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Notes on Homoclinic Solutions of the Steady Swift-Hohenberg Equation*

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Abstract This paper considers the steady Swift-Hohenberg equation

$$u'''' + \beta^2 u'' + u^3 - u = 0.$$

Using the dynamic approach, the authors prove that it has a homoclinic solution for each $\beta \in [\sqrt[4]{8} - \epsilon_0, \sqrt[4]{8})$, where ϵ_0 is a small positive constant. This slightly complements Santra and Wei's result [Santra, S. and Wei, J., Homoclinic solutions for fourth order traveling wave equations, *SIAM J. Math. Anal.*, 41, 2009, 2038–2056], which stated that it admits a homoclinic solution for each $\beta \in (0, \beta_0)$ where $\beta_0 = 0.9342 \cdots$.

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1 Introduction

The well-known Swift-Hohenberg (SH) equation

$$u_t = \mu u - (1 + \partial_{xx})^2 u - u^3 \tag{1.1}$$

was originally derived in the context of Rayleigh-Bénard convection with thermal fluctuations in the limit of infinite Prandtl numbers (see [11]). It captures much of the observed physical behavior and has now become a general tool used to investigate not only the Rayleigh-Bénard convection, but also other pattern-forming systems (see [2]). This equation has been studied intensively by many authors. Mielke and Schneider [7] proved the existence of the global attractor in a weighted Sobolev space on the whole real line. Hsieh et al [4–5] remarked that the elemental instability mechanism is the negative diffusion term $-u_{xx}$. Yari [13] discussed the bifurcation and asymptotic behavior of its solutions with the Dirichlet boundary condition and Peletier and Rottschäfer [8] gave some numerical results.

If we consider only the steady solutions of (1.1) for $\mu > 1$, it can be written as

$$u'''' + \beta^2 u'' + u^3 - u = 0 \tag{1.2}$$

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after scaling, where the prime $'$ means taking the derivative with respect to x. Van den Berg, Peletier and Troy [12] investigated the existence of periodic solutions. Bonheure [1] established the existence of multitransition kinks and pulses obtained as local minima of the associated functional. Smets and Van den Berg [10] used the mountain-pass lemma and Struwe's monotonicity trick to prove that the equation (1.2) has a homoclinic solution for almost all $\beta \in (0, \sqrt[4]{8})$ while Santra and Wei [9] employed the energy and the Morse index to show that it admits a homoclinic solution for each $\beta \in (0, \beta_0)$, where $\beta_0 = \sqrt{\frac{\sqrt{2}}{k_0}} \approx 0.9342\cdots$, and k_0 is a solution of $4k_0^2 - 2k_0 - 3 = 0$.

For each fixed $\beta \in [\beta_0, \sqrt[4]{8})$, whether (1.2) has a homoclinic solution is still an open question. In this paper, we use the dynamic approach and the norm form analysis in particular to obtain that the equation (1.2) has a homoclinic solution for each β less than but close to $\sqrt[4]{8}$. This partially complements the result in [9].

2 Main Results

Note that (1.2) has three stationary solutions: 0, 1 and -1. We are here interested in the last two stationary solutions 1 and -1 which correspond to the homoclinic solutions. By symmetry, it is sufficient to consider the solutions homoclinic to 1. Let $v = u - 1$ which changes (1.2) into

$$v'''' + \beta^2 v'' + v^3 + 3v^2 + 2v = 0. \tag{2.1}$$

The linear equation of (2.1) is

$$v'''' + \beta^2 v'' + 2v = 0, \tag{2.2}$$

whose eigenvalues satisfy

$$\lambda_\pm^2 = \frac{-\beta^2 \pm \sqrt{\beta^4 - 8}}{2}. \tag{2.3}$$

Thus, the threshold $\beta = \sqrt[4]{8}$ corresponds to the upper limit for saddle-focus equilibriums. If $|\beta| < \sqrt[4]{8}$, then four eigenvalues $\pm\lambda_\pm$ are complex and have nonzero real parts. We have the following theorem.

Theorem 2.1 *There exists a positive constant ϵ_0 such that for each $\beta \in [\sqrt[4]{8} - \epsilon_0, \sqrt[4]{8})$, the equation* (1.2) *has a homoclinic solution exponentially approaching* 1.

Remark 2.1 In the same way, we can prove that (1.2) has a homoclinic solution exponentially approaching -1.

The proof of Theorem 2.1 is based on the normal form analysis and the results given by Iooss and Pérouème [6]. In the following, we will give the proof.

Firstly we focus on the normal form of (2.1) for β near $\sqrt[4]{8}$ and assume that for simplicity

$$\beta^2 = \sqrt{8} - a\mu, \tag{2.4}$$

where a is a positive constant and $\mu > 0$ is a small parameter. Let $U = (v, u_1, u_2, u_3)^{\mathrm{T}}$, where $u_1 = v_x$, $u_2 = v_{xx}$ and $u_3 = v_{xxx}$, which changes (2.1) into

$$U' = \mathcal{A}U + \mu\mathcal{A}_\mu U + N(U), \tag{2.5}$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0 & -\sqrt{8} & 0 \end{pmatrix}, \quad \mathcal{A}_\mu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 \end{pmatrix}, \quad N(U) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -v^3 - 3v^2 \end{pmatrix}. \tag{2.6}$$

The system (2.5) is reversible with a reverser S defined by

$$S(v, u_1, u_2, u_3) = (v, -u_1, u_2, -u_3),$$

that is, $S(v, u_1, u_2, u_3)(-x)$ is also a solution whenever $(v, u_1, u_2, u_3)(x)$ is. A solution (v, u_1, u_2, u_3) is reversible if $S(v, u_1, u_2, u_3)(-x) = (v, u_1, u_2, u_3)(x)$.

$\mathcal{A}$ has two eigenvalues $\pm \mathrm{i}s_1$ which are double, where $s_1 = \sqrt[4]{2}$. The eigenvector U_1 and the generalized eigenvector U_2 of $\mathcal{A}$ corresponding to $\mathrm{i}s_1$ are given by

$$U_1 = \Big(-\frac{1}{\sqrt[4]{2^3}}, -\frac{\mathrm{i}}{\sqrt{2}}, \frac{1}{\sqrt[4]{2}}, \mathrm{i} \Big)^{\mathrm{T}}, \qquad U_2 = \Big(-\frac{3}{2}\mathrm{i}, \sqrt[4]{2}, \frac{\mathrm{i}}{\sqrt{2}}, 0 \Big)^{\mathrm{T}}, \tag{2.7}$$

respectively, and they satisfy

$$SU_1 = \overline{U}_1, \quad SU_2 = -\overline{U}_2.$$

The eigenvector V_2 and the generalized eigenvector V_1 of the adjoint operator $\mathcal{A}^* = \mathcal{A}^{\mathrm{T}}$ corresponding to $-\mathrm{i}s_1$ are given by

$$V_1 = \Big(\frac{1}{2\sqrt[4]{2}}, 0, \frac{3}{2\sqrt[4]{2^3}}, \frac{\mathrm{i}}{2} \Big)^{\mathrm{T}}, \quad V_2 = \Big(-\frac{\mathrm{i}}{2}, \frac{1}{2\sqrt[4]{2}}, -\frac{\mathrm{i}}{2\sqrt{2}}, \frac{1}{2\sqrt[4]{2^3}} \Big)^{\mathrm{T}}, \tag{2.8}$$

respectively. They satisfy $(U_j, V_j) = 1$ and $(U_j, V_k) = 0$ for $j \neq k$, $j, k = 1, 2$, where $(\cdot, \cdot)$ denotes the scalar product in $\mathbf{C}^4$.

We can write

$$U = AU_1 + BU_2 + \overline{AU}_1 + \overline{BU}_2 \tag{2.9}$$

such that (2.5) is changed into

$$X' = LX + F(\mu, X), \tag{2.10}$$

where $X = (A, B, \overline{A}, \overline{B})^{\mathrm{T}}$, L is given by

$$L = \begin{pmatrix} \mathrm{i}s_1 & 1 & 0 & 0 \\ 0 & \mathrm{i}s_1 & 0 & 0 \\ 0 & 0 & -\mathrm{i}s_1 & 1 \\ 0 & 0 & 0 & -\mathrm{i}s_1 \end{pmatrix}, \tag{2.11}$$

$F(\mu, 0) = 0$, $D_X F(0, 0) = 0$ and $F(0, X) = O(|X|^2)$.

Note that the reverser S is given by

$$S(A, B) = (\overline{A}, -\overline{B}) \tag{2.12}$$

and

$$SF = -FS, \quad SL = -LS. \tag{2.13}$$

From the general theory of normal forms (see [3]), (2.10) can be written as (for the sake of convenience, we still use X)

$$\begin{aligned} A' &= \mathrm{i}s_1 A + B + \mathrm{i}AP_0(\mu, A\overline{A}, \mathrm{i}(A\overline{B} - \overline{A}B)) + R_1(\mu, X), \\ B' &= \mathrm{i}s_1 B + \mathrm{i}BP_0(\mu, A\overline{A}, \mathrm{i}(A\overline{B} - \overline{A}B)) + AQ_0(\mu, A\overline{A}, \mathrm{i}(A\overline{B} - \overline{A}B)) + R_2(\mu, X), \end{aligned} \tag{2.14}$$

and their complex conjugates, where P_0 and Q_0 are real polynomials of their arguments with degree n (arbitrary but fixed), and

$$\begin{aligned}
&P_0(\mu, A\overline{A}, \mathrm{i}(A\overline{B} - \overline{A}B)) = O(|(\mu, A, B)|^{n-1}),\\
&Q_0(\mu, A\overline{A}, \mathrm{i}(A\overline{B} - \overline{A}B)) = O(|(\mu, A, B)|^{n-1}),\\
&R_1(\mu, X) = O(|(A, B)||(\mu, A, B)|^n),\\
&R_2(\mu, X) = O(|(A, B)||(\mu, A, B)|^n).
\end{aligned}$$

Moreover,

$$S\begin{pmatrix} R_1(\mu, X) \\ R_2(\mu, X) \end{pmatrix} = -\begin{pmatrix} R_1(\mu, SX) \\ R_2(\mu, SX) \end{pmatrix}$$

and Q_0 has the form

$$Q_0(\mu, A\overline{A}, \mathrm{i}(A\overline{B} - \overline{A}B)) = q_0\mu + q_1 A\overline{A} + \cdots ,$$

where

$$q_0 = \frac{a}{4} > 0, \quad q_1 = -1.$$

The calculations of q_0 and q_1 are similar to the ones in [3] by (2.7) and (2.8). Note that the normal form of the system (2.14) is exactly the system (4) in [6]. If we replace μ by $-\mu$ and use the results in IV. 3 of [6], we have the following theorem.

Theorem 2.2 *There exists a positive constant* μ_0 *such that for* $0 < \mu \le \mu_0$ *the system* (2.14) *has a homoclinic solution which is reversible and exponentially tends to* 0 *as* $x \to \pm\infty$.

From this theorem, we easily obtain Theorem 2.1.

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