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A Note on Restricted Representations of the Witt Superalgebras*

Yufeng YAO[1] Bin SHU[2]

(*Dedicated to the memory of professor Guang-Yu Shen with deep respect and admiration*)

Abstract Let F be an algebraically closed field of prime characteristic $p > 3$, and $W(n)$ the Witt superalgebra over F, which is the Lie superalgebra of superderivations of the Grassmann algebra in n indeterminates. The dimensions of simple atypical modules in the restricted supermodule category for $W(n)$ are precisely calculated in this paper, and thereby the dimensions of all simple modules can be precisely given. Moreover, the restricted supermodule category for $W(n)$ is proved to have one block.

Keywords Restricted Lie superalgebra, Witt superalgebra, Restricted representation, Atypical weight
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1 Introduction and Preliminaries

This paper is a sequel to [3], where the authors determined all restricted simple modules and their character formulas along with the Cartan invariants for the Witt superalgebra $W(n)$. In this paper, we present precise dimension formulas of those simple modules, and give a new and interesting observation that the category of restricted supermodules of $W(n)$ has only one block.

Recall that Kac classified finite-dimensional simple Lie superalgebras over the field of complex numbers (cf. [1]). Although until now, the classification of finite-dimensional simple Lie superalgebras over a field of prime characteristic is unknown, one naturally expects that there would be some modular version of the classification of iso-classes of finite-dimensional complex simple Lie superalgebras. The Lie superalgebras of Cartan type in prime characteristic would be the main series of simple Lie superalgebras, apart from the classical series. Those Lie superalgebras were mainly studied in [5].

In this paper, we always assume that the ground field F is algebraically closed with characteristic $p > 3$, and that all modules (vector spaces) are over F.

Manuscript received September 16, 2011. Revised October 27, 2012.
[1]Department of Mathematics, Shanghai Maritime University, Shanghai 201306, China.
E-mail: yfyao@shmtu.edu.cn
[2]Department of Mathematics, East China Normal University, Shanghai 200241, China.
E-mail: bshu@math.ecnu.edu.cn
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1.1 Restricted Lie superalgebras and their restricted representations

Recall that a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is called a restricted one if $\mathfrak{g}_{\bar{0}}$ is a restricted Lie algebra and $\mathfrak{g}_{\bar{1}}$ is a restricted $\mathfrak{g}_{\bar{0}}$-module under the adjoint action (cf. [2]).

Let $(\mathfrak{g}, [p])$ be a restricted Lie superalgebra. As in the case of restricted Lie algebras, one can define the so-called restricted enveloping superalgebra $u(\mathfrak{g})$ to be the quotient of $U(\mathfrak{g})$ by the ideal generated by $\{x^p - x^{[p]} \mid x \in \mathfrak{g}_{\bar{0}}\}$, where $U(\mathfrak{g})$ denotes the universal enveloping superalgebra of $\mathfrak{g}$. A representation (V, ρ) of $\mathfrak{g}$ is said to be restricted if ρ satisfies that $\rho(x)^p = \rho(x^{[p]}),\ \forall x \in \mathfrak{g}_{\bar{0}}$. All restricted $\mathfrak{g}$-modules constitute a full subcategory of the $\mathfrak{g}$-module category, which coincides with the $u(\mathfrak{g})$-module category denoted by $u(\mathfrak{g})$-mod.

1.2 The Witt superalgebras

Let $\Lambda(n)$ be the free commutative superalgebra with n odd generators $\xi_1, \cdots, \xi_n$. Then $\Lambda(n)$ is isomorphic to the Grassmann algebra. Let $W(n)$ be the Lie superalgebra of superderivations of $\Lambda(n)$. Then $W(n) = \{\sum_{i=1}^{n} f_i D_i \mid f_i \in \Lambda(n)\}$, where D_i's are the superderivations of $\Lambda(n)$ defined via $D_i(\xi_j) = \delta_{ij}$. The superalgebra $W(n)$ is called the Witt superalgebra of rank n. The $\mathbb{Z}$-grading of $\Lambda(n)$ given by $\deg \xi_i = 1$ for all $1 \leq i \leq n$ induces the $\mathbb{Z}$-grading of the Witt superalgebra $W(n) = \bigoplus_{i=-1}^{n-1} W(n)_{[i]}$, where

$$W(n)_{[i]} = \operatorname{span}_F\{\xi_{t_1}\xi_{t_2}\cdots\xi_{t_{i+1}} D_s \mid 1 \leq t_1 < t_2 < \cdots < t_{i+1} \leq n,\ 1 \leq s \leq n\}.$$

Associated with this grading, there is a natural filtration:

$$W(n) = W(n)_{-1} \supseteq W(n)_0 \supseteq \cdots,$$

where $W(n)_i = \bigoplus_{j \geq i} W(n)_{[j]}$. It is easy to see that $W(n)$ is a restricted Lie superalgebra.

From now on, we always assume $\mathfrak{g} = W(n)$, unless otherwise indicated. Note that $\mathfrak{g}_{[0]} \cong \mathfrak{gl}(n)$ under the following map:

$$\mathfrak{g}_{[0]} \to \mathfrak{gl}(n),$$
$$\sum a_{ij}\xi_i D_j \mapsto \sum a_{ij} E_{ij}.$$

Furthermore, we have the standard triangular decomposition: $\mathfrak{g}_{[0]} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$, where $\mathfrak{n}^- = \sum_{i>j} F\xi_i D_j$, $\mathfrak{h} = \sum_{i=1}^{n} F h_i$ for $h_i = \xi_i D_i, 1 \leq i \leq n$, and $\mathfrak{n} = \sum_{i<j} F\xi_i D_j$. Set $\mathfrak{b}^{\pm} := \mathfrak{h} + \mathfrak{n}^{\pm}$ and $\mathfrak{b}^+$ is usually simply denoted by $\mathfrak{b}$. Set $N^- := \mathfrak{n}^- \oplus \mathfrak{g}_{[-1]}$, $N^+ := \mathfrak{n} \oplus \mathfrak{g}_1$, $B^- := \mathfrak{h} \oplus N^-$, $B^+ := \mathfrak{h} \oplus N^+$, $\mathfrak{g}^+ := \mathfrak{g}_{[0]} \oplus \mathfrak{g}_1 = \mathfrak{g}_0$ and $\mathfrak{g}^- := \mathfrak{g}_{[0]} \oplus \mathfrak{g}_{[-1]}$. It is easy to check that $\mathfrak{b}^{\pm}$, $N^{\pm}$, $B^{\pm}$ and $\mathfrak{g}^{\pm}$ are restricted subalgebras of $\mathfrak{g}$.

The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_{[0]}$ is also a Cartan subalgebra of $\mathfrak{g}$. We then have a root space decomposition: $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$, where

$$\Delta = \{\varepsilon_{i_1} + \cdots + \varepsilon_{i_k} - \varepsilon_j \mid 1 \leq i_1 < \cdots < i_k \leq n, 1 \leq j \leq n\} \cup \{-\varepsilon_i \mid 1 \leq i \leq n\},$$

and $\{\varepsilon_1, \cdots, \varepsilon_n\}$ is the standard basis in $\mathfrak{h}^*$ with $\varepsilon_i(h_j) = \delta_{ij}$.

By the well-known PBW theorem, we have the superspace isomorphism $u(\mathfrak{g}) \cong \bigwedge(\mathfrak{g}_{[-1]}) \otimes_F u(\mathfrak{g}^+)$.

2 Dimensions of Restricted Atypical Simple Modules for $W(n)$

Recall that the iso-classes of irreducible restricted $\mathfrak{g}_{[0]}$-modules are parameterized by the set of restricted weights $\Lambda := \{\lambda = (\lambda_1, \cdots, \lambda_n) \mid \lambda_i \in \mathbb{F}_p, i = 1, \cdots, n\}$. More precisely, for a given $\lambda \in \Lambda$, there is a one-dimensional restricted $\mathfrak{b}$-module F_λ on which $\mathfrak{h}$ acts as a scalar determined by λ, while $\mathfrak{n}$ acts trivially. Then one has the so-called baby Verma module $Z(\lambda) := u(\mathfrak{g}_{[0]}) \otimes_{u(\mathfrak{b})} F_\lambda$ which has a simple head denoted by $L^0(\lambda)$, where $u(\mathfrak{g}_{[0]})$ and $u(\mathfrak{b})$ are restricted enveloping algebras of $\mathfrak{g}_{[0]}$ and $\mathfrak{b}$ respectively. Then $\{L^0(\lambda) \mid \lambda \in \Lambda\}$ is the set of representatives of restricted simple $\mathfrak{g}_{[0]}$-modules. Since $\mathfrak{g}_1$ is a restricted nilpotent ideal of $\mathfrak{g}_0$, each simple restricted $\mathfrak{g}_0$-module is also a simple $\mathfrak{g}_{[0]}$-module with trivial action of $\mathfrak{g}_1$. Moreover, each simple restricted module of $\mathfrak{g}_{[0]}$ can be extended to the one of $\mathfrak{g}_0$ with trivial action of $\mathfrak{g}_1$.

For each $\lambda \in \Lambda$, the Kac module $K(\lambda)$ is by definition the induced $\mathfrak{g}$-module $u(\mathfrak{g}) \otimes_{u(\mathfrak{g}^+)} L^0(\lambda)$, where $L^0(\lambda)$ is considered canonically as a $\mathfrak{g}^+$-module with trivial $\mathfrak{g}_1$-action. Each Kac module $K(\lambda)$ has a unique maximal submodule $J(\lambda)$ which is the sum of all proper submodules of $K(\lambda)$. Therefore, $K(\lambda)$ has a unique simple quotient denoted by $L(\lambda)$. If $L(\lambda) = K(\lambda)$, then $L(\lambda)$ is called a typical simple $\mathfrak{g}$-module and λ is called a typical weight. If $L(\lambda) \neq K(\lambda)$, then $L(\lambda)$ is called an atypical simple $\mathfrak{g}$-module, in which case λ is called an atypical weight. According to [3, Proposition 2.2], the family $\{L(\lambda) \mid \lambda \in \Lambda\}$ is the set of iso-classes of irreducible restricted $\mathfrak{g}$-modules. Moreover, by [3, Proposition 2.6], $\lambda \in \Lambda$ is atypical if and only if λ is of the form $\lambda = a\varepsilon_i + \sum\limits_{j=i+1}^{n} \varepsilon_j$ for some $a \in \{0, 1, \cdots, p-1\}$ and $i \in \{1, \cdots, n\}$.

Combining [3, Proposition 2.11] with [3, Corollary 2.16], we have the following result.

Lemma 2.1 *The following statements hold:*
(1) *There exists the following exact sequence:*

$$0 \to L((p-1)\varepsilon_n) \to K(0) \to L(0) \to 0.$$

(2) *For $\tau = t\varepsilon_i + \varepsilon_{i+1} + \cdots + \varepsilon_n$ with $t \neq 0, 1$, there exists the following exact sequence:*

$$0 \to L(\tau - \varepsilon_i) \to K(\tau) \to L(\tau) \to 0.$$

(3) *For $\tau = \sum\limits_{j=i}^{n} \varepsilon_j$ with $i > 1$, there exist the following two exact sequences:*

$$0 \to J(\tau) \to K(\tau) \to L(\tau) \to 0,$$
$$0 \to L((p-1)\varepsilon_{i-1} + \tau) \to J(\tau) \to L(0) \to 0.$$

By Lemma 2.1, we get the following main theorem on the dimensions of restricted atypical simple $\mathfrak{g}$-modules.

Theorem 2.1 *The dimensions of restricted atypical simple $\mathfrak{g}$-modules are given as follows:*
(1) $\dim L(0) = 1$.
(2) $\dim L(t\varepsilon_i + \varepsilon_{i+1} + \cdots + \varepsilon_n)$

$$= 2^n \Big(\sum_{j=1}^{p-1-t} (-1)^{j+1} \dim L^0((t+j)\varepsilon_i + \varepsilon_{i+1} + \cdots + \varepsilon_n)$$
$$+ \sum_{s=i+1}^{n} \sum_{k=1}^{p-1} (-1)^{t+k+1} \dim L^0(k\varepsilon_s + \varepsilon_{s+1} + \cdots + \varepsilon_n) + (-1)^t \Big) + (-1)^{t+1}(n-i+1).$$

Proof (1) $K(0)=u(\mathfrak{g})\bigotimes_{u(\mathfrak{g}_0)}\mathbf{1}$, where $\mathbf{1}$ is the trivial $\mathfrak{g}_0$-module. It is obvious that $J(0)=\bigoplus\limits_{i\geq 1}\bigwedge^i\mathfrak{g}\otimes\mathbf{1}$ is the unique maximal $\mathfrak{g}$-submodule of $K(0)$. Then $L(0)\cong K(0)/J(0)$ which is one-dimensional.

(2) We use induction on i and t to verify the assertion.

By Lemma 2.1(1), $\dim L((p-1)\varepsilon_n)=\dim K(0)-\dim L(0)=2^n-1$. For any $1\leq l<p-1$, if the assertion holds for $i=n$ and $t>l$, we then show that it also holds for $i=n$ and $t=l$. According to the following exact sequence by Lemma 2.1(2):

$$0\to L(l\varepsilon_n)\to K((l+1)\varepsilon_n)\to L((l+1)\varepsilon_n)\to 0,$$

we have

$$\begin{aligned}&\dim L(l\varepsilon_n)\\&=\dim K((l+1)\varepsilon_n)-\dim L((l+1)\varepsilon_n)\\&=2^n\dim L^0((l+1)\varepsilon_n)-2^n\Big(\sum_{j=1}^{p-2-l}(-1)^{j+1}\dim L^0((l+1+j)\varepsilon_n)+(-1)^{l+1}\Big)+(-1)^l\\&=2^n\Big(\sum_{j=1}^{p-1-l}(-1)^{j+1}\dim L^0((l+j)\varepsilon_n)+(-1)^l\Big)+(-1)^{l+1},\end{aligned}$$

which implies that the assertion also holds for $i=n$ and $t=l$. Therefore, we have proved that the assertion holds for $i=n$ and $1\leq t\leq p-1$.

Furthermore, for any $1\leq l'<n$, if the assertion holds for $i>l'$ and $1\leq t\leq p-1$, next we will show that it also holds for $i=l'$ and $1\leq t\leq p-1$. For that, we have the following two exact sequences by Lemma 2.1(3):

$$\begin{aligned}&0\to J(\varepsilon_{l'+1}+\varepsilon_{l'+2}+\cdots+\varepsilon_n)\to K(\varepsilon_{l'+1}+\varepsilon_{l'+2}+\cdots+\varepsilon_n)\\&\quad\to L(\varepsilon_{l'+1}+\varepsilon_{l'+2}+\cdots+\varepsilon_n)\to 0,\\&0\to L((p-1)\varepsilon_{l'}+\varepsilon_{l'+1}+\cdots+\varepsilon_n)\to J(\varepsilon_{l'+1}+\varepsilon_{l'+2}+\cdots+\varepsilon_n)\to L(0)\to 0.\end{aligned}$$

So

$$\dim K(\varepsilon_{l'+1}+\varepsilon_{l'+2}+\cdots+\varepsilon_n)=\dim J(\varepsilon_{l'+1}+\varepsilon_{l'+2}+\cdots+\varepsilon_n)+\dim L(\varepsilon_{l'+1}+\varepsilon_{l'+2}+\cdots+\varepsilon_n)$$

and

$$\begin{aligned}&\dim J(\varepsilon_{l'+1}+\varepsilon_{l'+2}+\cdots+\varepsilon_n)\\&=\dim L((p-1)\varepsilon_{l'}+\varepsilon_{l'+1}+\cdots+\varepsilon_n)+\dim L(0)\\&=\dim L((p-1)\varepsilon_{l'}+\varepsilon_{l'+1}+\cdots+\varepsilon_n)+1,\end{aligned}$$

from which

$$\begin{aligned}&\dim L((p-1)\varepsilon_{l'}+\varepsilon_{l'+1}+\cdots+\varepsilon_n)\\&=\dim K(\varepsilon_{l'+1}+\varepsilon_{l'+2}+\cdots+\varepsilon_n)-\dim L(\varepsilon_{l'+1}+\varepsilon_{l'+2}+\cdots+\varepsilon_n)-1\\&=2^n\dim L^0(\varepsilon_{l'+1}+\varepsilon_{l'+2}+\cdots+\varepsilon_n)-\dim L(\varepsilon_{l'+1}+\varepsilon_{l'+2}+\cdots+\varepsilon_n)-1.\end{aligned}$$

By the induction hypotheses,

$$\begin{aligned}
&\dim L(\varepsilon_{l'+1}+\varepsilon_{l'+2}+\cdots+\varepsilon_n)\\
&=2^n\Big(\sum_{j=1}^{p-2}(-1)^{j+1}\dim L^0((j+1)\varepsilon_{l'+1}+\varepsilon_{l'+2}+\cdots+\varepsilon_n)\\
&\quad+\sum_{s=l'+2}^{n}\sum_{k=1}^{p-1}(-1)^{k+2}\dim L^0(k\varepsilon_s+\varepsilon_{s+1}+\cdots+\varepsilon_n)-1\Big)+n-l'.
\end{aligned}$$

Then

$$\begin{aligned}
&\dim L((p-1)\varepsilon_{l'}+\varepsilon_{l'+1}+\cdots+\varepsilon_n)\\
&=2^n\dim L^0(\varepsilon_{l'+1}+\varepsilon_{l'+2}+\cdots+\varepsilon_n)-\dim L(\varepsilon_{l'+1}+\varepsilon_{l'+2}+\cdots+\varepsilon_n)-1\\
&=2^n\Big(\sum_{s=l'+1}^{n}\sum_{k=1}^{p-1}(-1)^{k+1}\dim L^0(k\varepsilon_s+\varepsilon_{s+1}+\cdots+\varepsilon_n)+(-1)^{p-1}\Big)+(-1)^p(n-l'+1)
\end{aligned}$$

which implies that the assertion holds for $i=l'$ and $t=p-1$.

For any $1\le l<p-1$, if the assertion holds for $i=l'$ and $t>l$, we then show that it also holds for $i=l'$ and $t=l$. For that, we have the following exact sequence by Lemma 2.1(2):

$$\begin{aligned}
&0\to L(l\varepsilon_{l'}+\varepsilon_{l'+1}+\cdots+\varepsilon_n)\to K((l+1)\varepsilon_{l'}+\varepsilon_{l'+1}+\cdots+\varepsilon_n)\\
&\quad\to L((l+1)\varepsilon_{l'}+\varepsilon_{l'+1}+\cdots+\varepsilon_n)\to 0.
\end{aligned}$$

So

$$\begin{aligned}
&\dim L(l\varepsilon_{l'}+\varepsilon_{l'+1}+\cdots+\varepsilon_n)\\
&=\dim K((l+1)\varepsilon_{l'}+\varepsilon_{l'+1}+\cdots+\varepsilon_n)-\dim L((l+1)\varepsilon_{l'}+\varepsilon_{l'+1}+\cdots+\varepsilon_n)\\
&=2^n\dim L^0((l+1)\varepsilon_{l'}+\varepsilon_{l'+1}+\cdots+\varepsilon_n)\\
&\quad-2^n\Big(\sum_{j=1}^{p-2-l}(-1)^{j+1}\dim L^0((l+1+j)\varepsilon_{l'}+\varepsilon_{l'+1}+\cdots+\varepsilon_n)\\
&\quad+\sum_{s=l'+1}^{n}\sum_{k=1}^{p-1}(-1)^{l+k}\dim L^0(k\varepsilon_s+\varepsilon_{s+1}+\cdots+\varepsilon_n)+(-1)^{l+1}\Big)-(-1)^l(n-l'+1)\\
&=2^n\Big(\sum_{j=1}^{p-1-l}(-1)^{j+1}\dim L^0((l+j)\varepsilon_{l'}+\varepsilon_{l'+1}+\cdots+\varepsilon_n)\\
&\quad+\sum_{s=l'+1}^{n}\sum_{k=1}^{p-1}(-1)^{l+k+1}\dim L^0(k\varepsilon_s+\varepsilon_{s+1}+\cdots+\varepsilon_n)+(-1)^{l}\Big)+(-1)^{l+1}(n-l'+1),
\end{aligned}$$

which implies that the assertion also holds for $i=l'$ and $t=l$.

Summing up, by the induction principle, we complete the proof.

3 Blocks of $u(\mathfrak{g})$

For each $\lambda\in\Lambda$, we denote by $Q(\lambda)$ the projective cover of $L(\lambda)$ in the $u(\mathfrak{g})$-module category, $Q^0(\lambda)$ the projective cover of $L^0(\lambda)$ in the $u(\mathfrak{g}_{[0]})$-module category and $V^0(\lambda)$ the baby Verma module in the $u(\mathfrak{g}_{[0]})$-module category. We conclude this paper with the following result.

Theorem 3.1 *Let $\mathfrak{g} = W(n)$ with $n > 2$. Then $u(\mathfrak{g})$ is of one block.*

Proof For any $\tau, \eta \in \Lambda$, by [3, Proposition 3.8],

$$[Q(\tau) : L(\eta)] = p^{s-n}2^t \sum_{\upsilon,\, \upsilon_1 \in \Lambda} [K(\upsilon_1 - \mathcal{E}) : L(\tau)][Q^0(\upsilon_1) : V^0(\sigma(\upsilon) - \mathcal{E})] \times \sum_{\omega,\, \omega_1 \in \Lambda} [V^0(\omega) : L^0(\omega_1)][K(\omega_1) : L(\eta)], \tag{3.1}$$

where $\mathcal{E} = \sum\limits_{j=1}^{n} \varepsilon_j$, s, t are defined as in [3, Lemma 3.5], and σ is a bijection on Λ which maps any $\tau \in \Lambda$ to $\tau + \mathcal{E} + 2(p-1)\rho \in \Lambda$, where ρ is the half-sum of all positive roots of $\mathfrak{g}_{[0]}$.

Note that there appears a nonzero summand corresponding to $\upsilon_1 = \tau + \mathcal{E}$, $\upsilon = \tau + \mathcal{E} - 2(p-1)\rho$, and $\omega = \omega_1 = \eta$ in the right-hand side of (3.1). Therefore, $[Q(\tau) : L(\eta)] \neq 0$, $\forall \tau, \eta \in \Lambda$. It follows that all $L(\eta)$ $(\eta \in \Lambda)$ belong to the same block. We complete the proof.

Remark 3.1 The result above on the number of blocks in the category of restricted representations is significantly different from the one in the case of the field of complex numbers. In the later case, one can know from [4] that there are infinitely many blocks.

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