for $j = 1, \cdots, m$, then we have*

$$\|T^*_{\Pi b}(\vec{f})\|_{L^p(\nu)} \le C\prod_{j=1}^{m} \|b_j\|_{\mathrm{BMO}(\omega)} \|f_j\|_{L^{p_j}(\mu)},$$

where $1 < p_j < \infty$, $1 < p < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$.

From Theorem 1.3, we can easily get Theorem 1.4.

Theorem 1.4 *Let T be an m-linear Calderón-Zygmund operator with the kernel K satisfying* (1.1) *and* (1.2). *Suppose that $1 < q_1, \cdots, q_m, q < \infty$ are given numbers satisfying $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, and T maps $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$. If $b_j \in \mathrm{BMO}(\omega)$, for $j = 1, \cdots, m$, then we have*

$$\|T^*_{\Pi b}(\vec{f})\|_{L^p} \le C \prod_{j=1}^{m} \|b_j\|_{\mathrm{BMO}} \|f_j\|_{L^{p_j}},$$

where $1 < p_j < \infty$, $1 < p < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$.

This article is arranged as follows. In Section 2, we present some definitions and lemmas. Some propositions will be listed and proved in Section 2. The proofs of Theorems 1.1–1.3 can be found in Section 3.

2 Preliminaries and Some Lemmas

A non-negative function μ defined on $\mathbb{R}^n$ is called a weight if it is locally integral. A weight μ is said to belong to the Muckenhoupt class $A_p(\mathbb{R}^n)$, $1 < p < \infty$, if there exists a constant C such that

$$\sup_B \Big(\frac{1}{|B|}\int_B \mu(x)\mathrm{d}x\Big)\Big(\frac{1}{|B|}\int_B \mu(x)^{-\frac{1}{p-1}}\mathrm{d}x\Big)^{p-1} \le C < \infty$$

for every ball $B \subset \mathbb{R}^n$. A weight μ is said to belong to class $A_1(\mathbb{R}^n)$ defined by

$$\Big(\frac{1}{|B|}\int_B \mu(x)\mathrm{d}x\Big) \le C \inf_{x\in B} \mu(x), \quad \text{a.e. } x \in \mathbb{R}^n$$

for every ball $B \ni x$. The class $A_\infty(\mathbb{R}^n)$ can be characterized as $A_\infty = \bigcup_{1\le p<\infty} A_p$.

Many properties of weights can be found in the book [9], and we only collect some of them in the following properties of weights which are the A_∞ condition and the Reverse Hölder condition:

(a) A_∞ condition: ω is in the class A_∞ if there exist constants C and $\sigma > 0$, such that, for every cube B and measurable set $E \subset B$ we have

$$\frac{\omega(E)}{\omega(B)} \le C\Big(\frac{|E|}{|B|}\Big)^{\sigma}.$$

(b) Reverse Hölder condition: $\omega \in A_p$ and there exist constants C and $\varepsilon > 0$ such that,

$$\Big(\frac{1}{|B|}\int_B \omega(x)^{1+\varepsilon}\mathrm{d}x\Big)^{\frac{1}{1+\varepsilon}} \le \frac{C}{|B|}\int_B \omega(x)\mathrm{d}x$$

for all cubes B.

The important properties of the weights are the weighted estimates for the maximal function, the sharp maximal function and their variants. We first recall the maximal function defined by

$$M(f)(x) = \sup_{B\ni x} \frac{1}{|B|}\int_B |f(y)|\mathrm{d}y.$$

It is well known that for $1 < p < \infty$, M maps $L^p(\mu)$ into itself if and only if $\mu \in A_p$ (see [10]).

The sharp maximal function is defined by

$$M^\sharp(f)(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(x) - f_B| \mathrm{d}y \approx \sup_{B \ni x} \inf_c \frac{1}{|B|} \int_B |f(x) - c| \mathrm{d}y.$$

We also recall the variants $M_\delta(f)(x) = (M(|f|^\delta)(x))^{\frac{1}{\delta}}$ and $M^\sharp_\delta(f)(x) = (M^\sharp(|f|^\delta)(x))^{\frac{1}{\delta}}$. The weighted maximal operator is defined by

$$M_\mu(f)(x) = \sup_{B \ni x} \Big(\frac{1}{\mu(B)} \int_B |f(y)| \mu(y) \mathrm{d}y \Big).$$

And we denote

$$\|f\|_{L^p(\mu)} = \Big(\frac{1}{\mu(B)} \int_B |f|^p \mu(y) \mathrm{d}y \Big)^{\frac{1}{p}}.$$

Lemma 2.1 (see [11]) (Kolmogorov's Inequality) *Let (X, μ) be a probability measure space and let $0 < p < q < \infty$, and then there exsits a constant $C = C_{p,q}$ such that*

$$\|f\|_{L^p(\mu)} \le C \|f\|_{L^{q,\infty}(\mu)}$$

for any measurable function f.

Lemma 2.2 (see [11]) *Let $0 < p, \delta < \infty$, and $\mu \in A_\infty(\mathbb{R}^n)$, and there exists $C > 0$ depending on the $A_\infty(\mathbb{R}^n)$ constant of μ such that*

$$\|M_\delta(f)\|_{L^p(\mu)} \le C \|M^\sharp_\delta(f)\|_{L^p(\mu)}$$

for any function f whose left side is finite.

The following definitions can be found in [8]. Let ω be a weight and b be an L^1 locally integrable function. Then b is in the weighted BMO class $\mathrm{BMO}(\omega)$ provided that

$$\|b\|_{\mathrm{BMO}(\omega)} = \sup_B \frac{1}{\omega(B)} \int_B |b(x) - b_B| \mathrm{d}x \le C < \infty.$$

We can easily know that

$$b \in \mathrm{BMO}(\omega) \Leftrightarrow \int_B |b - b_B| \mathrm{d}x \le C\omega(B)$$

for all cubes B.

Let $r \ge 1$ and ω be a weight. Define

$$S_r(b, \omega, B) = \Big(\frac{1}{|B|} \int_B |b - b_B|^r \omega(x)^r \mathrm{d}x \Big)^{\frac{1}{r}},$$

$$\Lambda_r(f, \omega, B) = \Big(\frac{1}{|B|} \int_B |f(x)\omega(x)|^r \mathrm{d}x \Big)^{\frac{1}{r}},$$

$$K^*_r(b, f, \omega)(x) = \sup_{x \in B} S_{rq'}(b, \omega, B) \Lambda_{rq}(f, \omega^{-1}, B).$$

We denote $K^* = K^*_1$.

Lemma 2.3 (see [8]) *For an appropriate choice of $q < p$ and for any r with $1 \le r < \frac{p}{q}$, there exists a weight ω depending on r such that*

(1) $\omega^{rq'} \in A_{q'}$;

(2) $\int_{\mathbb{R}^n} (K_r^*(b, f, \omega)(x))^p \nu(x) \mathrm{d}x \le C\|b\|^p_{\mathrm{BMO}(\omega)} \int_{\mathbb{R}^n} |f(x)|^p \mu(x) \mathrm{d}x$.

Lemma 2.4 *For any $1 < q < \infty$, let $\mu, \nu \in A_p$, $\omega = (\mu\nu^{-1})^{\frac{1}{p}}$, and (μ, ν) satisfy (1.6). Then there exists a constant $C > 0$ such that*

$$\frac{1}{|B|}\int_B |f(y)|\mathrm{d}y \le C(M_\nu(|f\omega|^q))^{\frac{1}{q}}(x).$$

Proof Thanks to Hölder's inequality, we obtain

$$\begin{aligned}\frac{1}{|B|}\int_B |f(y)|\mathrm{d}y &\le \Big(\frac{1}{|B|}\int_B |f(y)\omega(y)|^q \nu(y)\mathrm{d}y\Big)^{\frac{1}{q}}\Big(\frac{1}{|B|}\int_B \omega(y)^{-q'}\nu(y)^{-\frac{q'}{q}}\mathrm{d}y\Big)^{\frac{1}{q'}}\\ &\le \Big(\frac{\nu(B)}{|B|}\Big)^{\frac{1}{q}}\Big(\frac{1}{\nu(B)}\int_B |f\omega|^q\nu(y)\mathrm{d}y\Big)^{\frac{1}{q}}\Big(\frac{1}{|B|}\int_B \omega(y)^{-q'}\nu(y)^{-\frac{q'}{q}}\mathrm{d}y\Big)^{\frac{1}{q'}}\\ &\le (M_\nu(|f\omega|^q))^{\frac{1}{q}}(x)\Big(\frac{\nu(B)}{|B|}\Big)^{\frac{1}{q}}\Big(\frac{1}{|B|}\int_B \omega(y)^{-q'}\nu(y)^{-\frac{q'}{q}}\mathrm{d}y\Big)^{\frac{1}{q'}}.\end{aligned}$$

Since $\omega = (\mu\nu^{-1})^{\frac{1}{p}}$, then

$$\omega^{-q'}\nu^{\frac{-q'}{q}} = \mu^{\frac{-q'}{p}}\nu^{-q'(\frac{1}{q}-\frac{1}{p})}.$$

Choose s such that $sq'(\frac{1}{q} - \frac{1}{p}) = \frac{p'}{p}$, and s is so large, for q near p. And apply the reverse Hölder's inequality to $\mu^{-\frac{p'}{p}}$ with an exponent $\frac{q's'}{p'}$. So we can get

$$\begin{aligned}&\Big(\frac{\nu(B)}{|B|}\Big)^{\frac{1}{q}}\Big(\frac{1}{|B|}\int_B \omega(y)^{-q'}\nu(y)^{-\frac{q'}{q}}\mathrm{d}y\Big)^{\frac{1}{q'}}\\ &= \Big(\frac{\nu(B)}{|B|}\Big)^{\frac{1}{q}}\Big(\frac{1}{|B|}\int_B \mu^{\frac{-q'}{p}}\nu^{-q'(\frac{1}{q}-\frac{1}{p})}\mathrm{d}y\Big)^{\frac{1}{q'}}\\ &\le (C_0)^{\frac{1}{p}}(C_0)^{-\frac{1}{p}}\Big(\frac{\nu(B)}{|B|}\Big)^{\frac{1}{q}}\Big(\frac{1}{|B|}\int_B \mu^{-\frac{q's'}{p}}\mathrm{d}y\Big)^{\frac{1}{s'q'}}\Big(\frac{1}{|B|}\int_B \nu^{-\frac{p'}{p}}\mathrm{d}y\Big)^{\frac{1}{sq'}}\\ &\le C\Big(\frac{1}{|B|}\int_B \mu(y)\mathrm{d}y\Big)^{\frac{1}{p}}\Big(\frac{1}{|B|}\int_B \nu(y)^{-\frac{1}{p-1}}\mathrm{d}y\Big)^{\frac{p-1}{p}}\\ &\quad\cdot\Big(\frac{1}{|B|}\int_B \nu(y)\mathrm{d}y\Big)^{\frac{1}{q}}\Big(\frac{1}{|B|}\int_B \mu^{-\frac{q's'}{p}}\mathrm{d}y\Big)^{\frac{1}{s'q'}}\Big(\frac{1}{|B|}\int_B \nu^{-\frac{1}{p-1}}\mathrm{d}y\Big)^{\frac{1}{sq'}}\\ &\le C\Big(\frac{1}{|B|}\int_B \mu(y)\mathrm{d}y\Big)^{\frac{1}{p}}\Big(\frac{1}{|B|}\int_B \nu(y)^{-\frac{1}{p-1}}\mathrm{d}y\Big)^{\frac{1}{p'}+\frac{1}{sq'}}\\ &\quad\cdot\Big(\frac{1}{|B|}\int_B \nu(y)\mathrm{d}y\Big)^{\frac{1}{q}}\Big(\frac{1}{|B|}\int_B (\mu^{-\frac{p'}{p}})^{\frac{q's'}{p'}}\mathrm{d}y\Big)^{\frac{p'}{q's'}\cdot\frac{1}{p'}}\\ &\le C\Big(\frac{1}{|B|}\int_B \mu(y)\mathrm{d}y\Big)^{\frac{1}{p}}\Big(\frac{1}{|B|}\int_B \mu^{-\frac{p'}{p}}\mathrm{d}y\Big)^{\frac{1}{p'}}\Big[\Big(\frac{1}{|B|}\int_B \nu(y)\mathrm{d}y\Big)\Big(\frac{1}{|B|}\int_B \nu(y)^{-\frac{1}{p-1}}\mathrm{d}y\Big)^{p-1}\Big]^{\frac{1}{q}}\\ &\le C,\end{aligned}$$

where we have used the assumption (1.6) in the fourth inequality. Thus

$$\frac{1}{|B|}\int_B |f(y)|\mathrm{d}y \le C(M_\nu(|f\omega|^q))^{\frac{1}{q}}(x).$$

The following lemma was established by Bloom [8].

Lemma 2.5 (see [8]) *Let $\mu,\nu\in A_p$ and put $\omega=(\mu\nu^{-1})^{\frac{1}{p}}, 1<p<\infty$. Then for any cube B, there exists a constant $C>0$, such that*

$$\frac{1}{|B|}\int_B\omega(y)\mathrm{d}y\Big(\frac{1}{|B|}\int_B\nu(y)\mathrm{d}y\Big)^{\frac{1}{q}}\Big(\frac{1}{|B|}\int_B\omega^{-q'}\nu^{-\frac{q'}{q}}\mathrm{d}y\Big)^{\frac{1}{q'}}\le C<\infty.$$

Lemma 2.6 (see [8]) *Let $\omega\in A_\infty$, $b\in\mathrm{BMO}(\omega)$. There exists a constant $\sigma>0$, such that*

$$|b_B-b_{2^{k+1}B}|\le C2^{kn(1-\sigma)}\|b\|_{\mathrm{BMO}(\omega)}\frac{\omega(2^{k+1}B)}{|2^{k+1}B|}.$$

3 Weighted Estimates for Maximal Commutators

We will prove our theorems in this section. To begin with, we prepare another two maximal commutators to control the commutators.

Let $\varphi,\psi\in C^\infty([0,+\infty))$ such that $|\varphi'(t)|\le\frac{C}{t}$, $|\psi'(t)|\le\frac{C}{t}$ and they satisfy

$$\chi_{[2,\infty)}(t)\le\varphi(t)\le\chi_{[1,\infty)}(t),\ \ \chi_{[1,2]}(t)\le\varphi(t)\le\chi_{[\frac{1}{2},3]}(t).$$

We define the maximal operators

$$\Phi^*(\vec{f})(x)=\sup_{\eta>0}\Big|\int_{(\mathbb{R}^n)^m}K(x,y_1,\cdots,y_m)\varphi\Big(\frac{\sqrt{|x-y_1|+\cdots+|x-y_m|}}{\eta}\Big)\prod_{i=1}^m f_i(y_i)\mathrm{d}\vec{y}\Big|,$$
$$\Psi^*(\vec{f})(x)=\sup_{\eta>0}\Big|\int_{(\mathbb{R}^n)^m}K(x,y_1,\cdots,y_m)\psi\Big(\frac{\sqrt{|x-y_1|+\cdots+|x-y_m|}}{\eta}\Big)\prod_{i=1}^m f_i(y_i)\mathrm{d}\vec{y}\Big|.$$

For simplicity, we denote

$$K_{\varphi,\eta}(x,y_1,\cdots,y_m)=K(x,y_1,\cdots,y_m)\varphi\Big(\frac{\sqrt{|x-y_1|+\cdots+|x-y_m|}}{\eta}\Big),$$
$$K_{\psi,\eta}(x,y_1,\cdots,y_m)=K(x,y_1,\cdots,y_m)\psi\Big(\frac{\sqrt{|x-y_1|+\cdots+|x-y_m|}}{\eta}\Big),$$
$$\Phi_\eta(\vec{f})(x)=\int_{(\mathbb{R}^n)^m}K_{\varphi,\eta}(x,y_1,\cdots,y_m)\prod_{i=1}^m f_i(y_i)\mathrm{d}\vec{y},$$
$$\Psi_\eta(\vec{f})(x)=\int_{(\mathbb{R}^n)^m}K_{\psi,\eta}(x,y_1,\cdots,y_m)\prod_{i=1}^m f_i(y_i)\mathrm{d}\vec{y}.$$

The kernels of Φ_η and Ψ_η satisfy conditions (1.1) and (1.2) uniformly in η, respectively. And by the same argument as that in [4], the maximal operators Φ^* and Ψ^* have the same weighted estimates to T^* that appeared in Theorems A and B.

It is easy to see that $T^*(\vec{f})\le\Phi^*(\vec{f})+\Psi^*(\vec{f})$, and

$$T^*_{\Sigma b}(\vec{f})\le\Phi^*_{\Sigma b}(\vec{f})+\Psi^*_{\Sigma b}(\vec{f}),\quad T^*_{\Pi b}(\vec{f})\le\Phi^*_{\Pi b}(\vec{f})+\Psi^*_{\Pi b}(\vec{f}),\tag{3.1}$$

where

$$\Phi^*_{\Sigma b}(\vec{f})(x)=\sup_{\eta>0}\Big|\sum_{j=1}^m\int_{(\mathbb{R}^n)^m}K_{\varphi,\eta}(x,y_1,\cdots,y_m)f_1(y_1)\cdots(b_j(x)-b_j(y_j))f_j(y_j)\cdots f_m(y_m)\mathrm{d}\vec{y}\Big|,$$

$$\Psi^*_{\Sigma b}(\vec{f})(x)=\sup_{\eta>0}\Big|\sum_{j=1}^m\int_{(\mathbb{R}^n)^m}K_{\psi,\eta}(x,y_1,\cdots,y_m)f_1(y_1)\cdots(b_j(x)-b_j(y_j))f_j(y_j)\cdots f_m(y_m)\mathrm{d}\vec{y}\Big|,$$

$$\Phi^*_{\Pi b}(\vec{f})(x)=\sup_{\eta>0}\Big|\int_{(\mathbb{R}^n)^m}K_{\varphi,\eta}(x,y_1,\cdots,y_m)\prod_{j=1}^m(b_j(x)-b_j(y_j))\prod_{i=1}^m f_i(y_i)\mathrm{d}\vec{y}\Big|,$$

$$\Psi^*_{\Pi b}(\vec{f})(x)=\sup_{\eta>0}\Big|\int_{(\mathbb{R}^n)^m}K_{\psi,\eta}(x,y_1,\cdots,y_m)\prod_{j=1}^m(b_j(x)-b_j(y_j))\prod_{i=1}^m f_i(y_i)\mathrm{d}\vec{y}\Big|.$$

For simplicity, we will only prove the case $m=2$. The arguments for the case $m>2$ are similar. For the similarity of the two commutators $\Phi^*_{\Sigma b}(\vec{f})$ and $\Psi^*_{\Sigma b}(\vec{f})$, we might as well consider the former. And we establish the following crucial lemma.

Lemma 3.1 *Let $b_j\in\mathrm{BMO}(\omega)$ $(j=1,2)$, $0<\delta<\frac{1}{2}$, and $\overline{\omega}^{q'}\in A_{q'}$. Supposing that $\mu,\nu\in A_p$, $\omega=(\mu\nu^{-1})^{\frac{1}{p}}$, $1<p<\infty$, and that (u,v) satisfies* (1.6)*, then*

$$\begin{aligned}M^\sharp_\delta[\Phi^{*,j}_{b_j}(f_1,f_2)](x)\le{}& CK^*(b_j,\Phi^*(f_1,f_2),\overline{\omega})(x)+C(M_\nu|f_2\omega|^q(x))^{\frac{1}{q}}K^*(b_1,f_1,\overline{\omega})(x)\\&+C\|b_j\|_{\mathrm{BMO}(\omega)}(M_\nu|f_2\omega|^q(x))^{\frac{1}{q}}(M_\nu|f_1\omega|^q(x))^{\frac{1}{q}}\\&+C(M_\nu|f_1\omega|^q(x))^{\frac{1}{q}}K^*(b_2,f_2,\overline{\omega})(x)\end{aligned}$$

and

$$\begin{aligned}M^\sharp_\delta[\Psi^{*,j}_{b_j}(f_1,f_2)](x)\le{}& CK^*(b_j,\Psi^*(f_1,f_2),\overline{\omega})(x)+C(M_\nu|f_2\omega|^q(x))^{\frac{1}{q}}K^*(b_1,f_1,\overline{\omega})(x)\\&+C\|b_j\|_{\mathrm{BMO}(\omega)}(M_\nu|f_2\omega|^q(x))^{\frac{1}{q}}(M_\nu|f_1\omega|^q(x))^{\frac{1}{q}}\\&+C(M_\nu|f_1\omega|^q(x))^{\frac{1}{q}}K^*(b_2,f_2,\overline{\omega})(x).\end{aligned}$$

Proof Without loss of generality, we only consider the case: $j=1$ and denote b_1 by b for convenience. Fix $x\in\mathbb{R}^n$ and let $B=B(x,R)$ with $R>0$, and $\lambda=b_B$ be the average of b on B. To proceed with, we decompose $f_i=f_i^0+f_i^\infty$, where $f_i^0=f_i\chi_{B^*}$, $i=1,2$, and $B^*=B(x,2R)$. Let c be a constant to be fixed along the proof.

Since $0<\delta<1$, we have

$$\begin{aligned}&\Big(\frac{1}{|B|}\int_B||\Phi^{*,1}_b(f_1,f_2)(y)|^\delta-|c|^\delta|\mathrm{d}y\Big)^{\frac{1}{\delta}}\le\Big(\frac{1}{|B|}\int_B|\Phi^{*,1}_b(f_1,f_2)(y)-c|^\delta\mathrm{d}y\Big)^{\frac{1}{\delta}}\\
&\le\Big(\frac{1}{|B|}\int_B|(b(y)-\lambda)\Phi^*(f_1,f_2)(y)|^\delta\mathrm{d}y\Big)^{\frac{1}{\delta}}+\Big(\frac{1}{|B|}\int_B|\Phi^*((b-\lambda)f_1^0,f_2^0)(y)|^\delta\mathrm{d}y\Big)^{\frac{1}{\delta}}\\
&\quad+\Big(\frac{1}{|B|}\int_B|\Phi^*((b-\lambda)f_1^0,f_2^\infty)(y)|^\delta\mathrm{d}y\Big)^{\frac{1}{\delta}}+\Big(\frac{1}{|B|}\int_B|\Phi^*((b-\lambda)f_1^\infty,f_2^0)(y)|^\delta\mathrm{d}y\Big)^{\frac{1}{\delta}}\\
&\quad+\Big(\frac{1}{|B|}\int_B|\Phi^*((b-\lambda)f_1^\infty,f_2^\infty)(y)-c|^\delta\mathrm{d}y\Big)^{\frac{1}{\delta}}\\
&:=\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}.\end{aligned}$$

For the first term I, since $0 < \delta < 1$, by the Hölder's inequality and noting that $\overline{\omega}^{q'} \in A_{q'}$, $\lambda = b_B$, we get

$$\begin{aligned}
\mathrm{I} &\le \frac{1}{|B|}\int_B |b(y)-b_B||\Phi^*(f_1,f_2)(y)|\mathrm{d}y\\
&\le \frac{1}{|B|}\Big(\int_{B^*}|b(y)-b_{B^*}||\Phi^*(f_1,f_2)(y)|\mathrm{d}y + |b_B-b_{B^*}|\frac{1}{|B|}\int_{B^*}|\Phi^*(f_1,f_2)(y)|\mathrm{d}y\Big)\\
&\le C\frac{1}{|B^*|}\int_{B^*}|b(y)-b_{B^*}|\overline{\omega}(y)|\Phi^*(f_1,f_2)(y)|\overline{\omega}(y)^{-1}\mathrm{d}y\\
&\quad + \Big(\frac{1}{|B|}\int_B |b(y)-b_{B^*}|\mathrm{d}y\Big)\Big(\frac{1}{|B|}\int_{B^*}|\Phi^*(f_1,f_2)(y)|\mathrm{d}y\Big)\\
&\le C\Big(\frac{1}{|B^*|}\int_{B^*}|b(y)-b_{B^*}|^{q'}\overline{\omega}(y)^{q'}\mathrm{d}y\Big)^{\frac{1}{q'}}\Big(\frac{1}{|B^*|}\int_{B^*}|\Phi^*(f_1,f_2)(y)|^q\overline{\omega}(y)^{-q}\mathrm{d}y\Big)^{\frac{1}{q}}\\
&\quad + C\Big(\frac{1}{|B^*|}\int_{B^*}|\Phi^*(f_1,f_2)(y)|^q\overline{\omega}(y)^{-q}\mathrm{d}y\Big)^{\frac{1}{q}}\Big(\frac{1}{|B^*|}\int_{B^*}\overline{\omega}(y)^{q'}\mathrm{d}y\Big)^{\frac{1}{q'}}\\
&\quad \cdot\Big(\frac{1}{|B^*|}\int_{B^*}|b(y)-b_{B^*}|^{q'}\overline{\omega}(y)^{q'}\mathrm{d}y\Big)^{\frac{1}{q'}}\Big(\frac{1}{|B^*|}\int_{B^*}\overline{\omega}(y)^{-q}\mathrm{d}y\Big)^{\frac{1}{q}}\\
&= CS_{q'}(b,\overline{\omega},B^*)\Lambda_q(\Phi^*(f_1,f_2)\overline{\omega}^{-1},B^*)\Big[1+\Big(\frac{1}{|B^*|}\int_{B^*}\overline{\omega}(y)^{q'}\mathrm{d}y\Big)^{\frac{1}{q'}}\Big(\frac{1}{|B^*|}\int_{B^*}\overline{\omega}(y)^{-q}\mathrm{d}y\Big)^{\frac{1}{q}}\Big]\\
&\le CK^*(b,\Phi^*(f_1,f_2),\overline{\omega})(x).
\end{aligned}$$

For the second term II, since $0 < \delta < \frac{1}{2}$, by Kolmogorov's inequality with $p = \delta$, $q = \frac{1}{2}$ and the $L^1(\mathbb{R}^n)\times\cdots\times L^1(\mathbb{R}^n)$ to $L^{\frac{1}{m},\infty}(\mathbb{R}^n)$-boundedness of Φ^*, it ensures that

$$\begin{aligned}
\mathrm{II} &\le \|\Phi^*((b-\lambda)f_1^0,f_2^0)\|_{L^{\frac{1}{2},\infty}(\frac{\mathrm{d}x}{|B|})}\\
&\le C\Big(\frac{1}{|B|}\int_{B^*}|b(y_1)-\lambda||f_1(y_1)|\mathrm{d}y_1\Big)\Big(\frac{1}{|B|}\int_{B^*}|f_2(y_2)|\mathrm{d}y_2\Big)\\
&= C\mathrm{II}_1\cdot\mathrm{II}_2.
\end{aligned}$$

For II_1, II_2, estimating these just as I and Lemma 2.4 gives

$$\mathrm{II}_1 \le K^*(b,f_1,\overline{\omega})(x)$$

and

$$\mathrm{II}_2 \le C(M_\nu(|f\omega|^q)(x))^{\frac{1}{q}}.$$

Therefore,

$$\mathrm{II} \le CK^*(b,f_1,\overline{\omega})(x)(M_\nu(|f_2\omega|^q)(x))^{\frac{1}{q}}.$$

For the third term III, using the fact $|y-y_2| \sim |y_2-x|$ for any $y_2 \in (B^*)^c$, $y \in B$, the size estimate on $K_{\varphi,\eta}$, and Lemma 2.4, we obtain

$$\begin{aligned}
\mathrm{III} &\le \frac{1}{|B|}\int_B \sup_{\eta>0}|\Phi_\eta((b-\lambda)f_1^0,f_2^\infty)(y)|\mathrm{d}y\\
&\le \frac{1}{|B|}\int_B\int_{B^*\times(\mathbb{R}^n\setminus B^*)}\frac{A|(b(y_1)-\lambda)f_1(y_1)||f_2(y_2)|}{(|y-y_1|+|y-y_2|)^{2n}}\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&\le C\int_{B^*}|(b(y_1)-\lambda)f_1(y_1)|\mathrm{d}y_1\int_{\mathbb{R}^n\setminus B^*}\frac{|f_2(y_2)|}{|y_2-x|^{2n}}\mathrm{d}y_2\\
&\le C\Big(\int_{B^*}|b(y_1)-b_{B^*}||f_1(y_1)|\mathrm{d}y_1+|b_B-b_{B^*}|\int_{B^*}|f_1(y_1)|\mathrm{d}y_1\Big)\\
&\quad\cdot\Big(\sum_{k=1}^{\infty}\int_{2^kB^*\setminus 2^{k-1}B^*}\frac{|f_2(y_2)|}{|y_2-x|^{2n}}\mathrm{d}y_2\Big)\\
&\le C\frac{1}{|B^*|}\Big(\int_{B^*}|b(y_1)-b_{B^*}||f_1(y_1)|\mathrm{d}y_1+|b_B-b_{B^*}|\int_{B^*}|f_1(y_1)|\mathrm{d}y_1\Big)\\
&\quad\cdot\Big(\sum_{k=1}^{\infty}2^{-kn}\frac{1}{|2^kB^*|}\int_{2^kB^*}|f_2(y_2)|\mathrm{d}y_2\Big)\\
&\le CK^*(b,f_1,\overline{\omega})(x)(M_\nu(|f_2\omega|^q)(x))^{\frac{1}{q}}.
\end{aligned}
$$

We use the same computational technique in I to get the last inequality.

For the fourth term IV, using the fact $|y-y_1|\sim|y_1-x|$ for any $y_1\in(B^*)^c$, $y\in B$, the size estimate on $K_{\varphi,\eta}$ and Lemma 2.4, we obtain

$$
\begin{aligned}
\mathrm{IV}&\le\frac{1}{|B|}\int_B|\Phi^*((b-\lambda)f_1^\infty,f_2^0)(y)|\mathrm{d}y\\
&\le\frac{1}{|B|}\int_B\int_{(\mathbb{R}^n\setminus B^*)\times B^*}\frac{A|(b(y_1)-\lambda)f_1(y_1)f_2(y_2)|}{(|y-y_1|+|y-y_2|)^{2n}}\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y\\
&\le C\Big(\int_{\mathbb{R}^n\setminus B^*}\frac{|(b(y_1)-\lambda)f_1(y_1)|}{|y_1-x|^{2n}}\mathrm{d}y_1\Big)\Big(\int_{B^*}|f_2(y_2)|\mathrm{d}y_2\Big)\\
&\le C\Big(\sum_{k=1}^{\infty}2^{-kn}\frac{1}{|2^kB^*|}\int_{2^kB^*}|(b(y_1)-b_B)f_1(y_1)|\mathrm{d}y_1\Big)\Big(\frac{1}{|B^*|}\int_{B^*}|f_2(y_2)|\mathrm{d}y_2\Big)\\
&\le C\Big(\sum_{k=1}^{\infty}2^{-kn}\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|(b(y_1)-b_{2^{k+1}B})f_1(y_1)|\mathrm{d}y_1\\
&\quad+\sum_{k=1}^{\infty}2^{-kn}\frac{|b_B-b_{2^{k+1}B}|}{|2^{k+1}B|}\int_{2^{k+1}B}|f_1(y_1)|\mathrm{d}y_1\Big)\Big(\frac{1}{\nu(B^*)}\int_{B^*}|f(y_2)\omega(y_2)|^q\nu(y_2)\mathrm{d}y_2\Big)^{\frac{1}{q}}\\
&\quad\cdot\Big(\frac{\nu(B^*)}{|B^*|}\Big)^{\frac{1}{q}}\Big(\frac{1}{|B^*|}\int_{B^*}\omega(y_2)^{-q'}\nu(y_2)^{-\frac{q'}{q}}\mathrm{d}y_2\Big)^{\frac{1}{q'}}\\
&\le C(M_\nu(|f_2\omega|^q)(x))^{\frac{1}{q}}\Big(K^*(b,f_1,\overline{\omega})(x)+\sum_{k=1}^{\infty}2^{-kn}\frac{|b_B-b_{2^{k+1}B}|}{|2^{k+1}B|}\int_{2^{k+1}B}|f_1(y_1)|\mathrm{d}y_1\Big).
\end{aligned}
$$

For simplicity, we bound

$$
\begin{aligned}
&\sum_{k=1}^{\infty}2^{-kn}\frac{|b_B-b_{2^{k+1}B}|}{|2^{k+1}B|}\int_{2^{k+1}B}|f_1(y_1)|\mathrm{d}y_1\\
&\le C\|b\|_{\mathrm{BMO}(\omega)}\sum_{k=1}^{\infty}2^{-kn}2^{kn(1-\sigma)}\frac{\omega(2^{k+1}B)}{|2^{k+1}B|}\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|f_1(y_1)|\mathrm{d}y_1\\
&\le C\|b\|_{\mathrm{BMO}(\omega)}\sum_{k=1}^{\infty}2^{-kn\sigma}\Big(\frac{1}{\nu(2^{k+1}B)}\int_{2^{k+1}B}|f(y_2)\omega(y_2)|^q\nu(y_2)\mathrm{d}y_2\Big)^{\frac{1}{q}}\\
&\quad\cdot\frac{\omega(2^{k+1}B)}{|2^{k+1}B|}\Big(\frac{\nu(2^{k+1}B)}{|2^{k+1}B|}\Big)^{\frac{1}{q}}\Big(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}\omega(y_2)^{-q'}\nu(y_2)^{-\frac{q'}{q}}\mathrm{d}y_2\Big)^{\frac{1}{q'}}
\end{aligned}
$$

$$\leq C\|b\|_{\mathrm{BMO}(\omega)}(M_\nu(|f_2\omega|^q)(x))^{\frac{1}{q}},$$

where in the third inequality we have used Lemmas 2.5–2.6.

Hence,

$$\mathrm{IV} \leq C(M_\nu(|f_2\omega|^q)(x))^{\frac{1}{q}}[K^*(b, f_1, \overline{\omega})(x) + \|b\|_{\mathrm{BMO}(\omega)}(M_\nu(|f_1\omega|^q)(x))^{\frac{1}{q}}].$$

For V, fixing the value of c by taking $c = \Phi^*((b-\lambda)f_1^\infty, f_2^\infty)(x_0)$, and recalling that $K_{\varphi,\eta}$ satisfies (1.2) uniformly in η, then we can obtain

$$\begin{aligned}
\mathrm{V} &\leq \frac{1}{|B|}\int_B |\Phi^*((b-\lambda)f_1^\infty, f_2^\infty)(y) - \Phi^*((b-\lambda)f_1^\infty, f_2^\infty)(x_0)|\mathrm{d}y \\
&\leq \frac{1}{|B|}\int_B \sup_{\eta>0} |\Phi_\eta((b-\lambda)f_1^\infty, f_2^\infty)(y) - \Phi_\eta((b-\lambda)f_1^\infty, f_2^\infty)(x_0)|\mathrm{d}y \\
&\leq \frac{1}{|B|}\int_B \int_{(\mathbb{R}^n\setminus B^*)^2} \sup_{\eta>0} |K_{\varphi,\eta}(y, y_1, y_2) - K_{\varphi,\eta}(x_0, y_1, y_2)||b(y_1-\lambda)||f_1(y_1)||f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y \\
&\leq \frac{C}{|B|}\int_B \int_{(\mathbb{R}^n\setminus B^*)^2} \frac{|x_0-y|^\varepsilon}{(|y-y_1|+|y-y_2|)^{2n+\varepsilon}}|(b(y_1)-b_B)f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y \\
&\leq \frac{C}{|B|}\int_B \sum_{k=0}^\infty \int_{(2^{k+1}B^*\setminus 2^kB^*)^2} \frac{|x_0-y|^\varepsilon}{|y-y_1|^{2n+\varepsilon}}|(b(y_1)-b_B)f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y \\
&\leq C\sum_{k=0}^\infty \frac{|B^*|^{\frac{\varepsilon}{n}}}{|2^kB^*|^{2+\frac{\varepsilon}{n}}} \int_{(2^{k+1}B^*)^2} |(b(y_1)-b_B)f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2 \\
&\leq C\sum_{k=0}^\infty 2^{-k\varepsilon}\Big(\frac{1}{|2^{k+1}B^*|}\int_{2^{k+1}B^*} |(b(y_1)-b_B)f_1(y_1)|\mathrm{d}y_1\Big)\Big(\frac{1}{|2^{k+1}B^*|}\int_{2^{k+1}B^*} |f_2(y_2)|\mathrm{d}y_2\Big) \\
&\leq C[K^*(b, f_1, \overline{\omega})(x) + \|b\|_{\mathrm{BMO}(\omega)}(M_\nu(|f_1\omega|^q)(x))^{\frac{1}{q}}](M_\nu(|f_2\omega|^q)(x))^{\frac{1}{q}}.
\end{aligned}$$

From the estimates of I, II, III, IV and V, one obtains the desired result.

Now we are ready to return to the proof of Theorem 1.1.

Proof of Theorem 1.1 By Lemma 2.3, we choose $\overline{\omega} \in A_{p'}$ such that

$$\int_{\mathbb{R}^n} K^*(b_j, f, \overline{\omega})(x)^p\nu(x)\mathrm{d}x \leq C\|b_j\|_{\mathrm{BMO}(\omega)}^p \int_{\mathbb{R}^n} |f|^p\mu(x)\mathrm{d}x.$$

By Hardy and Littlewood's theorem, we have

$$\int_{\mathbb{R}^n} (M_\nu|f\omega|^q)^{\frac{p}{q}}\nu(x)\mathrm{d}x \leq C\int_{\mathbb{R}^n} |f\omega|^p\nu(x)\mathrm{d}x = C\int |f|^p\mu(x)\mathrm{d}x.$$

With the weighted bounded operator Φ^*, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} K^*(b_j, \Phi^*(f_1, f_2), \overline{\omega})^p\nu(x)\mathrm{d}x \\
&\leq C\int_{\mathbb{R}^n} |\Phi^*(f_1, f_2)|^p\mu(x)\mathrm{d}x \\
&\leq C\int_{\mathbb{R}^n} |f_1|^{p_1}\mu(x)\mathrm{d}x \int_{\mathbb{R}^n} |f_2|^{p_2}\mu(x)\mathrm{d}x.
\end{aligned}$$

Thus, with the help of Lemma 2.2 and Hölder's inequality, we obtain

$$\begin{aligned}&\|\Phi^*_{\Sigma b}(f_1,f_2))\|_{L^p(\nu)}\le\|M^\sharp_\delta(\Phi^*_{\Sigma b}(f_1,f_2))(x)\|_{L^p(\nu)}\\&\le C\Big(\int_{\mathbb{R}^n}K^*(b,\Phi^*(f_1,f_2),\overline{\omega})^p\nu(x)\mathrm{d}x\Big)^{\frac1p}\\&\quad+C\Big(\int_{\mathbb{R}^n}(M_\nu(|f_2\omega|^q)(x))^{\frac{p_2}{q}}\nu(x)\mathrm{d}x\Big)^{\frac1{p_2}}\Big(\int_{\mathbb{R}^n}K^*(b_1,f_1,\overline{\omega})^{p_1}\nu(x)\mathrm{d}x\Big)^{\frac1{p_1}}\\&\quad+C\|b\|_{\mathrm{BMO}(\omega)}\Big(\int_{\mathbb{R}^n}(M_\nu|f_2\omega|^q(x))^{\frac{p_2}{q}}\nu(x)\mathrm{d}x\Big)^{\frac1{p_2}}\Big(\int_{\mathbb{R}^n}(M_\nu(|f_1\omega|^q)(x))^{\frac{p_1}{q}}\nu(x)\mathrm{d}x\Big)^{\frac1{p_1}}\\&\le C\|b\|_{\mathrm{BMO}(\omega)}\|f_1\|_{L^{p_1}(\mu)}\|f_2\|_{L^{p_2}(\mu)}.\end{aligned}$$

Similarly,

$$\|\Psi^*_{\Sigma b}(f_1,f_2))\|_{L^p(\nu)}\le\|M^\sharp_\delta(\Psi^*_{\Sigma}(f_1,f_2))(x)\|_{L^p(\nu)}\le C\|b\|_{\mathrm{BMO}(\omega)}\|f_1\|_{L^{p_1}(\mu)}\|f_2\|_{L^{p_2}(\mu)}.$$

Consequently, from (3.1), we conclude the proof of Theorem 1.1.

Now we turn to prove Theorem 1.3. As before, we only consider the case $m=2$. And the proof of Theorem 1.2 is based on the following estimate of the sharp maximal function. So first we establish the following lemma about the sharp maximal function for $\Phi^*_{\Pi b}$, since the proof for $\Psi^*_{\Pi b}$ is almost the same as for $\Phi^*_{\Pi b}$.

Lemma 3.2 *Let $b_j\in\mathrm{BMO}(\omega)$ $(j=1,2)$, $0<\delta<\frac16$, and $\overline{\omega}^{q'}\in A_{q'}$. Suppose that $\mu,\nu\in A_p$, $\omega=(\mu\nu^{-1})^{\frac1p}, 1<p<\infty$, and that (u,v) satisfies* (1.6)*. Then*

$$M^\sharp_\delta(\Phi^*_{\Pi b}(f_1,f_2))(x)\le C\prod_{j=1}^2[K^*(b_j,f_j,\overline{\omega})(x)+\|b_j\|_{\mathrm{BMO}(\omega)}(M_\nu(|f_j\omega|^q)(x))^{\frac1q}]$$

and

$$M^\sharp_\delta(\Psi^*_{\Pi b}(f_1,f_2))(x)\le C\prod_{j=1}^2[K^*(b_j,f_j,\overline{\omega})(x)+\|b_j\|_{\mathrm{BMO}(\omega)}(M_\nu(|f_j\omega|^q)(x))^{\frac1q}].$$

Proof Fix $x\in\mathbb{R}^n$ and let $B=B(x,R)$. Take $\lambda_i=(b_i)_B$ as the average of b_i on B, i=1, 2. Let c be a constant to be fixed along the proof. We split $\Phi^*_{\Pi b}(f_1,f_2)(y)$ in the following way,

$$\begin{aligned}&\Phi^*_{\Pi b}(f_1,f_2)(y)\\&=\sup_{\eta>0}|(b_1(y)-\lambda_1)(b_2(y)-\lambda_2)\Phi_\eta(f_1,f_2)(y)-(b_1(y)-\lambda_1)\Phi_\eta(f_1,(b_2-\lambda_2)f_2)(y)\\&\quad-(b_2(y)-\lambda_2)\Phi_\eta((b_1-\lambda_1)f_1,f_2)(y)+\Phi_\eta((b_1-\lambda_1)f_1,(b_2-\lambda_2)f_2)(y)|.\end{aligned}$$

Since $0<\delta<\frac16$, then we have

$$\begin{aligned}&\Big(\frac1{|B|}\int_B||\Phi^*_{\Pi b}(f_1,f_2)(y)|^\delta-|c|^\delta|\mathrm{d}y\Big)^{\frac1\delta}\\&\le\Big(\frac1{|B|}\int_B|\Phi^*_{\Pi b}(f_1,f_2)(y)-c|^\delta\mathrm{d}y\Big)^{\frac1\delta}\end{aligned}$$

$$
\begin{aligned}
&\le \Big(\frac{1}{|B|}\int_B |(b_1(y)-\lambda_1)(b_2(y)-\lambda_2)\Phi^*(f_1,f_2)(y)|^\delta \mathrm{d}y\Big)^{\frac{1}{\delta}}\\
&\quad + \Big(\frac{1}{|B|}\int_B \Big(\sup_{\eta>0}|(b_1(y)-\lambda_1)\Phi_\eta(f_1,(b_2-\lambda_2)f_2)(y)|\Big)^\delta \mathrm{d}y\Big)^{\frac{1}{\delta}}\\
&\quad + \Big(\frac{1}{|B|}\int_B \Big(\sup_{\eta>0}|(b_2(y)-\lambda_2)\Phi_\eta((b_1-\lambda_1)f_1,f_2)(y)|\Big)^\delta \mathrm{d}y\Big)^{\frac{1}{\delta}}\\
&\quad + \Big(\frac{1}{|B|}\int_B \sup_{\eta>0}|\Phi_\eta((b_1-\lambda_1)f_1,(b_2-\lambda_2)f_2)-c|^\delta \mathrm{d}y\Big)^{\frac{1}{\delta}}\\
&:= \mathrm{E}_1+\mathrm{E}_2+\mathrm{E}_3+\mathrm{E}_4.
\end{aligned}
$$

For the term E_1, we overcome it by restricting that $0<\delta<\frac{1}{6}$, and then by Hölder's inequality, the boundedness of Φ^*, and Lemma 2.4, noting that $\lambda_i=(b_i)_B$, we have

$$
\begin{aligned}
\mathrm{E}_1 &\le \Big(\frac{1}{|B|}\int_B |b_1(y)-\lambda_1|^{3\delta}\mathrm{d}y\Big)^{\frac{1}{3\delta}}\Big(\frac{1}{|B|}\int_B |b_2(y)-\lambda_2|^{3\delta}\mathrm{d}y\Big)^{\frac{1}{3\delta}}\Big(\frac{1}{|B|}\int_B |\Phi^*(f_1,f_2)(y)|^{3\delta}\mathrm{d}y\Big)^{\frac{1}{3\delta}}\\
&\le C\Big(\frac{1}{|B|}\int_B |b_1(y_1)-\lambda_1|\mathrm{d}y_1\Big)\Big(\frac{1}{|B|}\int_B |b_2(y_2)-\lambda_2|\mathrm{d}y_2\Big)\|\Phi^*(f_1,f_2)(y)\|_{L^{\frac{1}{2},\infty}(\frac{\mathrm{d}y}{|B|})}\\
&\le C\Big(\frac{1}{|B|}\int_B |b_1(y_1)-\lambda_1|\mathrm{d}y_1\Big)\Big(\frac{1}{|B|}\int_B |f_1(y_1)|\mathrm{d}y_1\Big)\\
&\quad\cdot\Big(\frac{1}{|B|}\int_B |b_2(y_2)-\lambda_2|\mathrm{d}y_2\Big)\Big(\frac{1}{|B|}\int_B |f_2(y_2)|\mathrm{d}y_2\Big)\\
&\le C\Big(\frac{1}{|B|}\int_B |b_1(y_1)-\lambda_1|^{q'}\overline{\omega}(y_1)^{q'}\mathrm{d}y_1\Big)^{\frac{1}{q'}}\Big(\frac{1}{|B|}\int_B |f_1(y_1)|^q\overline{\omega}(y_1)^{-q}\mathrm{d}y_1\Big)^{\frac{1}{q}}\\
&\quad\cdot\Big(\frac{1}{|B|}\int_B \overline{\omega}(y_1)^{-q}\mathrm{d}y_1\Big)^{\frac{1}{q}}\Big(\frac{1}{|B|}\int_B \overline{\omega}(y_1)^{q'}\mathrm{d}y_1\Big)^{\frac{1}{q'}}\\
&\quad\cdot\Big(\frac{1}{|B|}\int_B |b_2(y_2)-\lambda_2|^{q'}\overline{\omega}(y_2)^{q'}\mathrm{d}y_2\Big)^{\frac{1}{q'}}\Big(\frac{1}{|B|}\int_B |f_2(y_2)|^q\overline{\omega}(y_2)^{-q}\mathrm{d}y_2\Big)^{\frac{1}{q}}\\
&\quad\cdot\Big(\frac{1}{|B|}\int_B \overline{\omega}(y_2)^{-q}\mathrm{d}y_2\Big)^{\frac{1}{q}}\Big(\frac{1}{|B|}\int_B \overline{\omega}(y_2)^{q'}\mathrm{d}y_2\Big)^{\frac{1}{q'}}\\
&\le CK^*(b_1,f_1,\overline{\omega})(x)K^*(b_2,f_2,\overline{\omega})(x).
\end{aligned}
$$

For the term E_2, noting that $0<\delta<\frac{1}{6}$, we use the facts $1=\delta+3\delta+(1-4\delta)$, and then by Hölder's inequality, Komolgorov's inequality (Lemma 2.1) and the $(L^1\times L^1, L^{\frac{1}{2},\infty})$-boundedness of Φ^*, we have

$$
\begin{aligned}
\mathrm{E}_2 &\le C\Big(\frac{1}{|B|}\int_B |b_1(y)-(b_1)_B|\mathrm{d}y\Big)\Big(\frac{1}{|B|}\int_B \sup_{\eta>0}|\Phi_\eta(f_1,(b_2-\lambda_2)f_2)(y)|^{\frac{1}{3}}\mathrm{d}y\Big)^3\Big(\frac{1}{|B|}\int_B \mathrm{d}y\Big)^{\frac{1-4\delta}{\delta}}\\
&\le C\Big(\frac{1}{|B|}\int_B |b_1(y_1)-(b_1)_B|\mathrm{d}y_1\Big)\|\Phi^*(f_1,(b_2-\lambda_2)f_2)\|_{L^{\frac{1}{2},\infty}(\frac{\mathrm{d}y}{|B|})}\\
&\le C\Big(\frac{1}{|B|}\int_B |b_1(y_1)-(b_1)_B|\mathrm{d}y_1\Big)\Big(\frac{1}{|B|}\int_B |f_1(y_1)|\mathrm{d}y_1\Big)\Big(\frac{1}{|B|}\int_B |b_2(y_2)-(b_2)_B||f_2(y_2)|\mathrm{d}y_2\Big)\\
&\le CK^*(b_1,f_1,\overline{\omega})(x)K^*(b_2,f_2,\overline{\omega})(x).
\end{aligned}
$$

Similarly, for the term E_3, we have

$$
\mathrm{E}_3 \le \|b_2\|_{\mathrm{BMO}(\omega)}(M_\nu(|f_2\omega|^q)(x))^{\frac{1}{q}}K^*(b_1,f_1,\overline{\omega})(x).
$$

Now we turn to estimate the last term E_4. To proceed with, we denote that $f_i = f_i^0 + f_i^\infty$, where $f_i^0 = f_i\chi_{B^*}$, $i = 1, 2$ and $B^* = B(x, 2R)$. Let $c = c_1 + c_2 + c_3$, where

$$c_1 = \Phi_\eta((b_1 - \lambda_1)f_1^0, (b_2 - \lambda_2)f_2^\infty)(x),$$
$$c_2 = \Phi_\eta((b_1 - \lambda_1)f_1^\infty, (b_2 - \lambda_2)f_2^0)(x),$$
$$c_3 = \Phi_\eta((b_1 - \lambda_1)f_1^\infty, (b_2 - \lambda_2)f_2^\infty)(x).$$

We split E_4 in the following way:

$$\mathrm{E}_4 \le \mathrm{E}_{41} + \mathrm{E}_{42} + \mathrm{E}_{43} + \mathrm{E}_{44},$$

where

$$\begin{aligned}
\mathrm{E}_{41} &= \Big(\frac{1}{|B|}\int_B \sup_{\eta>0} |\Phi_\eta((b_1 - \lambda_1)f_1^0, (b_2 - \lambda_2)f_2^0)(y)|^\delta \mathrm{d}y\Big)^{\frac{1}{\delta}},\\
\mathrm{E}_{42} &= \Big(\frac{1}{|B|}\int_B \sup_{\eta>0} |\Phi_\eta((b_1 - \lambda_1)f_1^0, (b_2 - \lambda_2)f_2^\infty)(y)\\
&\qquad - \Phi_\eta((b_1 - \lambda_1)f_1^0, (b_2 - \lambda_2)f_2^\infty)(x)|^\delta \mathrm{d}y\Big)^{\frac{1}{\delta}},\\
\mathrm{E}_{43} &= \Big(\frac{1}{|B|}\int_B \sup_{\eta>0} |\Phi_\eta((b_1 - \lambda_1)f_1^\infty, (b_2 - \lambda_2)f_2^0)(y)\\
&\qquad - \Phi_\eta((b_1 - \lambda_1)f_1^\infty, (b_2 - \lambda_2)f_2^0)(x)|^\delta \mathrm{d}y\Big)^{\frac{1}{\delta}},\\
\mathrm{E}_{44} &= \Big(\frac{1}{|B|}\int_B \sup_{\eta>0} |\Phi_\eta((b_1 - \lambda_1)f_1^\infty, (b_2 - \lambda_2)f_2^\infty)(y)\\
&\qquad - \Phi_\eta((b_1 - \lambda_1)f_1^\infty, (b_2 - \lambda_2)f_2^\infty)(x)|^\delta \mathrm{d}y\Big)^{\frac{1}{\delta}}.
\end{aligned}$$

For the term E_{41}, by Kolmogorov's inequality and the boundedness of Φ^*, choosing $1 < p_0 < \frac{1}{2\delta}$, and estimating this just as I in the proof of Lemma 3.1, we deduce that

$$\begin{aligned}
\mathrm{E}_{41} &\le \Big(\frac{1}{|B|}\int_B |\Phi^*((b_1 - \lambda_1)f_1^0, (b_2 - \lambda_2)f_2^0)(y)|^{p_0\delta} \mathrm{d}y\Big)^{\frac{1}{p_0\delta}}\\
&\le \|\Phi^*((b_1 - \lambda_1)f_1^0, (b_2 - \lambda_2)f_2^0)\|_{L^{\frac{1}{2},\infty}(\frac{\mathrm{d}y}{|B|})}\\
&\le \Big(\frac{1}{|B|}\int_B |(b_1(y_1) - \lambda_1)f_1^0(y_1)|\mathrm{d}y_1\Big)\Big(\frac{1}{|B|}\int_B |(b_2(y_2) - \lambda_2)f_2^0(y_2)|\mathrm{d}y_2\Big)\\
&\le C\Big(\frac{1}{|B^*|}\int_{B^*} |(b_1(y_1) - (b_1)_B)f_1(y_1)|\mathrm{d}y_1\Big)\Big(\frac{1}{|B^*|}\int_{B^*} |(b_2(y_2) - (b_2)_B)f_2(y_2)|\mathrm{d}y_2\Big)\\
&\le CK^*(b_1, f_1, \overline{\omega})(x)K^*(b_2, f_2, \overline{\omega})(x).
\end{aligned}$$

For E_{42}, since $K_{\varphi,\eta}$ satisfies (1.2) uniformly in η, using Lemma 2.4, we deduce that

$$\begin{aligned}
\mathrm{E}_{42} &\le \Big(\frac{1}{|B|}\int_B \sup_{\eta>0} |\Phi_\eta((b_1 - \lambda_1)f_1^0, (b_2 - \lambda_2)f_2^\infty)(y) - \Phi_\eta((b_1 - \lambda_1)f_1^0, (b_2 - \lambda_2)f_2^\infty)(x)|\mathrm{d}y\Big)\\
&\le \frac{C}{|B|}\int_B \Big(\int_{B^*\times(\mathbb{R}^n\setminus B^*)} \frac{|y - x_0|^\varepsilon}{(|y - y_1| + |y - y_2|)^{2n+\varepsilon}} |(b_1(y_1) - \lambda_1)f_1(y_1)|\\
&\quad \cdot |b_2(y_2) - \lambda_2||f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\Big)\mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&\le C\Big(\int_{B^*}|(b_1(y_1)-(b_1)_B)f_1(y_1)|\mathrm{d}y_1\Big)\Big(\sum_{k=0}^{\infty}\int_{2^{k+1}B^*\setminus 2^kB^*}\frac{|y-x_0|^{\varepsilon}}{|y-y_2|^{2n+\varepsilon}}\\
&\quad\cdot|b_2(y_2)-(b_2)_B||f_2(y_2)|\mathrm{d}y_2\Big)\\
&\le C\Big(\int_{B^*}|(b_1(y_1)-(b_1)_B)f_1(y_1)|\mathrm{d}y_1\Big)\\
&\quad\cdot\Big(\sum_{k=0}^{\infty}\frac{|B^*|^{\frac{\varepsilon}{n}}}{|2^{k+1}B^*|^{2+\frac{\varepsilon}{n}}}\int_{2^{k+1}B^*}|b_2(y_2)-(b_2)_B||f_2(y_2)|\mathrm{d}y_2\Big)\\
&\le C\Big(\frac{1}{|B^*|}\int_{B^*}|(b_1(y_1)-(b_1)_B)f_1(y_1)|\mathrm{d}y_1\Big)\\
&\quad\cdot\Big(\sum_{k=0}^{\infty}2^{-k(n+\varepsilon)}\frac{1}{|2^{k+1}B^*|}\int_{2^{k+1}B^*}|b_2(y_2)-(b_2)_B||f_2(y_2)|\mathrm{d}y_2\Big)\\
&\le CK^*(b_1,f_1,\overline{\omega})(x)\sum_{k=0}^{\infty}2^{-k(n+\varepsilon)}\frac{1}{|2^{k+1}B^*|}\int_{2^{k+1}B^*}|b_2(y_2)-(b_2)_B||f_2(y_2)|\mathrm{d}y_2\\
&\le CK^*(b_1,f_1,\overline{\omega})(x)\sum_{k=1}^{\infty}2^{-k(n+\varepsilon)}\Big(\frac{1}{|2^{k+1}B^*|}\int_{2^{k+1}B^*}|b_2(y_2)-(b_2)_{2^{k+1}B^*}||f_2(y_2)|\mathrm{d}y_2\\
&\quad+\frac{1}{|2^{k+1}B^*|}\int_{2^{k+1}B^*}|(b_2)_B-(b_2)_{2^{k+1}B^*}||f_2(y_2)|\mathrm{d}y_2\Big)\\
&\le CK^*(b_1,f_1,\overline{\omega})(x)(K^*(b_2,f_2,\overline{\omega})(x)+\|b_2\|_{\mathrm{BMO}(\omega)}(M_\nu(|f_2\omega|^q)(x))^{\frac{1}{q}}).
\end{aligned}
$$

Similar to E_{42}, we can get the estimates for E_{43},

$$
\mathrm{E}_{43}\le CK^*(b_2,f_2,\omega)(x)(K^*(b_1,f_1,\omega)(x)+\|b_1\|_{\mathrm{BMO}(\omega)}(M_\nu(|f_1\omega|^q)(x))^{\frac{1}{q}}).
$$

Now we turn to estimate E_{44}. Since $K_{\varphi,\eta}$ satisfies (1.2) uniformly in η, using Lemma 2.4, we deduce that

$$
\begin{aligned}
&|\Phi_\eta((b_1-\lambda_1)f_1^\infty,(b_2-\lambda_2)f_2^\infty)(y)-\Phi_\eta((b_1-\lambda_1)f_1^\infty,(b_2-\lambda_2)f_2^\infty)(x)|\\
&\le C\sum_{k=0}^{\infty}\int_{(2^{k+1}B^*\setminus 2^kB^*)^2}\frac{|x_0-y|^{\varepsilon}}{(|y-y_1|+|y-y_2|)^{2n+\varepsilon}}|(b_1(y_1)-\lambda_1)f_1(y_1)\\
&\quad\cdot(b_2(y_2)-\lambda_2)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\\
&\le\sum_{k=0}^{\infty}\frac{|B^*|^{\frac{\varepsilon}{n}}}{|2^kB^*|^{2+\frac{\varepsilon}{n}}}\int_{(2^{k+1}B^*)^2}|(b_1(y_1)-(b_1)_B)||f_1(y_1)||b_2(y_2)-(b_2)_B||f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\\
&\le C\sum_{k=0}^{\infty}2^{-k\varepsilon}\prod_{j=1}^{2}\Big(\frac{1}{|2^{k+1}B^*|}\int_{2^{k+1}B^*}|b_j(y_j)-(b_j)_B||f_j(y_j)|\mathrm{d}y_j\Big)\\
&\le C\prod_{j=1}^{2}(K^*(b_j,f_j,\omega)(x)+\|b_j\|_{\mathrm{BMO}(\omega)}(M_\nu(|f_j\omega|^q)(x))^{\frac{1}{q}}).
\end{aligned}
$$

Therefore,

$$
\mathrm{E}_{44}\le\frac{1}{|B|}\int_B\sup_{\eta>0}|\Phi_\eta((b_1-\lambda_1)f_1^\infty,(b_2-\lambda_2)f_2^\infty)(y)-\Phi_\eta((b_1-\lambda_1)f_1^\infty,(b_2-\lambda_2)f_2^\infty)(x)|\mathrm{d}y
$$

$$\leq C\prod_{j=1}^{2}(K^*(b_j,f_j,\overline{\omega})(x)+\|b_j\|_{\mathrm{BMO}(\omega)}(M_\nu(|f_j\omega|^q)(x))^{\frac{1}{q}}).$$

Consequently, combining the estimates for E_1, E_2, E_3 and E_4, we conclude the Lemma 3.2.

Now we are ready to return to the proof of Theorem 1.3.

Proof of Theorem 1.3 By Lemma 2.3, we choose $\overline{\omega}\in A_{p'}$ such that

$$\int_{\mathbb{R}^n}K^*(b_j,f,\overline{\omega})^p\nu(x)\mathrm{d}x\leq C\|b_j\|^p_{\mathrm{BMO}(\omega)}\int_{\mathbb{R}^n}|f|^p\mu(x)\mathrm{d}x.$$

By Hardy and Littlewood's theorem, we have

$$\int_{\mathbb{R}^n}(M_\nu|f\omega|^q)^{\frac{p}{q}}\nu(x)\mathrm{d}x\leq C\int_{\mathbb{R}^n}|f\omega|^p\nu(x)\mathrm{d}x=C\int_{\mathbb{R}^n}|f|^p\mu(x)\mathrm{d}x.$$

Thus, thanks to Lemma 2.2 and Hölder's inequality, we get

$$\begin{aligned}
&\|\Phi^*_{\Pi b}(f_1,f_2))(x)\|_{L^p(\nu)}\leq\|M^\sharp_\delta(\Phi^*_{\Pi b}(f_1,f_2))(x)\|_{L^p(\nu)}\\
&\leq C\Big(\int_{\mathbb{R}^n}K^*(b_1,f_1,\overline{\omega})^pK^*(b_2,f_2,\overline{\omega})^p\nu(x)\mathrm{d}x\Big)^{\frac{1}{p}}\\
&\quad+C\|b_2\|_{\mathrm{BMO}(\omega)}\Big(\int_{\mathbb{R}^n}(M_\nu(|f_2\omega|^q)(x))^{\frac{p_2}{q}}\nu(x)\mathrm{d}x\Big)^{\frac{1}{p_2}}\Big(\int_{\mathbb{R}^n}K^*(b_1,f_1,\overline{\omega})(x)^{p_1}\nu(x)\mathrm{d}x\Big)^{\frac{1}{p_1}}\\
&\quad+\|b_1\|_{\mathrm{BMO}(\omega)}\Big(\int_{\mathbb{R}^n}(M_\nu(|f_1\omega|^q)(x))^{\frac{p_1}{q}}\nu(x)\mathrm{d}x\Big)^{\frac{1}{p_2}}\Big(\int_{\mathbb{R}^n}K^*(b_2,f_2,\overline{\omega})(x)^{p_2}\nu(x)\mathrm{d}x\Big)^{\frac{1}{p_1}}\\
&\quad+C\|b_1\|_{\mathrm{BMO}(\omega)}\|b_2\|_{\mathrm{BMO}(\omega)}\Big\|(M_\nu(|f_2\omega|^q)(x))^{\frac{1}{q}}\Big\|_{L^{p_2}(\nu)}\Big\|(M_\nu(|f_1\omega|^q)(x))^{\frac{1}{q}}\Big\|_{L^{p_1}(\nu)}\\
&\leq C\|b_1\|_{\mathrm{BMO}(\omega)}\|b_2\|_{\mathrm{BMO}(\omega)}\|f_1\|_{L^{p_1}(\mu)}\|f_2\|_{L^{p_2}(\mu)}.
\end{aligned}$$

Similarly,

$$\|\Psi^*_{\Pi b}(f_1,f_2))(x)\|_{L^p(\nu)}\leq\|M^\sharp_\delta(\Psi^*_{\Pi}(f_1,f_2))(x)\|_{L^p(\nu)}\leq C\prod_{j=1}^{2}\|b_j\|_{\mathrm{BMO}(\omega)}\|f_j\|_{L^{p_j}(\mu)}.$$

Consequently, from (3.1), we conclude the proof of Theorem 1.3.

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