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Reflection of Shock Fronts in a van der Waals Fluid*

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Abstract In this paper, the reflection phenomenon of a vapor shock front (both sides of the front are in the vapor phase) in a van der Waals fluid is considered. Both the 1-dimensional case and the multidimensional case are investigated. The authors find that under certain conditions, the reflected wave can be a single shock, or a single subsonic phase boundary, or one weak shock together with one subsonic phase boundary, which depends on the strength of the incident shock. This is different from the known result for the reflection of shock fronts in a gas dynamical system due to Chen in 1989.

Keywords Shock, Subsonic phase boundary, Reflection, Boundary value problem
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1 Introduction

The phase transition is an important phenomenon in physics, mechanics and fluid dynamics. For a fluid or a material, if the density-pressure relation or the stress-strain function is non-monotonic, multiple phases coexist in general, such as in a van der Waals fluid and elastic-plastic materials. There has been rich literature devoted to the study of the existence and the stability of phase transitions in one space variable (cf. [8, 12, 16–17] and the references therein). In [8], Slemrod and Fan generally reviewed the works on the van der Waals fluid and phase transitions. For multidimensional phase transitions, by mode analysis, Benzoni-Gavage studied the linear stability of subsonic phase transitions in a van der Waals fluid under the capillarity admissibility criterion in [2] and obtained a sufficient condition on the uniform stability of subsonic phase transitions under the viscosity-capillarity criterion in [3]. Recently, Wang and Xin [18] studied the uniform stability and obtained the local existence of single multidimensional subsonic phase transitions in a van der Waals fluid under the viscosity-capillarity criterion. Considering the general conservation laws, Freistühler [9] studied the stability of the under-compressive shock fronts in multidimensional spaces under Majda's frame work (cf. [13–14]). The author in [19] proved the local existence of solutions consisting of one subsonic phase transition and one shock wave issuing from an initial discontinuity in several space variables.

The purpose of this paper is to investigate the reflection of shock fronts in a van der Waals fluid. So far as we know, even in the 1-dimensional space, to establish a theory for the general Riemann problem in a van der Waals fluid, as the one for strictly convex conservation laws (cf.

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[11]), is almost impossible due to the non-uniqueness of the solution to the Riemann problem arising in the meta-stable region (cf. [8, 16]). Hence it is hard to consider the reflection of shock fronts in a van der Waals fluid in general. That is why a new model (cf. [6]) is introduced for the retrograde fluid and it works well in the study of the shock reflection problem (cf. [7]) when the state behind the reflected shock enters the meta-stable region. But the analytic theory of this kind of model is not quite complete. Here we expect to get some insights into the shock reflection problem when the states are away from the meta-stable region via the model of the van der Waals fluid.

In this paper, we are concerned with the reflection of a shock front, both sides of which are in the stable vapor phase, against a rigid wall. We study both of the cases when the rigid wall is a plane and a non-flat surface. For the first case, both sides of the incident shock are constants, which is the 1-dimensional problem essentially, and we obtain that under certain conditions, which exclude the situation that the states behind the reflection enter the meta-stable region, the reflection phenomena are different when the strength of the incident shock varies. More precisely, the reflection can be a single shock, or a single subsonic phase boundary, or one weak shock together with a subsonic phase boundary. We shall prove that the reflection pattern is unique by solving algebraic equations. For the case that the boundary is not flat, we have two kinds of reflection pattern which also depend on the strength of the incident shock. The existence of reflection waves in this multidimentsional case will be obtained by developing the arguments of Chen [4] and Métivier [15]. Comparing the problem of double shocks in [15] and the problem of reflection shocks in gas dynamics studied in [4], the main novelty and difficulty of our problem is that it involves two free boundaries, the phase boundary and the shock front, and the physical boundary which is characteristic. This requires us to modify the arguments of [4, 15] in order to study the stability estimates for the linearized problem. Another important point is that the stability condition on the edge of the dihedral is different from the one for the double shock problem in [15]. As we showed in [19], this condition holds for sufficiently weak shock which coin-sides with our reflection pattern.

The remainder of this paper is arranged as follows. In the rest part of this section, we first recall the admissible criterion for subsonic phase boundaries and formulate our problem. In Section 2, we study the reflection on a plane wall. We shall prove that under certain conditions the reflected wave can be a single shock wave, or a single phase boundary, or one shock together with one phase boundary. In Section 3, we formulate the problem of high-dimensional reflection, and give the assumptions and the main results. We establish the linear estimate for the linearized problem in Section 4, and the nonlinear problem is studied in Section 5.

1.1 Admissibility criterion for subsonic phase transitions

For simplicity, we shall only study the problem in two space variables, i.e., $x = (x_1, x_2) \in \mathbb{R}^2$. It is easy to carry out the same discussion for the problem in higher dimensional spaces.

For a compressible inviscid isentropic fluid, the following well-known Euler equations:

$$\partial_t \begin{pmatrix} \rho \\ \rho u \\ \rho v \end{pmatrix} + \partial_{x_1} \begin{pmatrix} \rho u \\ \rho u^2 + p(\rho) \\ \rho u v \end{pmatrix} + \partial_{x_2} \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p(\rho) \end{pmatrix} = 0 \tag{1.1}$$

represent the conservation of mass and momentum, where ρ and (u, v) are the density and the velocity of the fluid, respectively. Denote $U = (\rho, u, v)^{\mathrm{T}}$,

$$F_0(U) = \begin{pmatrix} \rho \\ \rho u \\ \rho v \end{pmatrix}, \quad F_1(U) = \begin{pmatrix} \rho u \\ \rho u^2 + p(\rho) \\ \rho uv \end{pmatrix}, \quad F_2(U) = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p(\rho) \end{pmatrix}$$

and

$$A_1(U) = (F_0'(U))^{-1} F_1'(U) = \begin{pmatrix} u & \rho & 0 \\ \dfrac{c^2}{\rho} & u & 0 \\ 0 & 0 & u \end{pmatrix},$$

$$A_2(U) = (F_0'(U))^{-1} F_2'(U) = \begin{pmatrix} v & 0 & \rho \\ 0 & v & 0 \\ \dfrac{c^2}{\rho} & 0 & v \end{pmatrix},$$

where $c = (p'(\rho))^{\frac{1}{2}}$ is the sound speed. For smooth solutions, the system (1.1) is equivalent to

$$\partial_t U + A_1(U)\partial_{x_1} U + A_2(U)\partial_{x_2} U = 0.$$

In the following discussion, the notations of the density and the specific volume $\tau \equiv \rho^{-1}$ shall be used simultaneously. In the van der Waals fluid, the pressure law $P(\tau) = p(\frac{1}{\tau})$ is given by

$$P(\tau) = \frac{RT}{\tau - b} - \frac{a}{\tau^2}, \quad \tau > b, \tag{1.2}$$

where T denotes the temperature assumed to be a positive constant, R is the perfect gas constant, and a, b are positive constants. When the temperature $\frac{a}{4bR} < T < \frac{8a}{27bR}$ is fixed, there are $\tau_* < \tau^*$, such that

$$\begin{cases} P'(\tau) < 0, & \text{if } b < \tau < \tau_* \text{ or } \tau > \tau^*, \\ P'(\tau) > 0, & \text{if } \tau_* < \tau < \tau^*. \end{cases} \tag{1.3}$$

The state of $\tau \in (b, \tau_*)$ represents the liquid phase while that of $\tau \in (\tau^*, +\infty)$ is the vapor phase. Generally, these two phases are likely to coexist and one may observe the propagation of a phase boundary.

As usual, the Maxwell equilibrium $\{\tau_m, \tau_M\}$ of a phase transition is defined by the equal area rule:

$$P(\tau_m) = P(\tau_M), \quad \int_{\tau_m}^{\tau_M} (P(\tau) - P(\tau_m))\mathrm{d}\tau = 0, \tag{1.4}$$

and $\tau_m < \tau_*$, $\tau_M > \tau^*$. It is obvious that there is a unique point $\tau_1 > \tau_M$ at which the tangent to the graph of $p = P(\tau)$ passes through τ_m (cf. [3, Figure 3]). Denote

$$j_0^2 = -P'(\tau_1), \tag{1.5}$$

which equals to $\frac{P(\tau_m) - P(\tau_1)}{\tau_1 - \tau_m}$.

Let us recall the concept of subsonic phase transitions. A piecewise smooth function

$$U(t,x) = \begin{cases} U_+(t,x) & \text{for } x_1 > \psi(t,x_2), \\ U_-(t,x) & \text{for } x_1 < \psi(t,x_2), \end{cases} \tag{1.6}$$

satisfying that $U_\pm \in C^1\{\pm(x_1-\psi(t,x_2))>0\}$ belong to different phases with $\psi \in C^2$ being the phase boundary, is said to be a subsonic phase transition, if it satisfies the system (1.1) in the regions where $U(t,x)$ is smooth and satisfies the following Rankine-Hugoniot condition:

$$\psi_t[F_0(U)]-[F_1(U)]+\psi_{x_2}[F_2(U)]=0, \quad \text{on } \{x_1=\psi(t,x_2)\}, \tag{1.7}$$

where $[\cdot]$ denotes the jump of a function on the phase boundary $\{x_1=\psi(t,x_2)\}$. Moreover, the Mach numbers satisfy

$$M_\pm=\frac{1}{c_\pm}\Big|\frac{u_\pm-\psi_{x_2}v_\pm-\psi_t}{(1+\psi_{x_2}^2)^{\frac12}}\Big|<1, \tag{1.8}$$

where $c_\pm=(p'(\rho_\pm))^{\frac12}$ are the sound speeds. Due to (1.8), the Lax entropy condition is violated at $\{x_1=\psi(t,x_2)\}$. More precisely, the Rankine-Hugoniot condition (1.7) is not an enough one on the phase boundary to guarantee the well-posedness of the problem. Additional boundary conditions are needed to select proper candidate which is admissible in physics. In [17], Slemrod introduced the viscosity-capillarity admissibility criterion to determine the subsonic phase boundaries in one space variable, i.e., a phase boundary is called the viscosity-capillarity admissible if the states on both sides of the phase boundary can be connected by a travelling wave in a system by adding viscosity and capillarity terms to (1.1). This viscosity-capillarity criterion was studied recently in [3] for the multidimensional subsonic phase boundaries. More precisely, the phase boundary (1.6) satisfies the viscosity-capillarity admissibility criterion, if on the boundary $\{x_1=\psi(t,x_2)\}$ the following relation:

$$\Big[e'(\rho)+\frac{(u-\psi_{x_2}v-\psi_t)^2}{2(1+\psi_{x_2}^2)}\Big]=-\nu a(j,\nu), \quad \text{on } \{x_1=\psi(t,x_2)\} \tag{1.9}$$

holds, where ν is the ratio of the viscosity coefficient comparing with the square root of the capillarity coefficient, $e(\rho)=\rho E(\rho)$ is the free energy per unit volume with $E(\rho)$ being the specific free energy and $d_\rho E(\rho)=\frac{p(\rho)}{\rho^2}$,

$$j=\frac{\rho_\pm(u_\pm-\psi_{x_2}v_\pm-\psi_t)}{(1+\psi_{x_2}^2)^{\frac12}} \tag{1.10}$$

is the mass transfer flux across the phase boundary, which is assumed to be non-zero, and

$$a(j,\nu)=j\int_{-\infty}^{+\infty}\tau'^2(\xi;j,\nu)\mathrm{d}\xi, \tag{1.11}$$

where $\tau(\xi;j,\nu)$ is the viscosity-capillarity profile satisfying

$$\begin{cases}\tau''=\nu j\tau'+\pi-P(\tau)-j^2\tau,\\ \lim\limits_{\xi\to-\infty}\tau=\frac{1}{\rho_-}\Big|_{x_1=\psi}, \quad \lim\limits_{\xi\to+\infty}\tau=\frac{1}{\rho_+}\Big|_{x_1=\psi}\end{cases} \tag{1.12}$$

with τ', τ'' being the first and second order derivatives of τ with respect to ξ, respectively, $\pi=p(\rho_\pm)+\frac{j^2}{\rho_\pm}$ valued at $\{x_1=\psi\}$. In [3], Benzoni-Gavage proved the existence of the viscosity-capillarity profile $\tau(\xi;j,\nu)$ to (1.12) when $0<\nu\le\nu_0$ for some small $\nu_0>0$ and $0<j^2\le j_0^2$ with j_0 being given in (1.5). Moreover, Benzoni-Gavage showed that $U_\pm|_{x_1=\psi(t,x_2)}$ depend smoothly only on (j,ν), and as a consequence, $a(j,\nu)$ is a smooth function of (j,ν) in $\{0<j^2\le j_0^2,\ 0<\nu\le\nu_0\}$. As in [3], we call the subsonic phase transition (1.7) ν-admissible, if it satisfies (1.1) in the smooth regions and satisfies (1.8), (1.10) on the discontinuity $\{x_1=\psi(t,x_2)\}$.

1.2 The problem of reflection

Let $\Sigma = \{x_1 = \varphi_0(x_2)\}$ be a physical boundary in $\mathbb{R}^2$. For simplicity, suppose that $\varphi_0 \in C^\infty$ and $\varphi_0(0) = \varphi_0'(0) = 0$. The outside domain Σ, $\{x_1 > \varphi_0(x_2)\}$, is filled with a van der Waals fluid. On Σ, the normal velocity of the fluid vanishes, so the boundary condition on Σ can be written as

$$u - \varphi_0' v = 0. \tag{1.13}$$

Let $\mathscr{S} = \{x_1 = Vt\}$ be a planar shock front moving towards Σ when $t < 0$, and on both sides of $\mathscr{S}$ the flow fields are constants. We denote the incident shock by

$$U(t,x) = \begin{cases} U_A = (\rho_A, u_A, v_A), & x_1 < Vt, \\ U_B = (\rho_B, u_B, v_B), & x_1 > Vt, \end{cases} \tag{1.14}$$

where we assume $\tau_A, \tau_B \in (\tau_M, +\infty)$, $\tau_A > \tau_B$, namely, the states on both sides of the shock front are in the stable vapor phase. Due to (1.13), it is easy to see that the incident shock satisfies

$$u_A = 0 \quad \text{and} \quad v_A = v_B = 0. \tag{1.15}$$

At $t = 0$, $\mathscr{S}$ meets Σ at the point $(0, 0)$. Obviously, before the shock front $\mathscr{S}$ intersects with the surface Σ, the velocity of $\mathscr{S}$ and the flow fields on both sides of $\mathscr{S}$ remain constants. Therefore, the intersection of $\mathscr{S}$ with Σ is

$$\Gamma = \Big\{x_1 = \varphi_0(x_2),\ t = \frac{1}{V}\varphi_0(x_2)\Big\}.$$

For $t > \frac{1}{V}\varphi_0(x_2)$, the normal velocity of the fluid at the boundary Σ should vanish, which is incompatible with the state behind the shock front $\mathscr{S}$, so there will be some waves issuing from the curve $\{x_1 = \varphi_0(x_2), t = \frac{1}{V}\varphi_0(x_2)\}$ when $t > \frac{1}{V}\varphi_0(x_2)$. The purpose of this paper is to study such reflected waves. When the domain $\{x_1 > \varphi_0(x_2)\}$ is filled with a single-phased fluid, whose state function $p = p(\rho)$ is strictly increasing, Chen [4] established the existence of a reflected shock front after the planar shock hits the boundary. Due to the possibility of coexisting multiple phases, we shall see that there will be several interesting phenomena appearing after the reflection of a shock front in a van der Waals fluid.

2 Reflection in 1-Dimensional Problem

Let us begin with the simple 1-dimensional case where the physical boundary is a plane, namely, $\varphi_0(x_2) = 0$. Essentially, this is a 1-dimensional problem.

For two fields (ρ_1, u_1), (ρ_2, u_2) satisfying

$$(\tau_1 - \tau_2)(P(\tau_1) - P(\tau_2)) < 0,$$

we denote

$$\langle \tau_1, \tau_2 \rangle = \sqrt{(\tau_1 - \tau_2)(P(\tau_2) - P(\tau_1))}.$$

It is easy to see from the Rankine-Hugoniot conditions that if the two fields can be connected by a single discontinuity, then

$$\langle \tau_1, \tau_2 \rangle = |u_1 - u_2|$$

is the strength of the discontinuity.

The following is a basic fact of the reflection problem.

Theorem 2.1 *The sum of the strength of all the reflected discontinuities equals the strength of the incident shock. Namely, after the incident shock* (1.14) *meets the wall* Σ*, we suppose that there are* N *discontinuities,* R_k $(k = 1, \cdots, N)$*, i.e.,*

$$(\rho, u) = \begin{cases} (\rho_k, u_k), & x_1 > \sigma_k t, \\ (\rho_{k+1}, u_{k+1}), & x_1 < \sigma_k t, \end{cases} \qquad k = 1, \cdots, N,$$

where $\sigma_1 > \sigma_2 > \cdots > \sigma_N > 0$*. Then we have*

$$\sum_{k=1}^{N} \langle \tau_k, \tau_{k+1} \rangle = \langle \tau_A, \tau_B \rangle. \tag{2.1}$$

The relation (2.1) can be easily deduced from the Rankine-Hugoniot conditions and the facts

$$(\tau_1, u_1) = (\tau_B, u_B), \quad u_{N+1} = u_A = 0.$$

As in [3], we denote by $(\tau_l(j,\nu), \tau_r(j,\nu))$ the unique pair satisfying $\tau_l \in (b, \tau_*)$, $\tau_r \in (\tau^*, +\infty)$, which can be connected by a ν-admissible phase boundary for a fixed mass transfer flux j. For the van der Waals fluid, we propose the following hypothesis. There is a $\overline{j}$ $(0 < \overline{j}^2 \le j_0^2)$, such that the ν-admissible phase transition $(\tau_l(\overline{j},\nu), \tau_r(\overline{j},\nu))$ satisfies the following:

(H1) The strength of the phase transition $(\tau_l(\overline{j},\nu), \tau_r(\overline{j},\nu))$ is large enough, such that the estimate

$$\langle \tau_l(\overline{j},\nu), \tau_r(\overline{j},\nu) \rangle > \langle \tau_r(\overline{j},\nu), \tau^* \rangle \tag{2.2}$$

holds.

(H2) The state function (1.2) is convex at $\tau = \tau_r(\overline{j},\nu)$, i.e.,

$$P''(\tau_r(\overline{j},\nu)) > 0. \tag{2.3}$$

Remark 2.1 The hypothesis (H1) can help us exclude the situation that the state behind the reflection enters the meta-stable region, which will break the symmetry of planar wave (cf. [7]). In fact, if the strength of the incident shock (1.14) is the closed $\langle \tau_l(\overline{j},\nu), \tau_r(\overline{j},\nu) \rangle$ and the specific volume behind the shock is specified as $\tau_r(\overline{j},\nu)$, then the state behind the reflection can not be in the interval $(\tau^*, \tau_r(\overline{j},\nu))$, since the total strength of the reflected discontinuities is smaller than that of the incident shock.

Due to the continuity of τ_r, τ_l with respect to j (cf. [3]), the estimates (2.2)–(2.3) can be extended to

$$\langle \tau_l(j,\nu), \tau_r(j,\nu) \rangle + \langle \tau_r(j,\nu), \tau_r(\overline{j},\nu) \rangle > \langle \tau_r(\overline{j},\nu), \tau^* \rangle, \tag{2.4}$$

$$P''(\tau_r(j,\nu)) > 0 \tag{2.5}$$

for all j satisfying $\underline{j}^2 \le j^2 \le \overline{j}^2$ with a $\underline{j}^2 < \overline{j}^2$.

The following example shows that the hypotheses (H1)–(H2) are valid when the temperature is close to the critical value $T_c = \frac{8a}{27bR}$, i.e., the assumptions (2.2)–(2.3) are valid.

Example 2.1 When $T = T_c = \frac{8a}{27bR}$, the state function $P(\tau)$ is decreasing in τ. Obviously, the estimate (2.2) holds. When $T = T_c$ and $\nu = 0$, by straightforward computation, we find that $\tau_r(j_0, 0) \doteq 10.2024b$ is greater than the inflection point $\tau \doteq 6.6338b$, which implies

$P''(\tau_r(j_0,0)) > 0$. Hence the conditions (2.2)–(2.3) are valid when the temperature T is close to $\frac{8a}{27bR}$ and ν is small.

As we mentioned in Remark 2.1, we set that the state behind the incident shock (1.14) satisfies

$$\tau_B = \tau_r(\overline{j},\nu). \tag{2.6}$$

The result of 1-dimensional reflection is stated by the following theorem.

Theorem 2.2 (i) *When*

$$\langle\tau_A,\tau_B\rangle = \langle\tau_l(\overline{j},\nu),\tau_r(\overline{j},\nu)\rangle, \tag{2.7}$$

the reflection is a phase boundary.

(ii) *When*

$$\langle\tau_A,\tau_B\rangle < \langle\tau_l(\overline{j},\nu),\tau_r(\overline{j},\nu)\rangle, \tag{2.8}$$

$$\chi(\tau_A,\tau_B) > \chi(\tau_l(\underline{j},\nu),\tau_r(\underline{j},\nu)) + \chi(\tau_r(\underline{j},\nu),\tau_r(\overline{j},\nu)), \tag{2.9}$$

the reflection is one shock wave and one subsonic phase boundary.

Proof First we consider the case when there is only one discontinuity in the reflection. Denote by (ρ_+,u_+,v_+) and (ρ_-,u_-,v_-) the flow fields ahead of and behind the reflected discontinuity, respectively, and σ the speed of the discontinuity. We have that

$$\tau_+ = \tau_B, \quad u_+ = u_B, \quad u_- = 0, \quad v_- = v_+ = 0 \tag{2.10}$$

and the Rankine-Hugoniot conditions

$$\begin{cases} -\dfrac{\sigma}{\tau_-} = \dfrac{u_+ - \sigma}{\tau_+}, \\ P(\tau_-) + \dfrac{\sigma^2}{\tau_-} = P(\tau_+) + \dfrac{(u_+ - \sigma)^2}{\tau_+}. \end{cases} \tag{2.11}$$

From Theorem 2.1, we get

$$\chi(\tau_B,\tau_-) = \chi(\tau_B,\tau_A). \tag{2.12}$$

Set $F(\tau) = \chi(\tau_B,\tau) - \chi(\tau_B,\tau_A)$. We have $F'(\tau) < 0$ for $\tau \in (b,\tau_*] \cup [\tau^*,\tau_B)$.

(i) When (2.9) holds, we get

$$F(\tau_B) < 0, \quad F(\tau_I) > 0.$$

Therefore, one can find a unique τ_- in the interval (τ_I,τ_B) satisfying (2.12). It is easy to see that τ_- and τ_+ can be connected by a shock front. There would be another $\tau_- \in (b,\tau_*)$ satisfying (2.12), which implies that the reflection could also be a subsonic phase boundary. Due to the uniqueness of (τ_l,τ_r) with respect to j and the assumption (2.2), we find that this situation can not be true.

(ii) When (2.7) holds, obviously, $\tau_- = \tau_r$ is the unique point satisfying (2.12) for $\tau_- \in (b,\tau_*] \cup [\tau^*,\tau_B)$. Therefore, the reflection is a ν-admissible subsonic phase boundary with the mass transfer flux $\overline{j}$.

(iii) When (2.8)–(2.9) hold, one can find a unique $\tau_- \in (\tau_l(\overline{j},\nu),\tau_*)$, which implies that the reflection could be a subsonic phase boundary. Due to the uniqueness of $(\tau_l(j,\nu),\tau_r(j,\nu))$ (cf. [3]), τ_- should be $\tau_l(\overline{j},\nu)$ which is impossible by using (2.9). Therefore, in this case, there should appear more than one discontinuity in the reflection. By analyzing the characteristic, we find that the reflected wave should consist of one subsonic phase boundary and one shock front. The phase boundary is behind the shock front. Denote by $(\rho_\pm,u_\pm,v_\pm)$ the flow fields ahead of and behind the phase boundary, σ_1 and σ_2 the speeds of the phase boundary and the reflected shock front, respectively. Noting that the flow fields ahead of and behind the shock are (ρ_B,u_B,v_B) and (ρ_+,u_+,v_+), respectively, we have

$$u_- = 0, \quad v_- = v_+ = 0, \tag{2.13}$$

$$0 < \sigma_1 < \sqrt{p'(\rho_-)}, \tag{2.14}$$

$$u_+ < \sigma_1 < \sigma_2 < u_+ + \sqrt{p'(\rho_+)}, \tag{2.15}$$

$$\sigma_2 > u_B + \sqrt{p'(\rho_B)}. \tag{2.16}$$

Denote by

$$j_1 = -\frac{\sigma_1}{\tau_-} = \frac{u_+ - \sigma_1}{\tau_+} \quad \text{and} \quad j_2 = \frac{u_+ - \sigma_2}{\tau_+} = \frac{u_B - \sigma_2}{\tau_B} \tag{2.17}$$

the mass transfer fluxes acrossing the phase boundary and the shock front, respectively. To determine the reflected waves, we need to find $(\tau_+,\tau_-,\sigma_1,\sigma_2,u_+)$, such that (2.13)–(2.17) and the following equations:

$$\begin{cases} P(\tau_-) + j_1^2\tau_- = P(\tau_+) + j_1^2\tau_+, \\ (e'(\rho_-) - e'(\rho_+)) + \dfrac{j_1^2}{2}(\tau_-^2 - \tau_+^2) = -\nu a(j_1,\nu), \\ P(\tau_+) + j_2^2\tau_+ = P(\tau_B) + j_2^2\tau_B \end{cases} \tag{2.18}$$

hold. From Theorem 2.1, we get

$$\langle \tau_-,\tau_+\rangle + \langle \tau_B,\tau_+\rangle = \langle \tau_A,\tau_B\rangle. \tag{2.19}$$

Regarding τ_+ and τ_- as functions of j_1 and setting

$$f(j_1) = \langle \tau_-,\tau_+\rangle + \langle \tau_B,\tau_+\rangle - \langle \tau_A,\tau_B\rangle,$$

we have

$$\begin{aligned} f'(j_1) = &-\frac{1}{\langle \tau_-,\tau_+\rangle}([P'(\tau)\partial_{j_1}\tau][\tau] + [P(\tau)][\partial_{j_1}\tau]) \\ &+ \frac{P'(\tau_+)\partial_{j_1}\tau_+(\tau_B-\tau_+) - (P(\tau_+)-P(\tau_B))\partial_{j_1}\tau_+}{2\sqrt{(P(\tau_+)-P(\tau_B))(\tau_B-\tau_+)}}. \end{aligned}$$

According to the results in [2], we have

$$\partial_{j_1}\tau_+ = -j_1\frac{\tau_- - \tau_+}{C_+^2} + O(\nu) \quad \text{and} \quad \partial_{j_1}\tau_- = j_1\frac{\tau_- - \tau_+}{C_-^2} + O(\nu), \tag{2.20}$$

where $C_\pm^2 = -P'(\tau_\pm) - j_1^2$. Substituting (2.20) into $f'(j_1)$ and noticing $j_1 < 0$, we get $f'(j_1) < 0$ when ν is sufficiently small. It is easy to know that $f(-|\underline{j}|) < 0$ and $f(-|\overline{j}|) > 0$. Thus we can find a unique $j_1 \in (-|\overline{j}|, -|\underline{j}|)$ satisfying $f(j_1) = 0$ which gives τ_+ and τ_-. Therefore, we can determine u_+, σ_1 and σ_2 from (2.17).

3 Reflection in Multidimensional Spaces

In this section, we consider the case when a shock hits a curved rigid wall which is a perturbation of the planar case studied in Section 2 near the origin.

As in Section 2, here we assume that (2.3)–(2.4) are valid and the incident shock (1.14) satisfies (2.6). We have the following main results of this paper.

Theorem 3.1 (i) *When*

$$\chi(\tau_A, \tau_B) < \chi(\tau_B, \tau_I), \tag{3.1}$$

the reflection is a shock wave.

(ii) *When*

$$\chi(\tau_A, \tau_B) < \chi(\tau_l(\overline{j}, \nu), \tau_r(\overline{j}, \nu)), \tag{3.2}$$

$$\chi(\tau_A, \tau_B) > \chi(\tau_l(\underline{j}, \nu), \tau_r(\underline{j}, \nu)) + \chi(\tau_r(\underline{j}, \nu), \tau_r(\overline{j}, \nu)), \tag{3.3}$$

the reflection is one shock wave and one phase transition.

As we can see from this theorem, in this multidimensional reflection problem, we do not have the situation that the reflection wave contains only one subsonic phase boundary as in Theorem 2.2(ii). To explain this difference, let us first give the following result analogous to Theorem 2.1.

Theorem 3.2 *Suppose that after the incident shock* (1.14) *meets the wall* Σ*, there are* N *discontinuities* R_k $(k = 1, \cdots, N)$ *in the reflection which are denoted by*

$$U(t,x) = \begin{cases} (\rho_k, u_k, v_k), & x_1 > \varphi_k(t, x_2), \\ (\rho_{k+1}, u_{k+1}, v_{k+1}), & x_1 < \varphi_k(t, x_2), \end{cases}$$

satisfying

$$\partial_t \varphi_1 > \partial_t \varphi_2 > \cdots > \partial_t \varphi_N > 0,$$

and then on $\Gamma = \{x_1 = \varphi_0(x_2), t = \frac{1}{V}\varphi_0(x_2)\}$*, we have*

$$\chi(\tau_A, \tau_B) = \sum_{k=1}^{N} \chi(\tau_k, \tau_{k+1}) \sqrt{1 + \varphi_0'^2}. \tag{3.4}$$

This result can be obtained by computing the Rankine-Hugoniot condition as done in Theorem 2.1, and we omit the details here.

Remark 3.1 As a consequence of Theorem 3.2, we can see that in the multidimensional case the reflected wave can not be a single phase boundary. In fact, if the reflected wave contains only one phase boundary and we denote by (ρ_+, u_+, v_+) and (ρ_-, u_-, v_-) the flow fields ahead of and behind this phase boundary, then from Theorem 3.2, we get

$$\tau_+ = \tau_B = \tau_r(\overline{j}, \nu), \tag{3.5}$$

$$\chi(\tau_+, \tau_-) \sqrt{1 + \varphi_0'^2} = \chi(\tau_A, \tau_B). \tag{3.6}$$

Due to the uniqueness of $(\tau_l(j, \nu), \tau(j, \nu))$ with respect to j (cf. [3]), we get $\tau_- = \tau_l(\overline{j}, \nu)$, which implies that the left-hand side of (3.6) is not a constant while the right-hand side is a constant. Therefore, the reflection can not be a single phase boundary.

The case that the reflection wave is only a shock front was studied by Chen [4] already. So we shall focus on the problem that the reflected wave contains one shock front and one subsonic phase boundary. Suppose that (3.2)–(3.3) hold. By continuity, there is a neighborhood $\mathscr{O}$ of the origin $(t,x)=(0,0)$ such that

$$\chi(\tau_A,\tau_B)<\chi(\tau_l(\bar{j},\nu),\tau_r(\bar{j},\nu))\sqrt{1+\varphi_0'^2}, \tag{3.7}$$

$$\chi(\tau_A,\tau_B)>(\chi(\tau_l(\underline{j},\nu),\tau_r(\underline{j},\nu))+\chi(\tau_r(\underline{j},\nu),\tau_r(\bar{j},\nu)))\sqrt{1+\varphi_0'^2} \tag{3.8}$$

are satisfied in $\mathscr{O}$.

By carrying out the same calculation as in the proof of Theorem 2.2(iii), we see that if we froze the data of the fluid field behind the incident shock (1.14) at any fixed point of Γ, there should appear one planar shock and one planar phase boundary in the reflection. Inspired by this observation, let us construct the reflection wave in this 2-dimensional case containing one shock and one phase boundary as follows:

$$U(t,x)=\begin{cases}(\rho_-,u_-,v_-), & \varphi_0(x_2)<x_1<\varphi_1(t,x_2),\\ (\rho_+,u_+,v_+), & \varphi_1(t,x_2)<x_1<\varphi_2(t,x_2),\\ (\rho_B,u_B,v_B), & x_1>\varphi_2(t,x_2)\end{cases}\quad \text{for } t>\frac{1}{V}\varphi_0(x_2),$$

where $\Gamma_1=\{x_1=\varphi_2(t,x_2)\}$ and $\Gamma_2=\{x_1=\varphi_1(t,x_2)\}$ are the shock front and the subsonic phase boundary, respectively, satisfying

$$\varphi_1\Big(\frac{\varphi_0(x_2)}{V},x_2\Big)=\varphi_2\Big(\frac{\varphi_0(x_2)}{V},x_2\Big)=\varphi_0(x_2).$$

Therefore, the unknowns $(\rho_\pm,u_\pm,v_\pm)$ and (φ_1,φ_2) satisfy the following problem:

$$\begin{cases}\partial_t U_\pm+A_1(U_\pm)\partial_{x_1}U_\pm+A_2(U_\pm)\partial_{x_2}U_\pm=0, & \text{in } G_\pm,\\ u_- -\varphi_0' v_-=0, & \text{on } \Gamma_0,\\ \partial_t\varphi_1[F_0(U)]_1-[F_1(U)]_1+\partial_{x_2}\varphi_1[F_2(U)]_1=0, & \text{on } \Gamma_1,\\ \Big[e'(\rho)+\dfrac{(u-\partial_{x_2}\varphi_1 v-\partial_t\varphi_1)^2}{2(1+(\partial_{x_2}\varphi_1)^2)}\Big]_1=-\nu a(j,\nu), & \text{on } \Gamma_1,\\ \partial_t\varphi_2[F_0(U)]_2-[F_1(U)]_2+\partial_{x_2}\varphi_2[F_2(U)]_2=0, & \text{on } \Gamma_2,\end{cases} \tag{3.9}$$

where $G_-=\{(t,x)\mid \varphi_0(x_2)<x_1<\varphi_1(t,x_2),\ t>0\}$, $G_+=\{(t,x)\mid \varphi_1(t,x_2)<x_1<\varphi_2(t,x_2),\ t>0\}$, $\Gamma_0=\{(t,x)\mid x_1=\varphi_0(x_2),\ t>0\}$, $[\cdot]_i$ $(i=1,2)$ denotes the jump of a function on Γ_i $(i=1,2)$, and the notations in the 4th equation are defined in the same way as in (1.10)–(1.12).

Remark 3.2 In the above problem all functions are defined on $\{t>0\}$ instead of $\big\{t>\frac{\varphi_0(x_2)}{V}\big\}$, this is just for simplicity of presentation since one can replace t by $t-\frac{\varphi_0(x_2)}{V}$ without changing the discussion by noticing that $\widetilde{t}=t-\frac{\varphi_0(x_2)}{V}$ is still the time direction.

As we did in the proof of Theorem 2.2(iii), we shall show that the data of $(U_\pm,\varphi_1,\varphi_2)$ on $\Gamma=\{t=0,x_1=\varphi_0(x_2)\}$ satisfy the corresponding boundary condition on Γ_i $(i=0,1,2)$ and they are uniquely determined. Let $\sigma_1=\partial_t\varphi_1(t,x_2)$ and $\sigma_2=\partial_t\varphi_2(t,x_2)$. The data of

$(U_\pm, \varphi_1, \varphi_2)$ on Γ should satisfy

$$u_- - \varphi_0' v_- = 0, \tag{3.10}$$
$$0 < \sigma_1 < \sqrt{p'(\rho_-)}, \tag{3.11}$$
$$u_+ - \varphi_0' v_+ < \sigma_1 < \sigma_2 < u_+ - \varphi_0' v_+ + \sqrt{p'(\rho_+)}, \tag{3.12}$$
$$\sigma_2 > u_B + \sqrt{p'(\rho_B)} \tag{3.13}$$

and the following jump conditions:

$$\begin{cases} \sigma_1[\rho]_1 - [\rho u]_1 + \varphi_0'[\rho v]_1 = 0, \\ \sigma_1[\rho u]_1 - [\rho u^2 + p]_1 + \varphi_0'[\rho uv]_1 = 0, \\ \sigma_1[\rho v]_1 - [\rho uv]_1 + \varphi_0'[\rho v^2 + p]_1 = 0, \\ \Big[e'(\rho) + \dfrac{(u - \varphi_0' v - \sigma_1)^2}{2(1 + \varphi_0'^2)^2}\Big]_1 = -\nu a(j_1, \nu), \\ \sigma_2[\rho]_2 - [\rho u]_2 + \varphi_0'[\rho v]_2 = 0, \\ \sigma_2[\rho u]_2 - [\rho u^2 + p]_2 + \varphi_0'[\rho uv]_2 = 0, \\ \sigma_2[\rho v]_2 - [\rho uv]_2 + \varphi_0'[\rho v^2 + p]_2 = 0, \end{cases} \tag{3.14}$$

where $[\,\cdot\,]_i$ $(i = 1, 2)$ denotes the jump of a function on Γ_i $(i = 1, 2)$ as the (t, x) approaches Γ, $a(j_1, \nu) = j_1 \int_{-\infty}^{+\infty} \tau'^2(\xi; j_1, \nu) \mathrm{d}\xi$ with $\tau(\xi; j_1, \nu)$ being the viscosity capillarity profile satisfying

$$\begin{cases} \tau'' = \nu j_1 \tau' + \pi - P(\tau) - j_1^2 \tau, \\ \lim\limits_{\xi \to -\infty} \tau = \dfrac{1}{\rho_-}\Big|_{x_1 = \varphi_0}, \quad \lim\limits_{\xi \to +\infty} \tau = \dfrac{1}{\rho_+}\Big|_{x_1 = \varphi_0} \end{cases} \tag{3.15}$$

with $j_1 = \frac{\rho_\pm(u_\pm - \varphi_0' v_\pm - \sigma_1)}{\sqrt{1 + \varphi_0'^2}}$ and $\pi = P(\tau_\pm) + j_1^2 \tau_\pm$ valued at $\{x_1 = \varphi_0\}$. From Theorem 3.2, we have

$$(\chi(\tau_+, \tau_-) + \chi(\tau_+, \tau_B))\sqrt{1 + \varphi_0'^2} = \chi(\tau_A, \tau_B). \tag{3.16}$$

Regarding (τ_+, τ_-) as functions of j_1 and noticing (3.7)–(3.8), similar to the proof of Theorem 2.2(iii), we can find a unique $j_1(x_2) \in (-|\overline{j}|, -|\underline{j}|)$ satisfying (3.16). Then from (3.10) and (3.14), all data of $(U_\pm, \partial_t \varphi_1, \partial_t \varphi_2)$ on Γ are determined uniquely.

Remark 3.3 Since $\varphi_0 \in C^\infty$, from the above discussion, we see that the data of $(\rho_\pm, u_\pm, v_\pm)$, σ_1 and σ_2 on Γ are smooth in x_2.

Obviously, the problem (3.9) is the one with two free boundaries and one fixed boundary. To transform the free boundaries into fixed ones, we introduce the following transformation

$$\widetilde{x}_1 = \begin{cases} \dfrac{t(x_1 - \varphi_0)}{\varphi_1 - \varphi_0}, & \text{in } G_-, \\ \dfrac{t(x_1 + \varphi_2 - 2\varphi_1)}{\varphi_2 - \varphi_1}, & \text{in } G_+, \end{cases} \quad \widetilde{x}_2 = x_2, \quad \widetilde{t} = t, \quad \widetilde{U}(\widetilde{t}, \widetilde{x}) = U(t, x). \tag{3.17}$$

The transformation (3.17) changes G_+, G_- and Γ_i $(i = 0, 1, 2)$ into

$$\widetilde{G}_+ = \{(\widetilde{t}, \widetilde{x}) \mid \widetilde{t} < \widetilde{x}_1 < \widetilde{2}t,\ \widetilde{t} > 0\}, \quad \widetilde{G}_- = \{(\widetilde{t}, \widetilde{x}) \mid 0 < \widetilde{x}_1 < \widetilde{t},\ \widetilde{t} > 0\}$$

and

$$\widetilde{\Gamma}_i = \{(\widetilde{t}, \widetilde{x}) \mid \widetilde{x}_1 = it,\ \widetilde{t} > 0\}, \quad i = 0, 1, 2,$$

respectively. Denote $\widetilde{A}_1^{\pm}(U_\pm, \psi) = \frac{\partial \widetilde{x}_1}{\partial t} I + A_1(U_\pm)\frac{\partial \widetilde{x}_1}{\partial x} + A_2(U_\pm)\frac{\partial \widetilde{x}_1}{\partial x_2}$ and $\widetilde{A}_2^{\pm}(U_\pm) = A_2(U_\pm)$. The problem (3.9) now becomes

$$\begin{cases} \partial_t U_\pm + A_1^\pm \partial_{x_1} U_\pm + A_2^\pm \partial_{x_2} U_\pm = 0, & \text{in } G_\pm, \\ u_- - \varphi_0' v_- = 0, & \text{on } \Gamma_0, \\ \partial_t \varphi_1 [F_0(U)]_1 - [F_1(U)]_1 + \partial_{x_2}\varphi_1 [F_2(U)]_1 = 0, & \text{on } \Gamma_1, \\ \Big[e'(\rho) + \dfrac{(u - \partial_{x_2}\varphi_1 v - \partial_t \varphi_1)^2}{2(1 + (\partial_{x_2}\varphi_1)^2)}\Big]_1 = -\nu a(j, \nu), & \text{on } \Gamma_1, \\ \partial_t \varphi_2 [F_0(U)]_2 - [F_1(U)]_2 + \partial_{x_2}\varphi_2 [F_2(U)]_2 = 0, & \text{on } \Gamma_2, \end{cases} \tag{3.18}$$

where we have dropped the tildes for simplicity.

Remark 3.4 From (3.17), it is easy to have

$$\lim_{t\to 0}\Big|\frac{\partial(\widetilde{t}, \widetilde{x})}{\partial(t, x)}\Big| = \begin{cases} \lim\limits_{t\to 0} \dfrac{t}{\varphi_1 - \varphi_0}, & \text{in } G_-, \\ \lim\limits_{t\to 0} \dfrac{t}{\varphi_2 - \varphi_1}, & \text{in } G_+. \end{cases}$$

Noticing that $\varphi_1|_{t=0} = \varphi_2|_{t=0} = \varphi_0$ and $0 < \partial_t \varphi_1|_{t=0} < \partial_t \varphi_2|_{t=0}$, we know that the transformation (3.17) is nonsingular in $G_\pm$ when t is small.

Denote by $\gamma_i\cdot$ the trace operator on Γ_i $(i = 0, 1, 2)$, $\gamma U = (\gamma_1 U_-, \gamma_2 U_-, \gamma_2 U_+)$ and $l = (0, -1, \varphi_0')$. For simplicity of notations, we denote the problem (3.18) by

$$\begin{cases} L_\pm(U_\pm, \psi) U_\pm = 0, & \text{in } G_\pm, \\ \gamma_0 U_- \cdot l = 0, & \text{on } \Gamma_0, \\ \mathscr{F}_1(\gamma U, \varphi) = 0, & \text{on } \Gamma_1, \\ \mathscr{F}_2(\gamma U, \varphi) = 0, & \text{on } \Gamma_2. \end{cases} \tag{3.19}$$

Next, we propose the following assumptions for the problem (3.19).

(A1) The planar phase boundary

$$U(t, x) = \begin{cases} U_-(0, 0, 0) & \text{for } x_1 < \sigma_1(0)t, \\ U_+(0, 0, 0) & \text{for } x_1 > \sigma_1(0)t \end{cases}$$

and the planar shock front

$$U(t, x) = \begin{cases} U_+(0, 0, 0) & \text{for } x_1 < \sigma_2(0)t, \\ U_B & \text{for } x_1 > \sigma_2(0)t \end{cases}$$

are uniformly stable in the sense of [18] and [13], respectively.

(A2) $U_+(0, 0, 0)$ satisfies a stability condition on the edge of the dihedral in connection with the one proposed by Alinhac [1] for solving the linear Goursat problem. This will be described in detail by Remark 4.1 in the coming section.

4 Linear Problems

To study the nonlinear problem (3.19), let us first study its linearized problem in this section. Denote $U=(U_+,U_-)$ and $\varphi=(\varphi_1,\varphi_2)$. Let (V,ψ) be the perturbation of (U,φ). The linearized problem of (3.19) is as follows:

$$\begin{cases} L_\pm(U_\pm,\varphi)V_\pm=\partial_t V_\pm+A_1^\pm(U_\pm,\varphi)\partial_{x_1}V_\pm+A_2^\pm(U_\pm)\partial_{x_2}V_\pm=f_\pm, & \text{in } G_\pm,\\ \gamma_0 V_-\cdot l=0, & \text{on } \Gamma_0,\\ F_{1,(\gamma U,\varphi)}(\gamma V,\psi)=b_1\partial_t\psi_1+c_1\partial_{x_2}\psi_1+\mathscr{M}_1\gamma_1 V_++\mathscr{N}_1\gamma_1 V_-=g_1, & \text{on } \Gamma_1,\\ F_{2,(\gamma U,\varphi)}(\gamma V,\psi)=b_2\partial_t\psi_2+c_2\partial_{x_2}\psi_2+\mathscr{N}_2\gamma_2 V_+=g_2, & \text{on } \Gamma_2, \end{cases} \tag{4.1}$$

where $F_{i,(\gamma U,\varphi)}(\gamma V,\psi)$ $(i=1,2)$ are the Fréchet derivatives of $\mathscr{F}_i(\gamma U,\varphi)$ with respect to $(\gamma U,\varphi)$, i.e.,

$$F_{i,(\gamma U,\varphi)}(\gamma V,\psi)=\frac{\mathrm{d}}{\mathrm{d}\epsilon}\mathscr{F}_i(\gamma U+\epsilon\gamma V,\varphi+\epsilon\psi)\Big|_{\epsilon=0}.$$

The coefficients in the boundary conditions of (4.1) read

$$b_1=\begin{pmatrix} [F_0(U)]_1\\ \dfrac{[v-\partial_{x_2}\varphi_1 v]_1}{1+(\partial_{x_2}\varphi_1)^2}+\nu\partial_j a\dfrac{\rho_-}{\sqrt{1+(\partial_{x_2}\varphi_1)^2}} \end{pmatrix},$$

$$c_1=\begin{pmatrix} [F_2(U)]_1\\ \nu\partial_j a\dfrac{\rho_-v_-+\partial_{x_2}\varphi_1\rho_-(u_--\partial_t\varphi_1)}{(1+(\partial_{x_2}\varphi_1)^2)^{\frac32}}+\dfrac{[v(u-\partial_{x_2}\varphi_1 v-\partial_t\varphi_1)]_1}{1+(\partial_{x_2}\varphi_1)^2}+\dfrac{\partial_{x_2}\varphi_1[(\partial_t\varphi_1+\partial_{x_2}\varphi_1 v-u)^2]_1}{(1+\partial_{x_2}\varphi_1)^2} \end{pmatrix},$$

$$\mathscr{M}_1=\begin{pmatrix} \partial_t\varphi_1 F_0'(U_+)+\partial_{x_2}\varphi_1 F_2'(U_+)-F_1'(U_+)\\ l_+ \end{pmatrix},$$

$$\mathscr{N}_1=\begin{pmatrix} F_1'(U_-)-\partial_t\varphi_1 F_0'(U_-)-\partial_{x_2}\varphi_1 F_2'(U_-)\\ l_- \end{pmatrix}$$

with $\partial_j a=\frac{\partial a}{\partial j}$,

$$\begin{aligned} l_+&=\Big(-e''(\rho_+),\frac{\partial_t\varphi_1-u_++\partial_{x_2}\varphi_1 v_+}{1+(\partial_{x_2}\varphi_1)^2},\frac{\partial_{x_2}\varphi_1(u_+-\partial_{x_2}\varphi_1 v_+-\partial_t\varphi_1)}{1+(\partial_{x_2}\varphi_1)^2}\Big),\\ l_-&=\Big(\nu\partial_j a\frac{\partial_t\varphi_1-u_-+\partial_{x_2}\varphi_1 v_-}{\sqrt{1+(\partial_{x_2}\varphi_1)^2}}+e''(\rho_-),-\nu\partial_j a\frac{\rho_-}{\sqrt{1+(\partial_{x_2}\varphi_1)^2}}+\frac{u_--\partial_{x_2}\varphi_1 v_--\partial_t\varphi_1}{1+(\partial_{x_2}\varphi_1)^2},\\ &\qquad\partial_{x_2}\varphi_1\Big(\nu\partial_j a\frac{\rho_-}{\sqrt{1+(\partial_{x_2}\varphi_1)^2}}+\frac{\partial_t\varphi_1+\partial_{x_2}\varphi_1 v_--u_-}{1+(\partial_{x_2}\varphi_1)^2}\Big)\Big) \end{aligned}$$

and

$$\begin{aligned} b_2&=[F_0(U)]_2,\\ c_2&=[F_2(U)]_2,\\ \mathscr{N}_2&=F_1'(U_+)-\partial_t\varphi_2 F_0'(U_+)-\partial_{x_2}\varphi_2 F_2'(U_+). \end{aligned}$$

Remark 4.1 With all the coefficients given in the above, we can describe the assumption (A2) precisely. Freeze all the coefficients at the origin $(0,0,0)$. Denote by $\lambda_1^\pm<\lambda_2^\pm<\lambda_3^\pm$ the eigenvalues of $A_1^\pm$, and $r_1^\pm,r_2^\pm,r_3^\pm$ the corresponding right eigenvectors. Let

$$\begin{aligned} \mathscr{R}_1&=(\mathscr{M}_1 r_3^+,\mathscr{N}_1 r_1^-,\mathscr{N}_1 r_2^-,b_1)^{-1}(\mathscr{M}_1 r_1^+,\mathscr{M}_1 r_2^+,\mathscr{N}_1 r_3^-)\in\mathbb{R}^{4\times3},\\ \mathscr{R}_2&=(\mathscr{N}_2 r_1^+,\mathscr{N}_2 r_2^+,b_2)^{-1}\mathscr{N}_1 r_3^+\in\mathbb{R}^{3\times1}. \end{aligned}$$

The stability condition is as follows:

$$|(\mathscr{R}_1)_{11}(\mathscr{R}_2)_{11}| + |(\mathscr{R}_1)_{12}(\mathscr{R}_2)_{21}| < 1, \tag{4.2}$$

where $(\,\cdot\,)_{ij}$ denotes the (i,j)-th element of a matrix.

The above stability condition is similar to the one for the double shock problem given in [15]. It is satisfied when the shock front is sufficiently weak while the strength of the phase boundary is fixed. In fact, through a tedious calculation as in the appendix of [19], we can get

$$|(\mathscr{R}_1)_{11}(\mathscr{R}_2)_{11}| + |(\mathscr{R}_1)_{12}(\mathscr{R}_2)_{21}| = M\frac{(\lambda_3^+ - \sigma_2)}{(\lambda_1^+ - \sigma_2)},$$

where M is bounded depending only on $[U]_1$. Therefore, if we fix $[U]_1$ and let $\lambda_3^+ - \sigma_2$ be sufficiently small, then (4.2) is valid.

To establish estimates of the solutions to the problem (4.1), as in [4, 15], we first introduce the weighted Sobolev spaces.

4.1 Weighted Sobolev spaces

Due to the transformation (3.9), the weighted Sobolev spaces that we use here are a little different from those in [4, 15]. Therefore, it is necessary for us to verify the equivalence of the spaces after the blow-up of $\Gamma = \{t = x_1 = 0\}$, the dyadic partition of unity and the dilation as in [4, 15]. Denote $\beta = (\beta_1, \beta_2)$, $G_\pm^T = \{(t,x) \mid (t,x) \in G_\pm, t \in (0,T)\}$, $\Gamma_i^T = \{(t,x) \mid (t,x) \in \Gamma_i,\ t \in (0,T)\}$ $(i = 0,1,2)$, $V_1 = x_1\partial_{x_1}$, $V_2 = \partial_{x_2}$ and $\partial_x = (\partial_{x_1}, \partial_{x_2})$. We introduce the following spaces and the corresponding norms:

$$\begin{aligned}
L^2_\lambda(\Omega) &= \{u \mid t^{-\lambda}u \in L^2(\Omega)\},\\
\|u\|_{L^2_\lambda(\Omega)} &= \|t^{-\lambda}u\|_{L^2(\Omega)},\\
H^r_\lambda(G_\pm^T) &= \{u \mid \partial_t^l\partial_x^\beta u \in L^2_{\lambda-l-\beta_1}(G_\pm^T),\ l+|\beta| \le r\},\\
\|u\|_{H^r_\lambda(G_\pm^T)} &= \Big\{\sum_{l+|\beta|\le r} \lambda^{2(r-l-\beta_1)}\|\partial_t^l\partial_x^\beta u\|^2_{L^2_{\lambda-l-\beta_1}(G_\pm^T)}\Big\}^{\frac12},\\
H^{r,k}_\lambda(G_-^T) &= \{u \mid \partial_t^l V^\beta \partial_{x_1}^m u \in L^2_{\lambda-l-m}(G_-^T),\ l+|\beta|+m \le r+k, m \le r\},\\
\|u\|_{H^{r,k}_\lambda(G_-^T)} &= \Big\{\sum_{\substack{l+|\beta|+m\le r+k\\ m\le r}} \lambda^{2(r+k-l-m)}\|\partial_t^l V^\beta \partial_{x_1}^m u\|^2_{L^2_{\lambda-l-m}(G_-^T)}\Big\}^{\frac12},\\
B^k_\lambda(G_-^T) &= \bigcap_{r\le\frac{k}{2}} H^{r,k-2r}_\lambda(G_-^T),\\
\|u\|_{B^k_\lambda(G_-^T)} &= \Big\{\sum_{r\le\frac{k}{2}} \|u\|^2_{H^{r,k-2r}_\lambda(G_-^T)}\Big\}^{\frac12},\\
H^r_\lambda(\Gamma_i^T) &= \{f \mid \partial_t^l\partial_{x_2}^m f \in L^2_{\lambda-l}(\Gamma_i^T),\ l+m \le r\}, \quad i = 1,2,\\
\|f\|_{H^r_\lambda(\Gamma_i^T)} &= \Big\{\sum_{l+m\le r} \lambda^{2(r-l)}\|\partial_t^l\partial_{x_2}^m f\|^2_{L^2_{\lambda-l}(\Gamma_i^T)}\Big\}^{\frac12}, \quad i = 1,2.
\end{aligned}$$

Analogous to [15], we introduce the following transformation to blow up the $\Gamma = \{t = x_1 = 0\}$:

$$j: \quad s = t, \quad y_1 = \frac{x_1}{t}, \quad y_2 = x_2, \tag{4.3}$$

which maps Γ_i^T $(i=0,1,2)$ into

$$\widehat{\Gamma}_i^T=\{(s,y)\mid y_1=i, s\in(0,T)\},\quad i=0,1,2$$

and $G_\pm^T$ into

$$\begin{aligned}\widehat{G}_+^T&=\{(s,y)\mid 1<y_1<2, s\in(0,T)\},\\ \widehat{G}_-^T&=\{(s,y)\mid 0<y_1<1, s\in(0,T)\}.\end{aligned}$$

Denote

$$\begin{aligned}J_\lambda u(s,y_1,y_2)&=s^{-\lambda}u(s,sy_1,y_2),\quad \text{in } G_\pm^T,\\ J_\lambda f(s,y_2)&=s^{-\lambda}f(s,y_2),\quad \text{on } \Gamma_i^T,\ i=0,1,2,\end{aligned}$$

$\widehat{V}_1=y_1\partial_{y_1}$, $\widehat{V}_2=\partial_{y_2}$, $\widehat{V}=(\widehat{V}_1,\widehat{V}_2)$, and $\partial_y=(\partial_{y_1},\partial_{y_2})$. Similar to [15], in the coordinates (s,y), we introduce the following spaces and norms:

$$\begin{aligned}
\widehat{H}^r(\widehat{G}_\pm^T)&=\{u\mid (s\partial_s)^l\partial_y^\beta u\in L^2(\widehat{G}_\pm^T), l+|\beta|\le r\},\\
\|u\|_{\widehat{H}_\lambda^r(\widehat{G}_\pm^T)}&=\Big\{\sum_{l+|\beta|\le r}\lambda^{2(r-l-\beta_1)}\|(s\partial_s)^l\partial_y^\beta u\|^2_{L^2(\widehat{G}_\pm^T)}\Big\}^{\frac12},\\
\widehat{H}^{r,k}(\widehat{G}_-^T)&=\{u\mid (s\partial_s)^l\widehat{V}^\beta\partial_{y_1}^m u\in L^2(\widehat{G}_-^T), l+|\beta|+m\le r+k, m\le r\},\\
\|u\|_{\widehat{H}_\lambda^{r,k}(\widehat{G}_-^T)}&=\Big\{\sum_{\substack{l+|\beta|+s\le r+k\\ m\le r}}\lambda^{2(r+k-l-m)}\|(s\partial_s)^l\widehat{V}^\beta\partial_{y_1}^m u\|^2_{L^2(\widehat{G}_-^T)}\Big\}^{\frac12},\\
\widehat{B}^k(\widehat{G}_-^T)&=\bigcap_{r\le\frac{k}{2}}\widehat{H}^{r,k-2r}(\widehat{G}_-^T),\\
\|u\|_{\widehat{B}_\lambda^k(\widehat{G}_-^T)}&=\Big\{\sum_{r\le\frac{k}{2}}\|u\|^2_{\widehat{H}_\lambda^{r,k-2r}(\widehat{G}_-^T)}\Big\}^{\frac12},\\
\widehat{H}_\lambda^r(\widehat{\Gamma}_i^T)&=\{f\mid (s\partial_s)^l\partial_{y_2}^m f\in L^2(\widehat{\Gamma}_i^T), l+m\le r\},\quad i=1,2,\\
\|f\|_{\widehat{H}_\lambda^r(\widehat{\Gamma}_i^T)}&=\Big\{\sum_{l+m\le r}\lambda^{2(r-l)}\|(s\partial_s)^l\partial_{y_2}^m f\|^2_{L^2(\widehat{\Gamma}_i^T)}\Big\}^{\frac12},\quad i=1,2.
\end{aligned}$$

Lemma 4.1 *J_λ is isomorphic from $H^r_{\lambda+\frac12}(G_\pm^T)$ to $\widehat{H}_\lambda^r(\widehat{G}_\pm^T)$ and from $H_\lambda^r(\Gamma_i^T)$ to $\widehat{H}_\lambda^r(\widehat{\Gamma}_i^T)$* $(i=1,2)$.

Proof Here we only prove that J_λ is isomorphic from $H^r_{\lambda+\frac12}(G_\pm^T)$ to $\widehat{H}_\lambda^r(\widehat{G}_\pm^T)$. The other conclusions can be proved in the same way. Noticing that the Jacobian of the transformation j in $G_\pm^T$ is

$$\left|\frac{\partial(s,y)}{\partial(t,x)}\right|=\frac{1}{t},\tag{4.4}$$

we see that J_λ is an isomorphic mapping from $L^2_{\lambda+\frac12}(G_\pm^T)$ to $L^2(\widehat{G}_\pm^T)$.

Considering the higher order norm, from (4.3), we have

$$\partial_t=\partial_s-\frac{y_1}{s}\partial_{y_1},\quad \partial_{x_1}=\frac{1}{s}\partial_{y_1},\quad \partial_{x_2}=\partial_{y_2}.$$

Then we have

$$\begin{aligned}
&|\lambda^{r-l-\beta_1}t^{-(\lambda-l-\beta_1)}\partial_t^l\partial_x^\beta u|\\
&=\Big|\lambda^{r-l-\beta_1}s^{-(\lambda-l-\beta_1)}\Big(\partial_s-\frac{y_1}{s}\partial_{y_1}\Big)^l\Big(\frac{1}{s}\partial_{y_1}\Big)^{\beta_1}\partial_{y_2}^{\beta_2}(s^\lambda J_\lambda u)\Big|\\
&\le O(1)\lambda^{r-l-\beta_1}\sum_{k\le l}|s^{-(\lambda-l-\beta_1)}\partial_s^k(s^{\lambda-l-\beta_1+k}\partial_y^{\beta+(l-k,0)}J_\lambda u)|\\
&\le O(1)\sum_{k\le l}\sum_{j\le k}\lambda^{r-\beta_1-k+j}|(s\partial_s)^{k-j}\partial_y^{\beta+(l-k,0)}J_\lambda u|,
\end{aligned}$$

which implies that there exists a constant $M>0$, such that

$$\|u\|_{H^r_{\lambda+\frac12}(G^T_\pm)}\le M\|J_\lambda u\|_{\widehat{H}^r_\lambda(\widehat{G}^T_\pm)}.$$

Similarly, we can prove that there exists a constant $m>0$, such that

$$\|J_\lambda u\|_{\widehat{H}^r_\lambda(\widehat{G}^T_\pm)}\le m\|u\|_{H^r_{\lambda+\frac12}(G^T_\pm)}.$$

Thus we have proved that J_λ is isomorphic from $H^r_{\lambda+\frac12}(G^T_-)$ to $\widehat{H}^r_\lambda(\widehat{G}^T_-)$.

Remark 4.2 Similar to Lemma 4.1, we can prove that J_λ is isomorphic from $B^k_{\lambda+\frac12}(G^T_-)$ to $\widehat{B}^k_\lambda(\widehat{G}^T_-)$.

Similar to [15], we introduce the following dyadic partition of unity and dilation. Set $\chi\in C_0^\infty(\mathbb{R}^1)$, such that

$$\mathrm{supp}\chi\subset\Big(\frac12,2\Big)\quad\text{and}\quad\sum_{j=-\infty}^{+\infty}\chi(2^js)=1,\quad s>0.$$

Let

$$\begin{aligned}
v^j(s,y)&=\chi(2^js)v(s,y),\\
\widetilde{v}^j(s,y)&=2^{-\frac{j}{2}}v^j(2^{-j}s,y),\\
T_j&=\min(2,2^jT),\\
\widetilde{\Gamma}_i^T&=\{(s,y)\mid y_1=i,s\in(-\infty,T)\},\quad i=0,1,2,\\
\widetilde{G}_-^T&=\{(s,y)\mid 0<y_1<1,s\in(-\infty,T)\},\\
\widetilde{G}_+^T&=\{(s,y)\mid y_1>1,s\in(-\infty,T)\}.
\end{aligned}$$

The corresponding spaces and norms are as follows:

$$\begin{aligned}
H^r(\widetilde{G}^T_\pm)&=\{v\mid\partial_s^l\partial_y^\beta v\in L^2(\widetilde{G}^T_\pm),\ l+|\beta|\le r\},\\
\|v\|_{H^r_\lambda(\widetilde{G}^T_\pm)}&=\Big\{\sum_{l+|\beta|\le r}\lambda^{2(r-l-\beta_1)}\|\partial_s^l\partial_y^\beta v\|^2_{L^2(\widetilde{G}^T_\pm)}\Big\}^{\frac12},\\
H^{r,k}(\widetilde{G}^T_-)&=\{v\mid\partial_s^l\widehat{V}^\beta\partial_{y_1}^m u\in L^2(\widetilde{G}^T_-),\ l+|\beta|+m\le r+k,\ m\le r\},\\
\|v\|_{H^{r,k}_\lambda(\widetilde{G}^T_-)}&=\Big\{\sum_{\substack{l+|\beta|+s\le r+k\\ m\le r}}\lambda^{2(r+k-l-m)}\|\partial_s^l\widehat{V}^\beta\partial_{y_1}^m v\|^2_{L^2(\widetilde{G}^T_-)}\Big\}^{\frac12},
\end{aligned}$$

$$
\begin{aligned}
B^k(\widetilde{G}^T_-) &= \bigcap_{r\le\frac{k}{2}} H^{r,k-2r}(\widetilde{G}^T_-),\\
\|v\|_{B^k_\lambda(\widetilde{G}^T_-)} &= \Big\{\sum_{r\le\frac{k}{2}} \|v\|^2_{H^{r,k-2r}_\lambda(\widetilde{G}^T_-)}\Big\}^{\frac12},\\
H^r_\lambda(\widetilde{\Gamma}^T_i) &= \{g \mid \partial^l_s\partial^m_{y_2} g\in L^2(\widetilde{\Gamma}^T_i),\ l+m\le r\},\quad i=1,2,\\
\|g\|_{H^r_\lambda(\widetilde{\Gamma}^T_i)} &= \Big\{\sum_{l+m\le r}\lambda^{2(r-l)}\|\partial^l_s\partial^m_{y_2}g\|^2_{L^2(\widetilde{\Gamma}^T_i)}\Big\}^{\frac12},\quad i=1,2.
\end{aligned}
$$

As in [15], we have the following lemma.

Lemma 4.2 (i) *If $v\in\widehat{H}^r(\widehat{G}^T_\pm)$, and $\widetilde{v}^j$ is defined as above, then $\widetilde{v}^j\in H^r(\widetilde{G}^{T_j}_\pm)$ and*

$$
\sum_j \|\widetilde{v}^j\|^2_{H^r_\lambda(\widetilde{G}^{T_j}_\pm)} \le C_1\|v\|^2_{\widehat{H}^r_\lambda(\widehat{G}^T_\pm)}
$$

for a constant $C_1>0$.

(ii) *If there exists a sequence $\{w^j\}$ satisfying $w^j\in H^r(\widetilde{G}^{T_j}_\pm)$, $\mathrm{supp} w^j\subset\{\gamma\le\tau\le T_j\}$ with $\gamma>0$, and $\sum_j \|w^j\|^2_{H^r_\lambda(\widetilde{G}^{T_j}_\pm)}<\infty$, then $v=\sum_j 2^{\frac{j}{2}}w^j(2^js,y)\in\widehat{H}^r(\widehat{G}^T_\pm)$ and*

$$
\|v\|^2_{\widehat{H}^r_\lambda(\widehat{G}^T_\pm)} \le C_2\sum_j \|w^j\|^2_{H^r_\lambda(\widetilde{G}^{T_j}_\pm)}
$$

for a constant $C_2>0$.

Remark 4.3 For the dyadic partition of unity and the dilation on $\widehat{\Gamma}^T_i$ $(i=1,2)$ with respect to s, the conclusion of Lemma 4.2 is also true.

4.2 Linear estimates

In order to avoid tedious notations in our coming estimates, we introduce the following notations. We denote $f=(f_-,f_+)$, $g=(g_1,g_2)$, $A_\pm=(A^\pm_1,A^\pm_2)$, $A=(A_+,A_-)$, $b=(b_1,b_2)$, $c=(c_1,c_2)$, $\mathscr{M}=\mathscr{M}_1$, $\mathscr{N}=(\mathscr{N}_1,\mathscr{N}_2)$ and

$$
\|V\|^2_{C^k_{\lambda,T}} = \|V_-\|^2_{B^k_\lambda(G^T_-)} + \|V_+\|^2_{H^k_\lambda(G^T_+)},\quad \|f\|^2_{C^k_{\lambda,T}} = \|f_-\|^2_{B^k_\lambda(G^T_-)} + \|f_+\|^2_{H^k_\lambda(G^T_+)},
$$

$$
\|\gamma V\|^2_{H^k_{\lambda,T}} = \sum_{i=1}^2 \|\gamma_i V_+\|^2_{H^k_\lambda(\Gamma^T_i)} + \|\gamma_1 V_-\|^2_{H^k_\lambda(\Gamma^T_1)},\quad \|g\|^2_{H^k_{\lambda,T}} = \sum_{i=1}^2 \|g_i\|^2_{H^k_\lambda(\Gamma^T_i)}.
$$

We also use $\|\cdot\|_{\widehat{C}^k_{\lambda,T}}$, $\|\cdot\|_{\widehat{H}^k_{\lambda,T}}$ and $\|\cdot\|_{\widetilde{C}^k_{\lambda,T}}$, $\|\cdot\|_{\widetilde{H}^k_{\lambda,T}}$ to denote the corresponding quantities of the norms that we defined in Subsection 4.1. We denote

$$
\begin{aligned}
|||(V,\psi)|||^2_{k,\lambda,T} &= \lambda\|V\|^2_{C^k_{\lambda+\frac12,T}} + \|\gamma V\|^2_{H^k_{\lambda,T}} + \|\psi\|^2_{H^{k+1}_{\lambda+1,T}},\\
\epsilon(L,F) &= \max\{\|A-A(0)\|_{L^\infty},\ \|b-b(0)\|_{L^\infty},\ \|c-c(0)\|_{L^\infty},\\
&\qquad \|\mathscr{M}-\mathscr{M}(0)\|_{L^\infty},\ \|\mathscr{N}-\mathscr{N}(0)\|_{L^\infty}\},\\
\|L,F\|_k &= \|A\|_{C^k_T} + \|b\|_{H^k_T} + \|c\|_{H^k_T} + \|\mathscr{M}\|_{H^k_T} + \|\mathscr{N}\|_{H^k_T},
\end{aligned}
$$

where $A(0), b(0), \cdots$ denote the coefficients frozen at the origin, and C_T^k, H_T^k denote the normal norms of the Sobolev spaces without weights. Denote

$$W_{\lambda,T}^k = \{(V,\psi) \mid V_+ \in H^k_{\lambda+\frac12}(G_+^T),\ V_- \in B^k_{\lambda+\frac12}(G_-^T),\ \psi_i \in H^{k+1}_{\lambda+1}(\Gamma_i^T),\ i=1,2\},$$
$$W'^k_{\lambda,T} = \{(f,g) \mid f_+ \in H^k_{\lambda-\frac12}(G_+^T),\ f_- \in B^k_{\lambda-\frac12}(G_-^T),\ g_i \in H^k_{\lambda}(\Gamma_i^T),\ i=1,2\}.$$

We have the following theorem on the linear estimate.

Theorem 4.1 *Suppose $s \geq 10$ and that the assumptions* (A1)–(A4) *are satisfied. There exist $\epsilon_0 > 0$, $\lambda_0(K)$, $C(K)$ such that under the condition*

$$\epsilon(L,F) \leq \epsilon_0, \quad \|L,F\|_s \leq K, \tag{4.5}$$

if $\partial_t^m f|_{t=0} = 0$, $\partial_t^m g|_{t=0} = 0$ $(0 \leq m \leq k-1)$ for $k \leq s$, then the problem (4.1) *has a unique solution $(V,\psi) \in W^k_{\lambda,T}$ for any $\lambda > \lambda_0(K)$, $T \leq T_0$, and $(f,g) \in W'^k_{\lambda,T}$. Moreover, the estimate*

$$|||(V,\psi)|||^2_{k,\lambda,T} \leq C(K)\Big(\frac{1}{\lambda}\|f\|^2_{k,\lambda-\frac12,T} + \|g\|^2_{k,\lambda,T}\Big) \tag{4.6}$$

holds.

Proof Denote $\widehat{V}_\pm = J_\lambda V_\pm$ and $\widehat{\psi} = J_{\lambda+1}\psi$. By employing the transformation (4.3) for the problem (4.1), it follows that $\widehat{V}_\pm$ and $\widehat{\psi}$ satisfy the following problem:

$$\begin{cases} \widehat{L}_{\lambda,\pm}\widehat{V}_\pm = (s\partial_s\widehat{V}_\pm + \lambda\widehat{V}_\pm) + \widehat{A}_1^\pm\partial_{y_1}\widehat{V}_\pm + \widehat{s}A_2^\pm\partial_{y_2}\widehat{V}_\pm = J_{\lambda-1}f_\pm, & \text{in } \widehat{G}_\pm^T,\\ \gamma_0\widehat{V}_-\cdot l = 0, & \text{on } \widehat{\Gamma}_0^T,\\ \widehat{F}_{1,\lambda}(\gamma\widehat{V},\widehat{\psi}) = (s\partial_s\widehat{\psi}_1 + (\lambda+1)\widehat{\psi}_1)b_1 + sc_1\partial_{y_2}\widehat{\psi}_1 + \mathscr{M}_1\gamma_1\widehat{V}_+ + \mathscr{N}_1\gamma_1\widehat{V}_- = J_\lambda g_1, & \text{on } \widehat{\Gamma}_1^T,\\ \widehat{F}_{2,\lambda}(\gamma\widehat{V},\widehat{\psi}) = (s\partial_s\widehat{\psi}_2 + (\lambda+1)\widehat{\psi}_2)b_2 + sc_2\partial_{y_2}\widehat{\psi}_2 + \mathscr{N}_2\gamma_2\widehat{V}_+ = J_\lambda g_2, & \text{on } \widehat{\Gamma}_2^T, \end{cases}$$

where we still use $\gamma_i\cdot$ to denote the trace operators on $\widehat{\Gamma}_i^T$ $(i=1,2)$, $\widehat{A}_1^\pm = A_1^\pm - y_1 I$ and $\widehat{A}_2^\pm = A_2^\pm$. From Lemma 4.1, we see that (4.6) is equivalent to the following estimate on $\widehat{V}_\pm$ and $\widehat{\psi}$:

$$\lambda\|\widehat{V}\|^2_{\widehat{C}^k_{\lambda,T}} + \|\gamma\widehat{V}\|^2_{\widehat{H}^k_{\lambda,T}} + \|\widehat{\psi}\|^2_{\widehat{H}^{k+1}_{\lambda+1,T}} \leq C\Big(\frac{1}{\lambda}\|\widehat{L}_\lambda\widehat{V}\|^2_{\widehat{C}^k_{\lambda-1,T}} + \|\widehat{F}_\lambda(\gamma\widehat{V},\widehat{\psi})\|^2_{\widehat{H}^k_{\lambda,T}}\Big), \tag{4.7}$$

where $\widehat{L}_\lambda\widehat{V} = (\widehat{L}_{\lambda,+}\widehat{V}_+, \widehat{L}_{\lambda,-}\widehat{V}_-)$ and $\widehat{F}_\lambda(\gamma\widehat{V},\widehat{\psi}) = (\widehat{F}_{1,\lambda}(\gamma\widehat{V},\widehat{\psi}), \widehat{F}_{2,\lambda}(\gamma\widehat{V},\widehat{\psi}))$. For simplicity, we still denote $\widehat{V}_\pm$, $\widehat{\psi}$ by $V_\pm$, ψ. Introducing $V_\pm^j$, ψ^j, $\widetilde{V}_\pm^j$, $\widetilde{\psi}^j$ by using the dyadic partition of unity and dilation as in Subsection 4.1, we get

$$\begin{cases} \widehat{L}_{\lambda,\pm}V_\pm^j = \chi(2^js)\widehat{L}_{\lambda,\pm}V_\pm + w_\pm^j, & \text{in } \widetilde{G}_\pm^{T_j},\\ \gamma_0 V_-^j\cdot l = 0, & \text{on } \widetilde{\Gamma}_0^{T_j},\\ \widehat{F}_{1,\lambda}(\gamma V^j,\psi^j) = \chi(2^js)\widehat{F}_{1,\lambda}(\gamma V,\psi) + g_1^j, & \text{on } \widetilde{\Gamma}_1^{T_j},\\ \widehat{F}_{2,\lambda}(\gamma_2 V^j,\psi^j) = \chi(2^js)\widehat{F}_{2,\lambda}(\gamma V,\psi) + g_2^j, & \text{on } \widetilde{\Gamma}_2^{T_j}, \end{cases}$$

where $w_\pm^j = s\chi'(2^js)2^jV_\pm$ and $g_i^j = s\chi'(2^js)2^j\psi_ib_i$ $(i=1,2)$. Furthermore, $\widehat{L}_{\lambda,\pm}V_\pm^j$ and $\widehat{F}_{i,\lambda}(\gamma V^j,\psi^j)$ (i=1,2) can be written as $\widetilde{L}^j_{\lambda,\pm}\widetilde{V}^j_\pm$, $\widetilde{F}^j_{i,\lambda}(\gamma\widetilde{V}^j,\widetilde{\psi}^j)$ by changing s into $2^{-j}s$ in all coefficients in the equations and boundary conditions. From Lemma 4.2, we see that (4.7) is equivalent to

$$\lambda\|\widetilde{V}^j\|^2_{\widetilde{C}^k_{\lambda,T_j}} + \|\gamma\widetilde{V}^j\|^2_{\widetilde{H}^k_{\lambda,T_j}} + \|\widetilde{\psi}^j\|^2_{\widetilde{H}^{k+1}_{\lambda+1,T_j}} \leq C\Big(\frac{1}{\lambda}\|\widetilde{L}^j_\lambda\widetilde{V}^j\|^2_{\widetilde{H}^k_{\lambda-1,T_j}} + \|\widetilde{F}^j_\lambda(\gamma\widetilde{V}^j,\widetilde{\psi}^j)\|^2_{\widetilde{H}^k_{\lambda,T_j}}\Big) \tag{4.8}$$

for all $j \in \mathbb{Z}$, where $\widetilde{L}^j_\lambda \widetilde{V}^j = (\widetilde{L}^j_{\lambda,+}\widetilde{V}^j_+, \widetilde{L}^j_{\lambda,-}\widetilde{V}^j_-)$, $\widetilde{F}^j_\lambda(\gamma\widetilde{V}^j, \widetilde{\psi}^j) = (\widetilde{F}^j_{1,\lambda}(\gamma\widetilde{V}^j, \widetilde{\psi}^j), \widetilde{F}^j_{2,\lambda}(\gamma\widetilde{V}^j, \widetilde{\psi}^j))$ and $T_j = \min(2, 2^j T)$.

Now we can separately establish estimates in the neighborhoods of the physical boundary $\widetilde{\Gamma}^{T_j}_0$, the phase boundary $\widetilde{\Gamma}^{T_j}_1$ and the shock front $\widetilde{\Gamma}^{T_j}_2$. Let κ be a smooth cut-off function with $\kappa \equiv 1$ on $[\frac{2}{3}, 1]$ and $\kappa \equiv 0$ in $[0, \frac{1}{3}]$. Let $W_1 = \kappa(y_1)\widetilde{V}^j_-$, $W_2 = (1-\kappa(y_1))\widetilde{V}^j_-$, $W_3 = (1-\kappa(y_1+1))\widetilde{V}^j_+$, $W_4 = \widetilde{\kappa}(y_1+1)\widetilde{V}^j_+$ and $\Psi_1 = \widetilde{\psi}^j_1$, $\Psi_2 = \widetilde{\psi}^j_2$. Then (W_1, W_3, Ψ_1) satisfies the following problem:

$$\begin{cases} \widetilde{L}^j_{\lambda,-}W_1 = \kappa(y_1)\widetilde{L}^j_{\lambda,-}\widetilde{V}^j_- + (\widetilde{L}^j_{\lambda,-}\kappa(y_1))\widetilde{V}^j_-, & \text{in } \widetilde{G}^{T_j}_-, \\ \widetilde{L}^j_{\lambda,+}W_3 = (1-\kappa(y_1+1))\widetilde{L}^j_{\lambda,+}\widetilde{V}^j_+ + (\widetilde{L}^j_{\lambda,+}(1-\kappa(y_1+1)))\widetilde{V}^j_+, & \text{in } \widetilde{G}^{T_j}_+, \\ \widehat{F}_{1,\lambda}(\gamma W, \Psi_1) = \widetilde{F}^j_{1,\lambda}(\gamma\widetilde{V}^j, \widetilde{\psi}^j), & \text{on } \widetilde{\Gamma}^{T_j}_1, \end{cases}$$

and (W_4, Ψ_2) satisfies the following problem:

$$\begin{cases} \widetilde{L}^j_{\lambda,+}W_4 = \kappa(y_1+1)\widetilde{L}^j_{\lambda,+}\widetilde{V}^j_+ + (\widetilde{L}^j_{\lambda,+}\kappa(y_1+1))\widetilde{V}^j_+, & \text{in } \widetilde{G}^{T_j}_+, \\ \widehat{F}_{2,\lambda}(\gamma W, \Psi_2) = \widetilde{F}^j_{1,\lambda}(\gamma\widetilde{V}^j, \widetilde{\psi}^j), & \text{on } \widetilde{\Gamma}^{T_j}_2. \end{cases}$$

From the assumption (A4), we can establish the following estimate by utilizing the results in [13, 18]:

$$\begin{aligned} &\lambda\|(\kappa\widetilde{V}^j_-, \widetilde{V}^j_+)\|^2_{C^k_{\lambda,T_j}} + \sum_{i=1}^{2}\|\gamma\widetilde{V}^j\|^2_{H^k_{\lambda,T_j}} + \|\widetilde{\psi}^j_i\|^2_{H^{k+1}_{\lambda+1,T_j}} \\ &\le C\Big(\frac{1}{\lambda}(\|(\kappa\widetilde{L}^j_{\lambda,-}\widetilde{V}^j_-, \widetilde{L}^j_{\lambda,+}\widetilde{V}^j_+)\|^2_{C^k_{\lambda-1,T_j}} + \|\widetilde{V}^j_-\|^2_{B^k_{\lambda-1}(\widetilde{G}^{T_j}_-)}) + \|\widetilde{F}^j_\lambda(\gamma\widetilde{V}^j, \widetilde{\psi}^j)\|^2_{H^k_{\lambda,T_j}}\Big). \end{aligned} \tag{4.9}$$

The problem for W_2 is as follows:

$$\begin{cases} \widetilde{L}^j_{\lambda,-}W_2 = (1-\kappa)\widetilde{L}^j_{\lambda,-}\widetilde{V}^j_- - (\widetilde{L}^j_{\lambda,-}\kappa)\widetilde{V}^j_-, & \text{in } \widetilde{G}^{T_j}_-, \\ \gamma_0 W_2 \cdot l = 0, & \text{on } \widetilde{\Gamma}^{T_j}_0. \end{cases}$$

By using the results of [5, 10] for the above problem with $\widetilde{\Gamma}^{T_j}_0$ being characteristic, we have

$$\lambda\|(1-\kappa)\widetilde{V}^j_-\|^2_{B^k_\lambda(\widetilde{G}^{T_j}_-)} \le C\cdot\frac{1}{\lambda}(\|(1-\kappa)\widetilde{L}^j_{\lambda,-}\widetilde{V}^j_-\|^2_{B^k_{\lambda-1}(\widetilde{G}^{T_j}_-)} + \|\widetilde{V}^j_-\|^2_{B^k_{\lambda-1}(\widetilde{G}^{T_j}_-)}). \tag{4.10}$$

Summing up (4.9) and (4.10) and letting λ be sufficiently large, we see that (4.8) holds. Thus we have proved (4.6).

4.3 1-dimensional linear problems

In this subsection, we study the linearized problem of (3.18) in the one space variable case, which is essential to the construction of an approximate solution to the original nonlinear problem (3.18). We shall mainly follow the arguments of Métivier's work (cf. [15]). The main difference from [15] is that the problem is posed on two areas and the unknowns are coupled on free boundaries. Considering the following linearized problem of (3.18) independent of x_2:

$$\begin{cases} L^0_\pm(U_\pm, \varphi)V_\pm = \partial_t V_\pm + A^\pm_1(U_\pm, \varphi)\partial_{x_1}V_\pm = f_\pm, & \text{in } G^T_\pm, \\ \gamma_0(V_-)_2 = 0, & \text{on } \Gamma^T_0, \\ F^0_1(\gamma V, \psi) = b_1\partial_t\psi_1 + \mathscr{M}_1\gamma_1 V_+ + \mathscr{N}_1\gamma_1 V_- = g_1, & \text{on } \Gamma^T_1, \\ F^0_2(\gamma V, \psi) = b_2\partial_t\psi_2 + \mathscr{N}_2\gamma_2 V_+ = g_2, & \text{on } \Gamma^T_2, \end{cases} \tag{4.11}$$

where, without confusion, we denote

$$\begin{aligned}
\Gamma_i^T &= \{(t,x_1) \mid x_1 = it,\ t\in(0,T)\}, \quad i=0,1,2,\\
G_+^T &= \{(t,x_1) \mid t<x_1<2t,\ t\in(0,T)\},\\
G_-^T &= \{(t,x_1) \mid 0<x_1<t,\ t\in(0,T)\},
\end{aligned}$$

and $\gamma_i\cdot$ $(i=1,2,3)$ are the corresponding trace operators, i.e.,

$$\gamma_1 V(t) = V(t,0), \quad \gamma_2 V(t) = V(t,t), \quad \gamma_3 V(t) = V(t,2t).$$

As in [15], for $\lambda>0,\ \tau>0$, we define the following norms:

$$\begin{aligned}
\|v\|_\tau &= \|\mathrm{e}^{-\tau t} v\|_{L^\infty(G_\pm^T)},\\
|||v|||_{k,\tau} &= \sum_{|\alpha|\le k} \tau^{k-|\alpha|}\|\partial^\alpha v\|_\tau,\\
\|v\|_{\tau,\lambda} &= \|t^{-\lambda}\mathrm{e}^{-\tau t} v\|_{L^\infty(G_\pm^T)}.
\end{aligned}$$

For $\lambda\ge 1$, we define

$$\|v\|'_{\tau,\lambda} = \|(\lambda t^{\lambda-1}+\tau t^\lambda)^{-1}\mathrm{e}^{-\tau t} v\|_{L^\infty(G_\pm^T)}.$$

The norms of a function on Γ_i^T $(i=1,2)$ can be defined similarly.

For the problem (4.11), we have the following theorem.

Theorem 4.2 *Under the assumptions* (A1)–(A2)*, there exist* $\epsilon_0>0$ *and* $T_0>0$ *such that if* $\epsilon(L_0,F_0)\le\epsilon_0$ *and* $f|_{t=0}=0,\ g|_{t=0}=0$*, then for* $T\le T_0$*,* (4.11) *has a unique solution* $V_\pm\in C^0(G_\pm^T)$*,* $\psi_i\in C^1(\Gamma_i^T)$ $(i=1,2)$ *for* $f_\pm\in C^0(G_\pm^T)$ *and* $g_i\in C^0(\Gamma_i^T)$ $(i=1,2)$*, and*

(i) *for* $\tau>0$*,*

$$\|V_-\|_\tau+\|V_+\|_\tau+\sum_{i=1}^2(\|\partial_t\psi_i\|_\tau+\tau\|\psi_i\|_\tau)\le C\Big(\frac{1}{\tau}(\|f_-\|_\tau+\|f_+\|_\tau)+\sum_{i=1}^2\|g_i\|_\tau\Big); \tag{4.12}$$

(ii) *for* $\tau>0,\ \lambda\ge1$*,*

$$\|V_-\|_{\tau,\lambda}+\|V_+\|_{\tau,\lambda}+\sum_{i=1}^2\|\partial_t\psi_i\|_{\tau,\lambda}\le C\Big(\frac{1}{\tau}(\|f_-\|'_{\tau,\lambda}+\|f_+\|'_{\tau,\lambda})+\sum_{i=1}^2\|g_i\|_{\tau,\lambda}\Big); \tag{4.13}$$

(iii) *If the coefficients in* (4.11) *are* C^k*,* $f_\pm\in C^k(G_\pm^T)$*,* $g_i\in C^k(\Gamma_i^T)$ $(i=1,2)$ *and* $\partial_t^j f|_{t=0}=0$*,* $\partial_t^j g|_{t=0}=0$ $(0\le j\le k)$*, then* $V_\pm\in C^k(G_\pm^T)$*,* $\psi_i\in C^{k+1}(\Gamma_i^T)$ $(i=1,2)$*. Moreover, for sufficiently large* τ*, we have*

$$\|V_-\|_{k,\tau}+\|V_+\|_{k,\tau}+\sum_{i=1}^2\|\psi_i\|_{k+1,\tau}\le C\Big(\frac{1}{\tau}(\|f_-\|_{k,\tau}+\|f_+\|_{k,\tau})+\sum_{i=1}^2\|g_i\|_{k,\tau}\Big). \tag{4.14}$$

Proof We only sketch the proof for the estimate (4.12). The estimates (4.13)–(4.14) can be established in the same way with a little more computation (cf. [15]). Without loss of generality, we only consider the case that $A_1^\pm$ $(i=2,3)$ is diagonal, i.e., $A_1^\pm=\mathrm{diag}(\lambda_1^\pm,\lambda_2^\pm,\lambda_3^\pm)$. In this case, the first equation of the problem (4.11) becomes

$$\partial_t(V_\pm)_k+\lambda_k^{(i)}\partial_{x_1}(V_\pm)_k=(f_\pm)_k, \quad \text{in } G_\pm^T,\ k=1,2,3, \tag{4.15}$$

where $(\cdot)_k$ denote the k-th component of a vector. When $\epsilon(L,F)$ is sufficiently small, due to the assumption (A1), we can rewrite the third and fourth equations of (4.11) as

$$\begin{pmatrix}\gamma_1(V_+)_3\\ \gamma_2(V_-)_1\\ \gamma_1(V_-)_2\\ \partial_t\psi_1\end{pmatrix} = \mathscr{R}_1 \begin{pmatrix}\gamma_1(V_+)_1\\ \gamma_1(V_+)_2\\ \gamma_1(V_-)_3\end{pmatrix} + h_1, \quad \text{on } \Gamma_1^T \tag{4.16}$$

and

$$\begin{pmatrix}\gamma_2(V_+)_1\\ \gamma_2(V_+)_2\\ \partial_t\psi_2\end{pmatrix} = \mathscr{R}_2\gamma_2(V_+)_3 + h_2, \quad \text{on } \Gamma_2^T, \tag{4.17}$$

where

$$h_1 = (m_1e_3, n_1e_1, n_1e_2, b_1)^{-1}g_1, \quad h_2 = (n_2e_1, n_2e_2, b_2)^{-1}g_2,$$

and $\mathscr{R}_1$, $\mathscr{R}_2$ are the same as defined in Remark 4.1. Here the detailed forms of $\mathscr{R}_1$ and $\mathscr{R}_2$ are as follows:

$$\mathscr{R}_1 = -(\mathscr{M}_1e_3, \mathscr{N}_1e_1, \mathscr{N}_1e_2, b_1)^{-1}(\mathscr{M}_1e_1, \mathscr{M}_1e_2, \mathscr{N}_1e_3), \quad \mathscr{R}_2 = -(\mathscr{N}_2e_1, \mathscr{N}_2e_2, b_2)^{-1}\mathscr{N}_2e_3$$

with $e_i \in \mathbb{R}^3$ $(i = 1,2,3)$ being the standard unit vector. Together with the boundary condition on Γ_0^T

$$\gamma_0(V_-)_2 = 0, \quad \text{on } \Gamma_3^{\mathrm{T}}, \tag{4.18}$$

we shall solve the problem (4.15)–(4.18). Since the problem is a 1-dimensional linear problem, we can solve it by integrating along the characteristics. The key point is to determine the boundary value of the unknowns. First, we are going to determine the data $\gamma_2(V_+)_3$. For this purpose, we first analyze the characteristics. For $(t_0,x_0) \in G_\pm^T$, we denote by

$$C_k^\pm(t_0,x_0) = \left\{(t, x_k^\pm(t)) \,\middle|\, \frac{\mathrm{d}x_k^\pm}{\mathrm{d}t} = \lambda_k^\pm,\ x_k^\pm(t_0) = x_0,\ 0 \le t \le t_0\right\}$$

the backward characteristic of $\partial_t + \lambda_k^\pm\partial_{x_1}$ passing (t_0,x_0). Due to our calculation in Section 2, we have the following list of intersection points between the boundaries and the characteristics:

$$\begin{aligned}
C_1^+(t_0,x_0) \cap \Gamma_2^T &= (T_1(t_0,x_0), 2T_1(t_0,x_0)),\\
C_2^+(t_0,x_0) \cap \Gamma_2^T &= (T_2(t_0,x_0), 2T_2(t_0,x_0)),\\
C_3^+(t_0,x_0) \cap \Gamma_1^T &= (T_3(t_0,x_0), T_3(t_0,x_0)),\\
C_3^-(t_0,x_0) \cap \Gamma_0^T &= (S_3(t_0,x_0), 0),
\end{aligned}$$

where, as we can see, in G_+^T, C_1^+ and C_2^+ start from Γ_2^T while C_3^+ starts from Γ_1^T. In G_-^T, C_3^- starts from Γ_0^T. Obviously, we have that $0 \le T_k(t,x) \le t$ and $S_3(t,x) \le t$.

By integrating along the characteristics, we get the solution to the problem (4.15)–(4.18) as follows:

$$\begin{cases}
(V_+)_1(t,x) = \gamma_2(V_+)_1(T_1(t,x)) + (F_+)_1(t,x),\\
(V_+)_2(t,x) = \gamma_2(V_+)_2(T_2(t,x)) + (F_+)_2(t,x),\\
(V_+)_3(t,x) = \gamma_1(V_+)_3(T_3(t,x)) + (F_+)_3(t,x),\\
(V_-)_3(t,x) = (F_-)_3(t,x),
\end{cases} \tag{4.19}$$

where $(F_\pm)_k$ are the integrals of $(f_\pm)_k$ along the corresponding characteristics, i.e.,

$$(F_+)_1(t,x) = \int_{T_1(t,x)}^t (f_+)_1(s, x_1^+(s))\mathrm{d}s, \quad (F_+)_2(t,x) = \int_{T_2(t,x)}^t (f_+)_2(s, x_2^+(s))\mathrm{d}s,$$
$$(F_+)_3(t,x) = \int_{T_3(t,x)}^t (f_+)_3(s, x_3^+(s))\mathrm{d}s, \quad (F_-)_3(t,x) = \int_{S_3(t,x)}^t (f_-)_3(s, x_3^-(s))\mathrm{d}s.$$

By substituting (4.19) and (4.17) into (4.16), we get the following functional equation of $\gamma_1(V_+)_3$:

$$\gamma_2(V_+)_3(t) = (\mathscr{R}_1)_{11}(\mathscr{R}_2)_{11}\gamma_2(V_+)_3(Z_1(t)) + (\mathscr{R}_1)_{12}(\mathscr{R}_2)_{21}\gamma_2(V_+)_3(Z_2(t)) + H(t), \tag{4.20}$$

where

$$Z_1(t) = T_1(T_3(t,2t), T_3(t,2t)), \quad Z_2(t) = T_2(T_3(t,2t), T_3(t,2t))$$

and

$$H(t) = (\mathscr{R}_1)_{11}(\gamma_1(F_+)_1(T_3(t,2t)) + (h_2)_1(Z_1(t))) + (\mathscr{R}_1)_{12}(\gamma_1(F_+)_2(T_3(t,2t)) + (h_2)_2(Z_2(t)))$$
$$+ (\mathscr{R}_1)_{13}\gamma_1(F_-)_3(T_3(t,2t)) + h_1(T_3(t,2t)).$$

Since $0 \le T_k(x,t) \le t$, it is easy to verify

$$0 \le Z_i(t) \le t, \quad i = 1, 2 \tag{4.21}$$

and

$$\|h\|_\tau \le C\|g\|_\tau, \quad \|F\|_\tau \le \frac{C}{\tau}\|f\|_\tau. \tag{4.22}$$

From (4.21)–(4.22), we have

$$\|H\|_\tau \le C(\tau^{-1}\|f\|_\tau + \|g\|_\tau). \tag{4.23}$$

Denote by Λ the operator as follows:

$$\Lambda : \quad v \to \Lambda v = (\mathscr{R}_1)_{11}(\mathscr{R}_2)_{12} v \circ Z_2 + (\mathscr{R}_2)_{12}(\mathscr{R}_2)_{21} v \circ Z_3.$$

Then from the assumption (A4) and (4.21), we can see that there exists a constant $0 < \rho < 1$ such that

$$\|\Lambda v\|_\tau \le \rho\|v\|_\tau. \tag{4.24}$$

Therefore, there is a unique solution v to the equation

$$v = \Lambda v + H \tag{4.25}$$

and the solution satisfies the following inequality:

$$\|v\|_\tau \le C\|H\|_\tau. \tag{4.26}$$

With the inequalities (4.22)–(4.23) and (4.26), we determine $\gamma_2(V_+)_3$ from (4.20) and

$$\|\gamma_2(V_+)_3\|_\tau \le C(\tau^{-1}\|f\|_\tau + \|g\|_\tau).$$

Since $\gamma_2(V_+)_3$ is determined, $(V_+)_3$ is solved. Then by (4.18), $\gamma_2(V_+)_1$, $\gamma_2(V_+)_2$ and $\partial_t\psi_2$ are determined. Therefore, $(V_+)_1$, $(V_+)_2$ and ψ_2 are solved. Then by (4.17) and (4.19), we can solve V_- and ψ_1. For the case when $A_1^{(i)}$ is not diagonal, we can easily diagonalize $A_1^\pm$, which yields that $V_\pm$ satisfy equations similar to (4.15) with low-order terms of $V_\pm$. These terms can be absorbed by the left-hand sides of the inequalities by letting τ be sufficiently large.

5 Nonlinear Problems

In this section, we establish the existence of the solution to the nonlinear problem (3.18) by using the fixed point argument. To do that, first, we need to give the compatibility conditions and construct the first approximate solution.

5.1 Higher order compatibility conditions

According to the calculation in Section 3, we see that the 0-th order compatibility conditions for the problem (3.18) are satisfied.

Now we compute the k-th order compatibility conditions for the problem (3.18). Differentiating the boundary condition on Γ_0 given in (3.18) with respect to t, we get

$$l \cdot \partial_t^k U_- = 0, \quad \text{at } \{x_1 = t = 0\}. \tag{5.1}$$

Differentiating the third and fourth equations in (3.18) in the direction $\tau_1 = (1, 1, 0)$, we have

$$\begin{cases} \partial_t^{k+1}\varphi_1[F_0(U)]_1 - F_0'(U_-)(\sigma_1 I - A_1(U_-) + \varphi_0' A_2(U_-))\partial_{\tau_1}^k U_- \\ \quad +F_0'(U_+)(\sigma_1 I - A_1(U_+) + \varphi' A_2(U_+))\partial_{\tau_1}^k U_+ = H_1^k, \\ a_0 \partial_t^{k+1}\varphi_1 + l_0^- \partial_{\tau_1}^k U_- + l_0^+ \partial_{\tau_1}^k U_+ = H_2^k, \end{cases} \tag{5.2}$$

at $\{x_1 = 0, t = 0\}$, where H_i^k $(i = 1, 2)$ depend smoothly on $\partial_{\tau_1}^l \varphi_1|_{t=0}$ $(0 \le l \le k)$ and $\partial_{\tau_1}^l U_\pm|_{t=0,x_1=0}$ $(0 \le l \le k-1)$,

$$\begin{aligned} a_0 &= \varphi_0' v_+ - u_+ - \nu\alpha(j_1, \nu)\rho_-^0 \sqrt{1 + \varphi_0'^2}, \\ l_+^0 &= ((1 + \varphi_0'^2)e''(\rho_+), u_+^0 - \varphi_0' v_+ - \sigma_1, \varphi_0'(\sigma_1 - u_+ + \varphi_0' v_+)), \\ l_-^0 &= (-\nu\alpha(j_1, \nu)\sigma_1\sqrt{1 + \varphi_0'^2} - (1 + \varphi_0'^2)e''(\rho_-), \nu\alpha(j_1, \nu)\rho_-\sqrt{1 + \varphi_0'^2} + \sigma_1, \\ &\quad - \varphi_0'(\sigma_1 + \nu\alpha(j_1, \nu)\rho_-\sqrt{1 + \varphi_0'^2})), \end{aligned}$$

where $\alpha(j_1, \nu) = \partial_{j_1} a(j_1, \nu)$. Differentiating the fifth equation in (3.18) in the direction $\tau_2 = (1, 2, 0)$, we have

$$\partial_t^{k+1}\varphi_2[F_0(U)]_2 - F_0'(U_+)(\sigma_2 I - A_1(U_+) + \varphi_0' A_2(U_0))\partial_{\tau_2}^k U_+ = H_3^k, \tag{5.3}$$

at $\{x_1 = 0, t = 0\}$, where H_3^k $(i = 1, 2)$ depend smoothly on $\partial_{\tau_2}^l \varphi_2|_{t=0}$ $(0 \le l \le k)$ and $\partial_{\tau_2}^l U_+|_{t=0,x_1=0}$ $(0 \le l \le k-1)$.

On the other hand, from the equations of $U_\pm$ given in (3.18), it follows that

$$\begin{cases} \partial_t^k U_- = \dfrac{1}{\sigma_1^k}(\varphi_0' A_2(U_-) - A_1(U_-))^k \partial_{x_1}^k U_- + I_2^k, \\ \partial_{\tau_1}^k U_- = \dfrac{1}{\sigma_1^k}(\sigma_1 I - A_1(U_-) + \varphi_0' A_2(U_-))^k \partial_{x_1}^k U_- + I_1^k, \\ \partial_{\tau_1}^k U_+ = \dfrac{1}{(\sigma_2 - \sigma_1)^k}(\sigma_1 I - A_1(U_+) + \varphi_0' A_2(U_+))^k \partial_{x_1}^k U_+ + I_3^k, \\ \partial_{\tau_2}^k U_+ = \dfrac{1}{(\sigma_2 - \sigma_1)^k}(\sigma_2 I - A_1(U_+) + \varphi_0' A_2(U_+))^k \partial_{x_1}^k U_+ + I_4^k, \end{cases} \tag{5.4}$$

at $\{x_1=0, t=0\}$, where I_i^k $(i=1,2)$ depend smoothly on $\{\partial_{x_2}^m \partial_{x_1}^l U_\pm|_{x_1=t=0}\}$ and $\{\partial_{x_2}^m \partial_t^l \varphi_i|_{t=0}\}$ $(i=1,2)$ for $0 \le l \le k-1, 0 \le m+l \le k$. Substituting (5.4) into (5.1)–(5.3), we get the k-th order compatibility conditions

$$\begin{cases} \dfrac{1}{(-\sigma_1)^k} l(A_1(U_-) - \varphi_0' A_2(U_-))^k \partial_{x_1}^k U_- = J_1^k, \\ \partial_t^{k+1}\varphi_1 [F_0(U)]_1 + \frac{1}{(-\sigma_1)^k} F_0'(U_-)(A_1(U_-) - \varphi_0' A_2(U_-) - \sigma_1 I)^{k+1} \partial_{x_1}^k U_- \\ \quad + \dfrac{1}{(\sigma_2-\sigma_1)^k} (\sigma_1 I - A_1(U_+) + \varphi_0' A_2(U_+))^{k+1} \partial_{x_1}^k U_+ = J_2^k, \\ a_0 \partial_t^{k+1}\varphi_1 + \dfrac{1}{(-\sigma_1)^k} l_-^0 (A_1(U_-) - \varphi_0' A_2(U_-) - \sigma_1 I)^k \partial_{x_1}^k U_- \\ \quad + \dfrac{1}{(\sigma_2-\sigma_1)^k} l_-^0 (\sigma_1 I - A_1(U_+) + \varphi_0' A_2(U_+))^k \partial_{x_1}^k U_+ = J_3^k, \\ \partial_t^{k+1}\varphi_2 [F_0(U)]_2 + \dfrac{1}{(\sigma_1-\sigma_2)^k} F_0'(U_+)(A_1(U_+) - \varphi_0' A_2(U_+) - \sigma_2 I)^{k+1} \partial_{x_1}^k U_+ = J_4^k, \end{cases} \tag{5.5}$$

at $\{x_1 = 0, t = 0\}$, where J_i^k $(i = 1,2,3,4)$ depend smoothly on $\{\partial_{x_2}^m \partial_{x_1}^l U_\pm|_{x_1=t=0}\}$ and $\{\partial_{x_2}^m \partial_t^l \varphi_i|_{t=0}\}$ $(i=1,2)$ for $0 \le l \le k-1,\ 0 \le m+l \le k$.

5.2 Validity of compatibility conditions and the construction of approximate solutions

In this subsection, first, we show that we can find the data satisfying the compatibility conditions (5.5). Denote by $\lambda_1^\pm < \lambda_2^\pm < \lambda_3^\pm$ the eigenvalues of $A_1(U_\pm) - \varphi_0' A_2(U_\pm)$ at $\{t = x_1 = 0\}$. We have

$$\begin{cases} \lambda_1^\pm = u_\pm - \varphi_0' v_\pm - (1+\varphi_0'^2)^{\frac12} c_\pm, \\ \lambda_2^\pm = u_\pm - \varphi_0' v_\pm, \\ \lambda_3^\pm = u_\pm - \varphi_0' v_\pm + (1+\varphi_0'^2)^{\frac12} c_\pm, \end{cases} \tag{5.6}$$

where $c_\pm = p'(\rho_\pm)^{\frac12}$. The corresponding eigenvectors are

$$\begin{cases} r_1^\pm = \Big(1,\ -\dfrac{c_\pm}{\rho_\pm}\sqrt{1+\varphi_0'^2},\ \dfrac{c_\pm \varphi_0'}{\rho_\pm}\sqrt{1+\varphi_0'^2}\Big)^{\mathrm T}, \\ r_2^\pm = \Big(0,\ \dfrac{\varphi_0'}{\rho_\pm},\ \dfrac{1}{\rho_\pm}\Big)^{\mathrm T}, \\ r_3^\pm = \Big(1,\ \dfrac{c_\pm}{\rho_\pm}\sqrt{1+\varphi_0'^2},\ -\dfrac{c_\pm \varphi_0'}{\rho_\pm}\sqrt{1+\varphi_0'^2}\Big)^{\mathrm T}. \end{cases} \tag{5.7}$$

For constants β_1, β_2 and vector $(v_-, v_+) \in \mathbb{R}^3 \times \mathbb{R}^3$, we denote

$$\begin{aligned} M_1(\beta_1, v_-, v_+) &= \begin{pmatrix} 0 \\ [F_0(U)]_1 \\ a_0 \end{pmatrix} \beta_1 + \Big(-\frac{1}{\sigma_1}\Big)^k \begin{pmatrix} l(A_1(U_-) - \varphi_0' A_2(U_-))^k \\ F_0'(U_-)(A_1(U_-) - \varphi_0' A_2(U_-) - \sigma_1 I)^{k+1} \\ l_-^0 (A_1(U_-) - \varphi_0' A_2(U_-) - \sigma_1 I)^k \end{pmatrix} v_- \\ &\quad + \Big(\frac{1}{\sigma_2-\sigma_1}\Big)^k \begin{pmatrix} 0 \\ (\sigma_1 I - A_1(U_+) + \varphi_0' A_2(U_+))^{k+1} \\ l_-^0 (\sigma_1 I - A_1(U_+) + \varphi_0' A_2(U_+))^k \end{pmatrix} v_+, \\ M_2(\beta_2, v_+) &= [F_0(U)]_2 \beta_2 + \Big(\frac{1}{\sigma_1-\sigma_2}\Big)^k F_0'(U_+)(A_1(U_+) - \varphi_0' A_2(U_+) - \sigma_2 I)^{k+1} v_+. \end{aligned}$$

Similar to [14], we have the following results.

Theorem 5.1 *If constants β_1, β_2 and vectors $(v_-, v_+) \in \mathbb{R}^3 \times \mathbb{R}^3$ satisfy*

$$\begin{cases} M_1(\beta_1, v_-, v_+) = 0, \\ M_2(\beta_2, v_+) = 0, \end{cases} \tag{5.8}$$

then we have $\beta_1 = 0$, $\beta_2 = 0$, $v_- = 0$, $v_+ = 0$.

Proof For simplicity, we only consider the case when $\varphi_0' \equiv 0$. When φ_0' is sufficiently small, the conclusion still holds. Denote by P^+ the smoothly varying projections onto the subspace $\text{span}\{r_3^+\}$. Denote $w_+ = P^+ v_+$ and $u_+ = (I - P^+)v_+$. Considering $M_1(\beta_1, v_-, v_+) = 0$, we have that the basis of the set

$$\{(\beta_1, v_-, w_+) \mid w_+ = P^+ w_+\}$$

is given by

$$(1,0,0) \cup (0, r_1^-, 0) \cup (0, r_2^-, 0) \cup (0, r_3^-, 0) \cup (0,0,r_3^+).$$

We have

$$\begin{aligned} &\det(M_1(1,0,0), M_1(0,r_1^-,0), M_1(0,r_2^-,0), M_1(0,r_3^-,0), M_1(0,0,r_3^+)) \\ = &\frac{(\lambda_1^- - \sigma_1)(\lambda_2^- - \sigma_1)(\lambda_3^- - \sigma_1)(\sigma_1 - \lambda_3^+))^{k+1}}{(-\sigma_1)^{3k}(\sigma_2 - \sigma_1)^k} \\ &\times \begin{vmatrix} 0 & \dfrac{(\lambda_1^-)^k l \cdot r_1^-}{(\lambda_1^- - \sigma_1)^{k+1}} & 0 & \dfrac{(\lambda_3^-)^k l \cdot r_3^-}{(\lambda_3^- - \sigma_1)^{k+1}} & 0 \\ [F_0(U)]_1 & F_0'(U_-) r_1^- & F_0'(U_-) r_2^- & F_0'(U_-) r_3^- & F_0'(U_+) r_3^+ \\ a_0 & \dfrac{l_-^0 \cdot r_1^-}{\lambda_1^- - \sigma_1} & \dfrac{l_-^0 \cdot r_2^-}{\lambda_2^- - \sigma_1} & \dfrac{l_-^0 \cdot r_3^-}{\lambda_3^- - \sigma_1} & \dfrac{l_+^0 \cdot r_3^+}{\sigma_1 - \lambda_1^+} \end{vmatrix}. \end{aligned}$$

Denote the determinant on the right-hand side of the above equality by $\triangle$. We have

$$\begin{aligned} \triangle = &-\frac{(\lambda_1^-)^k l \cdot r_1^-}{(\lambda_1^- - \sigma_1)^{k+1}} \begin{vmatrix} [F_0(U)]_1 & F_0'(U_-) r_2^- & F_0'(U_-) r_3^- & F_0'(U_+) r_3^+ \\ a_0 & \dfrac{l_-^0 \cdot r_2^-}{\lambda_2^- - \sigma_1} & \dfrac{l_-^0 \cdot r_3^-}{\lambda_3^- - \sigma_1} & \dfrac{l_+^0 \cdot r_3^+}{\sigma_1 - \lambda_1^+} \end{vmatrix} \\ &- \frac{(\lambda_3^-)^k l \cdot r_3^-}{(\lambda_3^- - \sigma_1)^{k+1}} \begin{vmatrix} [F_0(U)]_1 & F_0'(U_-) r_1^- & F_0'(U_-) r_2^- & F_0'(U_+) r_3^+ \\ a_0 & \dfrac{l_-^0 \cdot r_1^-}{\lambda_1^- - \sigma_1} & \dfrac{l_-^0 \cdot r_2^-}{\lambda_2^- - \sigma_1} & \dfrac{l_+^0 \cdot r_3^+}{\sigma_1 - \lambda_1^+} \end{vmatrix}. \end{aligned}$$

Denote the first and the second determinants on the right-hand side of the above equality by $\triangle_1$ and $\triangle_2$, respectively. By using the Rankine-Hugoniot conditions, we get

$$\begin{aligned} \triangle_1 &= [u]_1^2 + \frac{c_- c_+}{\rho_- \rho_+}[\rho]_1^2 - \nu\alpha(j_1, \nu)\rho_-, \\ \triangle_2 &= [u]_1^2 + \frac{c_- c_+}{\rho_- \rho_+}[\rho]_1^2 + \nu\alpha(j_1, \nu)\rho_-. \end{aligned}$$

According to the result in [2], for sufficiently small ν, we have

$$\alpha(j_1, \nu) = \frac{\partial}{\partial j_1} a(j_1, \nu) \geq \alpha > 0.$$

Therefore, noticing $\lambda_1^- < 0 < \lambda_2^- < \sigma_1 < \lambda_3^-$ and $l \cdot r_1^- = -l \cdot r_3^- = \frac{c_-}{\rho_-}$, we get $\triangle > 0$. Thus, we has proved

$$\det(M_1(1,0,0), M_1(0,r_1^-,0), M_1(0,r_2^-,0), M_1(0,r_3^-,0), M_1(0,0,r_3^+)) \neq 0. \tag{5.9}$$

Similarly, we consider $M_2(\beta_2, v_+) = 0$. The basis of the set

$$\{(\beta_2, u_+) \mid P^+ u_+ = 0\}$$

is given by

$$(1,0) \cup (0, r_1^+) \cup (0, r_2^+).$$

By direct computation, we get

$$\det(M_2(1,0), M_2(0,r_1^+), M_2(0,r_2^+)) = -\frac{c_+[u]_2((\sigma_2-\lambda_1^+)(\sigma_2-\lambda_2^+))^{k+1}}{(\rho_+)^2(\sigma_2-\sigma_1)^{2k}} \neq 0. \tag{5.10}$$

Thus, we obtain the conclusion.

Remark 5.1 With Theorem 5.1, we can find the high order derivatives $(\partial_{x_1}^k U_\pm, \partial_t^{k+1}\varphi_1, \partial_t^{k+1}\varphi_2)$ from (5.5), once the 0-th order compatibility condition is satisfied.

Next, we shall construct the approximate solution to the problem (3.18). Denote

$$H\varphi = \Big(\varphi, \frac{\varphi}{t}, \nabla\varphi, x_1\frac{\nabla\varphi}{t}\Big),$$

$$\epsilon_{T_0}(U,\varphi) = \|U_+\|_{L^\infty(G_+^{T_0})} + \|U_-\|_{L^\infty(G_-^{T_0})} + \sum_{i=1}^{2}\|H\varphi_i\|_{L^\infty(\Gamma_i^{T_0})},$$

$$\|(U,\varphi)\|_{W_{T_0}} = \|U_+\|_{H^N(G_+^{T_0})} + \|U_-\|_{B^N(G_-^{T_0})} + \|\gamma_1 U_-\|_{H^N(\Gamma_1^{T_0})} + \sum_{i=1}^{2}\|\gamma_i U_+\|_{H^N(\Gamma_i^{T_0})}$$
$$+ \sum_{i=1}^{2}(\|\varphi_i\|_{H^{N+1}(\Gamma_i^{T_0})} + \|H\varphi_i\|_{H^N(\Gamma_i^{T_0})}).$$

Similar to [15], we construct the approximate solution as follows by using Theorem 4.2.

Theorem 5.2 *For $\epsilon > 0$, there exist an $M > 0$ and a C^∞ sequence $\{U^j, \varphi^j\}$, such that*

$$L_\pm(U_\pm^j, \varphi^j)U_\pm^j = O(t^j), \tag{5.11}$$
$$l \cdot \gamma_0 U_-^j = 0, \tag{5.12}$$
$$\mathscr{F}_1(\gamma U^j, \varphi^j) = O(t^{j+1}), \tag{5.13}$$
$$\mathscr{F}_2(\gamma U^j, \varphi^j) = O(t^{j+1}) \tag{5.14}$$

and

$$\epsilon_{T_0}(U^j - U^0, \varphi^j - \sigma t) \leq \epsilon, \tag{5.15}$$
$$\|(U^j, \varphi^j)\|_{W_{T_0}} \leq M. \tag{5.16}$$

Proof The proof of this theorem is similar to the one given in [15, Proposition 7.1.2] by iteration and by using Theorem 4.2. Here we sketch the proof for completeness. We take $U_\pm^0 =$

$V_\pm(x_2)$, $\varphi_i^0 = \sigma_i(x_2)t$ $(i = 1, 2)$, where $V_\pm(x_2)$ and $\sigma_i(x_2)$ satisfy the 0-th order compatibility conditions. We define successively U^j, φ^j by

$$U_\pm^{j+1} = U_\pm^j + V_\pm^j, \quad \varphi^{j+1} = \varphi^j + \psi^j,$$

with $(V_\pm^j, \psi^j)$ satisfying

$$\begin{aligned} L_\pm^0 V_\pm^j &= -L_\pm(U_\pm^j, \varphi^j)U_\pm^j, \\ \gamma_0 V_- \cdot l &= 0, \\ F_i^0(\gamma V^j, \psi^j) &= -\mathscr{F}_i(\gamma U^j, \varphi^j), \quad i = 1, 2, \end{aligned}$$

where the operators $L_\pm^0$, F_i^0 are the same as in (4.11) with the coefficients valued at U^0, φ^0.

It is convenient to add two conditions

$$U^j - U^0 = O(t), \quad \varphi^j - \varphi^0 = O(t^2) \tag{5.17}$$

to (5.11)–(5.14). Obviously, (5.11)–(5.14) and (5.17) are valid for $j = 0$. Now suppose that they are valid for $j \le n$, and we prove that they are also valid for $j = n + 1$. By using Theorem 4.2, from (5.11)–(5.14) for $j = n$, we have

$$V^n = O(t^{n+1}), \quad \psi^n = O(t^{n+2}), \tag{5.18}$$

which implies $H\psi^n = O(t^{n+1})$. Since $L_\pm(U_\pm^{n+1}, \varphi^{n+1}) - L_\pm(U_\pm^n, \varphi^n)$ can be written as a first order partial differential operator with coefficients containing $U^{n+1} - U^n$ and $H\psi^n$, we get

$$(L_\pm(U_\pm^{n+1}, \varphi^{n+1}) - L_\pm(U_\pm^n, \varphi^n))U_\pm^{n+1} = O(t^{n+1}). \tag{5.19}$$

Similarly, $L_\pm(U_\pm^n, \varphi^n) - L_\pm^0$ can be written as a sum of the first order operator with respect to x_2 and the first order operator with respect to (t, x_1) with coefficients containing the factors $U^n - U^0$ and $H(\varphi^n - \varphi^0)$. Therefore, by $\nabla_{(t,x_1)} V^n = O(t^n)$, $\partial_{x_2} V^n = O(t^{n+1})$ and (5.17), we have

$$(L_\pm(U_\pm^n, \varphi^n) - L_\pm^0)V_\pm^n = O(t^{n+1}). \tag{5.20}$$

Combining (5.19)–(5.20) with

$$f_\pm^{n+1} = (L_\pm(U_\pm^{n+1}, \varphi^{n+1}) - L_\pm(U_\pm^n, \varphi^n))U_\pm^{n+1} + (L_\pm(U_\pm^n, \varphi^n) - L_\pm^0)V_\pm^n,$$

we get $f_\pm^{n+1} = O(t^{n+1})$.

Considering

$$\begin{aligned} g_i^{n+1} &= \mathscr{F}_i(\gamma U^n + \gamma V^n, \varphi^n + \psi^n) \\ &= \mathscr{F}_i(\gamma U^n + \gamma V^n, \varphi^n + \psi^n) + F_{i,(\gamma U^n, \varphi^n)}(\gamma V^n, \psi^n) + O(|\gamma V^n|^2 + |\psi^n|^2) \\ &= (F_{i,(\gamma U^n, \varphi^n)} - F_i^0)(\gamma V^n, \psi^n) + O(|\gamma V^n|^2 + |\psi^n|^2), \end{aligned}$$

from (5.17), we know $\gamma U^n - \gamma U^0 = O(t)$, $\varphi^n - \varphi^0 = O(t^2)$. Then by $\gamma V^n = O(t^{n+1})$, $\psi^n = O(t^{n+1})$ and $\partial_{x_2}\psi^n = O(t^{n+1})$, we get $g_i^{n+1} = O(t^{n+2})$. Therefore, the relations (5.11)–(5.14) hold when $j = n + 1$.

To prove (5.15)–(5.16), we modify the sequence $\{(U^j, \varphi^j)\}$. Let $\zeta(t) \in C_0^\infty$ be a cut-off function, equal to 1 near the origin. δ_J is a constant, which will be determined later. For $j \le J$, set

$$\widetilde{U}^j = \zeta\Big(\frac{t}{\delta_J}\Big)(U^j - U^0) + U^0, \quad \widetilde{\varphi}^j = \zeta\Big(\frac{t}{\delta_J}\Big)(\varphi^j - \varphi^0) + \varphi^0.$$

Obviously, $(\widetilde{U}^j, \widetilde{\varphi}^j)$ still satisfy (5.11)–(5.14). We may choose δ_J sufficiently small such that finite pairs $(\widetilde{U}^j, \widetilde{\varphi}^j)$ for $j \le J$ satisfy (5.15). Moreover, letting $M = 1 + \max_{j \le J} \|(U^j, \varphi^j)\|$, the inequality (5.14) holds for $(\widetilde{U}^j, \widetilde{\varphi}^j)$.

For $j \ge J$, set

$$\widetilde{U}^j = \widetilde{U}^J + \zeta\Big(\frac{t}{\delta_j}\Big)(U^j - U^J), \quad \widetilde{\varphi}^j = \widetilde{\varphi}^J \zeta\Big(\frac{t}{\delta_j}\Big)(\varphi^j - \varphi^J).$$

As mentioned above, the equalities (5.11)–(5.14) are valid. Taking δ_j sufficiently small, we have $\epsilon_{T_0}(\widetilde{U}^j - \widetilde{U}^J, \widetilde{\varphi}^j - \widetilde{\varphi}^J) < \epsilon$, $\|\widetilde{U}^j - \widetilde{U}^J, \widetilde{\varphi}^j - \widetilde{\varphi}^J\|_{W_{T_0}} < 1$. Hence (5.15)–(5.16) are valid for all $(\widetilde{U}^j, \widetilde{\varphi}^j)$.

5.3 The iteration scheme and existence

In this subsection, we introduce the iteration scheme and establish the existence of the solution to the problem (3.18).

For $0 < T < \frac{T_0}{2}$, we denote by E_T and E'_T the two linear extension operators

$$E_T : B^k_{\lambda+\frac{1}{2}}(G^T_-) \times H^k_{\lambda+\frac{1}{2}}(G^T_+) \to B^k_{\lambda+\frac{1}{2}}(G^{T_0}_-) \times H^k_{\lambda+\frac{1}{2}}(G^{T_0}_+),$$
$$E'_T : H^k_\lambda(\Gamma^T_i) \to H^k_\lambda(\Gamma^{T_0}_i), \quad i = 1, 2$$

with norms less than C for any $k, \lambda \in \mathbb{R}^+$, and $\mathrm{supp} E_T U \subset G^{2T}_- \times G^{2T}_+$, $\mathrm{supp} E'_T g \subset \Gamma^{2T}_i$ for $U \in B^k_{\lambda+\frac{1}{2}}(G^T_-) \times H^k_{\lambda+\frac{1}{2}}(G^T_+)$ and $g \in \Gamma^T_i$ $(i = 1, 2)$.

The iteration scheme for solving the problem (3.18) is as follows, which is similar to [14]. For fixed $j_0 > \lambda_0(K)$, we choose (U^{j_0}, φ^{j_0}) constructed in Theorem 5.2 as (U^0, φ^0) and $(V^0, \psi^0) = (0, 0)$. We set

$$(U^{n+1}, \varphi^{n+1}) = (U^0, \varphi^0) + (E_T V^{n+1}, E'_T \psi^{n+1}),$$

where (V^{n+1}, ψ^{n+1}) is the solution of the following problem:

$$\begin{cases} L_\pm(U^n_\pm, \varphi^n) V^{n+1}_\pm = -L_\pm(U^n_\pm, \varphi^n) U^0_\pm, & \text{in } G^{T_0}_\pm, \\ l \cdot \gamma_0 V^{n+1}_- = 0, & \text{on } \Gamma^{T_0}_0, \\ F_{1,(\gamma U^n, \varphi^n)}(\gamma V^{n+1}, \psi^{n+1}) = -\mathscr{F}_1(\gamma U^n, \varphi^n) + F_{1,(\gamma U^n, \varphi^n)}(\gamma V^n, , \psi^n), & \text{on } \Gamma^{T_0}_1, \\ F_{2,(\gamma U^n, \varphi^n)}(\gamma V^{n+1}, \psi^{n+1}) = -\mathscr{F}_2(\gamma U^n, \varphi^n) + F_{2,(\gamma U^n, \varphi^n)}(\gamma V^n, \psi^n), & \text{on } \Gamma^{T_0}_2. \end{cases} \tag{5.21}$$

With the above iteration scheme, we now give the following result.

Theorem 5.3 (i) *There exist $\epsilon_1 > 0$, $M_1 > 0$ and $T \in (0, T_0)$, such that for $T \le T_1$ the sequence defined above satisfies*

$$\epsilon_{T_0}(U^n - U(0), \varphi^n - \sigma(0)t) \le \epsilon_1, \quad \|(U^n, \varphi^n)\|_{W_{T_0}} \le M_1. \tag{5.22}$$

(ii) *There exist $C_0 > 0$, $\lambda_0 > 0$, such that for $\lambda > \lambda_0$, $T \le T_1$, the sequence defined in the above satisfies*

$$\begin{aligned} &\|(U^{n+2} - U^{n+1}, \varphi^{n+2} - \varphi^{n+1})\|^2_{0,\lambda,T} \\ \le\ & C_0 T \|(U^{n+1} - U^n, \varphi^{n+1} - \varphi^n)\|^2_{0,\lambda,T}. \end{aligned} \tag{5.23}$$

The proof of this Theorem is analogous to that given in [4, 15]. Here we sketch the main steps for completeness.

Proof (i) The inequality (5.22) is valid for $n=0$ obviously. Supposing that it is true for n, we prove that it is valid for $n+1$.

Write $L_\pm(U^n_\pm,\varphi^n)U^0_\pm$ as $L_\pm(U^0_\pm,\varphi^0)U^0_\pm-(L_\pm(U^0_\pm,\varphi^0)-L_\pm(U^n_\pm,\varphi^n))U^0_\pm$. Noticing that

$$\|A(U^n,\varphi^n)-A(U^0,\varphi^0)\|^2_{H^N_{\lambda-\frac12,T}}\le C(\|V^n\|^2_{C^N_{\lambda-\frac12,T}}+\|H\psi^n\|^2_{H^N_{\lambda-\frac12,T}}),$$
$$\|H\psi^n\|^2_{H^N_{\lambda-\frac12,T}}\le C\|\psi^n\|^2_{H^{N+1}_{\lambda-\frac12,T}},$$

and $L_\pm(U^0_\pm,\varphi^0)U^0_\pm=O(t^{j_0})$ $(j_0>\lambda_0(K))$, we get

$$\|L(U^n,\varphi^n)U^0\|^2_{C^N_{\lambda-\frac12,T}}\le C_1T. \tag{5.24}$$

Similarly, we have

$$\|F_{(\gamma U^n,\varphi^n)}(\gamma V^{n+1},\psi^{n+1})\|^2_{H^N_{2\lambda-1,T}}\le C(\|\gamma V^n\|^2_{H^N_{\lambda,T}}+\|\psi^n\|^2_{H^{N+1}_{\lambda+1,T}}).$$

In the case of $2\lambda>\lambda+1$, the above inequality yields

$$\|F_{(\gamma U^n,\varphi^n)}(\gamma V^{n+1},\psi^{n+1})\|^2_{H^N_{2\lambda-1,T}}\le C_2T. \tag{5.25}$$

If we properly choose ϵ_1 and M_1, we can check the validity of the conditions in Theorem 4.1. Thus, for the problem (5.21), we obtain $(V^{n+1},\psi^n)\in W^N_{\lambda,T}$ and

$$|\!|\!|(V^n,\psi^n)|\!|\!|^2_{N,\lambda,T}\le C_3T. \tag{5.26}$$

When $N\ge 9$, we can use the embedding theorem to obtain

$$\epsilon_{T_0}(U^{n+1}-U(0),\varphi^{n+1}-\sigma(0)t)\le C_3C_4KT. \tag{5.27}$$

Taking $T_1=\min\left(\frac{1}{C_3},\frac{\epsilon_1}{C_3C_4K}\right)$, (5.22) is valid for (U^{n+1},φ^{n+1}). Thus the sequences $\{U^n,\varphi^n\}_{n\ge0}$ can be determined successively.

(ii) Denote $a^n=U^{n+1}-U^n$, $\alpha^n=\varphi^{n+1}-\varphi^n$ and

$$b^n_\pm=L_\pm(U^{n+1}_\pm,\varphi^n)a^{n+1}_\pm,$$
$$\beta^n_i=F_{i,(\gamma U^{n+1},\varphi^{n+1})}(\gamma a^{n+1},\alpha^{n+1}),\quad i=1,2.$$

From the iteration scheme (5.21), we have

$$b^n_\pm=(L_\pm(U^n_\pm,\varphi^n)-L_\pm(U^{n+1}_\pm,\varphi^{n+1}))U^{n+1}_\pm,$$
$$\beta^n_i=-\mathscr{F}_i(\gamma U^{n+1},\varphi^{n+1})+\mathscr{F}_i(\gamma U^n,\varphi^n)+F_{i,(\gamma U^n,\varphi^n)}(\gamma a^n,\alpha^n),\quad i=1,2.$$

When $N\ge 10$, we obtain

$$\|b^n\|^2_{C^{N-1}_{\lambda,T}}\le C_1T|\!|\!|(U^{n+1}-U^n,\varphi^{n+1}-\varphi^n)|\!|\!|^2_{N-1,\lambda-1,T},$$
$$\|\beta^n\|^2_{H^{N-1}_{2\lambda-1,T}}\le C_2|\!|\!|(U^{n+1}-U^n,\varphi^{n+1}-\varphi^n)|\!|\!|^2_{N-1,\lambda-1,T}.$$

Letting $2\lambda - 1 > \lambda + 1$ and noticing the uniform bound of $|||(U^n, \varphi^n)|||_{N-1,\lambda,T_0}$, we have

$$\|\beta^n\|^2_{H^{N-1}_{\lambda,T}} \leq C_2 T |||(U^{n+1} - U^n, \varphi^{n+1} - \varphi^n)|||^2_{N-1,\lambda-1,T}.$$

Employing Theorem 4.1 for the problem $(V^{n+2} - V^{n+1}, \psi^{n+2} - \psi^{n+1})$ from (5.21), one deduces

$$|||(U^{n+2} - U^{n+1}, \varphi^{n+2} - \varphi^{n+1})|||^2_{N-1,\lambda,T} \leq C_3 \lambda |||(b^n, \beta^n)|||^2_{N-1,\lambda,T},$$

which yields (5.23) by choosing λ properly.

From Theorem 5.3, it is easy to know that $\{U^n, \varphi^n\}_{n\geq 0}$ is bounded in $B^s_{\lambda+\frac{1}{2}}(G^{T_0}_-) \times H^s_{\lambda+\frac{1}{2}}(G^{T_0}_+)$ and convergent in $B^0_{\lambda+\frac{1}{2}}(G^{T_0}_-) \times H^0_{\lambda+\frac{1}{2}}(G^{T_0}_+)$, which implies that their limits (U, φ) are the solution to the problem (3.18).

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