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Generalized Symplectic Mean Curvature Flows in Almost Einstein Surfaces*

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Abstract The authors mainly study the generalized symplectic mean curvature flow in an almost Einstein surface, and prove that this flow has no type-I singularity. In the graph case, the global existence and convergence of the flow at infinity to a minimal surface with metric of the ambient space conformal to the original one are also proved.

Keywords Almost Einstein, Symplectic mean curvature flow, Singularity, Minimal surface
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1 Introduction

Suppose that $(M, J, \overline{\omega}, \overline{g})$ is a smooth Kähler manifold of complex dimension n. Let $\overline{\mathrm{Ric}}$ be the Ricci tensor of $\overline{g}$, and then the Ricci form $\overline{\rho}$ is defined by

$$\overline{\rho}(X, Y) = \overline{\mathrm{Ric}}(JX, Y).$$

Recently, T. Behrndt [1] proposed a generalized mean curvature flow. Instead of considering the flow in a Kähler-Einstein manifold, he considered the case that the ambient manifold is almost Einstein, that is, an n-dimensional Kähler manifold $(M, J, \overline{\omega}, \overline{g})$ with

$$\overline{\rho} = \lambda\overline{\omega} + ndd^c\psi$$

for some constant $\lambda \in \mathbb{R}$ and some smooth function ψ on M (see [1]).

Suppose that the Kähler manifold $(M, J, \overline{\omega}, \overline{g})$ is almost Einstein. Given an immersion $F_0 : \Sigma \to M$ of an n-dimensional manifold Σ into M, Behrndt [1] considered a generalized mean curvature flow

$$\begin{cases} \dfrac{\partial}{\partial t}F(x,t) = K(x,t), & (x,t) \in \Sigma \times (0,T), \\ F(x,0) = F_0(x), & x \in \Sigma, \end{cases} \tag{1.1}$$

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where

$$K = H - n\pi_{\nu\Sigma}(\overline{\nabla}\psi)$$

is a normal vector field along Σ which is called the generalized mean curvature vector field of Σ. As K is a differential operator differing from H just by lower order terms, it is easy to see that (1.1) has a unique solution on a short time interval (see [1]).

Behrndt [1] proved that if $\Sigma_0 = F_0(\Sigma)$ is Lagrangian in the almost Einstein manifold M, then along the generalized mean curvature flow (1.1), it remains Lagrangian for each time. Therefore, it is reasonable to call such a flow the generalized Lagrangian mean curvature flow.

As a special case, Behrndt [1] also considered the generalized Lagrangian mean curvature flow in an almost Calabi-Yau manifold (see [11]).

In [12], we studied the generalized Lagrangian mean curvature flow in an almost Einstein manifold. We proved that the singularity of this flow is characterized by the second fundamental form. We also proved that the type-I singularity of the generalized Lagrangian mean curvature flow in an almost Calabi-Yau manifold is a stationary cone. In particular, the generalized Lagrangian mean curvature flow has no type-I singularity.

Let $(M, J, \overline{\omega}, \overline{g})$ be a Kähler surface. For a compact oriented real surface Σ which is smoothly immersed in M, α is the Kähler angle of Σ in M (see [5]). We say that Σ is a symplectic surface if $\cos\alpha > 0$.

In this paper, we mainly study the generalized mean curvature flow in an almost Einstein surface with the initial surface symplectic. We show that if the initial surface Σ_0 is symplectic, then along the generalized mean curvature flow (1.1), it remains symplectic for each time. Therefore, we can call this flow the generalized symplectic mean curvature flow.

In general, the mean curvature flow may develop singularities as time evolves. According to the blow-up rate of the second fundamental form, Huisken [8] classified the singularities of the mean curvature flow into two types: type I and type II. Chen and Li [2] and Wang [13] independently proved that if M is a Kähler-Einstein surface, then the symplectic mean curvature flow has no type-I singularity. Following the idea in [8], we can also define type-I and type-II singularity for our flow. And we can also prove that if M is an almost Einstein surface, then the generalized symplectic mean curvature flow has no type-I singularity (see Theorem 5.1). Note that if $\psi = \text{const.}$, our flow is just the symplectic mean curvature flow in a Kähler-Einstein surface, so our result is a generalization of theirs.

In this paper, we also consider the graph case. Suppose that $M = M_1 \times M_2$, where $(M_1, \overline{g}_1, \omega_1)$ and $(M_2, \overline{g}_2, \omega_2)$ are Riemann surfaces with the same average scalar curvature r. Then M is an almost Einstein surface with $\overline{\rho} = r\omega + 2dd^c\psi$. Suppose that the initial surface is a graph with $\langle e_1 \times e_2, \omega_1\rangle > \frac{\sqrt{2}}{2}$, where $\{e_1, e_2\}$ is an orthonormal frame of the initial surface. We show that the generalized mean curvature flow (1.1) exists globally and the global solution $F(\cdot, t)$ sub-converges to F_∞ in C^2 as $t \to \infty$, possibly outside a finite set of points, and $\Sigma_\infty = F_\infty(\Sigma)$ is a minimal surface in $(M, \mathrm{e}^{2\psi}\overline{g})$. Chen, Li and Tian [4] and Wang [13] proved the global existence and convergence of the mean curvature flow in the graph case that

M_1 and M_2 are of the same constant curvature. Han and Li [7] proved a similar result for the Kähler-Ricci mean curvature flow. In [7] and this paper, we only assume that M_1 and M_2 have the same average scalar curvature.

2 Evolution Equations

In [12], we computed the evolution equations of the induced metric and the second fundamental form of Σ_t along the generalized mean curvature flow (1.1). We will omit the proof and state them here in this section.

Lemma 2.1 *Along the generalized mean curvature flow* (1.1)*, the induced metric evolves by*

$$\frac{\partial}{\partial t} g_{ij} = -2K^\alpha h_{ij}^\alpha.$$

Consequently, we have the following corollary.

Corollary 2.1 *The area element of Σ_t satisfies the following equation:*

$$\frac{\partial}{\partial t} \mathrm{d}\mu_t = -\langle K, H\rangle \mathrm{d}\mu_t, \tag{2.1}$$

and consequently,

$$\frac{\partial}{\partial t} \int_{\Sigma_t} \mathrm{d}\mu_t = -\int_{\Sigma_t} \langle K, H\rangle \mathrm{d}\mu_t. \tag{2.2}$$

We also have the following lemma.

Lemma 2.2 *Along the generalized mean curvature flow* (1.1)*, the norm of the second fundamental form satisfies*

$$\frac{\partial}{\partial t} |A|^2 \le \Delta |A|^2 - |\nabla A|^2 + C|A|^4 + C|A|, \tag{2.3}$$

where C depends on the ambient space M and $\|\psi\|_{C^2(M)}$.

Theorem 2.1 *If the second fundamental form of Σ_t is uniformly bounded under the generalized mean curvature flow* (1.1) *for all time $t \in [0, T)$, then the solution can be extended beyond T.*

3 The Evolution of the Kähler Angle Along the Flow

In this section, we consider the case $n = 2$. That is to say, M is an almost Einstein surface, and Σ_0 is a symplectic surface in M.

Let J_{Σ_t} be the almost complex structure in a tubular neighborhood of Σ_t on M with

$$\begin{cases} J_{\Sigma_t} e_1 = e_2, \\ J_{\Sigma_t} e_2 = -e_1, \\ J_{\Sigma_t} e_3 = e_4, \\ J_{\Sigma_t} e_4 = -e_3. \end{cases} \tag{3.1}$$

It is proved in [2] that

$$|\overline{\nabla} J_{\Sigma_t}|^2 = |h_{1k}^4 + h_{2k}^3|^2 + |h_{2k}^4 - h_{1k}^3|^2 \geq \frac{1}{2}|H|^2. \tag{3.2}$$

Choose an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ on $(M,\overline{g})$ along Σ_t such that $\{e_1, e_2\}$ is the basis of Σ_t and the symplectic form ω_t takes the form

$$\omega_t = \cos\alpha u_1 \wedge u_2 + \cos\alpha u_3 \wedge u_4 + \sin\alpha u_1 \wedge u_3 - \sin\alpha u_2 \wedge u_4, \tag{3.3}$$

where $\{u_1, u_2, u_3, u_4\}$ is the dual basis of $\{e_1, e_2, e_3, e_4\}$. Then along the surface Σ_t the complex structure on M takes the form (see [2])

$$J = \begin{pmatrix} 0 & \cos\alpha & \sin\alpha & 0 \\ -\cos\alpha & 0 & 0 & -\sin\alpha \\ -\sin\alpha & 0 & 0 & \cos\alpha \\ 0 & \sin\alpha & -\cos\alpha & 0 \end{pmatrix}.$$

Theorem 3.1 *The evolution equation for* $\cos\alpha$ *along* Σ_t *is*

$$\begin{aligned}\Big(\frac{\partial}{\partial t} - \Delta\Big)\cos\alpha = {} & |\overline{\nabla} J_{\Sigma_t}|^2 \cos\alpha + \lambda \sin^2\alpha\cos\alpha - 2\langle\overline{\nabla}\psi, \nabla\cos\alpha\rangle \\ & + 2\sin\alpha\cos\alpha[-\overline{\nabla}^2\psi(e_1, Je_3) + \overline{\nabla}^2\psi(e_2, Je_4)]. \end{aligned} \tag{3.4}$$

As a corollary, if the initial surface Σ_0 *is symplectic, then along the flow, at each time* t, Σ_t *is symplectic.*

Proof Using Lemma 2.1, (3.3), and the fact that $\overline{\nabla}\omega = 0$, we have

$$\begin{aligned}\frac{\partial}{\partial t}\cos\alpha &= \frac{\partial}{\partial t}\frac{\omega(e_1, e_2)}{\sqrt{\det(g_{ij})}} = \omega(\overline{\nabla}_{e_1}K, e_2) + \omega(e_1, \overline{\nabla}_{e_2}K) - \frac{1}{2}\cos\alpha g^{ij}\frac{\partial}{\partial t}g_{ij} \\ &= \omega(\overline{\nabla}_{e_1}K^4 e_4, e_2) - \omega(\overline{\nabla}_{e_2}K^3 e_3, e_1) + \omega(K^3\overline{\nabla}_{e_1}e_3, e_2) + \omega(K^4\overline{\nabla}_{e_1}e_4, e_2) \\ &\quad + \omega(e_1, K^3\overline{\nabla}_{e_2}e_3) + \omega(e_1, K^4\overline{\nabla}_{e_2}e_4) + \langle K, H\rangle\cos\alpha \\ &= \sin\alpha(K^4_{,1} + K^3_{,2}) + \omega(-K^3h^3_{11}e_1, e_2) + \omega(-K^4h^4_{11}e_1, e_2) \\ &\quad + \omega(e_1, -K^3h^3_{22}e_2) + \omega(e_1, -K^4h^4_{22}e_2) + \langle K, H\rangle\cos\alpha \\ &= \sin\alpha(K^4_{,1} + K^3_{,2}) - K^3h^3_{11}\cos\alpha - K^4h^4_{11}\cos\alpha - K^3h^3_{22}\cos\alpha \\ &\quad - K^4h^4_{22}\cos\alpha + \langle K, H\rangle\cos\alpha \\ &= \sin\alpha(K^4_{,1} + K^3_{,2}). \end{aligned} \tag{3.5}$$

Recall the equation in Proposition 3.1 and Lemma 3.2 in [6] for $\cos\alpha$ to have

$$\Delta\cos\alpha = -|\overline{\nabla} J_{\Sigma_t}|^2\cos\alpha + \sin\alpha(H^4_{,1} + H^3_{,2}) - \sin^2\alpha\overline{\mathrm{Ric}}(Je_1, e_2).$$

Thus we have

$$\Big(\frac{\partial}{\partial t} - \Delta\Big)\cos\alpha = |\overline{\nabla} J_{\Sigma_t}|^2\cos\alpha - \sin\alpha(V^4_{,1} + V^3_{,2}) + \sin^2\alpha\overline{\mathrm{Ric}}(Je_1, e_2). \tag{3.6}$$

Denote $V = 2\pi_{\nu\Sigma}(\overline{\nabla}\psi) = V^\alpha e_\alpha$. Then

$$K = H - V. \tag{3.7}$$

It is computed in [12] that

$$V^\alpha_{,i} = 2\langle \overline{\nabla}_{e_i}\overline{\nabla}\psi, e_\alpha\rangle + 2h^\alpha_{ij}\langle \overline{\nabla}\psi, e_j\rangle.$$

Recalling that (see [6])

$$\partial_1\alpha = -(h^4_{11} + h^3_{12})$$

and

$$\partial_2\alpha = -(h^3_{22} + h^4_{12}),$$

we get

$$\begin{aligned} V^4_{,1} + V^3_{,2} &= 2\langle \overline{\nabla}_{e_1}\overline{\nabla}\psi, e_4\rangle + 2h^4_{1j}\langle \overline{\nabla}\psi, e_j\rangle + 2\langle \overline{\nabla}_{e_2}\overline{\nabla}\psi, e_3\rangle + 2h^3_{2j}\langle \overline{\nabla}\psi, e_j\rangle \\ &= 2\langle \overline{\nabla}_{e_1}\overline{\nabla}\psi, e_4\rangle + 2\langle \overline{\nabla}_{e_2}\overline{\nabla}\psi, e_3\rangle - 2\langle \overline{\nabla}\psi, \partial_1\alpha e_1 + \partial_2\alpha e_2\rangle \\ &= 2\langle \overline{\nabla}_{e_1}\overline{\nabla}\psi, e_4\rangle + 2\langle \overline{\nabla}_{e_2}\overline{\nabla}\psi, e_3\rangle - 2\langle \overline{\nabla}\psi, \nabla\alpha\rangle. \end{aligned} \tag{3.8}$$

Since M is an almost Einstein surface, we have

$$\overline{\mathrm{Ric}}(Je_1, e_2) = \overline{\rho}(e_1, e_2) = \lambda\overline{\omega}(e_1, e_2) + 2dd^c\psi(e_1, e_2) = \lambda\cos\alpha + 2dd^c\psi(e_1, e_2).$$

Moreover,

$$d^c\psi(e_l) = -d\psi \circ J(e_l) = -\overline{g}(\overline{\nabla}\psi, Je_l) = -\langle \overline{\nabla}\psi, Je_l\rangle.$$

Hence

$$\begin{aligned} dd^c\psi(e_1, e_2) &= e_1(d^c\psi(e_2)) - e_2(d^c\psi(e_1)) - d^c\psi([e_1, e_2]) \\ &= -e_1\langle \overline{\nabla}\psi, Je_2\rangle + e_2\langle \overline{\nabla}\psi, Je_1\rangle - 0 \\ &= -\langle \overline{\nabla}_{e_1}\overline{\nabla}\psi, Je_2\rangle - \langle \overline{\nabla}\psi, J\overline{\nabla}_{e_1}e_2\rangle + \langle \overline{\nabla}_{e_2}\overline{\nabla}\psi, Je_1\rangle + \langle \overline{\nabla}\psi, J\overline{\nabla}_{e_2}e_1\rangle \\ &= -\langle \overline{\nabla}_{e_1}\overline{\nabla}\psi, Je_2\rangle - \langle \overline{\nabla}\psi, h^\alpha_{12}Je_\alpha\rangle + \langle \overline{\nabla}_{e_2}\overline{\nabla}\psi, Je_1\rangle + \langle \overline{\nabla}\psi, h^\alpha_{12}Je_\alpha\rangle \\ &= -\langle \overline{\nabla}_{e_1}\overline{\nabla}\psi, Je_2\rangle + \langle \overline{\nabla}_{e_2}\overline{\nabla}\psi, Je_1\rangle \\ &= -\langle \overline{\nabla}_{e_1}\overline{\nabla}\psi, e_1\rangle\langle Je_2, e_1\rangle - \langle \overline{\nabla}_{e_1}\overline{\nabla}\psi, e_4\rangle\langle Je_2, e_4\rangle \\ &\quad + \langle \overline{\nabla}_{e_2}\overline{\nabla}\psi, e_2\rangle\langle Je_1, e_2\rangle + \langle \overline{\nabla}_{e_2}\overline{\nabla}\psi, e_3\rangle\langle Je_1, e_3\rangle \\ &= \langle \overline{\nabla}_{e_1}\overline{\nabla}\psi, e_1\rangle\cos\alpha + \langle \overline{\nabla}_{e_1}\overline{\nabla}\psi, e_4\rangle\sin\alpha \\ &\quad + \langle \overline{\nabla}_{e_2}\overline{\nabla}\psi, e_2\rangle\cos\alpha + \langle \overline{\nabla}_{e_2}\overline{\nabla}\psi, e_3\rangle\sin\alpha. \end{aligned}$$

Thus we have

$$\begin{aligned} \overline{\mathrm{Ric}}(Je_1, e_2) = \lambda\cos\alpha &+ 2\langle \overline{\nabla}_{e_1}\overline{\nabla}\psi, e_1\rangle\cos\alpha + 2\langle \overline{\nabla}_{e_1}\overline{\nabla}\psi, e_4\rangle\sin\alpha \\ &+ 2\langle \overline{\nabla}_{e_2}\overline{\nabla}\psi, e_2\rangle\cos\alpha + 2\langle \overline{\nabla}_{e_2}\overline{\nabla}\psi, e_3\rangle\sin\alpha. \end{aligned} \tag{3.9}$$

Putting (3.8)–(3.9) into (3.6), we get

$$\begin{aligned}\Big(\frac{\partial}{\partial t}-\Delta\Big)\cos\alpha &= |\overline{\nabla}J_{\Sigma_t}|^2\cos\alpha+\lambda\sin^2\alpha\cos\alpha+2\sin\alpha\langle\overline{\nabla}\psi,\nabla\alpha\rangle\\ &\quad-2\sin\alpha\langle\overline{\nabla}_{e_1}\overline{\nabla}\psi,e_4\rangle-2\sin\alpha\langle\overline{\nabla}_{e_2}\overline{\nabla}\psi,e_3\rangle\\ &\quad+2\sin^2\alpha\cos\alpha\langle\overline{\nabla}_{e_1}\overline{\nabla}\psi,e_1\rangle+2\sin^3\alpha\langle\overline{\nabla}_{e_1}\overline{\nabla}\psi,e_4\rangle\\ &\quad+2\sin^2\alpha\cos\alpha\langle\overline{\nabla}_{e_2}\overline{\nabla}\psi,e_2\rangle+2\sin^3\alpha\langle\overline{\nabla}_{e_2}\overline{\nabla}\psi,e_3\rangle\\ &= |\overline{\nabla}J_{\Sigma_t}|^2\cos\alpha+\lambda\sin^2\alpha\cos\alpha-2\langle\overline{\nabla}\psi,\nabla\cos\alpha\rangle\\ &\quad-2\sin\alpha\cos^2\alpha\langle\overline{\nabla}_{e_1}\overline{\nabla}\psi,e_4\rangle-2\sin\alpha\cos^2\alpha\langle\overline{\nabla}_{e_2}\overline{\nabla}\psi,e_3\rangle\\ &\quad+2\sin^2\alpha\cos\alpha\langle\overline{\nabla}_{e_1}\overline{\nabla}\psi,e_1\rangle+2\sin^2\alpha\cos\alpha\langle\overline{\nabla}_{e_2}\overline{\nabla}\psi,e_2\rangle\\ &= |\overline{\nabla}J_{\Sigma_t}|^2\cos\alpha+\lambda\sin^2\alpha\cos\alpha-2\langle\overline{\nabla}\psi,\nabla\cos\alpha\rangle\\ &\quad+2\sin\alpha\cos\alpha\langle\overline{\nabla}_{e_1}\overline{\nabla}\psi,\sin\alpha e_1-\cos\alpha e_4\rangle\\ &\quad+2\sin\alpha\cos\alpha\langle\overline{\nabla}_{e_2}\overline{\nabla}\psi,\sin\alpha e_2-\cos\alpha e_3\rangle\\ &= |\overline{\nabla}J_{\Sigma_t}|^2\cos\alpha+\lambda\sin^2\alpha\cos\alpha-2\langle\overline{\nabla}\psi,\nabla\cos\alpha\rangle\\ &\quad+2\sin\alpha\cos\alpha\langle\overline{\nabla}_{e_1}\overline{\nabla}\psi,-Je_3\rangle+2\sin\alpha\cos\alpha\langle\overline{\nabla}_{e_2}\overline{\nabla}\psi,Je_4\rangle\\ &= |\overline{\nabla}J_{\Sigma_t}|^2\cos\alpha+\lambda\sin^2\alpha\cos\alpha-2\langle\overline{\nabla}\psi,\nabla\cos\alpha\rangle\\ &\quad+2\sin\alpha\cos\alpha[-\overline{\nabla}^2\psi(e_1,Je_3)+\overline{\nabla}^2\psi(e_2,Je_4)].\end{aligned}$$

This proves the theorem.

The above theorem motivates the following definition.

Definition 3.1 *A family of symplectic surfaces satisfying* (1.1) *is said to evolve by the generalized symplectic mean curvature flow.*

4 Monotonicity Formula

Let $H(X,X_0,t,t_0)$ be the backward heat kernel on $\mathbb{R}^4$. Let Σ_t be a smooth family of surfaces in $\mathbb{R}^4$ defined by $F_t:\Sigma\to\mathbb{R}^4$. Define

$$\rho(X,t)=(4\pi(t_0-t))H(X,X_0,t,t_0)=\frac{1}{4\pi(t_0-t)}\exp\Big(-\frac{|X-X_0|^2}{4(t_0-t)}\Big)$$

for $t<t_0$. We have along the generalized symplectic mean curvature flow (1.1)

$$\frac{\partial\rho}{\partial t}=\Big(\frac{1}{t_0-t}-\frac{\langle K,X-X_0\rangle}{2(t_0-t)}-\frac{|X-X_0|^2}{4(t_0-t)^2}\Big)\rho. \tag{4.1}$$

We also have

$$\Delta\rho=\Big(\frac{\langle F-X_0,\nabla F\rangle^2}{4(t_0-t)^2}-\frac{\langle F-X_0,H+g^{ij}\overline{\Gamma}^{\alpha}_{\rho\sigma}\frac{\partial F^\rho}{\partial x^i}\frac{\partial F^\sigma}{\partial x^j}e_\alpha\rangle}{2(t_0-t)}-\frac{1}{t_0-t}\Big)\rho. \tag{4.2}$$

Combining (4.1) with (4.2) gives us

$$\Big(\frac{\partial}{\partial t}+\Delta\Big)\rho=\Big(-\Big|K+\frac{(F-X_0)^{\perp}}{2(t_0-t)}\Big|^2+|K|^2-\frac{\langle g^{ij}\overline{\Gamma}^{\alpha}_{\rho\sigma}\frac{\partial F^{\rho}}{\partial x^i}\frac{\partial F^{\sigma}}{\partial x^j}e_{\alpha},F-X_0\rangle}{t_0-t}$$
$$-\frac{\langle\pi_{\nu\Sigma}(\overline{\nabla}\psi),F-X_0\rangle}{t_0-t}\Big)\rho. \tag{4.3}$$

Applying the evolution equation for $\cos\alpha$, we have

$$\Big(\frac{\partial}{\partial t}-\Delta\Big)\cos\alpha\geq|\overline{\nabla}J_{\Sigma_t}|^2\cos\alpha-2\langle\overline{\nabla}\psi,\nabla\cos\alpha\rangle-C\cos\alpha, \tag{4.4}$$

where C depends on $\|\psi\|_{C^2(M)}$ and λ.

On Σ_t, we set

$$v=\mathrm{e}^{Ct}\cos\alpha,$$

where C is the constant in (4.4). Denote the injectivity radius of $(M,\overline{g})$ by i_M. For $X_0\in M$, take a normal coordinate neighborhood U and let $\phi\in C_0^{\infty}(B_{2r}(X_0))$ be a cut-off function with $\phi\equiv 1$ in $B_r(X_0)$, $0<2r<i_M$. Using the local coordinates in U we may regard $F(x,t)$ as a point in $\mathbb{R}^k$ whenever $F(x,t)$ lies in U. We define

$$\Psi(F,X_0,t,t_0)=\int_{\Sigma_t}\frac{1}{v}\phi(F)\rho(F,t)\mathrm{d}\mu_t.$$

The following monotonicity formula generalizes Proposition 4.2 of [2] to the almost Einstein case. In [12], we got the similar monotonicity formula for the generalized Lagrangian mean curvature flow. Some of the estimates in the proof of the following proposition have appeared in [12]. For completeness, we sketch the proof below.

Proposition 4.1 *Let $F_t:\Sigma\to M$ be a generalized symplectic mean curvature flow in a compact almost Einstein surface M. Then there exist positive constants c_1, c_2, c_3 and c_4 depending only on M, F_0, t_0 and r which is the constant in the definition of Ψ, such that*

$$\frac{\partial}{\partial t}\Big(\mathrm{e}^{c_1\sqrt{t_0-t}}\int_{\Sigma_t}\frac{1}{v}\phi\rho\mathrm{d}\mu_t\Big)$$
$$\leq-\mathrm{e}^{c_1\sqrt{t_0-t}}\int_{\Sigma_t}\frac{1}{v}\phi\rho\Big(\frac{2|\nabla v|^2}{v^2}+\Big|K+\frac{(F-X_0)^{\perp}}{2(t_0-t)}\Big|^2+c_4|K|^2\Big)\mathrm{d}\mu_t$$
$$+\frac{c_2\mathrm{e}^{c_1\sqrt{t_0-t}}}{(t_0-t)^{\frac{3}{4}}}+c_3\mathrm{e}^{c_1\sqrt{t_0-t}}. \tag{4.5}$$

Proof By (3.4), we have

$$\Big(\frac{\partial}{\partial t}-\Delta\Big)\frac{1}{v}\leq-\frac{|\overline{\nabla}J_{\Sigma_t}|^2}{v}-2\Big\langle\overline{\nabla}\psi,\nabla\frac{1}{v}\Big\rangle-\frac{2|\nabla v|^2}{v^3}. \tag{4.6}$$

Note that

$$\frac{\partial\phi(F)}{\partial t}=\langle\overline{\nabla}\phi,K\rangle.$$

Using (2.1), (3.7), (4.3) and (4.6), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Psi(F,X_0,t,t_0)=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma_t}\frac{1}{v}\phi\rho\mathrm{d}\mu_t\\&=\int_{\Sigma_t}\Big(\frac{\partial}{\partial t}-\Delta\Big)\frac{1}{v}\phi\rho\mathrm{d}\mu_t+\int_{\Sigma_t}\Delta\frac{1}{v}\phi\rho\mathrm{d}\mu_t+\int_{\Sigma_t}\frac{1}{v}\rho\langle\overline{\nabla}\phi,K\rangle\mathrm{d}\mu_t+\int_{\Sigma_t}\frac{1}{v}\phi\Big(\frac{\partial}{\partial t}+\Delta\Big)\rho\mathrm{d}\mu_t\\&\quad-\int_{\Sigma_t}\frac{1}{v}\phi\Delta\rho\mathrm{d}\mu_t-\int_{\Sigma_t}\frac{1}{v}\phi\rho\langle K,K+V\rangle\mathrm{d}\mu_t\\&\le-\int_{\Sigma_t}\phi\rho\Big(\frac{|\overline{\nabla}J_{\Sigma_t}|^2}{v}+\frac{2}{v^3}|\nabla v|^2+\frac{1}{v}\Big|K+\frac{(F-X_0)^\perp}{2(t_0-t)}\Big|^2\Big)\mathrm{d}\mu_t+\int_{\Sigma_t}\Big(\phi\rho\Delta\frac{1}{v}-\frac{1}{v}\phi\Delta\rho\Big)\mathrm{d}\mu_t\\&\quad-\int_{\Sigma_t}\frac{1}{v}\phi\rho\frac{\langle g^{ij}\overline{\Gamma}^\alpha_{\rho\sigma}\frac{\partial F^\rho}{\partial x^i}\frac{\partial F^\sigma}{\partial x^j}e_\alpha,F-X_0\rangle}{t_0-t}\mathrm{d}\mu_t+\int_{\Sigma_t}\frac{1}{v}\rho\Big(\varepsilon^2\phi|K|^2+\frac{1}{4\varepsilon^2}\frac{|\overline{\nabla}\phi|^2}{\phi}\Big)\mathrm{d}\mu_t\\&\quad-\int_{\Sigma_t}2\Big\langle\overline{\nabla}\psi,\nabla\frac{1}{v}\Big\rangle\phi\rho\mathrm{d}\mu_t-\int_{\Sigma_t}\frac{1}{v}\phi\rho\frac{\langle V,F-X_0\rangle}{2(t_0-t)}\mathrm{d}\mu_t-\int_{\Sigma_t}\frac{1}{v}\phi\rho\langle K,V\rangle\mathrm{d}\mu_t.\end{aligned}\tag{4.7}$$

Again, by (2.1) and (3.7), we have

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{d}\mu_t&=-\langle K,H\rangle\mathrm{d}\mu_t=-\langle H,H-V\rangle\mathrm{d}\mu_t=(-|H|^2+\langle H,V\rangle)\mathrm{d}\mu_t\\&\le\frac{1}{4}|V|^2\mathrm{d}\mu_t\le C\mathrm{d}\mu_t,\end{aligned}$$

which implies that

$$\frac{\partial}{\partial t}\mathrm{Area}(\Sigma_t)\le C\mathrm{Area}(\Sigma_t).$$

Therefore, we have

$$\mathrm{Area}(\Sigma_t)\le\mathrm{e}^{Ct_0}\mathrm{Area}(\Sigma_0)\le C.\tag{4.8}$$

The same estimate as in [2] implies

$$\int_{\Sigma_t}\Big(\phi\rho\Delta\frac{1}{v}-\frac{1}{v}\phi\Delta\rho\Big)\mathrm{d}\mu_t\le C.\tag{4.9}$$

As $\phi\in C_0^\infty(B_{2r}(X_0),\mathbb{R}^4)$, we have (see [9, Lemma 6.6])

$$\frac{|\overline{\nabla}\phi|^2}{\phi}\le2\max_{\phi>0}|\overline{\nabla}^2\phi|.\tag{4.10}$$

By Young's inequality,

$$\begin{aligned}-\int_{\Sigma_t}\frac{1}{v}\phi\rho\langle K,V\rangle\mathrm{d}\mu_t&\le\varepsilon\int_{\Sigma_t}\frac{1}{v}\phi\rho|K|^2\mathrm{d}\mu_t+C(\varepsilon)\int_{\Sigma_t}\frac{1}{v}\phi\rho|V|^2\mathrm{d}\mu_t\\&\le\varepsilon\int_{\Sigma_t}\frac{1}{v}\phi\rho|K|^2\mathrm{d}\mu_t+C(\varepsilon).\end{aligned}\tag{4.11}$$

Using the fact that $|\overline{\nabla}\psi|\le C$, $|\nabla v|\le C|\nabla\alpha|\le C|\overline{\nabla}J_{\Sigma_t}|$ and Hölder's inequality, we have

$$\begin{aligned}\int_{\Sigma_t}\Big\langle\nabla\psi,\nabla\frac{1}{v}\Big\rangle\phi\rho\mathrm{d}\mu_t&\le C\int_{\Sigma_t}\Big|\Big\langle\nabla\psi,\frac{\nabla v}{v^2}\Big\rangle\Big|\phi\rho\mathrm{d}\mu_t\le C\Big(\int_{\Sigma_t}|\nabla v|^2\frac{1}{v}\phi\rho\mathrm{d}\mu_t\Big)^{\frac12}\Big(\int_{\Sigma_t}\frac{1}{v}\phi\rho\mathrm{d}\mu_t\Big)^{\frac12}\\&\le\varepsilon\int_{\Sigma_t}|\overline{\nabla}J_{\Sigma_t}|^2\frac{1}{v}\phi\rho\mathrm{d}\mu_t+C(\varepsilon).\end{aligned}\tag{4.12}$$

Since

$$|\overline{\nabla} J_{\Sigma_t}|^2 \geq \frac{1}{2}|H|^2 = \frac{1}{2}|K|^2 + \frac{1}{2}|V|^2 + \langle K, V\rangle \geq \frac{1}{4}|K|^2 - \frac{1}{2}|V|^2,$$

we have

$$-\int_{\Sigma_t} \frac{1}{v}\phi\rho|\overline{\nabla} J_{\Sigma_t}|^2 \mathrm{d}\mu_t \leq -\int_{\Sigma_t} \frac{1}{4v}\phi\rho|K|^2 \mathrm{d}\mu_t + C. \tag{4.13}$$

In a way similar to the proof of (13) in [3], we have

$$\frac{\langle F - X_0, g^{ij}\overline{\Gamma}^{\alpha}_{\rho\sigma}\frac{\partial F^\rho}{\partial x^i}\frac{\partial F^\sigma}{\partial x^j}e_\alpha\rangle}{2(t_0 - t)}\rho(F,t) \leq C_1\frac{\rho(F,t)}{\sqrt{t_0 - t}} + C. \tag{4.14}$$

Finally, we need to estimate the term $-\int_{\Sigma_t}\frac{1}{v}\phi\rho\frac{\langle V, F-X_0\rangle}{2(t_0-t)}\mathrm{d}\mu_t$. We claim that (see [12] for the proof)

$$\frac{|F - X_0|^2}{(t_0 - t)^\alpha}\rho(F,t) \leq C_1\frac{\rho(F,t)}{(t_0 - t)^\beta} + C, \quad 0 < \alpha - 1 < \beta < 1. \tag{4.15}$$

Especially, if we choose $\alpha = \frac{5}{4}$ and $\beta = \frac{1}{2}$, then we have

$$-\int_{\Sigma_t}\frac{1}{v}\phi\rho\frac{\langle V, F - X_0\rangle}{2(t_0 - t)}\mathrm{d}\mu_t \leq \frac{C}{\sqrt{t_0 - t}}\Psi + \frac{C}{(t_0 - t)^{\frac{3}{4}}}. \tag{4.16}$$

Putting (4.9), (4.12)–(4.14) and (4.16) into (4.7), we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\Psi \leq & -\int_{\Sigma_t}\frac{1}{v}\phi\rho\Big(\frac{2|\nabla v|^2}{v^2} + \Big|K + \frac{(F - X_0)^\perp}{2(t_0 - t)}\Big|^2 + c_4|K|^2\Big)\mathrm{d}\mu_t \\ & + \frac{c_1}{\sqrt{t_0 - t}}\Psi + \frac{c_2}{(t_0 - t)^{\frac{3}{4}}} + c_3.\end{aligned} \tag{4.17}$$

Rearranging (4.17) yields the desired inequality.

5 No Type-I Singularity

Using (2.3), we can argue in the same way as that of the mean curvature flow (for example, Lemma 4.6 of [2]) to obtain the lower bound of the blow-up rate of the maximal norm of the second fundamental form at a finite singular time T.

Lemma 5.1 *Let* $U_t = \max\limits_{\Sigma_t}|A|^2$*. If the generalized mean curvature flow* (1.1) *blows up at a finite time* $T > 0$*, there exists a positive* c *depending only on* M*, such that if* $0 < T - t < \frac{\pi}{32\sqrt{c}}$*, then the function* U_t *satisfies*

$$U_t \geq \frac{1}{8\sqrt{2}(T - t)}.$$

According to the lower bound of the blow-up rate, we can classify the singularities of the generalized symplectic mean curvature flow (1.1) into two types, which is similar to that of the mean curvature flow defined by Huisken [8]. This definition was given in [12].

Definition 5.1 *We say that the generalized mean curvature flow* (1.1) *develops type-I singularity at $T > 0$, if*

$$\limsup_{t\to T}(T-t)\max_{\Sigma_t}|A|^2 \leq C$$

for some positive constant C. Otherwise, we say that the generalized mean curvature flow (1.1) *develops type-II singularity.*

Arguing as in [2], we have

Theorem 5.1 *The generalized symplectic mean curvature flow has no type-I singularity at any $T > 0$.*

Proof Suppose that the generalized mean curvature flow develops a type-I singularity at a finite time $t_0 > 0$. Assume that

$$\lambda_k^2 = |A|^2(x_k, t_k) = \max_{t\leq t_k}|A|^2 \to \infty \quad as\ k \to \infty.$$

As Σ is closed, we may assume that $x_k \to p \in \Sigma$ and $t_k \to t_0$ as $k \to \infty$. We choose a local coordinate system on $(M, \overline{g})$ around $F(p, t_0)$ such that $F(p, t_0) = 0$. Then we rescale the generalized mean curvature flow to have

$$F_k(x,t) = \lambda_k(F(x, t_k + \lambda_k^{-2}t) - F(p, t_k)), \quad t \in [-\lambda_k^2 t_k, 0].$$

Denote by Σ_t^k the scaled surface $F_k(\cdot, t)$. Then the induced metric satisfies

$$g_{ij}^k = \lambda_k^2 g_{ij}, \quad (g^k)^{ij} = \lambda_k^{-2}g^{ij}.$$

The scaled surface satisfies

$$\frac{\partial F_k}{\partial t} = K_k = H_k - 2\lambda_k^{-1}\pi_{\nu\Sigma_k}(\overline{\nabla}\psi).$$

By Lemma 5.1, we have

$$\frac{C}{t_0 - t_k} \geq |A|^2(x_k, t_k) \geq \frac{c}{t_0 - t_k}$$

for some uniform constants c and C independent of k. We then have

$$|A_k|^2(x_k, 0) = \frac{1}{\lambda_k^2}|A|^2(x_k, t_k) = 1 \tag{5.1}$$

and

$$|A_k|^2(x,t) = \frac{1}{\lambda_k^2}|A|^2(x, \lambda_k^{-2}t + t_k) \leq 1,$$

so there exists a subsequence of F_k which we also denote by F_k, such that $F_k \to F_\infty$ in any ball $B_R(0) \subset \mathbb{R}^4$, and F_∞ satisfies

$$\frac{\partial F_\infty}{\partial t} = K_\infty = H_\infty$$

with

$$|A_\infty|^2(p,0)=1, \quad |A_\infty|^2 \le 1. \tag{5.2}$$

Set $v_k(F_k(x,t)) = v(F(x,\lambda_k^{-2}t+t_k))$ and $\phi_k(F_k(x,t)) = \phi(F(x,\lambda_k^{-2}t+t_k))$. It is easy to see that

$$\begin{aligned}&\int_{\Sigma_t^k} \frac{1}{v_k}\phi_k(F_k)\frac{1}{0-t}\exp\Big(-\frac{|F_k+\lambda_k F(p,t_k)|^2}{4(0-t)}\Big)\mathrm{d}\mu_t^k\\ =&\int_{\Sigma_{t_k+\lambda_k^{-2}t}} \frac{1}{v}\phi(F)\frac{1}{t_k-(t_k+\lambda_k^{-2}t)}\exp\Big(-\frac{|F(x,t_k+\lambda_k^{-2}t)|^2}{4(t_k-(t_k+\lambda_k^{-2}t))}\Big)\mathrm{d}\mu_t,\end{aligned}$$

where ϕ is the function defined in the definition of Ψ. Notice that $t_k+\lambda_k^{-2}t\to t_0$ for any fixed t. By Proposition 4.1,

$$\frac{\partial}{\partial t}(\mathrm{e}^{c_1\sqrt{t_0-t}}\Psi(F,X_0,t,t_0)) \le \frac{c_2\mathrm{e}^{c_1\sqrt{t_0-t}}}{(t_0-t)^{\frac34}} + c_3\mathrm{e}^{c_1\sqrt{t_0-t}},$$

and it then follows that $\lim\limits_{t\to t_0}\mathrm{e}^{c_1\sqrt{t_0-t}}\Psi$ exists. This implies that, for any fixed s_1 and s_2 with $-\infty<s_1<s_2<0$, we have

$$\begin{aligned}&\mathrm{e}^{c_1\sqrt{t_k-(t_k+\lambda_k^{-2}s_2)}}\int_{\Sigma_{s_2}^k}\frac{1}{v_k}\phi_k\frac{1}{0-s_2}\exp\Big(-\frac{|F_k+\lambda_k F(p,t_k)|^2}{4(0-s_2)}\Big)\mathrm{d}\mu_{s_2}^k\\ &-\mathrm{e}^{c_1\sqrt{t_k-(t_k+\lambda_k^{-2}s_1)}}\int_{\Sigma_{s_1}^k}\frac{1}{v_k}\phi_k\frac{1}{0-s_1}\exp\Big(-\frac{|F_k+\lambda_k F(p,t_k)|^2}{4(0-s_1)}\Big)\mathrm{d}\mu_{s_1}^k\\ \to& 0 \quad \text{as } k\to\infty.\end{aligned} \tag{5.3}$$

Integrating (4.5) from s_1 to s_2, we obtain

$$\begin{aligned}&-\mathrm{e}^{c_1\sqrt{-\lambda_k^{-2}s_2}}\int_{\Sigma_{s_2}^k}\frac{1}{v_k}\phi_k\frac{1}{0-s_2}\exp\Big(-\frac{|F_k+\lambda_k F(p,t_k)|^2}{4(0-s_2)}\Big)\mathrm{d}\mu_{s_2}^k\\ &+\mathrm{e}^{c_1\sqrt{-\lambda_k^{-2}s_1}}\int_{\Sigma_{s_1}^k}\frac{1}{v_k}\phi_k\frac{1}{0-s_1}\exp\Big(-\frac{|F_k+\lambda_k F(p,t_k)|^2}{4(0-s_1)}\Big)\mathrm{d}\mu_{s_1}^k\\ \ge&\int_{s_1}^{s_2}\mathrm{e}^{c_1\sqrt{-\lambda_k^{-2}t}}\int_{\Sigma_t^k}\frac{1}{v_k}\phi_k\rho(F_k,t)\Big|K_k+\frac{(F_k+\lambda_k F(p,t_k))^\perp}{2(0-t)}\Big|^2\mathrm{d}\mu_t^k\mathrm{d}t\\ &+\int_{s_1}^{s_2}\mathrm{e}^{c_1\sqrt{-\lambda_k^{-2}t}}\int_{\Sigma_t^k}\frac{2|\nabla v_k|^2}{v_k^3}\phi_k\rho(F_k,t)\mathrm{d}\mu_t^k\mathrm{d}t\\ &+\int_{s_1}^{s_2}\mathrm{e}^{c_1\sqrt{-\lambda_k^{-2}t}}\int_{\Sigma_t^k}\frac{1}{v_k}\phi_k\rho(F_k,t)c_4|K_k|^2\mathrm{d}\mu_t^k\mathrm{d}t\\ &-4\frac{c_2}{\sqrt{\lambda_k}}((-s_1)^{\frac14}-(-s_2)^{\frac14})\mathrm{e}^{c_1\lambda_k^{-1}\sqrt{-s_1}}-c_3\lambda_k^{-2}(s_2-s_1)\mathrm{e}^{c_1\lambda_k^{-1}\sqrt{-s_1}}.\end{aligned} \tag{5.4}$$

Since the singularity is of type I and the vector field $V=2\pi_{\nu\Sigma}(\overline{\nabla}\psi)$ is bounded, we know that there exists a constant $C>0$ such that for t closed to t_0,

$$|K|\le|H|+|V|\le\frac{C}{\sqrt{t_0-t}}.$$

Therefore,

$$|F(p,t_k)| \le \int_{t_k}^{t_0} \Big|\frac{\partial F}{\partial t}\Big| \mathrm{d}t = \int_{t_k}^{t_0} |K| \mathrm{d}t \le C\sqrt{t_0 - t_k} \le \frac{C}{\lambda_k},$$

where the last inequality follows from the type-I singularity assumption. Without loss of generality, we can assume that $\lambda_k F(p,t_k) \to Q$ as $k \to \infty$. Letting $k \to \infty$ in (5.4) and using (5.3), we get that

$$H_\infty = K_\infty \equiv 0,$$
$$(F_\infty + Q)^\perp \equiv 0.$$

That is,

$$\langle F_\infty + Q, e_\alpha \rangle = 0.$$

It follows that for $\alpha = 3, 4$,

$$\det((h_\infty)^\alpha_{ij}) = 0.$$

Since $H_\infty = 0$, we also have for $\alpha = 3, 4$,

$$\mathrm{tr}((h_\infty)^\alpha_{ij}) = 0.$$

Thus, $(h_\infty)^\alpha_{ij} = 0$ for all $i, j = 1, 2$, $\alpha = 3, 4$, which yields that $|A_\infty| \equiv 0$. This contradicts (5.2).

This finishes the proof of the theorem.

6 The Graph Case

In this section we study the generalized symplectic mean curvature flow (1.1) in a special case. Suppose that M is a product of compact Riemann surfaces M_1 and M_2, i.e. $(M, \overline{g}) = (M_1 \times M_2, \overline{g}_1 \oplus \overline{g}_2)$. We denote by r_1 and r_2 the average scalar curvature of M_1 and M_2, respectively. We assume that $r_1 = r_2$. Suppose that Σ is a graph in $M = M_1 \times M_2$. Recall the definition of the graph in [4]. A surface Σ is a graph in $M_1 \times M_2$ if $v = \langle e_1 \wedge e_2, \omega_1 \rangle \ge c_0 > 0$, where ω_1 is a unit Kähler form on M_1, and $\{e_1, e_2\}$ is an orthonormal frame on Σ. In this section, we use some ideas in [4, 7, 13]. We first prove a proposition.

Proposition 6.1 *Each Riemann surface $(N, \overline{g}, \omega)$ is an almost Einstein curve with $\overline{\rho} = r\overline{\omega} + dd^c\varphi$ for some smooth function φ on N, where r is the average scalar curvature of N.*

Proof Since $r = \frac{1}{\mathrm{vol}(N)} \int_N \mathrm{d}\mu$, $\int_N (r - R)\mathrm{d}\mu = 0$. By the Hodge theorem, there exists a smooth function φ such that $R = r + \Delta\varphi$. Since the complex dimension of N is 1, we have $\overline{\rho} = r\omega + dd^c\varphi$. This finishes the proof of the proposition.

Then we can get the following theorem.

Theorem 6.1 *Let $(M_1,\overline{g}_1,\omega_1)$ and $(M_2,\overline{g}_2,\omega_2)$ be Riemann surfaces which have the same average scalar curvature. Suppose that Σ_0 evolves along the generalized mean curvature flow in $M_1\times M_2$. If $v(\cdot,0)>\frac{\sqrt{2}}{2}$, then the generalized mean curvature flow exists for all time.*

Proof Set $r\equiv r_1=r_2$. By the above proposition, there exist smooth functions ψ_1 on M_1 and ψ_2 on M_2 such that $\overline{\rho}_1=r\omega_1+2dd^c\psi_1$ and $\overline{\rho}_2=r\omega_2+2dd^c\psi_2$. For each point (p_1,p_2) on $M_1\times M_2$, let $\psi(p_1,p_2)=\psi_1(p_1)+\psi_2(p_2)$. It follows that ψ is a smooth function on $M_1\times M_2$, and $\overline{\rho}=r\omega+2dd^c\psi$, which means that $M_1\times M_2$ is an almost Einstein surface.

Choose an orthonormal basis $\{e_1,e_2,e_3,e_4\}$ on M along Σ_t such that $\{e_1,e_2\}$ is the basis of Σ_t. Set $u_1=\langle e_1\wedge e_2,\omega_1+\omega_2\rangle$ and $u_2=\langle e_1\wedge e_2,\omega_1-\omega_2\rangle$, where ω_2 is a unit Kähler form on M_2. Since both $\omega_1+\omega_2$ and $\omega_1-\omega_2$ are parallel Kähler forms on $M_1\times M_2$, we see that Theorem 3.1 is applicable. Therefore,

$$\begin{aligned}\Big(\frac{\partial}{\partial t}-\Delta\Big)u_1&=J_1u_1+r(1-u_1^2)u_1-2\langle\overline{\nabla}\psi,\nabla u_1\rangle\\&\quad+2\sqrt{1-u_1^2}\,u_1[-\overline{\nabla}^2\psi(e_1,Je_3)+\overline{\nabla}^2\psi(e_2,Je_4)]\\&\geq J_1u_1-2\langle\overline{\nabla}\psi,\nabla u_1\rangle-c_1u_1,\end{aligned}\tag{6.1}$$

where

$$J_1=|h_{11}^4+h_{12}^3|^2+|h_{21}^4+h_{22}^3|^2+|h_{12}^4-h_{11}^3|^2+|h_{22}^4-h_{21}^3|^2.$$

By switching e_3 and e_4, we get that

$$\begin{aligned}\Big(\frac{\partial}{\partial t}-\Delta\Big)u_2&=J_2u_2+r(1-u_2^2)u_2-2\langle\overline{\nabla}\psi,\nabla u_2\rangle\\&\quad+2\sqrt{1-u_2^2}\,u_2[-\overline{\nabla}^2\psi(e_1,Je_4)+\overline{\nabla}^2\psi(e_2,Je_3)]\\&\geq J_2u_2-2\langle\overline{\nabla}\psi,\nabla u_2\rangle-c_2u_2,\end{aligned}\tag{6.2}$$

where

$$J_2=|h_{11}^4-h_{12}^3|^2+|h_{21}^4-h_{22}^3|^2+|h_{12}^4+h_{11}^3|^2+|h_{22}^4+h_{21}^3|^2.$$

It is clear that

$$\langle e_1\wedge e_2,\omega_1\rangle^2+\langle e_1\wedge e_2,\omega_2\rangle^2\leq 1.$$

The initial condition $v(x,0)>\frac{\sqrt{2}}{2}$ implies that $u_i(x,0)\geq v(x,0)-\frac{\sqrt{2}}{2}\geq c_0>0,\ i=1,2$. By (6.1)–(6.2),

$$\Big(\frac{\partial}{\partial t}-\Delta\Big)(\mathrm{e}^{c_it}u_i)\geq -2\langle\overline{\nabla}\psi,\nabla(\mathrm{e}^{c_it}u_i)\rangle.$$

Applying the maximum principle for parabolic equations, we obtain that $u_i(x,t)$ have positive lower bounds at any finite time. Suppose that $u_i\geq\delta$ for $0\leq t<t_0$. Then we claim that the flow F can be extended smoothly to $t_0+\varepsilon$ for some ε.

Set $u = u_1 + u_2$. Adding (6.1) to (6.2), we get

$$\Big(\frac{\partial}{\partial t} - \Delta\Big)u \geq u|A|^2 + 2(u_1 - u_2)h^3_{2k}h^4_{1k} - 2(u_1 - u_2)h^3_{1k}h^4_{2k} - 2\langle\overline{\nabla}\psi, \nabla u\rangle - Cu. \tag{6.3}$$

Since $u \geq 2\delta + |u_1 - u_2|$, using the Cauchy-Schwarz inequality, we get

$$\Big(\frac{\partial}{\partial t} - \Delta\Big)u \geq 2\delta|A|^2 - 2\langle\overline{\nabla}\psi, \nabla u\rangle - C. \tag{6.4}$$

Assume that (X_0, t_0) is a singularity point. As in the proof of Proposition 4.1, we can derive a weighted monotonicity formula for $\int_{\Sigma_t} \phi\frac{1}{u}\rho(F, X_0, t, t_0)\mathrm{d}\mu_t$, where ϕ is the cut-off function in Proposition 4.1.

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_{\Sigma_t}\frac{1}{u}\phi\rho(F, X_0, t, t_0)\mathrm{d}\mu_t\\
&= \int_{\Sigma_t}\Big(\frac{\partial}{\partial t} - \Delta\Big)\frac{1}{u}\phi\rho\mathrm{d}\mu_t + \int_{\Sigma_t}\Delta\frac{1}{u}\phi\rho\mathrm{d}\mu_t + \int_{\Sigma_t}\frac{1}{u}\langle\nabla\phi, K\rangle\rho\mathrm{d}\mu_t\\
&\quad + \int_{\Sigma_t}\frac{1}{u}\phi\Big(\frac{\partial}{\partial t} + \Delta\Big)\rho\mathrm{d}\mu_t - \int_{\Sigma_t}\frac{1}{u}\phi\Delta\rho\mathrm{d}\mu_t - \int_{\Sigma_t}\frac{1}{u}\phi\rho\langle K + V, K\rangle\mathrm{d}\mu_t\\
&\leq -\int_{\Sigma_t}\phi\rho\Big(\delta\frac{|A|^2}{u^2} + \frac{\delta}{2}\frac{|H|^2}{u^2} + \frac{1}{u}\Big|K + \frac{(F - X_0)^\perp}{2(t_0 - t)}\Big|^2 + \frac{2|\nabla u|^2}{u^3}\Big)\mathrm{d}\mu_t\\
&\quad + n\int_{\Sigma_t}\frac{1}{u^2}\langle\overline{\nabla}\psi, \nabla u\rangle\phi\rho\mathrm{d}\mu_t + \int_{\Sigma_t}\frac{C}{u^2}\phi\rho\mathrm{d}\mu_t\\
&\quad + \int_{\Sigma_t}\frac{1}{u}\langle\nabla\phi, K\rangle\rho\mathrm{d}\mu_t - \int_{\Sigma_t}\frac{1}{u}\phi\rho\frac{\langle g^{ij}\overline{\Gamma}^\alpha_{\rho\sigma}\frac{\partial F^\rho}{\partial x^i}\frac{\partial F^\sigma}{\partial x^j}e_\alpha, F - X_0\rangle}{2(t_0 - t)}\mathrm{d}\mu_t\\
&\quad - \int_{\Sigma_t}\frac{1}{u}\phi\rho\frac{\langle V, F - X_0\rangle}{2(t_0 - t)}\mathrm{d}\mu_t - \int_{\Sigma_t}\frac{1}{u}\phi\rho\langle K, V\rangle\mathrm{d}\mu_t + \int_{\Sigma_t}\Delta\phi\frac{1}{u}\rho\mathrm{d}\mu_t + 2\int_{\Sigma_t}\frac{1}{u}\langle\nabla\phi, \nabla\rho\rangle\mathrm{d}\mu_t\\
&\leq -\int_{\Sigma_t}\phi\rho\Big(\delta\frac{|A|^2}{u^2} + \frac{\delta}{2}\frac{|K|^2}{u^2} + \frac{2|\nabla u|^2}{u^3}\Big)\mathrm{d}\mu_t + \varepsilon\int_{\Sigma_t}\frac{|K|^2}{u^2}\phi\rho\mathrm{d}\mu_t + \frac{c_1}{\sqrt{t_0 - t}}\int_{\Sigma_t}\frac{1}{u}\phi\rho\mathrm{d}\mu_t\\
&\quad + \frac{c_2}{(t_0 - t)^{\frac{3}{4}}} + c_3\\
&\leq -\delta\int_{\Sigma_t}\phi\rho\frac{|A|^2}{u^2}\mathrm{d}\mu_t + \frac{c_1}{\sqrt{t_0 - t}}\int_{\Sigma_t}\frac{1}{u}\phi\rho\mathrm{d}\mu_t + \frac{c_2}{(t_0 - t)^{\frac{3}{4}}} + c_3.
\end{aligned}$$

It follows that

$$\frac{\partial}{\partial t}\Big(\mathrm{e}^{c_1\sqrt{t_0 - t}}\int_{\Sigma_t}\frac{1}{u}\phi\rho\mathrm{d}\mu_t\Big) \leq -\delta\mathrm{e}^{c_1\sqrt{t_0 - t}}\int_{\Sigma_t}\phi\rho\frac{|A|^2}{u^2}\mathrm{d}\mu_t + \frac{c_2\mathrm{e}^{c_1\sqrt{t_0 - t}}}{(t_0 - t)^{\frac{3}{4}}} + c_3\mathrm{e}^{c_1\sqrt{t_0 - t}}. \tag{6.5}$$

From this we see that $\lim\limits_{t \to t_0}\int_{\Sigma_t}\frac{1}{u}\phi\rho\mathrm{d}\mu_t$ exists.

Let $0 < \lambda_i \to \infty$ and let F_i be the blow-up sequence:

$$F_i(x, s) = \lambda_i(F(x, t_0 + \lambda^{-2}s) - X_0).$$

Let $\mathrm{d}\mu^i_s$ denote the induced volume form on Σ^i_s by F_i. It is obvious that

$$\int_{\Sigma_t}\frac{1}{u}\phi\rho(F, X_0, t, t_0)\mathrm{d}\mu_t = \int_{\Sigma^i_s}\frac{1}{u_i}\phi_i\rho(F_i, 0, s, 0)\mathrm{d}\mu^i_s,$$

where

$$t = t_0 + \lambda^{-2} s.$$

Therefore we get that

$$\begin{aligned}
&\frac{\partial}{\partial s}\Big(\mathrm{e}^{c_1\lambda_i^{-1}\sqrt{-s}}\int_{\Sigma_s^i}\frac{1}{u}\phi\rho(F_i,0,s,0)\mathrm{d}\mu_s^i\Big)\\
&=\lambda_i^{-2}\frac{\partial}{\partial t}\Big(\mathrm{e}^{c_1\sqrt{t_0-t}}\int_{\Sigma_t}\frac{1}{u}\phi\rho(F,X_0,t,t_0)\mathrm{d}\mu_t\Big)\\
&\le -\frac{\delta}{\lambda_i^2}\mathrm{e}^{c_1\sqrt{t_0-t}}\int_{\Sigma_t}\phi\rho\frac{|A|^2}{u^2}\mathrm{d}\mu_t+\frac{c_2}{\lambda_i^2}\frac{\mathrm{e}^{c_1\sqrt{t_0-t}}}{(t_0-t)^{\frac{3}{4}}}+\frac{c_3}{\lambda_i^2}\mathrm{e}^{c_1\sqrt{t_0-t}}\\
&=-\delta\mathrm{e}^{c_1\lambda_i^{-1}\sqrt{-s}}\int_{\Sigma_s^i}\phi\frac{|A_i|^2}{u^2}\rho(F_i,0,s,0)\mathrm{d}\mu_s^i+\frac{c_2\mathrm{e}^{c_1\lambda_i^{-1}\sqrt{-s}}}{\sqrt{\lambda_i}(-s)^{\frac{3}{4}}}+\frac{c_3}{\lambda_i^2}\mathrm{e}^{c_1\lambda_i^{-1}\sqrt{-s}}.
\end{aligned}$$

Note that $t_0 + \lambda_i^{-2}s \to t_0$ for any fixed s as $i \to \infty$ and that $\lim\limits_{t\to t_0} \mathrm{e}^{c_1\sqrt{t_0-t}}\int_{\Sigma_t}\frac{1}{u}\phi\rho\mathrm{d}\mu_t$ exists. By the above monotonicity formula, we have, for any fixed s_1 and s_2,

$$\begin{aligned}
0 \leftarrow\ &\mathrm{e}^{c_1\lambda_i^{-1}\sqrt{-s_1}}w\int_{\Sigma_{s_1}^i}\frac{1}{u_i}\phi_i\rho(F_i,0,s_1,0)\mathrm{d}\mu_{s_1}^i-\int_{\Sigma_{s_2}^i}\frac{1}{u_i}\phi_i\rho(F_i,0,s_2,0)\mathrm{d}\mu_{s_2}^i\\
&=-\int_{s_1}^{s_2}\frac{\mathrm{d}}{\mathrm{d}s}\Big(\mathrm{e}^{c_1\lambda_i^{-1}\sqrt{-s}}\int_{\Sigma_s^i}\frac{1}{u_i}\phi_i\rho(F_i,0,s,0)\mathrm{d}\mu_s^i\Big)\mathrm{d}s\\
&\ge\delta\int_{s_1}^{s_2}\mathrm{e}^{c_1\lambda_i^{-1}\sqrt{-s}}\int_{\Sigma_s^i}\phi_i\frac{|A_i|^2}{u_i^2}\rho(F_i,0,s,0)\mathrm{d}\mu_s^i\mathrm{d}s\\
&\quad-4\frac{c_2}{\sqrt{\lambda_i}}\mathrm{e}^{c_1\lambda_i^{-1}\sqrt{-s_1}}((-s_1)^{\frac{1}{4}}-(-s_2)^{\frac{1}{4}})-\frac{c_3}{\lambda_i^2}\mathrm{e}^{c_1\lambda_i^{-1}\sqrt{-s_1}}(s_2-s_1).
\end{aligned}$$

Since u_i is bounded below, we have

$$\int_{s_1}^{s_2}\int_{\Sigma_s^i}\phi|A_i|^2\rho(F_i,0,s,0)\to 0\quad\text{as } i\to\infty.$$

Therefore, for any ball $B_R(0)\subset\mathbb{R}^4$,

$$\int_{\Sigma_{s_i}^i\cap B_R(0)}|A_i|^2\rho(F_i,0,s,0)\to 0\quad\text{as } i\to\infty. \tag{6.6}$$

Because u has a positive lower bound, we see that Σ_t can locally be written as the graph of a map $f_t : \Omega\subset M_1\to M_2$ with uniformly bounded $|\mathrm{d}f_t|$. Consider the blow up of $f_{t_0+\frac{s_i}{\lambda_i^2}}$, as

$$f_i(y)=\lambda_i f_{t_0+\lambda_i^{-2}s_i}(\lambda_i^{-1}y).$$

It is clear that $|\mathrm{d}f_t|$ is also uniformly bounded and $\lim\limits_{i\to\infty} f_i(0) = 0$. By the Arzela's theorem, $f_i \to f_\infty$ in C^α on any compact set. By the inequality (29) in [10], we have

$$|A_i|\le|\nabla\mathrm{d}f_i|\le C(1+|\mathrm{d}f_i|^3)|A_i|,$$

where $\nabla \mathrm{d} f_i$ is measured with respect to the induced metric on $\Sigma^i_{s_i}$. From (6.6) it follows that, for any ball $B_R(0) \subset \mathbb{R}^4$,

$$\int_{\Sigma^i_{s_i} \cap B_R(0)} |\nabla \mathrm{d} f_i|^2 \rho(F_i, 0, s, 0) \to 0 \quad \text{as } i \to \infty,$$

which implies that $f_i \to f_\infty$ in $C^\alpha \cap W^{1,2}_{\mathrm{loc}}$ and the second derivative of f_∞ is 0. It is then clear that $\Sigma^i_{s_i} \to \Sigma^\infty$ and Σ^∞ is the graph of a linear function. Therefore,

$$\lim_{i\to\infty} \int_{B_{\lambda_i r}(X_0) \cap \Sigma^i_{S_i}} \rho(F_i, 0, s_i, 0) \mathrm{d}\mu^i_{s_i} = \int_{\Sigma^\infty} \rho(F_\infty, 0, -1, 0) \mathrm{d}\mu^\infty = 1.$$

We therefore have

$$\lim_{t\to t_0} \int_{B_r(X_0)\cap\Sigma_t} \rho(F, X_0, t, t_0) \mathrm{d}\mu_t = \lim_{i\to\infty} \int_{B_{\lambda_i r}(X_0) \cap \Sigma^i_{S_i}} \rho(F_i, 0, s_i, 0) \mathrm{d}\mu^i_{s_i} = 1.$$

By [14, Theorem 4.1] (note that $\beta(\mathcal{M})$ in this theorem for our flow is $\beta(\mathcal{M})(X, V) = n\pi_{V^\perp}(\overline{\nabla}\psi) -$ trace $II(x)|V$, where $X = (x, t)$ and $II(x)$ is the second fundamental form of M in $\mathbb{R}^N$ at x), we know that (X_0, t_0) is a regular point. This proves the theorem.

Now we consider the convergence of the generalized mean curvature flow. We follow the idea in [4]. We do not require the ambient space M to have a product structure in the following Theorem 6.2.

Theorem 6.2 *Let M be a Kähler surface. Suppose that the smooth solution of the generalized mean curvature flow* (1.1) *exists on* $[0, \infty)$. *Then there exists a finite set of points S and a sequence of $t_i \to \infty$ such that Σ_{t_i} converges to a surface satisfying $H = 2\pi_{\nu\Sigma}(\overline{\nabla}\psi)$, and the convergence is in C^2 outside S. In particular, if $(M, \overline{g})$ is an almost Einstein surface with $\overline{\rho} = \lambda\overline{\omega} + 2dd^c\psi$, then the limit surface is a minimal surface in $(M, \mathrm{e}^{2\psi}\overline{g})$.*

Proof By the Gauss equation, we have

$$\begin{aligned}\int_{\Sigma_t} |A|^2 \mathrm{d}\mu_{\widetilde{g}} &\le \int_{\Sigma_t} |H|^2 \mathrm{d}\mu_{\widetilde{g}} + C\widetilde{\mu}_t(\Sigma_t) + 4g - 4 \\ &\le \int_{\Sigma_t} |K|^2 \mathrm{d}\mu_{\widetilde{g}} + C\widetilde{\mu}_t(\Sigma_t) + 4g - 4,\end{aligned}$$

where g is the genus of the initial surface Σ_0. Because Σ_t is a continuous deformation of Σ_0, so its genus is also g. Define two conformally rescaled Riemannian metrics $\widetilde{g}$ and $\overline{g}$ on M by

$$\widetilde{g} = \mathrm{e}^{2\psi}\overline{g} \quad \text{and} \quad \widehat{g} = \mathrm{e}^{\psi}\overline{g}.$$

Proposition 2 in [1] gives

$$\frac{\partial}{\partial t} \int_{\Sigma_t} \mathrm{d}\mu_{\widetilde{g}} = -\int_{\Sigma_t} |K|^2_{\widehat{g}} \mathrm{d}\mu_{\widehat{g}},$$

from which we get

$$\widetilde{\mu}_t(\Sigma_t) \le \widetilde{\mu}_0(\Sigma_0) \quad \text{and} \quad \int_0^\infty \int_{\Sigma_t} |K|^2_{\widehat{g}} \mathrm{d}\mu_{\widehat{g}} \mathrm{d}t \le \widetilde{\mu}_0(\Sigma_0).$$

So,

$$\int_{\Sigma_t} |A|^2 \mathrm{d}\mu_{\widetilde{g}} \le \int_{\Sigma_t} |K|^2 \mathrm{d}\mu_{\widetilde{g}} + C,$$

and there exists a sequence $t_i \to \infty$ such that

$$\int_{\Sigma_{t_i}} |K|^2_{\widehat{g}} \mathrm{d}\mu_{\widehat{g}} \to 0 \quad \text{as } i \to \infty.$$

Hence,

$$\int_{\Sigma_{t_i}} |K|^2 \mathrm{d}\mu_{\widetilde{g}} = \int_{\Sigma_{t_i}} \mathrm{e}^{-\psi} |K|^2_{\widehat{g}} \mathrm{e}^{\psi} \mathrm{d}\mu_{\widehat{g}} = \int_{\Sigma_{t_i}} |K|^2_{\widehat{g}} \mathrm{d}\mu_{\widehat{g}} \to 0 \quad \text{as } i \to \infty. \tag{6.7}$$

It follows that

$$\int_{\Sigma_{t_i}} |A|^2 \mathrm{d}\widetilde{\mu}_{t_i} \le C,$$

and then

$$\int_{\Sigma_{t_i}} |A|^2 \mathrm{d}\mu_{t_i} = \int_{\Sigma_{t_i}} \mathrm{e}^{-2\psi} |A|^2 \mathrm{d}\widetilde{\mu}_{t_i} \le C \int_{\Sigma_{t_i}} |A|^2 \mathrm{d}\widetilde{\mu}_{t_i} \le C. \tag{6.8}$$

Suppose that Σ_{t_i} blows up around a point $p \in M$. We have

$$\lambda_i^2 = \max_{\Sigma_{t_i} \cap \overline{B}^M_r(p)} |A|^2 \to \infty.$$

Assume that $\lambda_i = |A|(x_i)$ and that $F(x_i, t_i) \to p$ as $i \to \infty$. Considering the blow-up sequence

$$F_i = \lambda_i (F(x + x_i, t_i) - F(x_i, t_i)),$$

we can see that $F_i \to F_\infty$ as $i \to \infty$ and F_∞ is a minimal surface in $\mathbb{R}^4$ with $|A| \le |A(0)| = 1$.

By [4, Lemma 5.3], we have

$$\varepsilon_0 \le \int_{B^4_1(0) \cap \Sigma_i} |A^i|^2 \mathrm{d}\mu_i = \int_{B^4_{\lambda_i^{-1}}(0) \cap \Sigma_i} |A|^2 \mathrm{d}\mu_{t_i}.$$

By (6.8), one can see that the blow-up set is at most a finite set of points which we denote by S. We can see from (6.7) that Σ_∞ is a surface with $K = 0$, i.e., $H = 2\pi_{\nu N}(\overline{\nabla}\psi)$. As mentioned in [1], given a surface Σ in $(M, \overline{g})$, $\widetilde{H} = \mathrm{e}^{-2\psi}(H - 2\pi_{\nu N}(\overline{\nabla}\psi)) = \mathrm{e}^{-2\psi} K$, where $\widetilde{H}$ is the mean curvature vector field on Σ with respect to the metric on Σ which is induced by $\widetilde{g} = \mathrm{e}^{2\psi}\overline{g}$. Consequently, $K = 0$ is equivalent to $\widetilde{H} = 0$. This proves the theorem.

Combining Theorem 6.1 and Theorem 6.2, we have the following corollary.

Corollary 6.1 *Assume that $M = M_1 \times M_2$, M_1 and M_2 are Riemann surfaces with the same average scalar curvature r. Then M is an almost Einstein surface with $\overline{\rho} = r\overline{\omega} + 2dd^c\psi$. Let Σ_0 be a graph in M. If $v(x, 0) \ge v_0 > \frac{1}{\sqrt{2}}$, then the glabal solution $F(\cdot, t)$ of (1.1) exists and sub-converges to F_∞ in C^2 as $t \to \infty$, possibly outside a finite set of points, and $\Sigma_\infty = F_\infty(\Sigma)$ is a minimal surface in $(M, \mathrm{e}^{2\psi}\overline{g})$.*

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