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Limit Cycles Bifurcating from a Quadratic Reversible Lotka-Volterra System with a Center and Three Saddles*

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Abstract This paper is concerned with limit cycles which bifurcate from a period annulus of a quadratic reversible Lotka-Volterra system with sextic orbits. The authors apply the property of an extended complete Chebyshev system and prove that the cyclicity of the period annulus under quadratic perturbations is equal to two.

Keywords Reversible Lotka-Volterra systems, Abelian integrals, Limit cycles
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1 Introduction

The second part of Hilbert's 16th problem (see [6]) asks about the maximum number and the location of limit cycles of a planar polynomial vector fields of degree n. If the quadratic centers belong to the Hamiltonian class, then the study of the number of limit cycles bifurcating from a period annulus or annuli (i.e., the weak Hilbert's 16th problem for $n = 2$) is finished, and the study of the number of limit cycles bifurcating from a singular loop, or from infinity is partially finished (see [3–4, 8–9, 12, 15, 17] and the references therein). If the quadratic centers belong to the reversible class (and do not belong to the Hamiltonian class), then the study seems very difficult, and the known results are very limited.

A weaker version of this problem is proposed by Arnold (see [1]) to study the zeros of Abelian integrals, that is the weak Hilbert's 16th problem or infinitesimal Hilbert's 16th problem. The problem is related to in the following way. Consider the perturbed system of a Hamiltonian vector field $X_\varepsilon = X_H + \varepsilon Y$, where

$$X_H = -H_y\partial_x + H_x\partial_y, \quad Y = P\partial_x + Q\partial_y.$$

A closed connected component of a level curve $\{H = h\}$ is denoted by γ_h and called an oval of H. The Abelian integral is

$$I(h) = \oint_{\gamma_h} Q\mathrm{d}x - P\mathrm{d}y.$$

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Therefore the number of isolated zeros of $I(h)$, counted with multiplicities, provides an upper bound for the number of ovals of $\{H = h\}$ that generates limit cycles of X_ε for $\varepsilon \approx 0$ (see [5, 12] for details). If the unperturbed system is integrable and non-Hamiltonian, then one has to consider pseudo-Abelian integrals (see [1] for details). As far as we know, most of the papers investigate the Hamiltonian centers and few papers study the non-Hamiltonian centers (see [3, 8, 17] for instance). Recently, Zhao [16] proved that the cyclicity of the period annulus of Q_4 is less than or equal to five.

It is well-known by Iliev [8] that any quadratic polynomial reversible Lotka-Volterra system can be written in the complex form

$$\dot{z} = -\mathrm{i}z + z^2 + b|\overline{z}|^2, \quad z = x + \mathrm{i}y,$$

or in the real form

$$\begin{cases} \dot{x} = y + (b+1)x^2 - (b+1)y^2, \\ \dot{y} = -x + 2(1-b)xy, \end{cases} \tag{1.1}$$

where b is a real parameter. Arnold [2] declared that the infinitesimal problem for system (1.1) is still open. For the system (1.1) of genus one, Gautier et al. [4] classify this kind of systems into 6 cases (rlv1)–(rlv6). Until now, [7, 11] have studied the cases (rlv1) and (rlv2), respectively, and [5, 14] have studied the cases (rlv3) and (rlv4), respectively.

The cyclicity of system (1.1) under quadratic perturbations for $b = 0,\ \frac{1}{2}$, was studied in [3–4], respectively. In this paper we study the case $b = \frac{5}{3}$, that is, the number of limit cycles bifurcates from the period annulus of the following quadratic reversible Lotka-Volterra system:

$$\begin{cases} \dot{x} = -y(1+x), \\ \dot{y} = x - 2x^2 + 2y^2. \end{cases} \tag{1.2}$$

System (1.2) has a first integral

$$H(x,y) = -\frac{1}{6}x^2(3+2x)(-2+3x^2+2x^3) + (1+x)^4y^2$$

with the integrating factor

$$M(x) = 2(1+x)^2.$$

Note that system (1.1) for $b = \frac{5}{3}$ can be reduced to system (1.2) by using a linear transformation. There is a period annulus surrounding the center at $(x, y) = (0, 0)$ bounded by three straight lines

$$\Big\{x = -1,\ y = \pm\frac{2x-1}{\sqrt{6}}\Big\},$$

corresponding to $H(x,y) = \frac{1}{6}$. The intersection points $(x,y) = \big(-1, -\frac{\sqrt{6}}{2}\big)$, $(x,y) = \big(-1, \frac{\sqrt{6}}{2}\big)$, and $(x,y) = \big(\frac{1}{2}, 0\big)$ of the three straight lines are three saddles of system (1.2). Therefore, the period annulus can be expressed by $\big\{\gamma_h,\ h \in \big(0, \frac{1}{12}\big)\big\}$, where the periodic orbit $\gamma_h \subset \{H(x,y) = h\}$.

The next result is a particular case of Theorem 3 in [8]. For convenience, we state it in the present paper.

Lemma 1.1 *The exact upper bound for the number of limit cycles produced by the period annulus of system* (1.2) *under quadratic perturbations is equal to the maximal number of zeros in* $h \in \left(0,\ \frac{1}{12}\right)$ (*counting multiplicities*) *of the Abelian integral*

$$I(h) = \oint_{\gamma_h} (1+x)^3 \Big(ay + b\frac{y}{1+x} + c\frac{x}{y} \Big) \mathrm{d}x, \tag{1.3}$$

where a, b and c are arbitrary constant.

The main result of this paper is the following theorems.

Theorem 1.1 *The cyclicity of the period annulus of system* (1.2) *under quadratic perturbations is two.*

This paper is organized in the following way. In Section 2 we introduce the definitions and the notations that we shall use. In Section 3 we give the proof of Theorem 1.1 by applying Theorem B in [5]. We shall rewrite the Abelian integral (1.3) as a linear combination of $\{I_0(h), I_1(h), I_2(h)\}$ and prove that $(I_0(h), I_1(h), I_2(h))$ forms an extended complete Chebychev system. Due to Theorem B in [5], we turn the problem of the number of zeros of the Abelian integral into a pure algebraic problem, namely, counting zeros of a polynomial. To solve the latter problem we shall use the notion of a resultant. The interested reader is referred to the appendix in [5] for details.

2 The Definitions of Chebyshev Systems

In order to prove the main results, some definitions and lemmas are needed.

Definition 2.1 *Let* $f_0(x), f_1(x), \cdots, f_{n-1}(x)$ *be analytic functions on an open interval* L *of* $\mathbb{R}$. *Then we have the following*:

(a) $(f_0(x), f_1(x), \cdots, f_{n-1}(x))$ *is a Chebyshev system* (*in short, T-system*) *on* L *if any nontrivial linear combination*

$$a_0 f_0(x) + a_1 f_1(x) + \cdots + a_{n-1} f_{n-1}(x)$$

has at most $n-1$ *isolated zeros on* L.

(b) $(f_0(x), f_1(x), \cdots, f_{n-1}(x))$ *is a complete Chebyshev system* (*in short, CT-system*) *on* L *if* $(f_0(x), f_1(x), \cdots, f_{k-1}(x))$ *is a T-system for all* $k = 1, 2, \cdots, n$.

(c) $(f_0(x), f_1(x), \cdots, f_{n-1}(x))$ *is an extended complete Chebyshev system* (*in short, ECT-system*) *on* L *if, for all* $k = 1, 2, \cdots, n$, *any nontrivial linear combination*

$$a_0 f_0(x) + a_1 f_1(x) + \cdots + a_{k-1} f_{k-1}(x)$$

has at most $k-1$ *isolated zeros on* L *counted with multiplicities.*

It is clear that if $(f_0(x), f_1(x), \cdots, f_{n-1}(x))$ is an ECT-system on L, then $(f_0(x), f_1(x), \cdots, f_{n-1}(x))$ is a CT-system on L. However, the reverse implication is not true. Moreover, if $(f_0(x), f_1(x), \cdots, f_{n-1}(x))$ is a T-system on L, and $f(x)$ is an analytic function and has a constant sign on L, then $(f(x)f_0(x), f(x)f_1(x), \cdots, f(x)f_{n-1}(x))$ is a T-system on L.

Remark 2.1 If $(f_0(x), f_1(x), \cdots, f_{n-1}(x))$ is an ECT-system on L, then, for each $k = 0,\ 1,\ 2, \cdots,\ n-1$, there exists a linear combination with exactly k simple zeros on L (see [10, 13] for instance).

Definition 2.2 *Let $f_0(x)$, $f_1(x), \cdots, f_{n-1}(x)$ be analytic functions on an open interval L of $\mathbb{R}$. The Wronskian of $(f_0(x), f_1(x), \cdots, f_{n-1}(x))$ at $x \in L$ is given by*

$$W[f_0, f_1, \cdots, f_{n-1}](x) = \operatorname{Det}(f_j^{(i)}(x))_{0\le i,j\le n-1} = \begin{vmatrix} f_0(x) & \cdots & f_{n-1}(x) \\ f_0'(x) & \cdots & f_{n-1}'(x) \\ \vdots & & \vdots \\ f_0^{(n-1)}(x) & \cdots & f_{n-1}^{(n-1)}(x) \end{vmatrix}.$$

The following result is well-known (see [10, 13] for instance).

Lemma 2.1 *The system $(f_0(x), f_1(x), \cdots, f_{n-1}(x))$ is an ECT-system on L if and only if, for each $k = 1, 2, \cdots, n$,*

$$W[f_0, f_1, \cdots, f_{k-1}](x) \neq 0 \quad \text{for all } x \in L.$$

3 Proof of Main Results

Recall that $H(x, y) = A(x) + B(x)y^2$ with

$$A(x) = -\frac{x^2(3 + 2x)(-2 + 3x^2 + 2x^3)}{6}, \quad B(x) = (1 + x)^4.$$

It is clear that $H(x, y)$ has a local minimum at $(x, y) = (0, 0)$, $B(x) > 0$ and $A(x)$ have a local minimum at $x = 0$. Denote the period annulus associated to the center origin by $\mathcal{P}$ and the projection of $\mathcal{P}$ on the x-axis by $(x_l, x_r) = \left(-1,\ \frac{1}{2}\right)$. Then there exists a unique analytic involution function $\sigma(x)$, such that $A(x) = A(\sigma(x))$ for all $x \in (x_l, x_r) = \left(-1,\ \frac{1}{2}\right)$. The next result is a particular case of Theorem B in [5]. For convenience, we state it in the present paper.

Theorem 3.1 (see [5]) *Let us consider the Abelian integrals*

$$I_i(h) = \oint_{\gamma_h} f_i(x)y^{2s-1}\mathrm{d}x, \quad i = 0, 1, \cdots, n-1,$$

where, for each $h \in (0,\ h_0)$, γ_h is the oval surrounding the origin inside the level curve $\{A(x) + B(x)y^2 = h\}$. Let σ be the involution associated to $A(x)$ and we define

$$l_i(x) = \frac{f_i(x)}{A'(x)B(x)^{\frac{2s-1}{2}}} - \frac{f_i(\sigma(x))}{A'(\sigma(x))B(\sigma(x))^{\frac{2s-1}{2}}}.$$

Then $(I_0, I_1, \cdots, I_{n-1})$ is an ECT-system on $(0, h_0)$ if $s > n - 2$ and $(l_0, l_1, \cdots, l_{n-1})$ is a CT-system on $(0, x_r)$.

Recall that a mapping σ is an involution if $\sigma \circ \sigma = \mathrm{Id}$ and $\sigma \neq \mathrm{Id}$. An involution σ is a diffeomorphism with a unique fixed point. Noting that $l_i(x) = -l_i(\sigma(x))$, we have that $(l_0, l_1, \cdots, l_{n-1})$ is a CT-system on $(0, x_r)$ if and only if $(l_0, l_1, \cdots, l_{n-1})$ is a CT-system on $(x_l, 0)$.

We rewrite the Abelian integral (1.3) as $I(h) = aI_0(h) + bI_1(h) + cI_2(h)$, where

$$I_0(h) = \oint_{\Gamma_h} (1 + x)^3 y\mathrm{d}x, \quad I_1(h) = \oint_{\Gamma_h} (1 + x)^2 y\mathrm{d}x, \quad I_2(h) = \oint_{\Gamma_h} \frac{(1 + x)^3 x}{y}\mathrm{d}x. \tag{3.1}$$

The following lemma, proved in [5], establishes a formula to change the integrand of an Abelian integral into other Abelian integrals that we want.

Lemma 3.1 (see [5]) *Let γ_h be an oval inside the level curve $\{A(x)+B(x)y^2=h\}$, and consider a function $F(x)$ such that $\frac{F(x)}{A'(x)}$ is analytic at $x=0$. Then, for any $k\in\mathbb{N}$,*

$$\oint_{\gamma_h} F(x)y^{2k-1}\mathrm{d}x = \oint_{\gamma_h} G(x)y^{2k+1}\mathrm{d}x,$$

where

$$G(x)=\frac{2}{2k+1}\Big(\frac{BF}{A'}\Big)'(x)-\Big(\frac{B'F}{A'}\Big)(x).$$

In what follows, we shall apply Theorem 3.1 to prove that $(I_0(h),I_1(h),I_2(h))$ is an ECT-system. By Lemma 3.1, it yields that

$$I_2(h)=\oint_{\Gamma_h}\frac{(1+x)^3x}{y}\mathrm{d}x=-2\oint_{\Gamma_h}\frac{(-2+x)(1+x)^3}{(-1+2x)^2}\mathrm{d}x.$$

However, we discover that $n=3$ and $s=1$ in the integrand of $(I_0(h),I_1(h),I_2(h))$, so that the condition $s>n-2$ is not fulfilled. Therefore we must take $s=3$ and apply Lemma 3.1 to overcome the shortcomings. Applying Lemma 3.1, we have that

$$\begin{aligned}I_0(h)&=\oint_{\Gamma_h}(1+x)^3y\mathrm{d}x=\frac{1}{h}\oint_{\Gamma_h}(A(x)+B(x)y^2)(1+x)^3y\mathrm{d}x\\&=\frac{1}{h}\oint_{\Gamma_h}f_0(x)y^3\mathrm{d}x=\frac{1}{h}\widetilde{I}_0(h),\end{aligned}$$

where

$$f_0(x)=\frac{(1+x)^3(24+2x-119x^2-85x^3+190x^4+260x^5+88x^6)}{18(-1+2x)^2}.$$

Exactly in the same way we obtain

$$\begin{aligned}I_1(h)&=\frac{1}{h}\oint_{\Gamma_h}f_1(x)y^3\mathrm{d}x=\frac{1}{h}\widetilde{I}_1(h),\\I_2(h)&=\frac{1}{h}\oint_{\Gamma_h}f_2(x)y^3\mathrm{d}x=\frac{1}{h}\widetilde{I}_2(h),\end{aligned}$$

where

$$\begin{aligned}f_1(x)&=\frac{(1+x)^2(24-4x-111x^2-68x^3+184x^4+240x^5+80x^6)}{18(-1+2x)^2},\\f_2(x)&=\frac{2(1+x)^3(-24-11x+91x^2+53x^3-164x^4-130x^5+40x^6+40x^7)}{9(-1+2x)^4}.\end{aligned}$$

It is clear that $(I_0(h),I_1(h),I_2(h))$ is an ECT-system in the interval $\big(0,\frac{1}{12}\big)$ if and only if $(\widetilde{I}_0(h),\widetilde{I}_1(h),\widetilde{I}_2(h))$ is an ECT-system in the interval $\big(0,\frac{1}{12}\big)$.

Setting

$$F_i(x)=\frac{f_i(x)}{A'(x)B(x)^{\frac{3}{2}}},$$

then

$$\begin{aligned}F_0(x)&=-\frac{24+2x-119x^2-85x^3+190x^4+260x^5+88x^6}{36x(1+x)^6(-1+2x)^3},\\F_1(x)&=\frac{24-4x-111x^2-68x^3+184x^4+240x^5+80x^6}{36x(1+x)^7(-1+2x)^3},\\F_2(x)&=-\frac{-24-11x+91x^2+53x^3-164x^4-130x^5+40x^6+40x^7}{9x(1+x)^6(-1+2x)^5}.\end{aligned}$$

Denote by σ the involution associated to $A(x)$, i.e., $A(x) = A(\sigma(x))$. In order to compute Wronskians, we set $z = \sigma(x)$ and

$$L_i(x,z) = \frac{f_i(z)}{2\sqrt{2}A'(z)B(z)^{\frac{3}{2}}} - \frac{f_i(x)}{A'(x)B(x)^{\frac{3}{2}}} = F(z) - F(x).$$

Then $l_i(x) = L_i(x,z)$. We only need to prove that system $(L_0(x,z), L_1(x,z), L_0(x,z))$ is an ECT-system on $\left(0, \frac{1}{2}\right)$. On account of

$$\begin{aligned} A(z) - A(x) &= -\frac{1}{6}(z-x)(3z + 2z^2 + 3x + 2zx + 2x^2) \\ &\quad \times (-2 + 3z^2 + 2z^3 + 3x^2 + 2x^3) \\ &= 0, \end{aligned}$$

it turns out that $z = \sigma(x)$ is defined by

$$q(x,z) = 3x + 2x^2 + 3z + 2xz + 2z^2 = 0 \tag{3.2}$$

and

$$\sigma'(x) = -\frac{3 + 4x + 2z}{3 + 2x + 4z}.$$

We shall depend on Wolfram Mathematica to compute three Wronskians and the resultant between two polynomials to show the nonexistence of zeros of a polynomial on the interval. In the following, we show the following lemma.

Lemma 3.2 *System $\{L_0(x,z), L_1(x,z), L_2(x,z)\}$ is an ECT-system on the interval $\left(0, \frac{1}{2}\right)$, i.e., system $\{l_0(z), l_1(z), l_2(z)\}$ is an ECT-system on the interval $\left(0, \frac{1}{2}\right)$, where z is defined by* (3.2).

Proof By Lemma 2.1, we split the proof into three cases to show that the three Wronskians have no zeros on $\left(0, \frac{1}{2}\right)$.

First, note that $W[L_0(x,z)] = L_0(x,z)$. By the common denominator and the factorization, we have

$$W[L_0(x,z)] = \frac{(x-z)\alpha_0(x,z)}{36x(1+x)^6(-1+2x)^3 z(1+z)^6(-1+2z)^3},$$

where $\alpha_0(x,z)$ is a polynomial of degree 15 in (x,z) with a very long expression. It follows from direct computations that the resultant with respect to z between $\alpha_0(x,z)$ and $q(x,z)$ is $512(1+x)^{12}(-1+2x)^6\zeta_0(x)$ (i.e., eliminating z from $\alpha_0(x,z) = 0$ and $q(x,z) = 0$), where

$$\begin{aligned} \zeta_0(x) = {} & 576 + 144x - 5808x^2 - 4866x^3 + 23377x^4 + 34643x^5 - 33266x^6 - 96116x^7 \\ & - 29704x^8 + 85432x^9 + 102448x^{10} + 46112x^{11} + 7744x^{12}. \end{aligned}$$

Note that $\zeta_0(0) = 576$, $\zeta_0\left(\frac{1}{2}\right) = 36$, and $\zeta_0(x)$ has a local minimum 35.0652 at $x \approx 0.486258$ on the interval $\left(0, \frac{1}{2}\right)$. Therefore, $\alpha_0(x,z) = 0$ and $q(x,z) = 0$ have no common roots for any $z \in \left(0, \frac{1}{2}\right)$, which implies that $W[L_0(x,z)] \neq 0$ for any $z \in \left(0, \frac{1}{2}\right)$ or $z \in (-1, 0)$.

Secondly, by the definition of $W[L_0(x,z), L_1(x,z)]$, it follows that

$$W[L_0, L_1] = \frac{(x-z)^3\alpha_1(x,z)}{1296x^2(1+x)^{14}(-1+2x)^5 z^2(1+z)^{14}(-1+2z)^5(3+2x+4z)},$$

where $\alpha_1(x,z)$ is a polynomial of degree 20 in (x,z) with a very long expression. It follows from direct computations that the resultant with respect to z between $\alpha_1(x,z)$ and $q(x,z)$ is $131072(1+x)^{22}(-1+2x)^{10}\zeta_1(z)$, where

$$\begin{aligned}\zeta_1(x) = {} & 2799360 - 4105728x - 46503072x^2 + 45785088x^3 + 400338504x^4 - 183937680x^5\\ & - 2275672545x^6 - 129790572x^7 + 9130566970x^8 + 5218540697x^9 - 25610976347x^{10}\\ & - 29571499544x^{11} + 45089620496x^{12} + 94543100152x^{13} - 25802210680x^{14}\\ & - 185339941376x^{15} - 92210899744x^{16} + 193050497152x^{17} + 262418215808x^{18}\\ & - 8054306816x^{19} - 261232193536x^{20} - 208431021056x^{21} + 15675223040x^{22}\\ & + 146996101120x^{23} + 128615911424x^{24} + 60615639040x^{25} + 17178050560x^{26}\\ & + 2775449600x^{27} + 198246400x^{28}.\end{aligned}$$

By calculating, we have that $\zeta_1(0) = 2799360$, $\zeta_1\left(\frac{1}{2}\right) = 6561$, and $\zeta_1(x)$ has a local minimum 6128.2996 at $x \approx 0.487723$ on the interval $\left(0, \frac{1}{2}\right)$. Therefore, $\alpha_1(x,z) = 0$ and $q(x,z) = 0$ have no common roots on the interval $\left(0, \frac{1}{2}\right)$, and $W[L_0(x,z), L_1(x,z)] \neq 0$ for any $x \in \left(0, \frac{1}{2}\right)$.

Finally, let us compute the third Wronskian and we have that

$$W[L_0, L_1, L_2] = \frac{(x-z)^6\alpha_2(x,z)}{648x^3(1+x)^{21}(-1+2x)^12z^3(1+z)^{21}(-1+2z)^12(3+2x+4z)^3},$$

where $\alpha_2(x,z)$ is a polynomial of degree 50 in (x,z) with a very long expression. The resultant with respect to z between $\alpha_2(x,z)$ and $q(x,z)$ is $618475290624(1+x)^{37}(-1+2x)^{19}\zeta_2(x)$, where

$$\begin{aligned}\zeta_2(x) = {} & 314424115200 - 338610585600x - 7197591260160x^2 + 3682188564480x^3\\ & + 82047833521920x^4 + 5210989511040x^5 - 592497119129472x^6\\ & - 372635968302720x^7 + 2885867540768232x^8 + 3558716269442172x^9\\ & - 9292142328963813x^{10} - 19140007898353035x^{11} + 16482330221454718x^{12}\\ & + 67114041663802748x^{13} + 5453813359662526x^{14} - 154918512493381814x^{15}\\ & - 126209494807367500x^{16} + 208708533197557384x^{17} + 381340040113757184x^{18}\\ & - 48210981342553392x^{19} - 597836271817833312x^{20} - 401835362862302080x^{21}\\ & + 403848032675763712x^{22} + 774435670832268032x^{23} + 266221550147915264x^{24}\\ & - 467143726426274816x^{25} - 712552216353261568x^{26} - 398681792365246464x^{27}\\ & + 197229693031993344x^{28} + 703625042364088320x^{29} + 693897335772938240x^{30}\\ & + 81980316555083776x^{31} - 593575661879296000x^{32} - 701169891281207296x^{33}\\ & - 226210840396169216x^{34} + 319417039476752384x^{35} + 534653451975524352x^{36}\\ & + 433636456305524736x^{37} + 236139850292527104x^{38} + 92931145817128960x^{39}\\ & + 26870184295792640x^{40} + 5633450904125440x^{41} + 819325409689600x^{42}\\ & + 74705587404800x^{43} + 3248069017600x^{44},\end{aligned}$$

$\zeta_2(0) = 314424115200$, $\zeta_2\left(\frac{1}{2}\right) = 76527504$, and $\zeta_2(x)$ has a local minimum 7.07214×10^7 at $x \approx 0.490262$ on the interval $\left(0, \frac{1}{2}\right)$. Hence, we can assert that $W[L_0, L_1, L_2] \neq 0$ for any $z \in \left(0, \frac{1}{2}\right)$. By Lemma 2.1, the proof of the result is completed.

Proof of Theorem 1.1 By Lemma 1.1, Lemma 3.2 and Theorem 3.1, we obtain Theorem 1.1.

Remark 3.1 The proof depends on the symbolic computations by Wolfram mathematica and some very long expressions are omitted for the sake of briefness, while the derivative process can be done precisely.

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