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Chen's Conjecture and Its Generalization*

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Abstract Let l_1, l_2, $\cdots$, l_g be even integers and x be a sufficiently large number. In this paper, the authors prove that the number of positive odd integers $k \le x$ such that $(k+l_1)^2$, $(k+l_2)^2$, $\cdots$, $(k+l_g)^2$ can not be expressed as $2^n + p^\alpha$ is at least $c(g)x$, where p is an odd prime and the constant $c(g)$ depends only on g.

Keywords Chen's conjecture, Powers of 2, Primes, Selberg's sieve method
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1 Introduction

In 1849, de Polignac [12] conjectured that every odd number larger than 3 can be written as the sum of an odd prime and a power of 2. He found a counterexample 959 soon. In 1934, Romanoff [13] proved that there are positive proportion natural numbers which can be expressed in the form $2^k + p$, where k is a positive integer and p is an odd prime. On the other hand, in 1950, van der Corput [16] proved that the counterexamples of de Polignac's conjecture form a set of positive lower density. By employing a covering system, Erdős [7] proved that there is an infinite arithmetic progression of positive odd numbers, each of which has no representation of the form $2^k + p$. Usually, nonlinear problems are difficult. Chen [1] firstly proved that if $(r, 12) \le 3$, then the set of positive odd integers k such that $k^r - 2^n$ has at least two distinct prime factors for all positive integers n contains an infinite arithmetic progression. For general r, Chen [1] gave the following conjecture.

Conjecture 1.1 For any positive integer r, there exist infinitely many positive odd numbers k such that $k^r - 2^n$ has at least two distinct prime factors for all positive integers n.

Conjecture 1.1 is particularly difficult when r is a high power of 2. Filaseta, Finch and Kozek [9] confirmed Conjecture 1.1 for $r = 4, 6$. Wu and Sun [17] proved that for any positive integer m divisible by none of $3, 5, 7, 11, 13$, there exists an infinite arithmetic progression of positive odd integers, the mth powers of whose terms are never of the form $2^n \pm p^\alpha$. Chen [2] proved that there exists an arithmetic progression of positive odd numbers for each term M of which none of the eight consecutive odd numbers M, $M-2$, $M-4$, $M-6$, $M-8$, $M-10$, $M-12$ and $M-14$ can be expressed in the form $2^n + p^\alpha$, where p is a prime and n, α are nonnegative integers.

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Up to now, all the arguments to the above problems used the method of covering congruences. Recently, using Selberg's sieve method, Tao [15] proved that for any $K \geq 1$ and sufficiently large x, the number of primes $p \leq x$ such that $|kp \pm ja^n|$ is composite for all $1 \leq a, j, k \leq K$ and $1 \leq i \leq K \log x$, is at least $C_K \frac{x}{\log x}$, where C_K is a constant depending only on K. Using Tao's idea, in this paper we shall prove the following theorem.

Theorem 1.1 *Let $l_1, l_2, \cdots, l_g$ be even integers and x be a sufficiently large number. Then the number of positive odd integers $k \leq x$ such that $(k+l_1)^2, (k+l_2)^2, \cdots, (k+l_g)^2$ can not be expressed as $2^n + p^\alpha$ is at least $c(g)x$, where p is an odd prime and the constant $c(g)$ depends only on g.*

Remark 1.1 With a similar discussion, we can prove that for any even integers $l_1, l_2, \cdots, l_g$ and a sufficiently large number x, the number of primes $q \leq x$ such that $q + l_1, q + l_2, \cdots, q + l_g$ can not be expressed as $2^n + p^\alpha$ is at least $c_1(g)\frac{x}{\log x}$, where p is a prime and the constant $c_1(g)$ depends only on g.

2 Proofs

Lemma 2.1 (see [10, Theorem 4.1]) *Let A be a finite set of integers. Let P be a set of some primes, $P(z) = \prod\limits_{\substack{p<z \\ p\in P}} p$ and $S(A, P, z) = \sum\limits_{\substack{a\in A \\ (a,\, P(z))=1}} 1$. Take a multiplicative function $\omega(d)$ satisfying $\omega(d) > 0$ for $d \mid P(z)$ and $\omega(d) = 0$ for otherwise. Let B_1, B_2 and κ be three positive constant numbers such that $0 \leq \frac{\omega(p)}{p} \leq 1-\frac{1}{B_1}$ and $\sum\limits_{w\leq p<z} \frac{\omega(p)\log p}{p} \leq \kappa \log \frac{z}{w} + B_2$. Let A_d be the number of elements of A which are divisible by d. Take a number X and let $R(d) = |A_d| - \frac{\omega(d)}{d}X$. Let $\nu(d)$ denote the number of distinct prime factors of d and $W(z) = \prod\limits_{p<z} \big(1 - \frac{\omega(p)}{p}\big)$. Then we have*

$$S(A, P, z) \leq B_3 XW(z) + \sum_{\substack{d\mid P(z) \\ d<z^2}} 3^{\nu(d)} \mid R(d)|,$$

where B_3 depends only on B_1, B_2 and κ.

Lemma 2.2 *Let x be a sufficiently large number. Then there exists an absolutely constant number c_2 such that*

$$\#\{k \leq x :\ k \equiv a \pmod m,\ k^2 = 2^n + p\} \leq c_2 \frac{x}{\phi(m)\log x} \prod_{2<p\mid m} \Big(1 + \frac{1}{p-2}\Big)$$

for any positive integer n.

Proof Suppose that $A(n) = \{k^2 - 2^n : k \leq x,\ k \equiv a \pmod m\}$. Take $P = \{p : p \equiv \pm 1 \pmod 8,\ (p, m) = 1,\ p \text{ is a prime}\}$. Take a multiplicative function $\omega(d)$ satisfying $\omega(p) = 2$ for $p \in P$ and $\omega(p) = 0$ for $p \notin P$. Let $B_1 = 3$, and we have $0 \leq \frac{\omega(p)}{p} \leq 1 - \frac{1}{B_1}$.

By the prime number theory in arithmetic progressions [6], we have

$$\sum_{\substack{p\leq x \\ p\equiv s \pmod r}} \log p = \frac{x}{\phi(r)} + O\Big(\frac{x}{\log^2 x}\Big).$$

Thus, we have

$$\begin{aligned}
&\sum_{w\le p<z}\frac{\omega(p)\log p}{p}\\
&\le 2\sum_{\substack{w\le p<z\\ p\equiv\pm1 \ (\mathrm{mod}\ 8)}}\frac{\log p}{p}\\
&\le 2\int_w^z\frac{1}{t}\mathrm{d}\sum_{\substack{w\le p<t\\ p\equiv\pm1 \ (\mathrm{mod}\ 8)}}\log p\\
&\le \log\frac{z}{w}+O(1).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&W(z)=\prod_{p<z}\Big(1-\frac{\omega(p)}{p}\Big)\\
&\le \prod_{\substack{p<z\\ p\equiv\pm1 \ (\mathrm{mod}\ 8)}}\Big(1-\frac{2}{p}\Big)\prod_{2<p|m}\Big(1-\frac{2}{p}\Big)^{-1}\\
&\le \frac{c_3}{\log z}\prod_{2<p|m}\Big(1-\frac{1}{p}\Big)^{-1}\prod_{2<p|m}\Big(1-\frac{2}{p}\Big)^{-1}(1-\frac{1}{p})\\
&\le \frac{c_3 m}{\phi(m)\log z}\prod_{2<p|m}\Big(1+\frac{1}{p-2}\Big).
\end{aligned}$$

Take $X=|A(n)|$. Noting $|R(d)|\le 2^{\nu(d)}$ for $d\mid P(z)$, we have

$$\sum_{\substack{d|P(z)\\ d<z^2}}3^{\nu(d)}|R(d)|\le\sum_{d<z^2}6^{\nu(d)}=O(z^{2(\frac{\log 6}{\log 2}+1)}).$$

By Lemma 2.1, taking $z=x^{\frac{1}{6}}$, we have

$$\begin{aligned}
&\#\{k\le x: k\equiv a \pmod m,\ k^2=2^n+p\}\\
&\le \#\{k\le x: k\equiv a \pmod m,\ k^2=2^n+p,\ p\ge z\}\\
&\quad+\#\{k\le x: k\equiv a \pmod m,\ k^2=2^n+p,\ p<z\}\\
&\le \#\{k\le x: k\equiv a \pmod m,\ (k^2-2^n,P(z))=1\}+z\\
&\le \#\{k^2-2^n: k\le x,\ k\equiv a \pmod m,\ (k^2-2^n,P(z))=1\}+z\\
&\le S(A(n),P,z)+z\\
&\le c_4\frac{x}{\phi(m)\log z}\prod_{2<p|m}\Big(1+\frac{1}{p-2}\Big)+\sum_{\substack{d|P(z)\\ d<z^2}}3^{\nu(d)}|R(d)|+z.\\
&\le c_5\frac{x}{\phi(m)\log z}\prod_{2<p|m}\Big(1+\frac{1}{p-2}\Big)+O(z^{2(\frac{\log 6}{\log 2}+1)})\\
&\le c_2\frac{x}{\phi(m)\log x}\prod_{2<p|m}\Big(1+\frac{1}{p-2}\Big).
\end{aligned}$$

This completes the proof of Lemma 2.2.

By Lemma 2.1, similar to Lemma 2.2, we have the following Lemma 2.3.

Lemma 2.3 *Let x be a sufficiently large number. Suppose that $p_1, p_2, \cdots, p_h$ are distinct primes less than $x^{\frac{1}{6}}$. Then*

$$\#\{1 \le n \le x : n \not\equiv 0 \pmod{p_j} \text{ for every } 1 \le j \le h\} \le c_6 x \prod_{j=1}^{h}\Big(1 - \frac{1}{p_j}\Big),$$

where c_6 is an absolute constant.

Lemma 2.4 (see [5]) *Let A, B and C be nonzero integers. Let p, q and r be positive integers for which $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. Then the generalized Fermat equation $Ax^p + By^q = Cz^r$ has only finitely many solutions in integers x, y, z with $(x, y, z) = 1$.*

Lemma 2.5 *Let r be a positive integer. Then the equation*

$$k^r = 2^n + p^\alpha$$

has $o\big(\frac{x}{\log x}\big)$ solutions in k, n, p, α with odd integers $k \le x$, positive integers n, primes p and integers $\alpha \ge 2$.

Proof Suppose that $\alpha > r$. Since $n \le \frac{r \log x}{\log 2}$, $\alpha \le \frac{r \log x}{\log 2}$ and $p \le x^{\frac{r}{\alpha}}$, there exist $O\big(x^{\frac{r}{\alpha}}\big(\frac{r \log x}{\log 2}\big)^2\big)$ solutions.

Suppose that $\alpha \le r$. For $r = 2$ and $\alpha = 2$, we have $k = 2^{n_1} + p$ and $k = 2^{n_2} - p$. Hence $2k = 2^{n_1} + 2^{n_2}$, and thus the number of k is $O(\log^2 x)$. So there exist $O(\log^3 x)$ solutions. For $r \ge 3$, let $n = 7l + t,\ t = 0, 1, 2, \cdots, 6$. Thus we have

$$k^r = 2^t (2^l)^7 + p^\alpha.$$

Noting $\frac{1}{r} + \frac{1}{7} + \frac{1}{\alpha} < 1$, and by Lemma 2.4, there exist finitely many solutions.

This completes the proof of Lemma 2.5.

Lemma 2.6 (see [9]) *The series*

$$\sum_{n=1}^{\infty} \frac{(\log n)^\alpha}{P(2^n - 1)}$$

converges for any $\alpha < \frac{1}{2}$, where $P(n)$ denotes the largest prime factor of n.

Proof of Theorem 1.1 By Lemma 2.6, we have that the series $\prod\limits_{n=3}^{\infty}\big(1 - \frac{1}{P(2^n-1)}\big)^{-1}$ and $\prod\limits_{n=3}^{\infty}\big(1 + \frac{1}{P(2^n-1)-2}\big)$ are both convergent, and we denote them by c_7 and c_8, respectively. Let C be a sufficiently large constant number. Take $3 \le p_{11}, \cdots, p_{1K_1}, \cdots, p_{g1}, \cdots, p_{gK_g} \le C$ to be primes satisfying

$$\prod_{j=1}^{K_i}\Big(1 - \frac{1}{p_{ij}}\Big) \le \frac{\log 2}{17 g c_2 c_6 c_7 c_8}.$$

Take q_{ij} to be the largest prime factor of $2^{p_{ij}} - 1$ and $W = \prod\limits_{i=1}^{g}\prod\limits_{j=1}^{K_i} q_{ij}$. Let M be an odd integer satisfying

$$M + l_i \equiv 1 \Big(\bmod\ \prod_{j=1}^{K_i} q_{ij}\Big), \quad i = 1, 2, \cdots, g.$$

Let

$$
\begin{aligned}
S &= \{k \le x : k \equiv M \pmod{2W}\},\\
T_{1i} &= \{k \le x : k \equiv M \pmod{2W}, (k+l_i)^2 \text{ can be expressed as } 2^n + p^\alpha\\
&\qquad \text{with } p, \alpha, n \text{ satisfying } n \equiv 0 \pmod{p_{ij}} \text{ for some } j \text{ with } 1 \le j \le K_i\}
\end{aligned}
$$

and

$$
\begin{aligned}
T_{2i} = \{k \le x : k \equiv M \pmod{2W},\ & (k+l_i)^2 \text{ can only be expressed as } 2^n + p^\alpha\\
&\text{with } p, \alpha, n \text{ satisfying } n \not\equiv 0 \pmod{p_{ij}} \text{ for all } j \text{ with } 1 \le j \le K_i\}.
\end{aligned}
$$

For $k \in T_{1i}$, we have $(k+l_i)^2 - 2^n \equiv (M+l_i)^2 - 2^n \pmod{q_{ij}}$, so $(k+l_i)^2 - 2^n \equiv 0 \pmod{q_{ij}}$. Thus $p = q_{ij}$. Hence $|T_{1i}| \le K_i(\frac{2\log(x+l_i)}{\log 2})^2$.

By Lemmas 2.2–2.3 and Lemma 2.5, we have

$$
\begin{aligned}
|T_{2i}| &\le c_6 \frac{2\log(x+l_i)}{\log 2} \prod_{j=1}^{K_i}\Big(1 - \frac{1}{p_{ij}}\Big)\Big(c_2 \frac{x+l_i}{\phi(2W)\log(x+l_i)} \prod_{p|W}\Big(1+\frac{1}{p-2}\Big)\Big) + o\Big(\frac{x}{\log x}\Big)\\
&\le \frac{3c_2c_6}{\log 2}\frac{x+l_i}{W}\prod_{j=1}^{K_i}\Big(1-\frac{1}{p_{ij}}\Big)\prod_{p|W}\Big(1+\frac{1}{p-2}\Big)\prod_{i=1}^{g}\prod_{j=1}^{K_i}\Big(1-\frac{1}{q_{ij}}\Big)^{-1}\\
&\le \frac{4c_2c_6}{\log 2}\frac{x}{W}\prod_{j=1}^{K_i}\Big(1-\frac{1}{p_{ij}}\Big)\prod_{i=1}^{g}\prod_{j=1}^{K_i}\Big(1+\frac{1}{q_{ij}-2}\Big)\prod_{i=1}^{g}\prod_{j=1}^{K_i}\Big(1-\frac{1}{q_{ij}}\Big)^{-1}\\
&\le \frac{x}{4gW}.
\end{aligned}
$$

Thus, we have

$$
|S| - \sum_{i=1}^{g}(|T_{1i}| + |T_{2i}|) \ge \frac{x}{2W} - 1 - \Big(\sum_{i=1}^{g} K_i \frac{2\log(x+l_i)}{\log 2}\Big)^2 - \frac{x}{4W} \ge c(g)x.
$$

This completes the proof of Theorem 1.1.

Now, considering Conjecture 1.1 and the proof of Theorem 1.1, we propose the following Conjecture 2.1.

Conjecture 2.1 Let r be a positive integer. Let x be a sufficiently large number. Then there exists an absolutely constant number c_9 such that

$$
\#\{k \le x :\ k \equiv a \pmod{m},\ k^r = 2^n + p\} \le c_9 \frac{x}{\phi(m)\log x}\prod_{2<p|m}\Big(1+\frac{1}{p-2}\Big)
$$

for any positive integer n.

Remark 2.1 Under Conjecture 2.1, we can prove the following result: Let $l_1, l_2, \cdots, l_g$ be even integers and x be a sufficiently large number. Then the number of positive odd integers $k \le x$ such that $(k+l_1)^r, (k+l_2)^r, \cdots, (k+l_g)^r$ can not be expressed as $2^n + p^\alpha$ is at least $c_{10}(g,r)x$, where p is an odd prime and the constant $c_{10}(g,r)$ depends on g and r.

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