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Asymptotics for the Tail Probability of Random Sums with a Heavy-Tailed Random Number and Extended Negatively Dependent Summands*

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Abstract Let $\{X, X_k : k \geq 1\}$ be a sequence of extended negatively dependent random variables with a common distribution F satisfying $EX > 0$. Let τ be a nonnegative integer-valued random variable, independent of $\{X, X_k : k \geq 1\}$. In this paper, the authors obtain the necessary and sufficient conditions for the random sums $S_\tau = \sum\limits_{n=1}^{\tau} X_n$ to have a consistently varying tail when the random number τ has a heavier tail than the summands, i.e.,

$$\frac{P(X > x)}{P(\tau > x)} \to 0$$

as $x \to \infty$.

Keywords Asymptotic behavior, Random sums, Heavy-Tailed distribution
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1 Introduction

Let $\{X, X_k : k \geq 1\}$ be a sequence of random variables with a common distribution F and let τ be a nonnegative integer-valued random variable with a distribution F_τ. For any distribution G and real number x, we let $G(x) = G((-\infty, x])$ and denote its tail by $\overline{G}(x) = G((x, \infty))$. The aim of the present paper is to investigate the asymptotic behavior of the tail probability of a random sum $S_\tau = \sum\limits_{k=1}^{\tau} X_k$ when the random number τ has a heavier tail than the summands, i.e., $\lim\limits_{x\to\infty} \frac{\overline{F}(x)}{\overline{F}_\tau(x)} = 0$. Random sums play important roles in many applied probability fields such as financial insurance, risk theory, teletraffic, queueing theory and so on. Generally speaking, it is hard to obtain the precise distribution of S_τ, so one possible approach is to discuss the asymptotic behavior of the tail probability $P(S_\tau > x)$ as $x \to \infty$.

Hereafter, all limit relationships are for $x \to \infty$ unless otherwise stated. For two positive functions $a(x)$ and $b(x)$, we write $a(x) \sim b(x)$ if $\lim \frac{a(x)}{b(x)} = 1$ and write $a(x) = o(b(x))$ if $\lim \frac{a(x)}{b(x)} = 0$.

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Next, we introduce some common distribution classes. A random variable X or its distribution F is said to be heavy-tailed if

$$Ee^{tX} = \int_{-\infty}^{\infty} e^{tx} F(dx) = \infty$$

for any positive number t, and is otherwise light-tailed. Below we list some of the commonly used subclasses of heavy-tailed distributions.

A random variable X or its distribution F is said to be long-tailed (denoted by $X \in \mathcal{L}$ or $F \in \mathcal{L}$) if $\lim\limits_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = 1$ for any fixed $y > 0$; to have a consistently varying tail (denoted by $X \in \mathcal{C}$ or $F \in \mathcal{C}$) if $\lim\limits_{y\nearrow 1} \limsup\limits_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1$; to have a dominatedly varying tail (denoted by $X \in \mathcal{D}$ or $F \in \mathcal{D}$) if $\limsup\limits_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty$ for any fixed $y \in (0, 1)$; to have a regularly varying tail with an index α for some $\alpha > 0$ (denoted by $X \in \mathcal{R}_{-\alpha}$ or $F \in \mathcal{R}_{-\alpha}$), if $\lim\limits_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha}$ for any fixed $y > 0$; and to be subexponential (denoted by $X \in \mathcal{S}$ or $F \in \mathcal{S}$) if $F \in \mathcal{L}$ and $\lim\limits_{x\to\infty} \frac{\overline{F^{*2}}(x)}{\overline{F}(x)} = 2$, where $F^{*2} = F * F$ denotes the convolution of F with itself.

It is well-known that $\mathcal{R}_{-\alpha} \subset \mathcal{C} \subset \mathcal{L} \cap \mathcal{D} \subset \mathcal{S} \subset \mathcal{L}$

for any $\alpha \geq 0$.

In many areas of applied probability, it is found that random sums often have a heavy tail. Many researchers are interested in the questions of what causes the heavy tail of a random sum and what is the relationship among the tail probabilities of S_τ, X and τ. In one case where the summands have a heavier tail than τ (i.e., $\overline{F_\tau}(x) = o(\overline{F}(x))$), it is found that the tail behavior of a random sum S_τ is decided by the tail of X and the mean of τ, and that S_τ and X belong to the same subclass of heavy-tailed distributions (see [4–5, 8, 10, 13–14] etc.).

Recently, other cases in which the tail of X is not heavier than that of τ have attracted a lot of academic attention. Faÿ et al. [7] gave sufficient conditions for $S_\tau \in \mathcal{R}_{-\alpha}$ when X has a lighter tail than τ (i.e., $\overline{F}(x) = o(\overline{F_\tau}(x))$), and gave necessary conditions for $S_\tau \in \mathcal{R}_{-\alpha}$ when X has a lighter tail than S_τ. It states that the tail behavior of a random sum S_τ is decided by the tail of τ and the mean of X, and that S_τ and τ belong to the same class $\mathcal{R}_{-\alpha}$ if $\overline{F}(x) = o(\overline{F_\tau}(x))$. Some sufficient conditions for $S_\tau \in \mathcal{C}$ have been obtained by many authors (see [1, 11, 15] etc.).

The purpose of this paper is to give necessary and sufficient conditions for $S_\tau \in \mathcal{C}$ when $\overline{F}(x) = o(\overline{F}_\tau(x))$ in which the summands X_k ($k \geq 1$) are extended negatively dependent random variables (see Definition 2.1 below) defined on $(-\infty, \infty)$.

We will introduce some definitions of the dependence structure and give the main results of this paper in Section 2. The proofs of the theorems are given in Section 3.

2 Main Results

First, we give some definitions of the dependence structure, which are introduced by Chen et al. [3] and Liu [9].

Definition 2.1 (see [3, 9]) *A finite family of random variables $\{X_k : 1 \leq k \leq n\}$ is said to be*

(1) *lower extended negatively dependent* (*LEND for short*) *if there exists a constant* $M \geq 1$, *such that for all real numbers* $x_1, x_2, \cdots, x_n$,

$$P(X_1 \leq x_1, \cdots, X_n \leq x_n) \leq M \prod_{k=1}^{n} P(X_k \leq x_k); \tag{2.1}$$

(2) *upper extended negatively dependent* (*UEND for short*) *if there exists a constant* $M \geq 1$, *such that for all real numbers* $x_1, x_2, \cdots, x_n$,

$$P(X_1 > x_1, \cdots, X_n > x_n) \leq M \prod_{k=1}^{n} P(X_k > x_k); \tag{2.2}$$

(3) *extended negatively dependent* (*END for short*) *if there exists a constant* $M \geq 1$, *such that both* (2.1) *and* (2.2) *hold for all real numbers* $x_1, x_2, \cdots, x_n, \cdots$.

The constant M in equations (2.1)–(2.2) is said to be dominating constant. A sequence of random variables $\{X_k : k \geq 1\}$ is said to be END (LEND, UEND) if each of its finite subfamilies is END (LEND, UEND) for some common dominating constant M.

The END structure covers many negative dependence structures and, more interestingly, it covers certain positive dependence structures. More detailed discussions and some examples can be found in Chen et al. [3] and Liu [9].

Now, we give the main results of this paper as follows.

Theorem 2.1 *Let* $\{X, X_k : k \geq 1\}$ *be a sequence of END random variables with a common distribution* F *satisfying* $EX > 0$. *Let* τ *be a nonnegative integer-valued random variable with a distribution* F_τ, *independent of* $\{X, X_k : k \geq 1\}$. *Suppose that one of the following two conditions holds*:

(i) $E\tau < \infty$ *and*

$$P(X > x) = o(P(\tau > x)), \tag{2.3}$$

or

(ii) $E\tau = \infty$ *and*

$$\lim_{x \to \infty} (x^r \vee x \ln^\delta x) P(X > x) = 0 \tag{2.4}$$

for some $r \geq 1$ *and* $\delta > 1$ *and*

$$\limsup_{x \to \infty} \frac{E[\tau I(\tau \leq x)]}{(x^r \vee x \ln^\delta x) P(\tau > x)} < \infty. \tag{2.5}$$

Then the following two assertions are equivalent:

(a) $\tau \in \mathcal{C}$;

(b) $S_\tau \in \mathcal{C}$.

Furthermore, each of them implies that

$$P(S_\tau > x) \sim P\Big(\tau > \frac{x}{EX}\Big). \tag{2.6}$$

Remark 2.1 The following question naturally occurs: Can (2.6) imply (a) or (b)?

The following example gives a negative answer.

Example 2.1 Let X be degenerate at $p > 0$ (so $F \notin \mathcal{C}$) and let τ be any nonnegative integer-valued random variable. Obviously, we have

$$P(S_\tau > x) = P(p\tau > x) = P\Big(\tau > \frac{x}{EX}\Big).$$

This shows that (2.6) may not imply (a) or (b).

Remark 2.2 If $F_\tau \in R_{-\alpha}$ for some $\alpha \in (0,1)$, then (2.5) holds for any $r \geq 1$ by Karamata's theorem (see [2, Propositions 1.5.8 and 1.5.9a]) and $E\tau = \infty$. Remark 4.5 of Faÿ et al. [7] gave an example in which $S_\tau \in R_{-1}$ and $\overline{F}(x) = o(\overline{F}_\tau(x))$ can not imply (2.6) if $E\tau = \infty$. Hence, some extra conditions are needed if $E\tau = \infty$. It is obvious that both (1.4) in [15] and (3.11) in [11] are stronger than (2.5) when $E\tau = \infty$.

Remark 2.3 If $E\tau < \infty$, then (2.3) implies that $EXI(X > 0) < \infty$, where $I(\cdot)$ is the indicator function of a set. If $E\tau = \infty$, then (2.5) implies that $EXI(X > 0) < \infty$. Hence, the conditions of Theorem 2.1 always imply that $E|X| < \infty$ since $EX > 0$.

3 Proof of Theorem 2.1

Before giving the proof of the main results, we first give several lemmas. The first lemma is a direct consequence of Definition 2.1 and was mentioned by Chen et al. [3].

Lemma 3.1 *If $\{X_k : 1 \leq k \leq n\}$ are UEND (or LEND) random variables for some dominating constant M and $\{h_k(\cdot) : 1 \leq k \leq n\}$ are non-decreasing functions, then $\{h_k(X_k) : 1 \leq k \leq n\}$ are still UEND (or LEND) random variables for the same dominating constant M.*

Lemma 3.2 *Let $\{X_k : 1 \leq k \leq n\}$ be UEND (or LEND) random variables for some dominating constant M. Let $\{Y_k : 1 \leq k \leq n\}$ be independent random variables, independent of $\{X_k : 1 \leq k \leq n\}$. Let*

$$Z_k = X_k + Y_k, \quad k = 1, 2, \cdots, n.$$

Then, $\{Z_k : 1 \leq k \leq n\}$ are UEND (or LEND) random variables for the same dominating constant M.

Proof We only prove the case that $\{X_k : 1 \leq k \leq n\}$ are UEND. For any real numbers $x_1, x_2, \cdots, x_n$,

$$\begin{aligned}
&P(Z_1 > x_1, \cdots, Z_n > x_n) \\
&= \int_{\mathbb{R}^n} P(X_1 > x_1 - y_1, \cdots, X_n > x_n - y_n) P(Y_1 \in \mathrm{d}y_1, \cdots, Y_n \in \mathrm{d}y_n) \\
&\leq M \int_{\mathbb{R}^n} P(X_1 > x_1 - y_1) \cdots P(X_n > x_n - y_n) P(Y_1 \in \mathrm{d}y_1) \cdots P(Y_n \in \mathrm{d}y_n) \\
&= M P(Z_1 > x_1) \cdots P(Z_n > x_n).
\end{aligned}$$

The next lemma is a slight adjustment of Corollary 3.1 of Tang [12].

Lemma 3.3 *Let $\{X, X_k : k \geq 1\}$ be a sequence of UEND random variables with a common distribution $F \in \mathcal{D}$ and a mean $\mu = EX$. Then for each fixed $\gamma > 0$ and some $C = C(\gamma)$ irrespective to x and n, the inequality*

$$P(S_n - n\mu > x) \leq Cn\overline{F}(x+\mu)$$

holds uniformly for all $x \geq \gamma n$ and $n = 1, 2, \cdots$.

Proof The proof is just similar to that of Corollary 3.1 of Tang [12] and hence is omitted.

The following three lemmas play key roles in the proof of Theorem 2.1.

Lemma 3.4 *Let $\{X, X_k : k \geq 1\}$ be a sequence of END random variables with a common distribution F satisfying $EX \in (0,\ \infty)$. Let τ be a nonnegative integer-valued random variable with a distribution F_τ, independent of $\{X, X_k : k \geq 1\}$. Then*

$$\liminf_{x\to\infty} \frac{P(S_\tau > x)}{P\Big(\tau > \dfrac{cx}{EX}\Big)} \geq 1 \tag{3.1}$$

for any $c > 1$.

Proof For any $c > 1$ and $x > 0$, we have

$$P(S_\tau > x) = \sum_{n=1}^{\infty} P(S_n > x)P(\tau = n) \geq \sum_{n > \frac{cx}{EX}} P(S_n > x)P(\tau = n).$$

For any fixed $\varepsilon > 0$, by Theorem 1 in [3], there exists $x_1 > 0$, such that

$$P\Big(\frac{S_n}{n} - EX > -\frac{(c-1)EX}{c}\Big) > 1 - \varepsilon$$

holds for all $x > x_1$ and $n \geq \frac{cx}{EX}$. Consequently, for any $x > x_1$, it follows that

$$\begin{aligned} P(S_\tau > x) &\geq \sum_{n > \frac{cx}{EX}} P\Big(\frac{S_n}{n} > \frac{EX}{c}\Big)P(\tau = n) \\ &= \sum_{n > \frac{cx}{EX}} P\Big(\frac{S_n}{n} - EX > -\frac{(c-1)EX}{c}\Big)P(\tau = n) \\ &> (1-\varepsilon)P\Big(\tau > \frac{cx}{EX}\Big). \end{aligned}$$

By the arbitrariness of ε, (3.1) holds for any $c > 1$.

Lemma 3.5 *Under the conditions of Theorem 2.1, if $S_\tau \in \mathcal{L} \cap \mathcal{D}$, then we have*

$$\limsup_{x\to\infty} \frac{P(S_\tau > x)}{P\Big(\tau > \dfrac{vx}{EX}\Big)} \leq 1 \tag{3.2}$$

for any $v < 1$.

Proof Obviously, (3.2) holds if $v \leq 0$, so we suppose $v \in (0, 1)$ in the later discussion. By Remark 2.3, it follows that $E|X| < \infty$. It is easy to see that

$$P(S_\tau > x) \leq \sum_{n \leq \frac{vx}{EX}} P(S_n > x)P(\tau = n) + P\Big(\tau > \frac{vx}{EX}\Big). \tag{3.3}$$

If we can prove that

$$p(x) \triangleq \sum_{n\le \frac{vx}{EX}} P(S_n > x)P(\tau = n) = o(P(S_\tau > x)), \tag{3.4}$$

then from (3.3) and (3.4), it immediately follows that

$$\liminf_{x\to\infty} \frac{P\Big(\tau > \dfrac{vx}{EX}\Big)}{P(S_\tau > x)} \ge 1,$$

which is equivalent to (3.2). So we only need to prove (3.4).

First we discuss case (i) where $E\tau < \infty$: By Lemma 3.4 and $v \in (0,1)$, it follows that

$$\limsup_{x\to\infty} \frac{P(\tau > x)}{P(S_\tau > (vEX)x)} \le 1.$$

Combining with (2.3) and $S_\tau \in \mathcal{D}$ yields that

$$\begin{aligned} P(X > x) &= \frac{P(X > x)}{P(\tau > x)} \cdot \frac{P(\tau > x)}{P(S_\tau > (vEX)x)} \cdot \frac{P(S_\tau > (vEX)x)}{P(S_\tau > x)} \cdot P(S_\tau > x) \\ &= o(P(S_\tau > x)). \end{aligned}$$

Let $\{Y, Y_k : k \ge 1\}$ be a sequence of independent identically distributed random variables with a common distribution V, where V is the uniform distribution on the interval $[0,1]$, independent of $\{X, X_k : k \ge 1\}$ and τ. Let $Z = X + Y,\ Z_k = X_k + Y_k,\ k \ge 1$. Then, by Lemma 2.2, $\{Z, Z_k : k \ge 1\}$ is a sequence of END random variables with a common distribution $F * V$. By $S_\tau \in \mathcal{L} \cap \mathcal{D}$ and (2.3), it is easy to see that $P(Z > x) = o(P(S_\tau > x))$. By Lemma 4.4 in [7], there exists a nondecreasing slowly varying function $L(x)$ satisfying

$$L(x) \to \infty \quad \text{and} \quad \frac{P(Z > x)}{P(S_\tau > x)} L(x) \to 0.$$

Hence there exists $x' > 0$, such that

$$P(Z > x) \le \frac{P(S_\tau > x)}{L(x)} \le 1$$

holds for all $x \ge x'$. Define a distribution G as follows:

$$G(x) = G((-\infty, x]) = \begin{cases} 0, & x < x', \\ 1 - \dfrac{P(S_\tau > x)}{L(x)}, & x \ge x', \end{cases}$$

and let

$$X' = G^{-1}(F * V(Z)), \quad X'_k = G^{-1}(F * V(Z_k)), \quad k = 1, 2, \cdots,$$

where

$$G^{-1}(y) = \inf\{t \in R : G(t) \ge y\}, \quad 0 \le y \le 1.$$

It is easy to see that $P(X' \le x) = G(x)$ for all real number x and

$$P(X' > x) = o(P(S_\tau > x)). \tag{3.5}$$

By Proposition A.16(d) in [6], it follows that $G(X'_k) \ge F * V(Z_k)$ for all $k \ge 1$, which implies that $X'_k \ge Z_k \ge X_k$ a.s. for all $k \ge 1$ since $G(x) \le F * V(x)$ for all real numbers x. Moreover, it follows that $EX \le EX' < \infty$ by the definition of X' and $E\tau < \infty$. Write $S'_n = \sum_{k=1}^n X'_k,\ n \ge 1$. Then $P(S_n > x) \le P(S'_n > x)$ holds for all $x \ge 0$ and $n \ge 1$ since $S_n \le S'_n$ a.s. holds for all $n \ge 1$.

For all $x > 0$, we split $p(x)$ into two parts as

$$\begin{aligned} p(x) &\le \sum_{n \le \frac{vx}{EX'}} P(S'_n > x)P(\tau = n) + \sum_{\frac{vx}{EX'} < n \le \frac{vx}{EX}} P(S_n > x)P(\tau = n) \\ &\triangleq p_1(x) + p_2(x). \end{aligned} \tag{3.6}$$

Note that $S_\tau \in \mathcal{L} \cap \mathcal{D}$ implies $G \in \mathcal{L} \cap \mathcal{D}$. By Lemma 3.3, there exists a positive constant $C = C(v)$ independent of x and n, such that

$$P(S'_n > x) \le P(S'_n - nEX' > (1-v)x) \le Cn\overline{G}((1-v)x + EX')$$

holds for all $n \le \frac{vx}{EX'}$ $\left(x \ge \frac{EX'}{v}n\right)$. Combining with (3.5) we have

$$\begin{aligned} p_1(x) &\le C \sum_{n \le \frac{vx}{EX'}} nP(\tau = n)\overline{G}((1-v)x + EX') \\ &\le CE\tau\overline{G}((1-v)x + EX') \\ &= o(P(S_\tau > x)). \end{aligned} \tag{3.7}$$

On the other hand, by Theorem 1 in [3], we have

$$\lim_{n\to\infty} P\Big(\frac{S_n}{n} - EX > \frac{(1-v)EX}{v}\Big) = 0.$$

It follows that

$$\begin{aligned} p_2(x) &\le \sum_{\frac{vx}{EX'} < n \le \frac{vx}{EX}} P\Big(\frac{S_n}{n} - EX > \frac{(1-v)EX}{v}\Big)P(\tau = n) \\ &\le o(1)\overline{F}_\tau\Big(\frac{vx}{EX'}\Big) = o(P(S_\tau > x)). \end{aligned} \tag{3.8}$$

Hence (3.4) follows for the case $E\tau < \infty$.

Now we discuss the case (ii) where $E\tau = \infty$: Let

$$g_r(x) = \begin{cases} x^r, & x \ge 1, \\ 0, & x < 1, \end{cases}$$

if $r > 1$; and let

$$g_r(x) = \begin{cases} x\ln^\delta x, & x > 1, \\ 0, & x \le 1, \end{cases}$$

if $r = 1$. The assumption (2.4) implies that

$$P(X > x) = o\Big(\frac{1}{g_r(x)}\Big).$$

By Lemma 4.4 in [7], there exists a nondecreasing slowly varying function $L(x)$ satisfying

$$L(x) \to \infty \quad \text{and} \quad P(X > x) = o\Big(\frac{1}{g_r(x)L(x)}\Big).$$

Thus there exists $x' > 1$, such that

$$P(X > x) \le \frac{1}{g_r(x)L(x)} \le 1$$

holds for all $x \ge x'$. Define a distribution G as follows:

$$G(x) = G((-\infty, x]) = \begin{cases} 0, & x < x', \\ 1 - \dfrac{1}{g_r(x)L(x)}, & x \ge x'. \end{cases}$$

It is obvious that $G \in \mathcal{R}_{-r} \subset \mathcal{L} \cap \mathcal{D}$. Without loss of generality, we assume that F is absolutely continuous, otherwise F can be replaced by $F * V$, where V is the uniform distribution on the interval $[0, 1]$, so then $F * V$ is absolutely continuous and $\overline{F * V}(x) = o(P(S_\tau > x))$. Let

$$X'_k = G^{-1}(F(X_k)), \quad k = 1, 2, \cdots$$

and

$$S'_n = \sum_{k=1}^{n} X'_k, \quad n \ge 1.$$

Similarly to the proof of (3.7), there exists a positive constant $C = C(v)$ independent of x and n, such that

$$\begin{aligned} p_1(x) &\le C \sum_{n \le \frac{vx}{EX'}} nP(\tau = n)\overline{G}((1 - v)x + EX') \\ &= CE\tau I\Big(\tau \le \frac{vx}{EX'}\Big)\overline{G}((1 - v)x + EX'). \end{aligned}$$

Hence, for sufficiently large x, we have

$$\begin{aligned} p_1(x) &\le 2CE\tau I\Big(\tau \le \frac{vx}{EX'}\Big)\overline{G}((1 - v)x) \\ &= \frac{C}{L((1 - v)x)} \frac{E\tau I\Big(\tau \le \frac{vx}{EX'}\Big)}{g_r\Big(\frac{vx}{EX'}\Big)P\Big(\tau > \frac{vx}{EX'}\Big)} \frac{g_r\Big(\frac{vx}{EX'}\Big)}{g_r\Big(\frac{(1 - v)x}{EX'}\Big)} \frac{P\Big(\tau > \frac{vx}{EX'}\Big)}{P(S_\tau > x)} P(S_\tau > x) \\ &= o(P(S_\tau > x)). \end{aligned}$$

Combining with (3.6) and (3.8), (3.4) is obtained.

Lemma 3.6 *Under the conditions of Theorem* 2.1*, if* $\tau \in \mathcal{L} \cap \mathcal{D}$*, then* (3.2) *holds for any* $v < 1$.

Proof The proof is similar to Lemma 3.5 and hence is omitted.

Proof of Theorem 2.1 Obviously, (a) and (2.6) imply (b); and (b) and (2.6) imply (a). Therefore, we need only to prove that either (a) or (b) implies (2.6). We first prove that (b) implies (2.6). It suffices to prove that

$$\liminf_{x\to\infty} \frac{P(S_\tau > x)}{P\Big(\tau > \dfrac{x}{EX}\Big)} \geq 1 \tag{3.9}$$

and

$$\limsup_{x\to\infty} \frac{P(S_\tau > x)}{P\Big(\tau > \dfrac{x}{EX}\Big)} \leq 1. \tag{3.10}$$

By Lemma 3.4, (3.1) holds for any $c > 1$. It follows that

$$\liminf_{x\to\infty} \frac{P(S_\tau > x)}{P\Big(\tau > \dfrac{x}{EX}\Big)} = \lim_{c\downarrow 1} \liminf_{x\to\infty} \frac{P(S_\tau > x)}{P\Big(S_\tau > \dfrac{x}{c}\Big)} \frac{P\Big(S_\tau > \dfrac{x}{c}\Big)}{P\Big(\tau > \dfrac{x}{EX}\Big)} \geq 1$$

since $S_\tau \in \mathcal{C}$. (3.9) is obtained.

The proof of (3.10) is similar to that of (3.9). By Lemma 3.5, (3.2) holds for all $v < 1$. It follows that

$$\limsup_{x\to\infty} \frac{P(S_\tau > x)}{P\Big(\tau > \dfrac{x}{EX}\Big)} = \lim_{v\uparrow 1} \limsup_{x\to\infty} \frac{P(S_\tau > x)}{P\Big(S_\tau > \dfrac{x}{v}\Big)} \frac{P\Big(S_\tau > \dfrac{x}{v}\Big)}{P\Big(\tau > \dfrac{x}{EX}\Big)} \leq 1.$$

The proof of the fact that (a) implies (2.6) is quite similar to the above. By Lemma 3.4, Lemma 3.6 and $F_\tau \in \mathcal{C}$, it follows that

$$\liminf_{x\to\infty} \frac{P(S_\tau > x)}{P\Big(\tau > \dfrac{x}{EX}\Big)} = \lim_{c\downarrow 1} \liminf_{x\to\infty} \frac{P(S_\tau > x)}{P\Big(\tau > \dfrac{cx}{EX}\Big)} \frac{P\Big(\tau > \dfrac{cx}{EX}\Big)}{P\Big(\tau > \dfrac{x}{EX}\Big)} \geq 1$$

and

$$\limsup_{x\to\infty} \frac{P(S_\tau > x)}{P\Big(\tau > \dfrac{x}{EX}\Big)} = \lim_{v\uparrow 1} \limsup_{x\to\infty} \frac{P(S_\tau > x)}{P(\tau > \dfrac{vx}{EX})} \frac{P\Big(\tau > \dfrac{vx}{EX}\Big)}{P\Big(\tau > \dfrac{x}{EX}\Big)} \leq 1.$$

This finishes the proof of Theorem 2.1.

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