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Quasi-sure Flows Associated with Vector Fields of Low Regularity*

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Abstract The authors construct a solution $U_t(x)$ associated with a vector field on the Wiener space for all initial values except in a 1-slim set and obtain the 1-quasi-sure flow property where the vector field is a sum of a skew-adjoint operator not necessarily bounded and a nonlinear part with low regularity, namely one-fold differentiability. Besides, the equivalence of capacities under the transformations of the Wiener space induced by the solutions is obtained.

Keywords Quasi-sure flows, Abstract Wiener space, Low regularity
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1 Introduction

One of the fundamental problems of the theory of dynamical systems concerns the existence and uniqueness of a global flow U_t generated by a vector field X. To be precise, consider the following integral equation:

$$U_t(x) = x + \int_0^t X(U_s(x))\mathrm{d}s,$$

where X is a measurable vector field on some topological vector space equipped with a positive Radon measure μ on its Borel σ-algebras. The problem was first treated by Cruzeiro [3–4] and she established the existence of a flow on the Wiener space associated to a general vector field valued in the Cameron-Martin space, under exponential integrability of the vector field as well as its gradient and divergence. Yun [14–15] refined the almost-sure existence to the quasi-sure existence of flows on the Wiener space associated to weakly differentiable vector fields. It was shown in [14] that if μ is the Wiener measure on an abstract Wiener space and X is a vector field taking values in the Cameron-Martin space H of μ, belonging to the Sobolev class W_∞^∞ over μ and satisfying the exponential integrability condition $\exp\big\{|\delta_\mu X| + \|X\| + \sum\limits_{k=1}^{\infty} \|\nabla^k X\|_{H^{\otimes n}}\big\} \in L^p(\mu)$ for all $p > 1$, where $H^{\otimes n}$ is the n-fold tensor product of H, then there exists a solution $\{U_t\}$ for all initial values except in an (r,p)-polar set for all $r \geq 2$ and $p > 1$. We call this (r,p)-quasi-sure existence for all $r \geq 2$ and $p > 1$.

Peters [11] obtained the almost-sure existence under some weaker conditions where only one-fold differentiability of X was required. Therefore, we expect that this situation can be refined

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to $(1,p)$-quasi-sure existence for any $p > 1$. Unfortunately, the method used by Yun [14] is associated with the Ornstein-Uhlenbeck semigroup which acts on the Banach-valued functional. Since only one-fold differentiability is required, we can not use this method associated with the Ornstein-Uhlenbeck semigroup to give the $(1,p)$-quasi-sure existence for any $p > 1$ through the procedure (see [14]). In fact, it is well-known that there is no relation between the derivative operator and the Ornstein-Uhlenbeck operator acting on Banach-valued functionals due to the lack of Meyer inequality; moreover, the fractional order Ornstein-Uhlenbeck operator is not easy to deal with. Consequently, we must take another way to construct a solution $U_t(x)$ associated to a vector field on the Wiener space for all initial values except in a $(1,p)$-polar set for any $p > 1$. After the establishment of $(1,p)$-quasi-sure existence for any $p > 1$, we can also obtain the $(1,p)$-quasi-sure flow property for any $p > 1$ and the equivalence of capacities under the transformations of the Wiener space induced by the solution.

The results in the present paper extend the previous results in [14–15] in the following way. First of all, we will consider the vector which is the sum of a skew-adjoint operator not necessarily bounded and a non-linear part. This situation is equivalent to considering a non-linear perturbation of a semigroup of rotations. Secondly, the condition on X only requires one-fold differentiability.

This paper is organized as follows. In Section 2, we recall some elements in the Malliavin calculus and deal with the capacity theory of the Sobolev space $W_1^p(B,H)$. In the last part of Section 2, we recall some results in [11] for later use. Our main work is in Section 3: We prove the $(1,p)$-quasi-sure existence of the solutions associated with the integral equations for any $p > 1$, and get the $(1,p)$-quasi-sure flow property of the solutions for any $p > 1$. At the end of Section 3, we also prove the equivalence of capacities under the transformations of the Wiener space induced by the solutions, which refines the property of mutual absolute continuity.

2 Preliminaries

2.1 Derivatives and Sobolev spaces

Now let us recall and fix some notations and notions. Let (B,H,μ) be an abstract Wiener space introduced by Gross [6], where

(1) B is a real, separable Banach space with the norm $\|\cdot\|$,

(2) H is a real, separable Hilbert space densely and continuously imbedded in B with the inner product $\langle x,y\rangle_H$,

(3) μ is the standard Gaussian measure, i.e., the Borel probability measure on B such that

$$\int_B \exp\{i(h,x)\}\mathrm{d}\mu = \exp\Big(-\frac{1}{2}\langle h,h\rangle_H\Big),$$

where $h \in B^* \subseteq H^*$ and $(\cdot,\cdot)$ is a natural pairing of B^* and B.

We refer to [10] (see also [7]) for the background in the Malliavin calculus. In the following we fix an orthonormal basis $\{h_i; i \ge 1\}$ of H with $h_i \in B^*$ for all $i \ge 1$. Let G be a separable Hilbert space. A G-valued functional F is called a cylindrical if there exists $N, M \ge 1$, $f_i \in C_b^\infty(\mathbb{R}^M)$ and $k_i \in G$ $(1 \le i \le N)$, such that

$$F = \sum_{i=1}^N F_i k_i = \sum_{i=1}^N f_i(h_1(\omega),\cdots,h_M(\omega))k_i.$$

We denote by $\mathrm{Cylin}(W,G)$ the space of G-valued cylindrical functions. For $F \in \mathrm{Cylin}(W,G)$, we define

$$DF = \sum_{i=1}^{N} DF_i \otimes k_i,$$

where

$$DF_i = \sum_{j=1}^{M} \partial_j f_i(h_1(\omega), \cdots, h_M(\omega))h_j, \quad 1 \le i \le N.$$

For any $p > 1$, we can define the seminorm on $\mathrm{Cylin}(W, G)$ as

$$\|X\|_{W_{1,p}}^p = \int_B ||X(x)||_G^p \mathrm{d}\mu + \int_B ||DX(x)||_{H\otimes G}^p \mathrm{d}\mu.$$

Now for any $p > 1$, we define $W_1^p(B, G)$ as the completion of a $\mathrm{Cylin}(W, G)$ with respect to the norm $\|\cdot\|_{W_{1,p}}$. If $G = \mathbb{R}$, we simply write $W_1^p(B)$ and $\mathrm{Cylin}(W)$. Furthermore, a Gaussian divergence operator $\delta(X)$ can be defined as the adjoint in $L^2(B, G)$ of the gradient along H:

$$\int_B \langle X, DF\rangle_H \mathrm{d}\mu = \int_B F\delta(X)\mathrm{d}\mu, \quad \forall F \in \mathrm{Cylin}(W, G).$$

Let $\{V^n\}_{n=1}^{\infty}$ be an increasing sequence of a finite dimensional subspace of B^*, such that the projection $P_{V^n} \uparrow Id|_H$ strongly. We denote by μ^n the Gaussian measure on V^n associated to the restriction of the inner product $\langle\cdot,\cdot\rangle_H$ to V^n. Also denote by $\sigma_n \subseteq \mathscr{B}_B$ the σ-algebra consisting of cylindrical sets based on V^n. Obviously, $\{\sigma_n\}_{n=1}^{\infty}$ is a filtration of sub σ-algebras of $\mathscr{B}_B$, where $\mathscr{B}_B$ is a σ-algebra generated by $\bigcup_{n=1}^{\infty} \sigma_n$. The Ornstein-Uhlenbeck semigroup T_ε on B is defined by the Mehler formula

$$T_\varepsilon X(x) = \int_B X(\mathrm{e}^{-\varepsilon}x + \sqrt{1-\mathrm{e}^{-2\varepsilon}}y)\mathrm{d}\mu(y).$$

This semigroup provides us with smooth approximations of a vector field X.

Definition 2.1 *For $X \in L(B, H)$, the finite dimensional approximations of X* w.r.t. *$\{V^n\}_{n=1}^{\infty}$ are defined as*

$$X^n = P_{V^n} \circ E[T_{\tau_n} X|\sigma_n], \quad n = 1, \cdots, \infty,$$

where $\{\tau_n\}_{n=1}^{\infty}$ is a sequence of positive numbers converging to zero.

The following results are due to G. Peters [11].

Proposition 2.1 *Let $1 < p < \infty$, and $\{X^n\}_{n=1}^{\infty}$ be the finite dimensional approximations of an $X \in W_1^p(B, H)$ defined in Definition* 2.1*. Then X^n are C^∞ cylindrical functionals based on the subspace V^n such that*

$$\|X^n - X\|_{W_{1,p}} \to 0, \quad \textit{as } n \to \infty.$$

2.2 Capacity theory on Wiener spaces

We fix $p > 1$. Given an open set O in B, its $(1, p)$-capacity is defined by

$$C_{1,p}(O) = \inf\{\|f\|_{W_{1,p}} : \ f \in W_1^p, f \ge 1 \text{ a.e. on } O\},$$

and for any subset $A \subseteq B$, its $(1,p)$-capacity is defined by

$$C_{1,p}(A) = \inf\{C_{1,p}(O) : \ O \text{ is open and } A \subseteq O\}.$$

If $C_{1,p}(A) = 0$, then A is called a $(1,p)$-polar set. If some properties hold except on a $(1,p)$-polar set, then we say that it holds $(1,p)$-quasi-everywhere. A subset A will be called 1-slim if $C_{1,p}(A) = 0$ for any $p > 1$. We also say that it holds 1-quasi-everywhere if some properties hold except on a 1-slim set.

For any H-valued Wiener functional $f : B \to H$, if for any $\varepsilon > 0$, there exists an open set O with $C_{1,p}(O) < \varepsilon$ such that $f : B \backslash O \to H$ is continuous, then we call this H-valued Wiener functional $(1,p)$-quasi-continuous. An H-valued Wiener functional f is said to possess a $(1,p)$-quasi-continuous modification $\widetilde{f}$ if among the equivalence classes of μ-measurable functions of f, we can choose a $(1,p)$-quasi-continuous function $\widetilde{f}$. An H-valued Wiener functional f is said to be 1-quasi-continuous if it is $(1,p)$-quasi-continuous for all $p > 1$.

We note that the following property holds for Sobolev spaces on an abstract Wiener space:

$$W_1^p(B,H) \cap C_b(B;H) \text{ is dense in } W_1^p(B,H) \text{ and } 1 \in W_1^p(B,H).$$

By Meyer inequality, we can get the equivalence between Sobolev spaces $W_1^p(B,H)$ and $\mathscr{F}_1^p(B,H)$, which is defined through the Ornstein-Uhlenbeck operator. Then it has been proved by Shigekawa [13] (see also Denis [5]) that any $f \in W_1^p(B,H)$ admits a $(1,p)$-quasi-continuous modification and this is denoted by $\widetilde{f}$, and the following Chebyshev type inequality holds:

$$C_{1,p}(||\widetilde{f}||_H \geq \lambda) \leq \frac{1}{\lambda^p}||f||_{W_{1,p}}.$$

Moreover, we can get a capacity version of Kolmogorov's criterion for path continuity. We refer the readers to Shigekawa [13] for a proof.

Theorem 2.1 *Let $X = \{X(t), t \in D\}$ be an H-valued process on a domain D of $\mathbb{R}^d$ and $p > 1$. Suppose that $X(t) \in W_1^p(B,H)$. Further, suppose that there exist constants $\alpha > 0$ and $c > 0$ such that for all $(s,t) \in D \times D$,*

$$\|X(t) - X(s)\|_{W_{1,p}}^p \leq c|t-s|^{d+\alpha}.$$

Then $X(t)$ admits a $(1,p)$-quasi-continuous modification $\widetilde{X}(t)$ for each $t \in D$, and for $(1,p)$-quasi-every $x \in B$, the sample paths of $\widetilde{X}(t)$ are continuous.

2.3 Anticipating flows on the Wiener space

Now we turn to the integral equation on the Wiener space and recall some results related to this article. We denote by $\mathscr{L}(H,H)$ the Banach space of a linear continuous operator $L : H \to H$ equipped with the norm

$$\|L\|_{\mathscr{L}(H,H)} = \sup_{\substack{v \in H \\ \|v\|_H = 1}} \|L(v)\|_H.$$

We also define a nice strongly continuous semigroup on H as follows.

Definition 2.2 *A strongly continuous semigroup Q_t of linear operators on H is said to be nice if there exists a measurable norm $\|x\|_{B_1}$ and a constant C_T such that*

$$\|x\|_B + \|Q_t x\|_B \leq C_T \|x\|_{B_1}, \quad x \in H, \ t \in [-T,T].$$

Proposition 2.2 summarizes the results of Peters in [11].

Proposition 2.2 *Let $X \in \bigcap_{p>1} L^p(B,H)$ be a vector field on B, Q_t be a nice strongly continuous semigroup of a unitary operator on H, and $\widetilde{Q}_t : B \to B$ denote the measurable linear extension of Q_t to B. Suppose that the vector field X satisfies the following conditions:*

$$\exists \lambda_0, \quad \int_B \exp(\lambda_0 |\delta X|) \mathrm{d}\mu < \infty,$$
$$\forall \lambda, \quad \int_B \exp(\lambda \|DX\|_{\mathscr{L}(H,H)}) \mathrm{d}\mu < \infty.$$

Then there exists a solution $U_t(x)$ of the integral equation

$$U_t(x) = \widetilde{Q}_t(x) + \int_0^t Q_{t-s} X(U_s(x)) \mathrm{d}s \quad \text{for } \mu\text{-a.e. } x \in B,$$

for all $t \in \mathbb{R}$.

Also, the image of Gaussian measure μ under U_t has the Radon-Nikodym density

$$J_{U_t}(x) = \exp\Big(\int_0^t \delta X(U_{-s}(x)) \mathrm{d}s \Big),$$

and for $T > 0$, there exist $\widetilde{p}_T > 1$ and $C(\widetilde{p}_T, T) > 0$ such that

$$\|J_{U_t}\|_{L^{\widetilde{p}_T}} \le C(\widetilde{p}_T, T), \quad |t| \le T.$$

The solution U_t enjoys the crude flow property, i.e., for every $s \in \mathbb{R}$ there exists a set $E_s \subseteq B$ such that $\mu(E_s) = 1$ and

$$U_t \circ U_s(x) = U_{t+s}(x), \quad \forall x \in E_s, \quad \forall t \in \mathbb{R}.$$

Remark 2.1 The measurable linear extension $\widetilde{Q}_t$ always exists and preserves the Gaussian measure μ (see [8]). As in [11, Section 4], under the condition that Q_t is nice, $\widetilde{Q}_t$ can be considered as a (possibly unbounded) linear operator on B with a domain in $\mathcal{D}(\widetilde{Q}_t)$ of full measure, where $\mathcal{D}(\widetilde{Q}_t)$ does not depend on t. Since $\widetilde{Q}_t$ is a linear operator and is H-differentiable, we obtain $D\widetilde{Q}_t(x) = Q_t$, $x \in B$, for each $t \in \mathbb{R}$.

The strategy for proving this proposition is to find a solution to approximate integral equations

$$U_t^n(x) = \widetilde{Q}_t(x) + \int_0^t Q_{t-s} X^n(U_s^n(x)) \mathrm{d}s, \quad x \in B,$$

where $\{X^n\}_{n=1}^{\infty}$ are approximations of X which are defined as in Definition 2.1, and then to show that the limit $\lim_{n\to\infty} U_t^n(x)$ exists μ-a.e. in $x \in B$ and prove the theorem. More details of the proof can be found in [11].

We give the following result from [11] which will be used later.

Proposition 2.3 *For $T > 0$, we can choose $p_T > 1$ and constants $C(p_T, T)$, that are independent of n, such that the finite dimensional flow U_t^n is a Radon-Nikodym derivative that satisfies*

$$\|J_{U_t^n}\|_{p_T} \le C(p_T, T), \quad t \in [-T, T].$$

Also, for each $s \in \mathbb{R}$ there exists a set E_s such that $\mu(E_s) = 1$ and

$$U_{t+s}^n(x) = U_t^n \circ U_s^n(x), \quad \forall x \in E_s, \ \forall t \in \mathbb{R}.$$

3 1-Quasi-sure Analysis of Integral Equations on the Wiener Space

We now give the main result concerning the 1-quasi-sure flows associated with a vector field of low regularity.

Theorem 3.1 *Let $X \in \bigcap_{p>1} L^p(B, H)$ be a vector field on B, Q_t be a nice strongly continuous semigroup of a unitary operators on H, and $\widetilde{Q}_t : B \to B$ denote the measurable linear extension of Q_t to B. Further, let the vector field X fulfill the following conditions*:

$$\exists \lambda_0, \quad \int_B \exp(\lambda_0 |\delta X|) \mathrm{d}\mu < \infty,$$
$$\forall \lambda, \quad \int_B \exp(\lambda \|DX\|_{H \otimes H}) \mathrm{d}\mu < \infty.$$

Then we can choose a 1-quasi-continuous modification $\widetilde{X}(x)$ of $X(x)$ defined everywhere on B, and we can construct $U_t(x)$, $t \in \mathbb{R}$ and $x \in B$, satisfying the following integral equation:

$$U_t(x) = \widetilde{Q}_t(x) + \int_0^t Q_{t-s} \widetilde{X}(U_s(x)) \mathrm{d}s \quad \text{for 1-quasi-every } x \in B \tag{3.1}$$

for all $t \in \mathbb{R}$.

Moreover, the solution U_t has the 1-quasi-sure flow property, i.e., for all $s \in \mathbb{R}$,

$$U_t \circ U_s(x) = U_{t+s}(x) \quad \text{for 1-quasi-every } x \in B$$

for all $t \in \mathbb{R}$.

Finally, the mapping $x \to U_t(x)$ preserves the class of 1-slims set for all $t \in \mathbb{R}$.

Remark 3.1 Thanks to the following lemma which shows that bounded continuous operators from B to H are of the Hilbert-Schmidt class when restricted to H, our results also hold under exponential integrability assumptions $\exp\{\|DX(x)\|_{\mathscr{L}(H,H)}\} \in L^p(\mu)$ for all $p > 1$. Hence our results refine the almost-sure existence in [11] to 1-quasi-sure existence.

Lemma 3.1 (see [2, Theorem 3.5.10]) *Let (B, H, μ) be an abstract Wiener space, and then one can find an orthonormal basis $\{e_n\}$ in H such that*

$$\sum_{n=1}^{\infty} \|e_n\|_B^2 < \infty.$$

We also need the following lemma (see [1]), and for the convenience of the readers, we include the proof.

Lemma 3.2 *Let $DX(x) \in \mathscr{L}(H, H)$ be a linear continuous operator, and then we have*

$$\|DX(x)\|_{H \otimes H} \leq C \|DX(x)\|_{\mathscr{L}(H,H)},$$

with C depending only on B and μ.

Proof By the above lemma, we can find a complete orthonormal system $\{e_n\}_{n=1}^{\infty}$ of H such that $\sum_{n=1}^{\infty} \|e_n\|_B^2 =: C < +\infty$. Then we obtain

$$\begin{aligned}\|DX(x)\|_{H \otimes H}^2 &= \sum_{i=1,j=1}^{\infty} (\langle DX(x)(e_i), e_j \rangle_H)^2 = \sum_{i=1}^{\infty} \|DX(x)(e_i)\|_H^2 \\ &\leq \|DX(x)\|_{\mathscr{L}(H,H)} \sum_{i=1}^{\infty} \|e_i\|_B^2 = C \|DX(x)\|_{\mathscr{L}(H,H)}.\end{aligned}$$

The rest of this article is devoted to proving Theorem 3.1. For convenience, we fixed $T > 0$. If we obtain the desired results, then since $T > 0$ is arbitrary, the results can also be extended to the case when $t \in \mathbb{R}$.

3.1 Existence

We divide the proof of existence into four steps. As before, we define X^n as $X^n = P_{V^n} \circ E[T_{\tau_n} X|\sigma_n]$, where $\{\tau_n\}_{n=1}^{\infty}$ is a sequence of positive numbers converging to zero.

Step 1 First we note that if $V_t(x)$ solves the integral equation:

$$\begin{cases} V_t(x) = \displaystyle\int_0^t Q_{-s} X(\widetilde{Q}_s \circ (V_s(x) + x)) \mathrm{d}s, \\ V_0(x) = 0, \end{cases} \tag{3.2}$$

then $U_t(x) = \widetilde{Q}_t \circ (V_t(x) + x)$ solves the original equation (3.1). By the results in [11], we can deduce that $V_t(x) \in H$. Therefore, we can use the theory of the Malliavin calculus and the capacity version of Kolmogorov's criterion to investigate the 1-quasi-sure property.

Step 2 We introduce the following process parameterized by $[0, 1] \times [0, T]$:

$$Z(s,t) = \begin{cases} V_t(x), & \text{if } s = 0, \\ V_t^n(x) + (s - \frac{1}{n}) \frac{V_t^{n+1}(x) - V_t^n(x)}{(n+1)^{-1} - n^{-1}}, & \text{if } \frac{1}{n+1} < s \le \frac{1}{n}, \end{cases}$$

where V_t^n solves the following approximating integral equation:

$$\begin{cases} V_t^n(x) = \displaystyle\int_0^t Q_{-s} X^n(\widetilde{Q}_s \circ (V_s^n(x) + x)) \mathrm{d}s, \\ V_0^n(x) = 0. \end{cases}$$

We have the following proposition.

Proposition 3.1 *$Z(s,t)$ has a 1-quasi-continuous modification $\widetilde{Z}(s,t)$ for each $(s,t) \in [0,1] \times [0,T]$. Moreover, the sample paths of $\widetilde{Z}(s,t)$ are continuous for 1-quasi-every $x \in B$.*

To prove this proposition, we need some lemmas. The following lemma can be found in [11].

Lemma 3.3 *For all $p > 1$, we have*

$$\sup_{0 \le t \le T} E[\|V_t^n - V_t\|_{H \otimes H}^p] \to 0, \quad \text{as } n \to \infty.$$

We also need the following simple lemma.

Lemma 3.4 *For all $p > 1$, we have*

$$\sup_{0 \le t \le T} \sup_{n \ge 1} E[\|DV_t^n(x)\|_{H \otimes H}^p] < \infty.$$

Proof Since

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} DV_t^n(x) = & Q_{-t} \cdot DX^n(\widetilde{Q}_t \circ (V_t^n(x) + x)) \cdot Q_t \cdot DV_t^n(x) \\ & + Q_{-t} \cdot DX^n(\widetilde{Q}_t \circ (V_t^n(x) + x)) \cdot Q_t, \end{aligned}$$

we have

$$DV_t^n(x) = \int_0^t Q_{-s} \cdot DX^n(\widetilde{Q}_s \circ (V_s^n(x) + x)) \cdot Q_s \cdot DV_s^n(x)\mathrm{d}s$$
$$+ \int_0^t Q_{-s} \cdot DX^n(\widetilde{Q}_s \circ (V_s^n(x) + x)) \cdot Q_s\mathrm{d}s.$$

Therefore,

$$\|DV_t^n(x)\|_{H\otimes H} \leq \int_0^t \|DX^n(\widetilde{Q}_s \circ (V_s^n(x) + x))\|_{H\otimes H}(\|DV_s^n(x)\|_{H\otimes H} + 1)\mathrm{d}s.$$

By Gronwall's lemma,

$$\|DV_t^n\|_{H\otimes H} \leq \exp\Big(\int_0^t \|DX^n(\widetilde{Q}_s \circ (V_s^n(x) + x))\|_{H\otimes H}\mathrm{d}s\Big) - 1.$$

Thus using Proposition 2.3, we have

$$\int_B \|DV_t^n\|_{H\otimes H}^p \mathrm{d}\mu \leq \int_B \exp(T\|DX^n(x)\|_{H\otimes H})J_{U_t^n}(x)\mathrm{d}\mu + C_0$$
$$\leq C(p_T, T)\Big(\int_B \exp(p \cdot q_T \cdot T \cdot \|DX(x)\|_{H\otimes H})\mathrm{d}\mu\Big)^{\frac{1}{q_T}} + C_0,$$

where p_T is as in Proposition 2.3 and $\frac{1}{p_T} + \frac{1}{q_T} = 1$. By the assumptions of Theorem 3.1 we can conclude that

$$\sup_{0\leq t\leq T} \sup_{n\geq 1} E[\|DV_t^n(x)\|_{H\otimes H}^p] < \infty.$$

The following two lemmas will play a crucial role.

Lemma 3.5 *For all $p > 1$, we have*

$$\sup_{0\leq t\leq T} E[\|DV_t^n - DV_t\|_{H\otimes H}^p] \to 0, \quad \text{as } n \to \infty.$$

Proof Since

$$\frac{\mathrm{d}}{\mathrm{d}t}DV_t^n(x) = Q_{-t} \cdot DX^n(\widetilde{Q}_t \circ (V_t^n(x) + x)) \cdot Q_t \cdot DV_t^n(x)$$
$$+ Q_{-t} \cdot DX^n(\widetilde{Q}_t \circ (V_t^n(x) + x)) \cdot Q_t$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}DV_t(x) = Q_{-t} \cdot DX(\widetilde{Q}_t \circ (V_t(x) + x)) \cdot Q_t \cdot DV_t(x) + Q_{-t} \cdot DX(\widetilde{Q}_t \circ (V_t(x) + x)) \cdot Q_t,$$

we put

$$\alpha_t^n = Q_{-t} \cdot DX^n(\widetilde{Q}_t \circ (V_t^n(x) + x)) \cdot Q_t,$$
$$\alpha_t = Q_{-t} \cdot DX(\widetilde{Q}_t \circ (V_t(x) + x)) \cdot Q_t.$$

We have

$$DV_t^n(x) - DV_t(x) = \int_0^t (\alpha_s^n - \alpha_s) \cdot (1 + DV_s^n(x))\mathrm{d}s + \int_0^t \alpha_s \cdot (DV_s^n - DV_s)\mathrm{d}s.$$

For fixed t, $0 \le t \le T$, we have

$$\begin{aligned}&\|DV_t^n(x) - DV_t(x)\|_{H\otimes H}\\ \le &\int_0^T \|\alpha_s^n - \alpha_s\|_{H\otimes H} \cdot (1 + \|DV_s^n\|_{H\otimes H})\mathrm{d}s + \int_0^t \|\alpha_s\|_{H\otimes H} \cdot \|DV_s^n - DV_s\|_{H\otimes H}\mathrm{d}s.\end{aligned}$$

Then by Gronwall's lemma,

$$\begin{aligned}&\|DV_t^n(x) - DV_t(x)\|_{H\otimes H}\\ \le &\Big(\int_0^T \|\alpha_s^n - \alpha_s\|_{H\otimes H} \cdot (1 + \|DV_s^n\|_{H\otimes H})\mathrm{d}s\Big) \cdot \exp\Big(\int_0^t \|\alpha_s\|_{H\otimes H}\mathrm{d}s\Big).\end{aligned}$$

By Jensen's inequality

$$\begin{aligned}&\|DV_t^n(x) - DV_t(x)\|_{H\otimes H}^p\\ \le &\Big(T\int_0^T \|\alpha_s^n - \alpha_s\|_{H\otimes H} \cdot (1 + \|DV_s^n\|_{H\otimes H})\frac{\mathrm{d}s}{T}\Big)^p \cdot \exp\Big(p\int_0^t \|\alpha_s\|_{H\otimes H}\mathrm{d}s\Big)\\ \le &T^{p-1}\int_0^T \|\alpha_s^n - \alpha_s\|_{H\otimes H}^p \cdot (1 + \|DV_s^n\|_{H\otimes H})^p\mathrm{d}s \cdot \exp\Big(p\int_0^t \|\alpha_s\|_{H\otimes H}\mathrm{d}s\Big).\end{aligned}$$

Thus

$$\begin{aligned}&E[\|DV_t^n(x) - DV_t(x)\|_{H\otimes H}^p]\\ \le &T^{p-1}E\Big[\Big(\int_0^T \|\alpha_s^n - \alpha_s\|_{H\otimes H}^p \cdot (1 + \|DV_s^n\|_{H\otimes H})^p\mathrm{d}s\Big)^2\Big]^{\frac{1}{2}} \cdot E\Big[\exp\Big(2p\int_0^t \|\alpha_s\|_{H\otimes H}\mathrm{d}s\Big)\Big]^{\frac{1}{2}}\\ \le &T^{p-\frac{1}{2}}E\Big[\int_0^T \|\alpha_s^n - \alpha_s\|_{H\otimes H}^{2p} \cdot (1 + \|DV_s^n\|_{H\otimes H})^{2p}\mathrm{d}s\Big]^{\frac{1}{2}} \cdot \Big(\int_0^t E[\exp(2p\cdot t\cdot \|\alpha_s\|_{H\otimes H})]\frac{\mathrm{d}s}{t}\Big)^{\frac{1}{2}}\\ \le &T^{p-\frac{1}{2}}\Big(\int_0^T E[\|\alpha_s^n - \alpha_s\|_{H\otimes H}^{2p} \cdot (1 + \|DV_s^n\|_{H\otimes H})^{2p}]\mathrm{d}s\Big)^{\frac{1}{2}} \cdot \Big(\int_0^t E[\exp(2p\cdot t\cdot \|\alpha_s\|_{H\otimes H})]\frac{\mathrm{d}s}{t}\Big)^{\frac{1}{2}}.\end{aligned}$$

Note that by the assumptions of Theorem 3.1, we have

$$\begin{aligned}E[\exp(2p\cdot t\cdot \|\alpha_s\|_{H\otimes H})] &= E[\exp(2p\cdot t\cdot \|DX(x)\|_{H\otimes H})J_{U_t}(x)]\\ &\le E[\exp(2p\cdot \widetilde{q}_T\cdot t\cdot \|DX(x)\|_{H\otimes H})]^{1/\widetilde{q}_T} \cdot \|J_{U_t}\|_{\widetilde{p}_T}\\ &< +\infty,\end{aligned}$$

where $\widetilde{p}_T$ is as in Proposition 2.2 and $\frac{1}{\widetilde{p}_T} + \frac{1}{\widetilde{q}_T} = 1$, and

$$E[\|\alpha_s^n - \alpha_s\|_{H\otimes H}^{2p} \cdot (1 + \|DV_s^n\|_{H\otimes H})^{2p}] \le E[\|\alpha_s^n - \alpha_s\|_{H\otimes H}^{4p}]^{\frac{1}{2}}E[(1 + \|DV_s^n\|_{H\otimes H})^{4p}]^{\frac{1}{2}}.$$

Thus it remains to show that

$$\sup_{0\le s\le T} E[\|\alpha_s^n - \alpha\|_{H\otimes H}^{4p}] \to 0, \quad n \to 0.$$

Since

$$\begin{aligned}&E[\|\alpha_s^n - \alpha_s\|_{H\otimes H}^{4p}]\\ \le &2^{4p-1}(E[\|Q_{-s}\cdot DX^n(\widetilde{Q}_s\circ(V_s^n(x) + x))\cdot Q_s - Q_{-s}\cdot DX^n(\widetilde{Q}_s\circ(V_s(x) + x))\cdot Q_s\|_{H\otimes H}^{4p}]\\ &+ E[\|Q_{-s}\cdot DX^n(\widetilde{Q}_s\circ(V_s(x) + x))\cdot Q_s - Q_{-s}\cdot DX(\widetilde{Q}_s\circ(V_s(x) + x))\cdot Q_s\|_{H\otimes H}^{4p}]), \quad (3.3)\end{aligned}$$

for the second term in (3.3) we have

$$
\begin{aligned}
&E[\|Q_{-s}\cdot DX^n(\widetilde{Q}_s\circ(V_s(x)+x))\cdot Q_s - Q_{-s}\cdot DX(\widetilde{Q}_s\circ(V_s(x)+x))\cdot Q_s\|^{4p}_{H\otimes H}]\\
&= E[\|DX^n(x)-DX(x)\|^{4p}_{H\otimes H}J_{U_t}(x)]\\
&\le C(\widetilde{p}_T,T)E[\|DX^n(x)-DX(x)\|^{4p\widetilde{q}_T}_{H\otimes H}]^{1/\widetilde{q}_T}\to 0,\quad n\to\infty,
\end{aligned}
$$

where $\widetilde{p}_T$ is as in Proposition 2.2 and $\frac{1}{\widetilde{p}_T}+\frac{1}{\widetilde{q}_T}=1$. The first term in (3.3) is handled as follows. First by [11, Section 7], taking a subsequence if necessary we obtain that there exists a subset $A\in B$ such that $\mu(A)=1$, and $U_t^n(x)\to U_t(x)$, $x\in A$. Thus from the smoothness of DX^n, we know that

$$
DX^n(U_t^n(x))\to DX^n(U_t(x)),\quad x\in A.
$$

Using Egoroff's theorem, for every $\varepsilon>0$, there exists a measurable subset K_ε such that $\mu(K_\varepsilon^c)<\varepsilon$ and $DX^n(U_t^n(x))$ converges to $DX^n(U_t^n)$ uniformly on K_ε. Thus we have

$$
\begin{aligned}
&E[\|Q_{-s}\cdot DX^n(\widetilde{Q}_s\circ(V_s^n(x)+x))\cdot Q_s - Q_{-s}\cdot DX^n(\widetilde{Q}_s\circ(V_s(x)+x))\cdot Q_s\|^{4p}_{H\otimes H}]\\
&=\Big(\int_{K_\varepsilon}+\int_{K_\varepsilon^c}\Big)\|Q_{-s}\cdot DX^n(\widetilde{Q}_s\circ(V_s^n(x)+x))\cdot Q_s\\
&\quad - Q_{-s}\cdot DX^n(\widetilde{Q}_s\circ(V_s(x)+x))\cdot Q_s\|^{4p}_{H\otimes H}\mathrm{d}\mu. \qquad (3.4)
\end{aligned}
$$

Since $DX^n(U_t^n(x))$ converges to $DX^n(U_t(x))$ uniformly on K_ε, the first part in (3.4) converges to zero as n tends to infinity. The second part is dealt with as follows:

$$
\begin{aligned}
&\int_{K_\varepsilon^c}\|Q_{-s}\cdot DX^n(\widetilde{Q}_s\circ(V_s^n(x)+x))\cdot Q_s - Q_{-s}\cdot DX^n(\widetilde{Q}_s\circ(V_s(x)+x))\cdot Q_s\|^{4p}_{H\otimes H}\mathrm{d}\mu\\
&\le 2^{4p-1}\Big(\int_{K_\varepsilon^c}(\|Q_{-s}\cdot DX^n(\widetilde{Q}_s\circ(V_s^n(x)+x))\cdot Q_s\|^{4p}_{H\otimes H}\\
&\quad +\|Q_{-s}\cdot DX^n(\widetilde{Q}_s\circ(V_s(x)+x))\cdot Q_s\|^{4p}_{H\otimes H})\mathrm{d}\mu\Big)\\
&=2^{4p-1}\Big(\int_{K_\varepsilon^c}\|DX^n(x)\|^{4p}_{H\otimes H}J_{U_t^n}(x)\mathrm{d}\mu+\int_{K_\varepsilon^c}\|DX^n(x)\|^{4p}_{H\otimes H}J_{U_t}(x)\mathrm{d}\mu\Big)\\
&\le 2^{4p-1}(\|DX^n\|^{4p}_{8pq_T}\cdot\mu(K_\varepsilon^c)^{\frac{1}{2q_T}}\|J_{U_t^n}\|_{p_T}+\|DX^n\|^{4p}_{8p\widetilde{q}_T}\cdot\mu(K_\varepsilon^c)^{\frac{1}{2\widetilde{q}_T}}\|J_{U_t}\|_{\widetilde{p}_T})\\
&\le C(p,p_T,\widetilde{p}_T)(\varepsilon^{\frac{1}{2q_T}}+\varepsilon^{\frac{1}{2\widetilde{q}_T}}),
\end{aligned}
$$

where p_T is as in Proposition 2.3 and $\frac{1}{p_T}+\frac{1}{q_T}=1$, and $\widetilde{p}_T$ is as in Proposition 2.2 and $\frac{1}{\widetilde{p}_T}+\frac{1}{\widetilde{q}_T}=1$. Thus we have

$$
\sup_{0\le t\le T}E[\|DV_t^n-DV_t\|^p_{H\otimes H}]\to 0,\quad n\to\infty.
$$

Lemma 3.6 *For any $t,s\in[0,T]$ and all $p>1$, we have*

$$
E[\|DV_t^n-DV_s^n\|^{2p}_{H\otimes H}]\le C(p,T)(t-s)^{2p}.
$$

Proof Since

$$
\begin{aligned}
&E[\|DV_t^n - DV_s^n\|_{H\otimes H}^{2p}] \\
&= E\Big[\Big\| \int_s^t (Q_{-u}\cdot DX^n(\widetilde{Q}_u \circ (V_u^n(x)+x))\cdot Q_u \cdot DV_u^n(x) \\
&\quad + Q_{-u}\cdot DX^n(\widetilde{Q}_u \circ (V_u^n(x)+x))\cdot Q_u)\mathrm{d}u\Big\|_{H\otimes H}^{2p}\Big] \\
&\le (t-s)^{2p-1}\int_s^t \mathrm{d}u E[\|Q_{-u}\cdot DX^n(\widetilde{Q}_u \circ (V_u^n(x)+x))\cdot Q_u \cdot DV_u^n(x) \\
&\quad + Q_{-u}\cdot DX^n(\widetilde{Q}_u \circ (V^n(x)+x))\cdot Q_u\|_{H\otimes H}^{2p}] \\
&= (t-s)^{2p-1}\int_s^t \mathrm{d}u E[\|Q_{-u}\cdot DX^n(\widetilde{Q}_u \circ (V_u^n(x)+x))\cdot Q_u \cdot (1+DV_u^n(x))\|_{H\otimes H}^{2p}] \\
&\le (t-s)^{2p-1}\int_s^t \mathrm{d}u E[\|Q_{-u}\cdot DX^n(\widetilde{Q}_u \circ (V_u^n(x)+x))\cdot Q_u\|_{H\otimes H}^{4p}]^{\frac{1}{2}} E[1+\|DV_u^n(x)\|_{H\otimes H}^{2p}]^{\frac{1}{2}} \\
&\le (t-s)^{2p-1}\int_s^t \mathrm{d}u E[\|DX^n(x)\|_{H\otimes H}^{4p} J_{U_t^n}(x)]^{\frac{1}{2}} E[1+\|DV_u^n(x)\|_{H\otimes H}^{2p}]^{\frac{1}{2}},
\end{aligned}
$$

then from the assumptions of Theorem 3.1, Proposition 2.3 and Lemma 3.4, we deduce that

$$E[\|DV_t^n - DV_s^n\|_{H\otimes H}^{2p}] \le C(p,T)(t-s)^{2p}.$$

Proof of Proposition 3.1 First by Lemma 3.3, we have

$$E[\|V_t^n - V_t\|_H^{2p}] \to 0, \quad n\to\infty \text{ uniformly for } t\in[0,T].$$

Therefore, extracting a subsequence still denoted by $\{n\}$, we have

$$E[\|V_t^n - V_t\|_H^{2p}] \le C_1(p,T)2^{-2np}. \tag{3.5}$$

On the other hand, together with the assumptions of Theorem 3.1 and Proposition 2.3, we have

$$
\begin{aligned}
E[\|V_t^n - V_s^n\|_H^{2p}] &= E\Big[\Big\|\int_s^t (Q_{-u}X^n(\widetilde{Q}_u\circ(V_u^n(x)+x)))\mathrm{d}s\Big\|_H^{2p}\Big] \\
&\le (t-s)^{2p-1}\int_s^t \mathrm{d}u E[\|Q_{-u}X^n(\widetilde{Q}_u\circ(V_u^n(x)+x))\|_H^{2p}] \\
&= (t-s)^{2p-1}\int_s^t \mathrm{d}u E[\|Q_{-u}X^n(x)\|_H^{2pq_T}]^{\frac{1}{2}}\|J_{U_u^n}\|_{p_T} \\
&\le C_2(p,T)(t-s)^{2p},
\end{aligned} \tag{3.6}
$$

where p_T is as in Proposition 2.3 and $\frac{1}{p_T}+\frac{1}{q_T}=1$. Then from inequalities (3.5)–(3.6), taking further subsequence if necessary, for all $p>1$, we have

$$
\begin{aligned}
E[\|Z(s,t_1)-Z(s,t_2)\|_H^{2p}] \le C_3(p,T)|t_1-t_2|^{2p}, \\
E[\|Z(s_1,t)-Z(s_2,t)\|_H^{2p}] \le C_4(p,T)|s_1-s_2|^{2p}.
\end{aligned}
$$

Hence we can take p large enough such that

$$E[\|(Z(s_1,t_1)-Z(s_2,t_2)\|_H^{2p}] \le C_5(p,T)(|t_1-t_2|^{2+\varepsilon}+|s_1-s_2|^{2+\varepsilon})$$

for some $\varepsilon > 0$ and $(s_1, s_2) \in [0,1] \times [0,1]$, $(t_1, t_2) \in [0,T] \times [0,T]$.

It remains to show that for some p, we have

$$E[\|D(Z(s_1,t_1) - Z(s_2,t_2))\|^{2p}_{H\otimes H}] \le C_6(p,T)(|t_1 - t_2|^{2+\varepsilon} + |s_1 - s_2|^{2+\varepsilon})$$

for some $\varepsilon > 0$ and $(s_1, s_2) \in [0,1] \times [0,1]$, $(t_1, t_2) \in [0,T] \times [0,T]$.

By Lemmas 3.5–3.6, taking further subsequence if necessary, we obtain

$$E[||DV^n_t - DV_t||^{2p}_{H\otimes H}] \le C_7(p,T)2^{-2np},$$
$$E[\|DV^n_t - DV^n_s\|^{2p}_{H\otimes H}] \le C_8(p,T)(t-s)^{2p}.$$

Then by the same procedure we can get that

$$E[\|D(Z(s_1,t_1) - Z(s_2,t_2))\|^{2p}_{H\otimes H}] \le C_9(p,T)(|t_1 - t_2|^{2+\varepsilon} + |s_1 - s_2|^{2+\varepsilon})$$

for some $\varepsilon > 0$ and $(s_1, s_2) \in [0,1] \times [0,1]$, $(t_1, t_2) \in [0,T] \times [0,T]$.

Therefore, the conclusion follows from Proposition 2.1.

We deduce from this proposition immediately the following proposition.

Proposition 3.2 *For each $t \in [0,T]$, $V_t(x)$ and $V^n_t(x)$ have a 1-quasi-continuous modification $\widetilde{V}_t(x)$ and $\widetilde{V}^n_t(x)$, respectively.*

Proposition 3.3 *There exists a 1-slim set A such that $\lim\limits_{n\to\infty} V^n_t(x) = \widetilde{V}_t(x)$ for all $x \in A^c$ and all $t \in [0,T]$.*

Step 3 Since $X \in \bigcap\limits_{p>1} W^p_1(B,H)$, we can take a 1-quasi-continuous modification $\widetilde{X}$ of X by

$$\widetilde{X}(x) = \begin{cases} \lim\limits_{n\to\infty} X^n(x), & \text{if it converges,} \\ 0, & \text{otherwise.} \end{cases}$$

Then for fixed s, there exists a subsequence still denoted by $\{n\}$ such that $Q_{-s}X^n(\widetilde{Q}_s \circ (\widetilde{V}^n_s(x) + x))$ converges to $Q_{-s}\widetilde{X}(\widetilde{Q}_s \circ (\widetilde{V}^n_s(x) + x))$, 1-quasi-every $x \in B$ and the limit is 1-quasi-continuous. Repeating the same argument as in Lemma 3.3 and Lemma 3.5, we can deduce the following lemma.

Lemma 3.7 *For all $p > 1$,*

$$\sup_{0\le s\le T} \|Q_{-s}X^n(\widetilde{Q}_s \circ (\widetilde{V}^n_s(x) + x)) - Q_{-s}\widetilde{X}(\widetilde{Q}_s \circ (\widetilde{V}_s(x) + x))\|_{W_{1,p}} \to 0, \quad \textit{as } n \to \infty.$$

We denote

$$F^n_t(x) = \int_0^t Q_{-s}X^n(\widetilde{Q}_s \circ (\widetilde{V}^n_s(x) + x))\mathrm{d}s$$

and

$$F_t(x) = \int_0^t Q_{-s}\widetilde{X}(\widetilde{Q}_s \circ (\widetilde{V}_s(x) + x))\mathrm{d}s.$$

By Lemma 3.7, for fixed t we have

$$\begin{aligned} &\|F^n_t - F_t\|^p_{W_{1,p}} \\ &\le t^{p-1} \cdot \int_0^t \|Q_{-s}X^n(\widetilde{Q}_s \circ (\widetilde{V}^n_s(x) + x)) - Q_{-s}\widetilde{X}(\widetilde{Q}_s \circ (\widetilde{V}_s(x) + x))\|^p_{W_{1,p}}\mathrm{d}s \\ &\le t^p \cdot \sup_{0\le s\le t} \|Q_{-s}X^n(\widetilde{Q}_s \circ (\widetilde{V}^n_s(x) + x)) - Q_{-s}\widetilde{X}(\widetilde{Q}_s \circ (\widetilde{V}_s(x) + x))\|^p_{W_{1,p}} \to 0, \quad n \to \infty. \end{aligned}$$

Then we can deduce that $\{F_t^n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence in $W_1^p(B,H)$ and therefore, we can take a subsequence $\{n_k\}_{k=1}^{\infty}$ such that for any $\varepsilon > 0$, there exists a closed set A with $C_{1,p}(A^c) < \varepsilon$ and $F_t^{n_k}(x)$ converges to $F_t(x)$ uniformly in $x \in A$. Thus we can deduce that $F_t(x)$ is 1-quasi-continuous. Therefore for fixed $t \in [0,T]$, we have

$$\lim_{n\to\infty} F_t^n(x) = F_t(x) \quad \text{for 1-quasi-every } x \in B, \tag{3.7}$$

and the limit is 1-quasi-continuous.

However, this is not our purpose because for different $s \in [0,T]$ we have different 1-slim sets. Thus we still need to show that there exists a common set A with $C_{1,p}(A) = 0$ for any $p > 1$, such that for all $x \in A^c$ and $t \in [0,T]$, $\lim\limits_{n\to\infty} F_t^n(x) = F_t(x)$. For this purpose, we need the following lemma.

Lemma 3.8 *For any $s,t \in [0,T]$ and all $p > 1$, taking further subsequence if necessary, we have*

$$\begin{aligned} &E[\|F_t^n - F_t\|_H^{2p}] \to 0, \quad \text{as } n \to 0, \\ &E[\|F_t^n - F_s^n\|_H^{2p}] \le C_1(p,T)(t-s)^{2p}, \\ &E[\|DF_t^n - DF_t\|_{H\otimes H}^{2p}] \to 0, \quad \text{as } n \to 0, \\ &E[\|DF_t^n - DF_s^n\|_{H\otimes H}^{2p}] \le C_2(p,T)(t-s)^{2p}. \end{aligned} \tag{3.8}$$

Proof The proof of the first formula can be seen in [11]. The proof of the other formulas is just a repetition of Lemma 3.5 and Lemma 3.6.

Hence using the same skills, we have the following proposition.

Proposition 3.4 *For each $t \in [0,T]$, $F_t(x)$ has a 1-quasi-continuous modification $\widetilde{F}_t$.*

Proposition 3.5 *There exist a 1-slim set A such that $\lim\limits_{n\to\infty} F_t^n(x) = \widetilde{F}_t(x)$ for all $x \in A^c$ and $t \in [0,T]$.*

Though we obtain $F_t^n(x) \to \widetilde{F}_t(x)$ as n tends to infinity for 1-quasi-every $x \in B$, we don't know the expression of $\widetilde{F}_t(x)$. For this, we proceed as follows. By (3.7), we know that for each $t \in [0,T]$, there exists a 1-slim set A_t such that for all $x \in A_t^c$, $\widetilde{F}_t(x) = F_t(x)$. However, both $\widetilde{F}_t(x)$ and $F_t(x)$ have continuous sample paths, and hence there exists a common 1-slim set A such that for all $x \in A^c$, $\widetilde{F}_t(x) = F_t(x) = \int_0^t Q_{-s}\widetilde{X}(\widetilde{Q}_s \circ (\widetilde{V}_s(x) + x))\mathrm{d}s$ for all $t \in \mathbb{R}$.

Therefore, we conclude that there exists a subsequence still denoted by $\{n\}$ and a 1-slim set A, such that

$$F_t^n(x) = \int_0^t Q_{-s}X^n(\widetilde{Q}_s \circ (\widetilde{V}_s^n(x) + x))\mathrm{d}s \to F_t(x) = \int_0^t Q_{-s}\widetilde{X}(\widetilde{Q}_s \circ (\widetilde{V}_s(x) + x))\mathrm{d}s$$

for all $x \in A^c$ and all $t \in [0,T]$.

Step 4 We first note that by Peters [11] there exists a solution $V^n(x)$ satisfying the following integral equation:

$$V_t^n(x) = \int_0^t Q_{-s}X^n(\widetilde{Q}_s \circ (V_s^n(x) + x))\mathrm{d}s, \quad \forall x \in A_0,$$

where $\mu(A_0) = 1$. Thus by Proposition 3.2, there exists a 1-quasi-continuous modification $\widetilde{V}_t^n(x)$ of $V_t^n(x)$ and

$$\widetilde{V}_t^n(x) = \int_0^t Q_{-s}X^n(\widetilde{Q}_s \circ (V_s^n(x) + x))\mathrm{d}s, \quad \forall x \in A_0. \tag{3.9}$$

Since $Q_t : H \to H$ leaves the subspace V^n invariant, from (3.9), we know $\widetilde{V}_t^n(x) : V^n \to V^n$. The fact that in finite dimensions the embedding $W_1^p(\mathbb{R}^n) \subset C_b(\mathbb{R}^n)$ for $p > n$ generates the implication: If $C_{1,p}(A) = 0$, then A is empty. This implies that $\widetilde{V}_t^n(x)$ is a continuous modification of $V_t^n(x)$.

For any $y \in B$, there exists a sequence $\{x_h\} \subseteq A_0$ converging to y. Since $\widetilde{V}_t^n(x)$ is continuous, $\widetilde{V}_t^n(x_h)$ converges to $\widetilde{V}_t^n(y)$ as h converges to infinity. By Proposition 2.1, we know that X^n are C^∞ cylindrical functionals based on the subspace V^n. Then together with the fact $V_s^n(x) = \widetilde{V}_s^n(x)$, if $x \in A_0$, we have

$$\begin{aligned}\int_0^t Q_{-s} X^n(\widetilde{Q}_s \circ (V_s^n(x_h) + x_h))\mathrm{d}s &= \int_0^t Q_{-s} X^n(\widetilde{Q}_s \circ (\widetilde{V}_s^n(x_h) + x_h))\mathrm{d}s \\ &\to \int_0^t Q_{-s} X^n(\widetilde{Q}_s \circ (\widetilde{V}_s^n(y) + y))\mathrm{d}s.\end{aligned}$$

Therefore we have

$$\widetilde{V}_t^n(y) = \int_0^t Q_{-s} X^n(\widetilde{Q}_s \circ (\widetilde{V}_s^n(y) + y))\mathrm{d}s,$$

and this implies that $V_t^n(x)$ exists for all $x \in B$ and satisfies

$$V_t^n(x) = \int_0^t Q_{-s} X^n(\widetilde{Q}_s \circ (V_s(x) + x))\mathrm{d}s, \quad \forall t \in \mathbb{R}. \tag{3.10}$$

As in Step 3, we see that there is a subset A_1 with $C_{1,p}(A_1) = 0$ for any $p > 1$, such that for all $x \in A_1^c$,

$$\int_0^t Q_{-s} X^n(\widetilde{Q}_s \circ (\widetilde{V}_s^n(x) + x))\mathrm{d}s \to \int_0^t Q_{-s} \widetilde{X}(\widetilde{Q}_s \circ (\widetilde{V}_s(x) + x))\mathrm{d}s, \quad \forall t \in \mathbb{R}.$$

Proposition 3.3 implies that there exists a subset A_2 with $C_{1,p}(A_2) = 0$ for any $p > 1$, such that for all $x \in A_2^c$ and all $t \in \mathbb{R}$, $V_t^n(x) \to \widetilde{V}_t(x)$. Thus combining this with (3.10), for all $x \in (A_1 \cap A_2)^c$ with $C_{1,p}(A_1 \cap A_2) = 0$ for any $p > 1$, we have

$$\widetilde{V}_t(x) = \int_0^t Q_{-s} \widetilde{X}(\widetilde{Q}_s \circ (\widetilde{V}_s(x) + x))\mathrm{d}s, \quad \forall t \in \mathbb{R}.$$

Hence $\widetilde{V}_t(x)$ satisfies the integral equation (3.2) for 1-quasi-every $x \in B$ and for all $t \in \mathbb{R}$. Then $\widetilde{U}_t(x) = \widetilde{Q}_t \circ (\widetilde{V}_t(x) + x)$ satisfies the following integral equation:

$$\widetilde{U}_t(x) = \widetilde{Q}_t(x) + \int_0^t Q_{t-s} \widetilde{X}(\widetilde{U}_s(x))\mathrm{d}s \quad \text{for1-quasi-every } x \in B$$

and for all $t \in \mathbb{R}$, and the 1-quasi-sure existence is established.

3.2 Quasi-sure flow property

Now we show that the solution $\widetilde{U}_t(x)$ has the 1-quasi-sure flow property, i.e., for each $s \in \mathbb{R}$ it satisfies

$$\widetilde{U}_t \circ \widetilde{U}_s(x) = \widetilde{U}_{t+s}(x)$$

for 1-quasi-every $x \in B$ and for all $t \in \mathbb{R}$.

First we need the following lemma.

Lemma 3.9 *For every $s \in \mathbb{R}$, we have that for all $p > 1$,*

$$\sup_{0\le t\le T} \|V_t^n \circ \widetilde{U}_s - \widetilde{V}_t \circ \widetilde{U}_s\|_{W_{1,p}} \to 0, \quad \text{as } n \to \infty.$$

Proof Since $\widetilde{U}_t(x) = \widetilde{Q}_t \circ (\widetilde{V}_t(x) + x)$, we have

$$\begin{aligned} &D(V_t^n \circ \widetilde{Q}_t \circ (\widetilde{V}_s(x) + x)) \\ &= DV_t^n \circ \widetilde{Q}_t \circ (\widetilde{V}_s(x) + x) \cdot Q_t \cdot D\widetilde{V}_s(x) + D\widetilde{V}_t^n \circ (\widetilde{Q}_t \cdot (\widetilde{V}_s(x) + x)) \cdot Q_t. \end{aligned}$$

Also we have

$$\begin{aligned} &D(V_t \circ \widetilde{Q}_t \circ (\widetilde{V}_s(x) + x)) \\ &= DV_t \circ \widetilde{Q}_t \circ (\widetilde{V}_s(x) + x) \cdot Q_t \cdot D\widetilde{V}_s(x) + DV_t \circ \widetilde{Q}_t \circ (\widetilde{V}_s(x) + x) \cdot Q_t. \end{aligned}$$

Thus by Lemma 3.5 we have

$$\begin{aligned} &E[\|D(V_t^n \circ \widetilde{U}_s(x)) - D(\widetilde{V}_t \circ \widetilde{U}_s(x))\|_{H\otimes H}^p] \\ &= E[\|(DV_t^n \circ \widetilde{Q}_t \circ (\widetilde{V}_s(x) + x) \cdot Q_t - D\widetilde{V}_t \circ \widetilde{Q}_t \circ (\widetilde{V}_s(x) + x) \cdot Q_t)(1 + D\widetilde{V}_s(x)))\|_{H\otimes H}^p] \\ &\le E[\|DV_t^n \circ \widetilde{Q}_t \circ (\widetilde{V}_s(x) + x) \cdot Q_t - D\widetilde{V}_t \circ \widetilde{Q}_t \circ (\widetilde{V}_s(x) + x) \cdot Q_t\|_{H\otimes H}^{2p}]^{\frac{1}{2}} E[\|1 + D\widetilde{V}_s(x))\|_{H\otimes H}^{2p}]^{\frac{1}{2}} \\ &\le E[\|DV_t^n(x) - D\widetilde{V}_t(x)\|_{H\otimes H}^{2p} J_{\widetilde{U}_t^n}(x)]^{\frac{1}{2}} E[1 + \|D\widetilde{V}_s(x)\|_{H\otimes H}^{2p}]^{\frac{1}{2}} \\ &\le C(p_T, T) E[\|DV_t^n(x) - D\widetilde{V}_t(x)\|_{H\otimes H}^{2pq_T}]^{\frac{1}{2q_T}} E[1 + \|D\widetilde{V}_s(x)\|_{H\otimes H}^{2p}]^{\frac{1}{2}} \to 0, \quad n \to \infty, \end{aligned}$$

where p_T is as in Proposition 2.3 and $\frac{1}{p_T} + \frac{1}{q_T} = 1$, and the lemma established.

Proposition 3.6 *The solution $\widetilde{U}_t(x)$ constructed in Subsection 3.1 enjoys the 1-quasi-sure flow properties, i.e., for every $s \in \mathbb{R}$, it satisfies*

$$\widetilde{U}_t \circ \widetilde{U}_s(x) = \widetilde{U}_{t+s}(x) \quad \text{for 1-quasi-every } x \in B$$

and for all $t \in \mathbb{R}$.

Proof We denote by $U_t(x)$ the solution constructed by Peters [11]. First note that by Proposition 3.2 $\widetilde{V}_t(x)$ is 1-quasi-continuous. Then, $\widetilde{U}_t(x)$ is also 1-quasi-continuous and $U_t(x) = \widetilde{U}_t(x)$ for almost-every $x \in B$. By the almost-everywhere flow property of $U_t(x)$, we have $\widetilde{U}_t \circ \widetilde{U}_s(x) = U_t \circ \widetilde{U}_s(x) = U_{t+s}(x) = \widetilde{U}_{t+s}(x)$ for almost-every $x \in B$. But $\widetilde{U}_{t+s}(x)$ is 1-quasi-continuous and hence, if we can show that $\widetilde{U}_t \circ \widetilde{U}_s(x)$ is 1-quasi-continuous, we have $\widetilde{U}_t \circ \widetilde{U}_s(x) = \widetilde{U}_{t+s}(x)$ for 1-quasi-every $x \in B$.

By Lemma 3.9, $\{V_t^n \circ \widetilde{U}_s(x)\}_{n=1}^{\infty}$ is a Cauchy sequence in $W_1^p(B, H)$ and therefore, we can take a subsequence $\{n_k\}_{k=1}^{\infty}$ such that for any $\varepsilon > 0$, there exists a closed set A with $C_{1,p}(A^c) < \varepsilon$ and $V_t^{n_k} \circ \widetilde{U}_s(x)$ converges to $\widetilde{V}_t \circ \widetilde{U}_s(x)$ uniformly in $x \in A$. Hence from the smoothness of the solution $V_t^{n_k}(x)$ and 1-quasi-continuous of $\widetilde{U}_s(x)$, we can deduce that $\widetilde{U}_t \circ \widetilde{U}_s(x)$ is 1-quasi-continuous.

3.3 Equivalence of capacities

Let $\widetilde{U}_t(x)$ be the solution constructed in Subsection 3.1.

Lemma 3.10 *For any $1 < p < p_1$ and all $t \in [0, T]$, there exists a constant C such that*

$$\sup_{0\le t\le T} \|\psi \circ \widetilde{U}_t\|_{W_{1,p}}^p \le C\|\psi\|_{W_{1,p_1}}^p$$

for all $\psi \in W_1^p(B, \mathbb{R})$.

Proof Combining Proposition 2.3 with Lemma 3.4, we obtain

$$\begin{aligned}
&\int_B \|D(\psi\circ\widetilde{Q}_t\circ(V_t^n(x)+x))\|_{H\otimes H}^p \mathrm{d}\mu\\
&=\int_B \|D\psi\circ\widetilde{Q}_t\circ(V_t^n(x)+x)\cdot Q_t\cdot D(V_t^n(x)+x)\|_{H\otimes H}^p \mathrm{d}\mu\\
&\le \Big(\int_B \|D\psi\circ\widetilde{Q}_t\circ(V_t^n(x)+x)\|_{H\otimes H}^{pq}\mathrm{d}\mu\Big)^{\frac{1}{q}}\cdot\Big(\int_B(\|DV_t^n(x)\|_{H\otimes H}^{pq'}+1)\mathrm{d}\mu\Big)^{\frac{1}{q'}}\\
&\le \Big(\int_B \|D\psi(x)\|_{Op}^{pq}J_{U_t^n(x)}(x)\mathrm{d}\mu\Big)^{\frac{1}{q}}\cdot\Big(\int_B(\|DV_t^n(x)\|_{H\otimes H}^{pq'}+1)\mathrm{d}\mu\Big)^{\frac{1}{q'}}\\
&\le C(p_T,T)\Big(\int_B \|D\psi(x)\|_{H\otimes H}^{pqq_T}(x)\mathrm{d}\mu\Big)^{\frac{1}{qq_T}}\cdot\Big(\int_B(\|DV_t^n(x)\|_{H\otimes H}^{pq'}+1)\mathrm{d}\mu\Big)^{\frac{1}{q'}}\\
&\le C(p,T)\|D\psi(x)\|_{p_1}^p,
\end{aligned}$$

where p_T is as in Proposition 2.3, $\frac{1}{q}+\frac{1}{q'}=1$, $\frac{1}{p_T}+\frac{1}{q_T}=1$ and $q_Tq=\frac{p_1}{p}$. Thus we have

$$\|\psi\circ U_t^n\|_{W_{1,p}}^p\le C\|\psi\|_{W_{1,p_1}}^p$$

for some constant C. Therefore, the proof is completed if we prove that $\psi\circ U_t^n$ converges to $\psi\circ\widetilde{U}_t$ in $W_1^p(B,\mathbb{R})$.

Since

$$D(\psi(\widetilde{Q}_t\circ(V_t^n(x)+x)))=D\psi\circ\widetilde{Q}_t\circ(V_t^n(x)+x)DV_t^n(x)+D\psi\circ\widetilde{Q}_t\circ(V_t^n(x)+x),$$

we can prove that

$$\|\psi\circ U_t^n-\psi\circ\widetilde{U}_t\|_{W_{1,p}}\to 0,\quad n\to\infty,$$

by the same method as in the proof of Lemma 3.5.

To show the equivalence, we need another lemma which has been proved by Yun [15].

Lemma 3.11 *There exists an increasing sequence $\{F_n\}_{n=1}^\infty$ of compact sets such that for all $p>1$,*

$$\lim_{n\to\infty}C_{1,p}(B\backslash F_n)=0,$$

and $\widetilde{U}_t|_{F_n}$, the restriction of $\widetilde{U}_t$ to F_n, is a homeomorphism.

With the above preparations, now we can show the equivalence of capacities between a set A in B and $U_t(A)$.

Proposition 3.7 *For $1<p_2<p<p_1$ and $r\ge 0$, there exist constants $C_1>0$ and $C_2>0$ such that*

$$C_2\cdot(C_{1,p_2}(D))^{\frac{p}{p_2}}\le C_{1,p}(\widetilde{U}_t(D))\le C_1\cdot(C_{1,p_1})^{\frac{p}{p_1}},\quad \forall D\in B. \tag{3.11}$$

Proof By Lemma 3.11, there exists an increasing sequence $\{F_n\}_{n=1}^\infty$ of compact sets such that $\widetilde{U}_t|_{F_n}$ is a homeomorphism. Let O be an open set in B. Then $O\cap F_n$ is open in F_n and $\widetilde{U}_t(O\cap F_n)$ is open in $\widetilde{U}_t(F_n)$. Thus there exists an open set O' in B such that

$$U_t(O\cap F_n)=O'\cap\widetilde{U}_t(F_n).$$

We can show that

$$\widetilde{U}_t(O) \subseteq [O' \cap \widetilde{U}_t(F_n)] \cup U_t(F_n^c),$$

and $[O' \cap \widetilde{U}_t(F_n)] \cup \widetilde{U}_t(F_n^c)$ is an open set. Then we have

$$\begin{aligned} C_{1,p}(\widetilde{U}_t(O)) &\le C_{1,p}([O' \cap \widetilde{U}_t(F_n)] \cup \widetilde{U}_t(F_n^c)) \\ &= \inf\{\|f\|_{W_{1,p}}^p; f \in W_1^p(B;\mathbb{R}), f \ge 1 \text{ a.e. on } [O' \cap \widetilde{U}_t(F_n)] \cup \widetilde{U}_t(F_n^c)\} \\ &\le \inf\{\|f\|_{W_{1,p}}^p; f \in W_1^p(B;\mathbb{R}), f \circ \widetilde{U}_t \ge 1 \text{ a.e. on } O \cup F_n^c\} \\ &= \inf\{\|\psi \circ \widetilde{U}_{-t}\|_{W_{1,p}}^p; \psi \circ \widetilde{U}_{-t} \in W_1^p(B;\mathbb{R}), \psi \ge 1 \text{ a.e. on } O \cup F_n^c\}. \end{aligned}$$

By Lemma 3.10, we have

$$\begin{aligned} C_{1,p}(\widetilde{U}_t(O)) &\le C_1 \inf\{\|\psi\|_{W_{1,p_1}}^p; \psi \in W_1^{p_1}(B;\mathbb{R}), \psi \ge 1 \text{ a.e. on } O \cup F_n^c\} \\ &= C_1 \cdot (C_{1,p_1}(O \cup F_n^c))^{\frac{p}{p_1}} \\ &\le C_1 \cdot (C_{1,p_1}(O)^{\frac{p}{p_1}} + C_{1,p_1}(F_n^c)^{\frac{p}{p_1}}) \\ &\to C_1 \cdot (C_{1,p_1}(O))^{\frac{p}{p_1}}, \quad \text{as } n \to \infty. \end{aligned}$$

Therefore, for an open set O in B, $C_{1,p}(\widetilde{U}_t(O)) \le C_1 \cdot (C_{1,p_1}(O))^{\frac{p}{p_1}}$ for some constant C_1. Then for an arbitrary set $A \subseteq B$, it is easy to show

$$C_{1,p}(\widetilde{U}_t(A)) \le C_1 \cdot (C_{1,p_1}(A))^{\frac{p}{p_1}}.$$

We note that for an arbitrary set A,

$$C_{1,p_2}(A) = C_{1,p_2}(\widetilde{U}_{-t} \circ \widetilde{U}_t(A)).$$

Thus we can easily get the first inequality of (3.11).

Corollary 3.1 *The flow $\widetilde{U}_t(x)$ constructed in Theorem* 3.1 *preserves the class of* 1*-slim sets, that is, if $A \subseteq B$ is a* 1*-slim set, then $\widetilde{U}_t(A)$ is also a* 1*-slim set for every $t \in \mathbb{R}$.*

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