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# On Reduced Lantern Relations in Mapping Class Groups*

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**Abstract** Let $S$ be a hyperbolic Riemann surface with a finite area. Let $G$ be the covering group of $S$ acting on the hyperbolic plane $\mathbf{H}$. In this paper, the author studies some algebraic relations in the mapping class group of $\dot{S}$ for $\dot{S} = S\backslash\{\text{a point}\}$. The author shows that the only possible relations between products of two Dehn twists and products of mapping classes determined by two parabolic elements of $G$ are the reduced lantern relations. As a consequence, a partial solution to a problem posed by J. D. McCarthy is obtained.

**Keywords** Dehn twists, Simple closed geodesics, Lantern relation
**2000 MR Subject Classification** 32G15, 30F60

## 1 Statement of Results

Let $S$ be a hyperbolic Riemann surface of type $(p, n)$ with a finite area, where $p$ is the genus and $n$ is the number of punctures of $S$. Assume throughout that $3p + n > 3$. Let $\mathbf{H}$ denote the hyperbolic plane. By the uniformization theorem (see [8]), there is a holomorphic covering map $\varrho : \mathbf{H} \to S$ from which we can obtain a covering group $G$ which acts on $\mathbf{H}$ as isometries and is a torsion free, finitely generated Fuchsian group of the first kind.

Denote by $\dot{S}$ the surface obtained from $S$ with one point $x$ removed. Let $\mathscr{F}$ denote the subgroup of the mapping class group on $\dot{S}$ that consists of mapping classes isotopic to the identity as $x$ is filled in. It is well-known (see [3, 5]) that $\mathscr{F}$ is the image of $G$ under the so-called "Bers isomorphism". In the literature, elements of $\mathscr{F}$ are called point-pushing mapping classes.

Let $\mathscr{F}_0$ be the subset of $\mathscr{F}$ consisting of elements with forms $t_a t_b^{-1}$ or $t_a^{-1} t_b$, where $a, b$ are simple closed geodesics on $\dot{S}$ and $t_c$ is the positive Dehn twist about a geodesic $c$. It is clear that if $t_a t_b^{-1} \in \mathscr{F}_0$ and both $a, b$ are non-trivial curves on $S$, then $\widetilde{a} = \widetilde{b}$. Here and hereafter, we use the symbols $\widetilde{a}$ and $\widetilde{b}$ to denote the geodesics on $S$ homotopic to $a$ and $b$, respectively. In the case where the pair $(a, b)$ fills $\dot{S}$, that is, $a \cup b$ intersects every simple closed geodesic on $\dot{S}$, then by Thurston's theorem (see [14]), $t_a t_b^{-1}$ and $t_a^{-1} t_b$ are pseudo-Anosov. The element $h$ of $G$ corresponding to $t_a t_b^{-1}$ or $t_a^{-1} t_b$ is called an essential hyperbolic element (see [10] for more information).

The main purpose of this article is to clarify the situation when the product of two parabolic elements of $G$ can be identified with an element of $\mathscr{F}_0$ (the product always belongs to $\mathscr{F}$). To state our results, we need some geometric and topological terms related to the mapping class group.

Let $\mathbf{D}$ be a thrice punctured disk with three punctures $x', y, z$. Denote by $D$ the boundary of $\mathbf{D}$. Let $a, b, c \subset \mathbf{D}$ be the boundaries of twice punctured disks enclosing $\{x', y\}, \{x', z\}$ and $\{y, z\}$, respectively, such that $a, b$ and $c$ pairwisely intersect twice (see Figure 1). Then the

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classical lantern relation (see [6, 11]) is simplified to the relation

$$t_D = t_c t_b t_a. \tag{1.1}$$

Usually, if $\dot{S}$ contains at least three punctures, $\mathbf{D}$ can be embedded into $\dot{S}$ in such a way that $x'$ is identified with $x$ and $\{y, z\}$ can be identified with two other punctures of $S$. Thus $a, b, c$ and $D$ can be considered simple closed geodesics on $\dot{S}$. In this situation we call these geodesics $a, b, c$ and $D$ geometrically related in Figure 1.

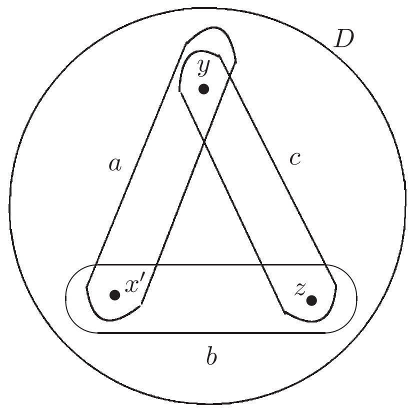


Figure 1 A reduced lantern relation

Assume that $S$ is non-compact. Then $G$ contains infinitely many parabolic elements and $\dot{S}$ contains at least two punctures. For each $h \in G$, let $h^*$ denote the corresponding element in $\mathscr{F}$. By Theorem 2 of [10], $h^*$ is the Dehn twist (positive or negative) along a geodesic $e$ on $\dot{S}$ that is the boundary of a twice punctured disk enclosing $x$ if and only if $h$ is a primitive parabolic element. In this case, $e$ is a trivial loop on $S$ and is called a preperipheral geodesic.

We first prove the following result.

**Theorem 1.1** *Let $h_1, h_2 \in G$ be parabolic elements such that $h_1^* = t_a$ and $h_2^* = t_b$. Assume that $(h_2h_1)^* \in \mathscr{F}_0$, which allows us to write $(h_2h_1)^*$ as $t_D t_c^{-1}$ or $t_c^{-1} t_D$ for some simple closed geodesics $c, D$ on $\dot{S}$. Then $\dot{S}$ contains at least three punctures* (*so $S$ contains at least two punctures*) *and $a, b, c, D$ are geometrically related in Figure* 1.

During a conference hosted by AMS in 2002, J. D. McCarthy asked a question about how to characterize geometric relations by means of algebraic relations among various Dehn twists. We use the symbol $i(a, b)$ to denote the geometric intersection number between $a$ and $b$. To the best knowledge of the author, only the following relations are well-known (see [9]):

(1) $t_a^j = t_b^k$ if and only if $j = k$ and $a = b$,

(2) $t_a^j t_b^k = t_b^k t_a^j$ if and only if $i(a, b) = 0$, and

(3) $t_b^j t_a^k t_b^j = t_a^k t_b^j t_a^k$ and $a \neq b$ if and only if $j = k = \pm 1$ and $i(a, b) = 1$.

Some results related to the classical lantern relation and the chain relation were found in Margalit [12] and Hamidi-Tehrani [7]. Their results rely on the strong hypothesis that some words generated by $t_a$ and $t_b$ are multi-twists (defined by finite collections of disjoint simple closed geodesics).

The proof of Theorem 1.1 leads to the following result, which does not impose any condition on commutativity and disjointness among simple closed geodesics and thereby gives a partial solution to the problem posed by McCarthy.

**Theorem 1.2** *Let $\mathcal{S}$ be a Riemann surface of type $(p, n)$ with a finite hyperbolic area. Assume that $3p + n > 3$ and $n \geq 1$. Let $a, b, c$ and $D$ be simple closed geodesics on $\mathcal{S}$. Then the relation* (1.1) *holds if and only if $\mathcal{S}$ contains at least three punctures and $a, b, c, D$ are geometrically related in Figure* 1.

Here is the outline of this paper. Section 2 is dedicated to preliminaries which include some definitions and well-known facts. In Section 3 we investigate parabolic loops in the fundamental group of $S$. In Section 4, we prove several related lemmas. In Section 5, we prove Theorem 1.1 and Theorem 1.2. Section 6 includes a technical lemma that handles the case where $(h_2h_1)^* \in \mathscr{F}_0$ is also represented by a product of two Dehn twists along non-preperipheral geodesics.

## 2 Background and Preliminaries

Let $G$ be a Fuchsian group of the first kind that acts on $\mathbf{H}$ as a group of isometries so that $\mathbf{H}/G \cong S$. Elements of $G$ are either hyperbolic or parabolic, and every hyperbolic element $g$ keeps invariant a unique oriented geodesic axis$(g)$ called the axis of $g$.

Let $\pi_1(S,x)$ denote the fundamental group of $S$. Then $\pi_1(S,x)$ is isomorphic to $G$. Let $\varepsilon : G \to \pi_1(S,x)$ be an isomorphism. An element $g \in G$ is hyperbolic if and only if $\varepsilon(g)$ is represented by a non-trivial closed geodesic; $g \in G$ is parabolic if and only if $\varepsilon(g)$ is represented by a loop around a puncture of $S$. More precisely, a hyperbolic element $g \in G$ is simple if and only if $\varrho(\text{axis}(g))$ is a simple closed geodesic; it is essential hyperbolic if and only if $\varrho(\text{axis}(g))$ is a filling closed geodesic (in the sense that every component of $S\backslash\varrho(\text{axis}(g))$ is either a polygon or a once punctured polygon); it is non-simple and non-essential if and only if $\varrho(\text{axis}(g))$ is a non-simple and non-filling closed geodesic.

Let $T(S)$ denote the Teichmüller space of $S$. That is, $T(S)$ is the space of all conformal structures $\mu(S)$ on $S$ quotient by an equivalent relation, where two conformal structures $\mu : S \to \mu(S)$ and $\mu' : S \to \mu'(S)$ are equivalent if and only if there is a conformal map $c : \mu(S) \to \mu'(S)$ such that $(\mu')^{-1}c\mu$ is isotopic to the identity. The equivalence class of $\mu$ is denoted by $[\mu]$. It is well-known that $T(S)$ is a complex manifold of dimension $3p+n-3$.

Let $V(S)$ be the fiber bundle over $T(S)$ so that any fiber of $V(S)$ over $[\mu] \in T(S)$ is the Riemann surface representing $[\mu]$. Then $V(S)$ is also a complex manifold of dimension $3p+n-2$, and its universal covering manifold $F(S)$ is called the Bers fiber space. The fiber over $[0] \in T(S)$ (represented by $S$) is the central fiber which is identified with the hyperbolic plane $\mathbf{H}$. Thus the covering group $G$ naturally acts on $F(S)$ that preserves each fiber in $F(S)$. A remarkable result of Bers [3] states that there exists an isomorphism $\varphi$ of $F(S)$ onto $T(\dot{S})$, which induces (by conjugation) an isomorphism $\varphi^*$ of $G$ onto $\mathscr{F}$.

By Theorem 2 of [10], $g \in G$ is a primitive parabolic element if and only if $g^*$ is a simple Dehn twist $t_a$ along the boundary $a$ of a twice punctured disk enclosing $x$; $g \in G$ is essential hyperbolic if and only if $g^*$ is pseudo-Anosov (in the sense of [14]); $g \in G$ is a simple hyperbolic element if and only if $g^*$ is a spin map $t_{c_1}^{-k}t_{c_2}^{k}$, where $k$ is an integer and $\{c_1, c_2\}$ are the boundary components of an $x$-punctured cylinder on $\dot{S}$. Finally, $g \in G$ is non-simple and non-essential if and only if $g^*$ is a pure mapping class that has a unique pseudo-Anosov component on $\dot{S}$ that contains the puncture $x$.

Let $Q(G)$ denote the group of quasiconformal automorphisms $w$ of $\mathbf{H}$ such that $wGw^{-1} = G$. Two such maps $w, w' \in Q(G)$ are said to be equivalent if $wgw^{-1} = w'g(w')^{-1}$ for every $g \in G$. It is well-known that $G$ can be regarded as a normal subgroup of $Q(G)/\sim$ and $\varphi^*$ extends to an isomorphism of $Q(G)/\sim$ onto the $x$-pointed mapping class group $\text{Mod}_S^x$ of $\dot{S}$. Let $[w]$ denote the equivalence class of an element $w \in Q(G)$ and $[w]^*$ denote the image of $[w] \in Q(G)/\sim$ under the isomorphism $\varphi^* : Q(G)/\sim \to \text{Mod}_S^x$.

Let $a \subset \dot{S}$ be a simple closed geodesic that is non-trivial on $S$ as $x$ is filled in. Let $\widetilde{a}$ denote the (non-trivial) simple closed geodesic homotopic to $a$ on $S$. Thus the positive Dehn twist $t_{\widetilde{a}}$ defines a special non-trivial reducible mapping class. Let $\widehat{a} \subset \mathbf{H}$ be a geodesic so that $\varrho(\widehat{a}) = \widetilde{a}$. Denote by $\{\Delta, \Delta'\}$ the components of $\mathbf{H}\backslash\{\widehat{a}\}$. Then $\widehat{a}$, $\Delta$ and $\Delta'$ are invariants under the action of a simple hyperbolic element of $G$. The Dehn twist $t_{\widetilde{a}}$ can be lifted to a map $\tau_a : \mathbf{H} \to \mathbf{H}$ with respect to $\Delta$, say, which satisfies the conditions

(i) $\tau_a G \tau_a^{-1} = G$;

(ii) $\varrho \circ \tau_a = t_{\widetilde{a}} \circ \varrho$.

In addition to (i) and (ii) above, $\tau_a$ defines a collection $\mathscr{U}_a$ of half planes in $\mathbf{H}$ in a partial order defined by inclusion. There are infinitely many maximal elements of $\mathscr{U}_a$, all maximal elements $\Delta_i$ ($\Delta$ is one of them) of $\mathscr{U}_a$ are mutually disjoint, and the complement

$$\Omega_a = \mathbf{H} \backslash \bigcup_i \Delta_i \subset \Delta'$$

is not empty. In fact, it is a convex region bounded by a collection of disjoint geodesics $\widehat{a}$ with $\varrho(\widehat{a}) = \widetilde{a}$. It is clear that $\Delta'$ contains infinitely many maximal elements of $\mathscr{U}_a$ and the map $\tau_a$ constructed above keeps each maximal element invariant and has the property that

$$\tau_a|_{\Omega_a} = \mathrm{id}.$$

The map $\tau_a$ so obtained depends on the choice of a geodesic $\widehat{a}$ with $\varrho(\widehat{a}) = \widetilde{a}$, but it does not depend on the choice of a boundary component of $\Omega_a$. Moreover, $\tau_a$ determines an element $[\tau_a] \in Q(G)/\sim$. By Lemma 3.2 of [15], we can properly choose $\widehat{a}$ (and $\Delta$) so that $[\tau_a]^* \in \mathrm{Mod}_S^x$ is represented by the Dehn twist $t_a$ along $a$. If we use $\Delta'$ to acquire a lifting map $\tau_{a'}$ of $t_{\tilde{a}}$, we have $[\tau_{a'}]^* = t_{a_0}$, where $a_0$ together with $a$ forms the boundary of an $x$-punctured cylinder. See [15, 18] for more details. In the rest of this paper we call the triple $(\tau_a, \Omega_a, \mathscr{U}_a)$ the configuration corresponding to $a$.

## 3 Products of Parabolic Elements

Let $x_0 = x, x_1, \cdots, x_n$ denote the punctures of $\dot{S}$. Let $\mathscr{T}(x, x_i)$ be the set of preperipheral geodesics enclosing $x$ and $x_i$. Let $\mathscr{T}(x) = \bigcup_i \mathscr{T}(x, x_i)$.

Assume that $a, b \in \mathscr{T}(x)$. Then $a$ and $b$ are trivial loops on $S$ as the puncture $x$ is filled in. This is equivalent to that $a$ and $b$ are preperipheral and thus are the boundaries of twice punctured disks $D(a)$ and $D(b)$ that enclose $x$.

By Theorem 2 of [10] and Theorem 2 of [13], there exist primitive parabolic elements $T_a, T_b \in G$ such that $T_a^* = t_a$ and $T_b^* = t_b$. Under the isomorphism $\varepsilon : G \to \pi_1(S, x)$, $T_a$ and $T_b$ correspond to parabolic loops $e_a$ and $e_b$ passing through $x$, respectively, such that $e_a$ goes around $x_1$, and $e_b$ also goes around a puncture $x_i$. Note that $e_a, e_b$ go around the same puncture if and only if $T_a$ and $T_b$ are conjugate to each other in $G$.

Let $d(a)$ be the deformation retract of $D(a)$, that is, $d(a)$ is a path on $S$ connecting $x$ and $x_1$ so that $D(a)$ can be reconstructed from fattening $d(a)$. Likewise, let $d(b)$ be the deformation retract of $D(b)$. Clearly, $d(a)$ and $d(b)$ determine the parabolic loops $e_a$ and $e_b$ on $S$ passing through $x$, respectively. Assume that $d(a)$ and $d(b)$ intersect in a minimum number of points. We say $d(a)$ and $d(b)$ are disjoint if they only meet at $x$. In this case, $(d(a), d(b))$ forms a binary tree with two leaves $x_1$ and $x_i$. If $D(a)$ and $D(b)$ share both punctures, then by our convention, $d(a)$ intersects $d(b)$.

**Lemma 3.1** *Let $[\sigma'] \in \pi_1(S, x)$ correspond to the product $T_b T_a$. Then any representative of $[\sigma']$ is freely homotopic to a trivial or simple closed geodesic $\sigma$ if and only if $d(a)$ and $d(b)$ are disjoint.*

**Proof** Obviously, if $d(a)$ and $d(b)$ are disjoint, i.e., $(d(a), d(b))$ forms a binary tree with two leaves $x_1$ and $x_i$, then $e_a \cdot e_b$ is homotopic to the boundary of a twice punctured disk enclosing $x_1$ and $x_i$. The converse can be proved by a geometric argument. Suppose that $d(a)$ and $d(b)$ intersect at a minimum number of intersection points $\mathscr{S} = \{u_i; 1 \leq i \leq k\}$. Figures 2(a)–(b) show the first two such points $u_1$ and $u_2$ in two different situations.

Note that each $u_i$ contributes four intersection points between $e_a$ and $e_b$ which form vertices of a quadrilateral $Q_i$. Figure 3 illustrates some details of the curve concatenation $e_a \cdot e_b$ at $u_1$ and $u_2$, and at $x$ based on Figure 2(a). Vertices $u_{ij}$, $1 \leq j \leq 4$, of each quadrilateral $Q_i$ are labeled counterclockwise.

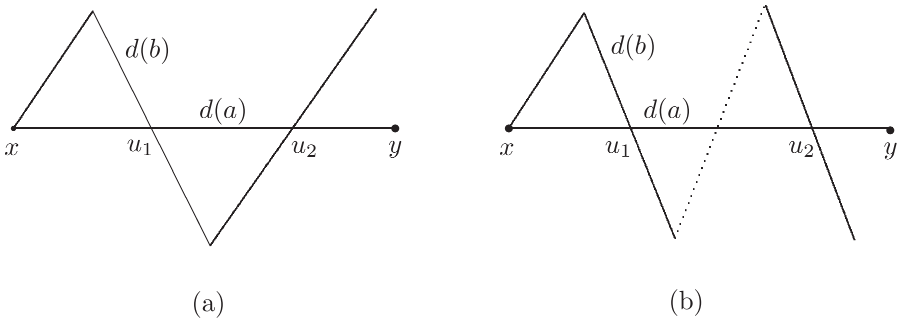


Figure 2 Paths $d(a)$ and $d(b)$ and their intersection points

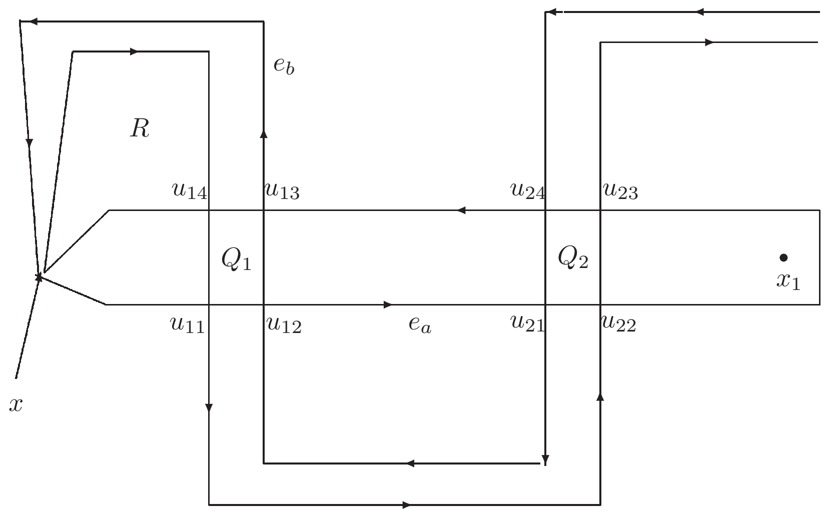


Figure 3 Fattenings of $d(a)$ and $d(b)$ produce $D(a)$ and $D(b)$ as well as the product $e_a \cdot e_b$

Since $\sigma$ is freely homotopic to $e_a \cdot e_b$, if $\sigma$ is a simple closed geodesic, then during the deformation, the points in $\mathscr{S}_0 = \{u_{ij};\ 1 \le i \le k \text{ and } 1 \le j \le 4\}$ are canceled in pairs, where at least one pair, according to the so-called bigon principle, constitutes vertices of a bigon. So it suffices to check if there exists a point $u_{ij}$ in $\mathscr{S}_0$ together with its neighboring point that forms vertices of a bigon. This can be done by examining each point $u_i$ for $1 \le i \le k$. The case where $i = 1$ and $j = 4$, is slightly different. If $u_{14}$ is the vertex of a monogon $R$, then $u_1$ is not in $\mathscr{S}$, which contradicts that $d(a)$ and $d(b)$ intersect at a minimum number of intersection points.

In the cases where $i > 1$, or $i = 1$ and $j \neq 4$, each vertex $u_{ij}$, $1 \le j \le 4$, of the quadrilateral $Q_i$ obtained from $u_i$ can not be canceled with any other vertex of $Q_i$. If $u_{i2}$ and $u_{(i+1)1}$ are also vertices of a bigon, then $u_i$ and $u_{i+1}$ are vertices of a bigon formed by $d(a)$ and $d(b)$. In this case, $u_i$ and $u_{i+1}$ can be removed from $\mathscr{S}$. This contradicts that $d(a)$ and $d(b)$ intersect at a minimum number of intersection points. After a finite number of steps, we see that there is no bigon in the complement of $e_a \cdot e_b$, that is to say, no points in $\mathscr{S}_0$ can be deleted. This leads to a contradiction. The case of Figure 2(b) can be handled in the same way.

Let $z_a, z_b$ denote the fixed points of $T_a$ and $T_b$, respectively. Conjugating by a Möbius transformation if necessary, we may assume without loss of generality that $z_a$ and $z_b$ are south and north poles on $\mathbf{S}^1$, respectively. Let $L$ and $R$ denote the left and right components of $\mathbf{S}^1 \backslash \{z_a, z_b\}$, respectively. See Figure 4.

For each point $z \in R$, one checks that $T_bT_a(z) \neq z$. Hence there are no fixed points of $T_bT_a$ on $R$. So the fixed point(s) of $T_bT_a$ must lie on $L$. The following lemma shows that there are actually two fixed points of $T_bT_a$ on $L$.

**Lemma 3.2** *If $3p+n>3$, then $T_bT_a \in G$ is hyperbolic.*

**Proof** By assumption, $a, b \in \mathscr{T}(x)$. If $d(a)$ and $d(b)$ intersect, then by Lemma 3.1, $T_bT_a$ is not parabolic. If $d(a)$ and $d(b)$ are disjoint, then by Lemma 3.1 again, $[\sigma']$ is represented by a trivial or simple closed geodesic $\sigma$. Note that $(p, n) \neq (0, 3)$, which implies $\dot{S}$ is not of type $(0, 4)$. Thus $\sigma$ is not trivial, which says that $\sigma$ is a non-trivial simple geodesic. So $T_bT_a$ is hyperbolic.

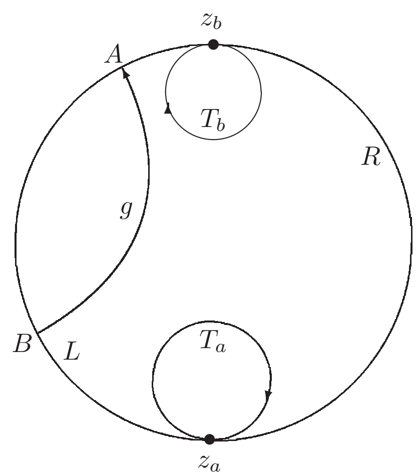


Figure 4 The product of $T_a$ and $T_b$ gives a hyperbolic element

By Lemma 3.2, $g := T_bT_a \in G$ is hyperbolic, whose axis axis($g$) meets $\mathbf{S}^1$ on the left component $L$ of $\mathbf{S}^1 \backslash \{z_a, z_b\}$. We also know that the orientation of the axis is as shown in Figure 4. Otherwise, suppose that axis($g$) takes an opposite orientation to the one shown in Figure 4. Then $g$ and $T_b^{-1}$ have the same relative motion direction. By the same proof of Lemma 7.1 of [16], $T_b^{-1}g$ is hyperbolic, which would contradict that $T_a = T_b^{-1}g$ and $T_a$ is a parabolic element of $G$.

## 4 Lantern Relation in a Reduced Form

In this section, we assume that $a, b, c$ and $D$ are simple closed geodesics on $\dot{S}$, such that

$$t_bt_a = t_c^{-1}t_D, \tag{4.1}$$

where $a, b \in \mathscr{T}(x)$ (in the case where $t_bt_a = t_Dt_c^{-1}$, the discussion is the same). As usual, we let $\widetilde{a}, \widetilde{b}, \widetilde{c}$ and $\widetilde{D}$ denote the geodesics homotopic to $a, b, c, D$ on $S$, respectively.

**Lemma 4.1** *With the above conditions, either $\widetilde{D}$ and $\widetilde{c}$ are trivial, or $\widetilde{D}$ and $\widetilde{c}$ are non-trivial.*

**Proof** If $\widetilde{D}$ is trivial and $\widetilde{c}$ is non-trivial, or $\widetilde{D}$ is non-trivial and $\widetilde{c}$ is trivial, then it quickly leads to a contradiction by filling in the puncture $x$ in (4.1).

**Lemma 4.2** *Assume that $a, b \in \mathscr{T}(x)$ and satisfy* (4.1)*. Then either $\widetilde{D}$ or $\widetilde{c}$ is non-trivial.*

**Proof** Suppose that both $\widetilde{D}$ and $\widetilde{c}$ are trivial. Then $a, b, c, D \in \mathscr{T}(x)$ satisfy (4.1). We claim that this does not occur, and the contradiction will complete the proof of the lemma.

Indeed, there are primitive parabolic elements $T_a, T_b, T_c, T_D \in G$ such that $T_a^* = t_a$, $T_b^* = t_b$, $T_c^* = t_c$ and $T_D^* = t_D$. Denote $\varepsilon(T_a) = e_a$, $\varepsilon(T_b) = e_b$, $\varepsilon(T_c) = e_c$ and $\varepsilon(T_D) = e_D$. These loops are parabolic in $\pi_1(S, x)$.

By Lemma 3.2, $T_bT_a \in G$ is a hyperbolic element. Thus the curve concatenation

$$\varepsilon(T_bT_a) = e_a \cdot e_b$$

is homotopic to a non-trivial closed geodesic $\sigma$. As discussed in the proof of Lemma 3.1, during the homotopy from $e_a \cdot e_b$ to $\sigma$, intersection points can be canceled only in pairs. Note that every interior intersection point between $d(a)$ and $d(b)$ contributes four intersection points between $a$ and $b$; near $x$, $a$ and $b$ intersect twice. In addition, if $d(a)$ and $d(b)$ intersect at the other endpoint $y$, then $a$ and $b$ intersect twice near $y$. We conclude that $a$ and $b$ intersect in an even number of points. So $\sigma$ has an even number of self-intersection points.

On the other hand, since $c, D \in \mathscr{T}(x)$, we have $i(c, D) > 0$. Thus from the above argument, $c = \partial D(c)$ and $D = \partial D(D)$ have an even number of intersection points, but the curve concatenation $e_D \cdot e_c^{-1}$ has an additional self-intersection point at $x$. So the number of self-intersection points of $e_D \cdot e_c^{-1}$ is odd. During the homotopy from $\varepsilon(T_c^{-1}T_D) = e_D \cdot e_c^{-1}$ to the geodesic $\sigma$, the self-intersection points could cancel only in pairs. We conclude that the number of self-intersection points of $\sigma$ is odd. It follows that $T_bT_a \neq T_c^{-1}T_D$. Thus via the Bers isomorphism, $t_bt_a \neq t_c^{-1}t_D$. Similarly, we can prove $t_bt_a \neq t_Dt_c^{-1}$.

From Lemmas 4.1–4.2, we conclude that both $\widetilde{c}$ and $\widetilde{D}$ are non-trivial. As a matter of fact, more is true.

**Lemma 4.3** *With the same conditions as in Lemma* 4.2*, $c$ and $D$ are disjoint, and hence $c$ and $D$ are the boundary components of an $x$-punctured cylinder on $\dot{S}$.*

**Proof** By Lemma 6.1, we assert that $i(c, D) = 0$. So either $c = D$ or $c$ and $D$ are disjoint. If $c = D$, then from (4.1), $t_bt_a$ is trivial. But this is impossible since $a, b \in \mathscr{T}(x)$ and thus $a$ and $b$ intersect. We assume that $c$ and $D$ are disjoint. By filling the puncture $x$, from (4.1), we see that $t_c^{-1}t_D$ projects to the trivial mapping class on $S$. But we know that $\widetilde{c}$ and $\widetilde{D}$ are non-trivial geodesics. So $\widetilde{c} = \widetilde{D}$. It follows that $c$ and $D$ are boundary components of an $x$-punctured cylinder on $\dot{S}$.

**Lemma 4.4** *Under the same notations and conditions as above, $D$ is disjoint from $a$ and $b$, or equivalently, $D$ is disjoint from $d(a) \cup d(b)$.*

**Proof** Let $\Delta$ denote the component of $\mathbf{H}\backslash\mathrm{axis}(T_bT_a)$ that does not include $z_a$ and $z_b$ (as shown in Figure 4). Note that $(T_bT_a)^* = t_c^{-1}t_D$. With the help of $\Delta$ one can construct a map $\tau \in Q(G)$, which is a lift of the Dehn twist $t_{\widetilde{D}} = t_{\widetilde{c}}$, where in fact $\widetilde{D} = \widetilde{c} = \varrho(\mathrm{axis}(T_bT_a))$. From Lemma 3.2 of [15], $[\tau]^* = t_D$ or $t_c$ (see Section 2 for more details).

Let $(\tau, \Omega, \mathscr{U})$ be the configuration obtained from $\tau$. By construction, $\Delta \in \mathscr{U}$. Note that $T_bT_a$ keeps the set of maximal elements of $\mathscr{U}$ invariant. If there is a maximal element $\Delta_0 \in \mathscr{U}$ that covers $z_a$ but not $z_b$, then $T_a(\Delta_0^*) \subset \Delta_0$ and thus $T_bT_a(\Delta_0^*) \subset \Delta_0^*$, which says that $\Delta_0$ is not a maximal element of $\mathscr{U}$. If $\Delta_0$ covers both $z_a$ and $z_b$, then either (i) $T_a(\Delta_0^*)$ is disjoint from $z_b$ or (ii) $T_a(\Delta_0^*)$ covers $z_b$. When (i) occurs, $T_bT_a(\Delta_0^*)$ is disjoint from $\Delta_0^*$; when (ii) occurs, we have $T_bT_a(\Delta_0) \subset \Delta_0$. All these would imply that $T_b(T_a(\Delta_0))$ is not a maximal element of $\mathscr{U}$. This contradiction tells us that $z_a$ can not belong to any maximal element of $\mathscr{U}$.

Now by considering the inverse $T_a^{-1}T_b^{-1}$ of $T_bT_a$, one can show that $z_b$ can not belong to any maximal element of $\mathscr{U}$. Hence both $z_a$ and $z_b \in \Omega \cap \mathbf{S}^1$. It follows that both $T_a$ and $T_b$ commute with $\tau$. If $[\tau]^* = t_c$, then $t_c$ commutes with $t_a$ and $t_b$, and so $t_c$ commutes with $t_bt_a$. But we have $t_ct_bt_a = t_D$, which implies that $c$ intersects $a \cup b$. This is absurd. We conclude that $[\tau]^* \neq t_c$. So $[\tau]^* = t_D$. Thus both $t_a$ and $t_b$ commute with $t_D$ (but $t_a$ and $t_b$ do not commute with each other). That is, $D$ does not intersect $a \cup b$.

We now proceed to study the properties of conjugate parabolic elements and their products. Assume that $a, b \in \mathscr{T}(x, x_1)$, which is equivalent to that $T_a$ and $T_b$ are conjugate in $G$.

**Lemma 4.5** *If $T_b$ is conjugate to $T_a$ in $G$, then $T_bT_a \in G$ is hyperbolic but not a simple hyperbolic element unless $a = b$.*

**Proof** From Lemma 3.2, $T_bT_a$ is hyperbolic. If $a \neq b$ and $T_bT_a$ is simple hyperbolic, then $T_bT_a$ corresponds to a simple closed geodesic $\gamma$ in $\pi_1(S, x)$.

By assumption, there is an element $h \in G$ such that $T_b = hT_ah^{-1}$. Thus $T_bT_a = hT_ah^{-1}T_a$. Note that $d(b) = h^*(d(a))$ determines a parabolic loop $e_b$, but $e_b$ is also defined by $hT_ah^{-1}$. We see that $hT_ah^{-1}T_a$ determines a loop $e_a \cdot e_b \in \pi_1(S, x)$. Since $d(b) = h^*(d(a))$, $d(a)$ and $d(b)$ share both endpoints $\{x, x_1\}$. This implies that the curve concatenation $e_a \cdot e_b$ is homotopic to a geodesic with at least two self-intersection points (two of which are near the puncture $x_1$). In other words, the axis of $hT_ah^{-1}T_a$ projects to a non-simple closed geodesic. It follows from the definition that $T_bT_a$ is not a simple hyperbolic element.

A mapping class $M$ is called a multi-twist if $M$ is represented by a finite product of Dehn twists about disjoint simple closed geodesics.

**Lemma 4.6** *If $T_b$ is conjugate to $T_a$ in $G$, then $(T_bT_a)^*$ is not a multi-twist unless $a = b$, in which case $T_b = T_a$ and $(T_bT_a)^*$ is a power of a Dehn twist.*

**Proof** Assume that $b \neq a$ and $(T_bT_a)^* = M$ is a multi-twist. Since $T_bT_a \in G$, by Theorem 2 of [10], if $T_bT_a$ is an essential hyperbolic element, or a non-simple non-essential hyperbolic element, then $(T_bT_a)^*$ can never be multi-twist. It follows that $(T_bT_a)^*$ is either parabolic or simple hyperbolic. By Lemma 3.2, $(T_bT_a)^*$ is not parabolic. So $(T_bT_a)^*$ must be simple hyperbolic. But this again contradicts Lemma 4.5. If $a = b$, then $T_a = T_b$. So $T_bT_a = T_a^2$ and hence $(T_bT_a)^* = (T_a^*)^2 = t_a^2$.

## 5 Proof of Theorems

**Proof of Theorem 1.1** We only handle the case where $(T_bT_a)^*$ is of the form (4.1). Suppose that $\dot{S}$ contains only two punctures $x$ and $x_1$. Then $a, b \in \mathscr{T}(x, x_1)$, and thus $T_a, T_b$ are conjugate in $G$. Since $a \neq b$, by Lemma 4.6, $(T_bT_a)^*$ is not a multi-twist, which implies that $c$ and $D$ are not disjoint. On the other hand, since $(T_bT_a)^* \in \mathscr{F}_0$ is of the form of (4.1), by Lemmas 4.1–4.2 and Lemma 6.1 in Appendix, we conclude that $c$ and $D$ do not intersect. This contradiction proves that $\dot{S}$ contains at least three punctures.

Assume that $a \in \mathscr{T}(x, x_1)$ and $b \in \mathscr{T}(x)$. By Lemmas 4.1–4.2, both $\widetilde{D}$ and $\widetilde{c}$ are non-trivial. Lemma 4.4 then asserts that $c$ is disjoint from $D$ and $\{c, D\}$ actually bounds an $x$-punctured cylinder on $S$. This implies that $t_bt_a = t_c^{-1}t_D$ is a multi-twist. By Theorem 2 of [10] and Theorem 2 of [13], there exists a simple hyperbolic element $h \in G$ such that $h^* = t_c^{-1}t_D$. But $(T_bT_a)^* = t_bt_a = t_c^{-1}t_D$. It follows that $h = T_bT_a$, which tells us that $T_bT_a$ is a simple hyperbolic element of $G$. Hence by Lemma 4.5, $T_a$ is not conjugate (in $G$) to $T_b$. As it turns out, $b \in \mathscr{T}(x, x_i)$ for some $x_i \neq x_1$. Moreover, by Lemma 3.1, $d(a)$ and $d(b)$ are disjoint, which says that $(d(a), d(b))$ forms a binary tree with two leaves $x_1$ and $x_i$.

By Lemma 4.4, $D$ is disjoint from $d(a) \cup d(b)$. This means that $D$ is disjoint from $a \cup b$. Finally, to see that $D$ bounds a thrice punctured disk on $S$, we observe that the curve concatenation $e_a \cdot e_b$ is homotopic to $\widetilde{D}$. But since $(d(a), d(b))$ forms a binary tree with leaves $\{x_1, x_i\}$, it is obvious that $e_a \cdot e_b$ bounds a twice punctured disk on $S$ which encloses $\{x_1, x_i\}$. From the above argument, $D$ is disjoint from $a \cup b$. If $D$ does not bound a thrice punctured disk, then $\widetilde{D}$ is not the boundary of any twice punctured disk, which leads to a contradiction.

We conclude that $D$ bounds a thrice punctured disk. Since $\{c, D\}$ bounds an $x$-punctured cylinder on $S$, $c$ bounds a twice punctured disk enclosing $\{x_1, x_i\}$. Thus Figure 1 has been reconstructed. This proves that $a, b, c$ and $D$ are geometrically related by Figure 1.

To prove Theorem 1.2, we need some preliminary results.

**Lemma 5.1** *Let $a, b, c \subset \dot{S}$ be simple closed geodesics. We have the following claims:*

(1) *If $t_a t_b$ is trivial, then both $a$ and $b$ are trivial.*

(2) *If $t_a t_b^{-1}$ is trivial, then either $a$ and $b$ are trivial, or $a = b$.*

(3) *If $t_a t_b = t_c$, then either $a, b$ and $c$ are trivial, or $a$ is trivial and $b = c$, or $b$ is trivial and $a = c$.*

(4) *If $t_a t_b^{-1} = t_c$, then either $a, b$ and $c$ are trivial, or $a$ and $b$ are non-trivial and $c$ is trivial, or $b$ is trivial and $a = c$, and*

(5) *If $t_a t_b = t_c^{-1}$, then $a, b$ and $c$ are trivial.*

**Proof** (1) If $a$ and $b$ are non-trivial, then $t_a \neq t_b^{-1}$. If $a$ is trivial and $b$ is non-trivial, or $a$ is non-trivial and $b$ is trivial, then $t_a t_b$ is a simple Dehn twist that is also non-trivial.

(2) If $t_a t_b^{-1}$ is trivial, then $t_a = t_b$, which implies that $a = b$ or both $a$ and $b$ are trivial.

(3) Suppose that not all $a, b$ and $c$ are trivial. If $a$ is trivial, then $t_b = t_c$ and thus $b = c$; otherwise $a$ is non-trivial. If $b$ is non-trivial, then $t_a t_b$ can not be a single Dehn twist. It follows that $b$ is trivial. Thus $t_a = t_c$ and so $a = c$.

(4) Suppose that not all $a, b$ and $c$ are trivial. If $b$ is trivial, then $t_a = t_c$, which says $a = c$. Otherwise, $b$ is non-trivial. If $a$ is also non-trivial, the only possibility is that $c$ is trivial and $a = b$. If $a$ is trivial, then $t_b^{-1} = t_c$, which is impossible.

(5) If only one of $a, b$ and $c$ is trivial, then $t_a t_b \neq t_c^{-1}$. If any two of $a, b$ and $c$ are trivial, the other one must also be trivial. If all $a, b$ and $c$ are non-trivial and $a$ and $b$ are disjoint, then $t_a t_b$ is multi-twist while $t_c^{-1}$ is a single Dehn twist. So $t_a t_b \neq t_c^{-1}$. If $a$ and $b$ intersect, then $t_a t_b$ can not be a single Dehn twist either.

**Proof of Theorem 1.2** We first assume that $\tilde{a}$ and $\tilde{b}$ are non-trivial (this is automatically true when $n = 1$; that is, $\mathcal{S}$ contains only one puncture). If $\tilde{c}$ or $\tilde{D}$ is trivial, then from (4.1), Lemma 5.1 (3) and (5), we assert that $\tilde{a}$ or $\tilde{b}$ or both are trivial. This is contradiction. If $\tilde{c}$ and $\tilde{D}$ are non-trivial, there are four subcases to consider: (i) $i(c, D) = 0$, $i(a, b) = 0$, (ii) $i(c, D) > 0$, $i(a, b) > 0$, (iii) $i(c, D) = 0$, $i(a, b) > 0$, and (iv) $i(c, D) > 0$, $i(a, b) = 0$.

If $i(c, D) = 0$, then $t_b t_a$ is either the square of a positive Dehn twist or a multi-twist with two positive components. Clearly, (i) does not hold (since $t_c^{-1} t_D$ is either trivial or a multi-twist with one positive and one negative components). (iv) says that $c$ and $D$ intersect. From Thurston's theorem [14], we see that on the surface supported by $c$ and $D$, $t_c^{-1} t_D$ is pseudo-Anosov. So (iv) can not happen either.

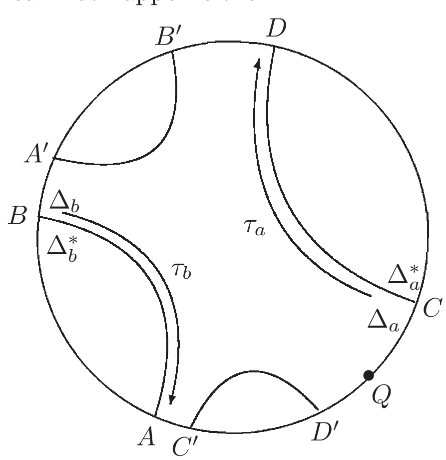


Figure 5 $\Omega_a \cap \Omega_b = \emptyset$

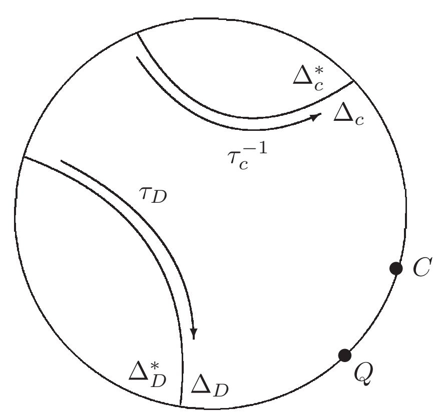


Figure 6 $\Omega_c \cap \Omega_D = \emptyset$

To handle the other two cases, we let $(\tau_a, \Omega_a, \mathscr{U}_a), (\tau_b, \Omega_b, \mathscr{U}_b), (\tau_c, \Omega_c, \mathscr{U}_c)$ and $(\tau_D, \Omega_D, \mathscr{U}_D)$ be the configurations corresponding to $a, b, c$ and $D$, respectively.

Suppose (ii) occurs with $\Omega_a \cap \Omega_b \neq \emptyset$. Note that $\tau_b\tau_a$ has no fixed points on $\mathbf{S}^1$, while $\tau_c^{-1}\tau_D$ has two or infinitely many fixed points on $\mathbf{S}^1$. We see that $\tau_b\tau_a \neq \tau_c^{-1}\tau_D$ on $\mathbf{S}^1$. Now assume that $\Omega_a \cap \Omega_b = \emptyset$. There are maximal elements $\Delta_a \in \mathscr{U}_a$ and $\Delta_b \in \mathscr{U}_b$ such that $\Delta_a \cup \Delta_b = \mathbf{H}$. We refer to Figure 5 where $\Delta_b^* = \mathbf{H}\backslash\Delta_b$ and $\tau_a(\Delta_b^*) \cap \mathbf{S}^1 = (A'B')$; likewise, $\Delta_a^* = \mathbf{H}\backslash\Delta_a$ and $\tau_b(\Delta_a^*) \cap \mathbf{S}^1 = (C'D')$ (here and hereafter we denote by $(AB)$ the minor arc on $\mathbf{S}^1$ connecting two non-antipodal points $A$ and $B$ on $\mathbf{S}^1$).

By examining the action of $\tau_b\tau_a$ on $\mathbf{S}^1$, we see that the fixed points for $\tau_b\tau_a$ (if exist) must lie on the arc $(D'C)$. Let $Q$ be the fixed point of $\tau_b\tau_a$ that is closest to $C$. Then $Q$ is also a fixed point of $\tau_c^{-1}\tau_D$. If $\Omega_c \cap \Omega_D = \emptyset$, there are maximal elements $\Delta_c \in \mathscr{U}_c$ and $\Delta_D \in \mathscr{U}_D$ such that $\Delta_c \cup \Delta_D = \mathbf{H}$. We have $Q \in (\Delta_c \cap \Delta_D) \cap \mathbf{S}^1$. See Figure 6.

For any $z \in \mathbf{S}^1$, let $d(\tau_b\tau_a(z), z)$ denote the Euclidean length of the arc of $\mathbf{S}^1\backslash\{z, \tau_b\tau_a(z)\}$ determined by the motion direction of $\tau_b\tau_a$ at $z$. Then $z$ is a fixed point of $\tau_b\tau_a$ if and only if $d(\tau_b\tau_a(z), z) = 2k\pi$ for an integer $k$. Similarly, we use $d(\tau_c^{-1}\tau_D(z), z)$ to denote the Euclidean length of the arc of $\mathbf{S}^1\backslash\{z, \tau_c^{-1}\tau_D(z)\}$ determined by the motion direction of $\tau_c^{-1}\tau_D$ at $z$. Since the motion directions of $\tau_c^{-1}$ and $\tau_D$ are opposite, $z$ is a fixed point of $\tau_c^{-1}\tau_D$ if and only if $d(\tau_c^{-1}\tau_D(z), z) = 0$. Now we choose a sequence $\{z_n\} \subset \mathbf{S}^1$ with $z_0 = C$, and $z_n \to Q$ from right.

Notice that $0 < d(\tau_b\tau_a(C), C) < 2\pi$ and $d(\tau_b\tau_a(z_n), z_n) > L$ (where $L$ is the arc length of $(C'D')$). We conclude that $d(\tau_b\tau_a(z_n), z_n) \to 2\pi$ and $d(\tau_c^{-1}\tau_D(z_n), z_n) \to 0$. It follows that $\tau_b\tau_a \neq \tau_c^{-1}\tau_D$ on $\mathbf{S}^1$.

Similarly, we can handle the case where $\Omega_c \cap \Omega_D \neq \emptyset$.

Suppose that (iii) occurs with $\Omega_a \cap \Omega_b \neq \emptyset$. In this case, $\Omega_c \cap \Omega_D \neq \emptyset$. It is clear that $\tau_b\tau_a$ has no fixed points on $\mathbf{S}^1$, while there are infinitely many fixed points for $\tau_c^{-1}\tau_D$. This is a contradiction. If $\Omega_a \cap \Omega_b = \emptyset$, a contradiction can also be derived by the similar argument as above (in this case, $z$ is a fixed point of $\tau_c^{-1}\tau_D$ if and only if $d(\tau_c^{-1}\tau_D(z), z) = 0$).

Note that for a surface with one puncture, $\tilde{a}, \tilde{b}$ are automatically non-trivial. We conclude that there is no relation (4.1) on $\mathcal{S}$ when $n = 1$.

It remains to consider the case where $S$ contains two or more punctures and $\tilde{a}$ or $\tilde{b}$ or both are trivial. Suppose that $a \in \mathscr{T}(x, x_1)$. Our first claim is that $c \notin \mathscr{T}(x, x_1)$. For otherwise, $t_c$ is conjugate to $t_a$ in $\mathscr{F}$ and from (4.1), we obtain

$$t_c t_b t_a t_b^{-1} = t_D t_b^{-1}.$$

Hence

$$t_c t_{t_b(a)} = t_D t_b^{-1}. \tag{5.1}$$

Since $a \in \mathscr{T}(x, x_1)$, we have $t_b(a) \in \mathscr{T}(x, x_1)$. This implies that $c$ and $t_b(a)$ intersect. So if $i(b, D) = 0$, then the right side of (5.1) is a multi-twist or the identity, while the left side of (5.1) is neither the identity nor a multi-twist. This leads to a contradiction. We conclude that $i(b, D) > 0$. But since $c, t_b(a) \in \mathscr{T}(x)$, by Lemma 5.1, either $\widetilde{D}$ and $\widetilde{b}$ are trivial, or both $\widetilde{D}$ and $\widetilde{b}$ are non-trivial and $\widetilde{D} = \widetilde{b}$. The former would contradict Lemma 4.2, and the latter would contradict Lemma 6.1.

Our next claim is $b \in \mathscr{T}(x)$. Indeed, by assumption, $a \in \mathscr{T}(x, x_1)$. There are four cases to be considered.

**Case 1** $\widetilde{D}$ and $\widetilde{c}$ are both non-trivial. By filling the puncture $x$, from (4.1) we obtain $t_{\widetilde{b}} = t_{\widetilde{c}}^{-1} t_{\widetilde{D}}$. By Lemma 5.1, $\widetilde{D} = \widetilde{c}$ and $\widetilde{b}$ is trivial. That is, $b \in \mathscr{T}(x)$.

**Case 2** $\widetilde{D}$ is trivial and $\widetilde{c}$ is non-trivial. By filling the puncture $x$, from (4.1) we obtain $t_{\widetilde{b}} = t_{\widetilde{c}}^{-1}$. This means that $t_{\widetilde{b}} t_{\widetilde{c}} = \mathrm{id}$. By Lemma 5.1, this is impossible unless $\widetilde{b} = \widetilde{c}$ is trivial. It follows that $b \in \mathscr{T}(x)$.

**Case 3** $\widetilde{D}$ is non-trivial and $\widetilde{c}$ is trivial. Again by filling in the puncture $x$, we see that $t_{\widetilde{b}} = t_{\widetilde{D}}$. Thus $\widetilde{b} = \widetilde{D}$. Let $\overline{a}, \overline{b}, \overline{c}$ and $\overline{D}$ denote the geodesics on $\dot{S} \cup \{x_1\}$ homotopic to $a, b, c$

and $D$ on $\dot{S} \cup \{x_1\}$, respectively. By filling in the puncture $x_1$, from (4.1) and $a \in \mathscr{T}(x, x_1)$, we obtain

$$t_{\overline{b}} = t_{\overline{c}}^{-1} t_{\overline{D}}. \tag{5.2}$$

If $\overline{c}$ is trivial, then $\overline{b} = \overline{D}$ and $c \in \mathscr{T}(x, x_1)$. This contradicts the fact that $c \notin \mathscr{T}(x, x_1)$.

Assume that $\overline{c}$ is non-trivial. Then by Lemma 5.1 and (5.2), $\overline{c} = \overline{D}$ and thus $\overline{b}$ is trivial. This says $b \in \mathscr{T}(x_1, x_2)$ ($b \notin \mathscr{T}(x, x_1)$ since $\widetilde{b}$ by assumption is non-trivial). Since $a \in \mathscr{T}(x, x_1)$, we have $a, b \in \mathscr{T}(x_1)$. By switching the roles of $x$ and $x_1$ and by Lemma 6.1, we conclude that $t_b t_a \neq t_c^{-1} t_D$.

**Case 4** Both $\widetilde{D}$ and $\widetilde{c}$ are trivial. In this case, by filling the puncture $x$ once again, from (4.1) we deduce that $t_{\widetilde{b}}$ is trivial. Thus $b \in \mathscr{T}(x)$. We are done.

We now use the same argument of Theorem 1.1 to complete the proof of Theorem 1.2.

## 6 Appendix

This section is devoted to the proof of a lemma which plays a key role in the proof of Theorem 1.1. With the same notations and terminology as in Section 4, we have the following Lemma.

**Lemma 6.1** *Let $a, b, c, D \subset \dot{S}$ be simple closed geodesics. Assume that $a, b \in \mathscr{T}(x)$, $i(c, D) > 0$ and $\widetilde{c}, \widetilde{D}$ are non-trivial on $S$ as $x$ is filled in. Then $t_b t_a \neq t_c^{-1} t_D$ and $t_b t_a \neq t_D t_c^{-1}$.*

**Proof** We only prove that $t_b t_a \neq t_c^{-1} t_D$. Suppose $t_b t_a = t_c^{-1} t_D$. By assumption, $a, b \in \mathscr{T}(x)$. Hence by filling the puncture $x$, we deduce that $t_{\widetilde{c}}^{-1} t_{\widetilde{D}}$ is the identity. Since $\widetilde{c}$ and $\widetilde{D}$ are non-trivial, by Lemma 5.1(2), we have $\widetilde{c} = \widetilde{D}$. Now $t_{\widetilde{c}}$ and $t_{\widetilde{D}}$ are well-defined non-trivial mapping classes on $S$. Let $(\tau_c, \Omega_c, \mathscr{U}_c)$ and $(\tau_D, \Omega_D, \mathscr{U}_D)$ be the configurations corresponding to $c$ and $D$, respectively (see Section 2 for an exposition).

Since $\widetilde{c} = \widetilde{D}$, all boundary geodesics of elements of $\mathscr{U}_c$ and $\mathscr{U}_D$ are disjoint. Hence by Theorem 1.2 of [19], there exist maximal elements $\Delta_c \in \mathscr{U}_c$ and $\Delta_D \in \mathscr{U}_D$ such that $\Delta_c \cup \Delta_D = \mathbf{H}$, $\partial\Delta_c \cap \partial\Delta_D = \varnothing$ and $\Delta_c \cap \Delta_D \neq \varnothing$. Denote $\Delta_c^* = \mathbf{H} \backslash \Delta_c$ and $\Delta_D^* = \mathbf{H} \backslash \Delta_D$. By Theorem 1.2 of [19], $\tau_c^{-1}\tau_D$ is a hyperbolic element of $G$ whose axis $\text{axis}(\tau_c^{-1}\tau_D)$ separates $\Delta_c^*$ from $\Delta_D^*$ (see Figure 7).

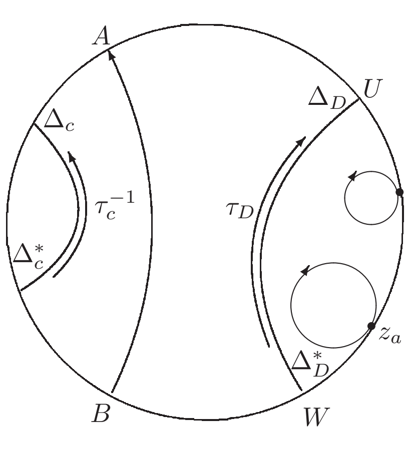


Figure 7 Both $z_a$ and $z_b$ are outside of $\Delta_D$

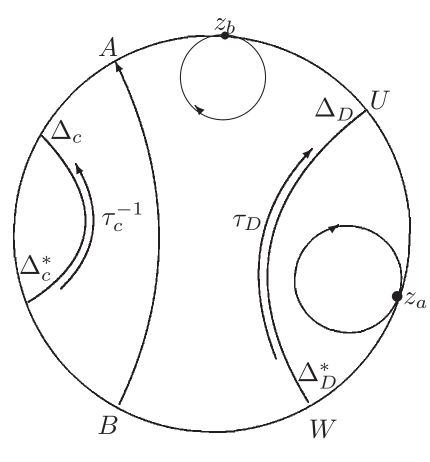


Figure 8 Only $z_a$ is outside of $\Delta_D$

By assumption, the equality (4.1) holds. This particularly implies that $\mathrm{axis}(\tau_c^{-1}\tau_D) = \mathrm{axis}(T_bT_a)$ (which is also denoted by $\overline{AB}$). Since $\partial\Delta_c$ and $\partial\Delta_D$ project (under $\rho$) to the simple closed geodesic $\widetilde{c} = \widetilde{D}$, $\tau_c^{-1}\tau_D(\Delta_D^*)$ is disjoint from $\Delta_D^*$. By hypothesis, $t_bt_a = t_c^{-1}t_D$. We conclude that $\mathrm{axis}(\tau_c^{-1}\tau_D) = \mathrm{axis}(T_bT_a) =$ geodesic $\overline{AB}$ connecting $A$ and $B$.

By combining Figure 4 and the remark thereafter, we deduce that $z_a$ and $z_b$ must lie on the right component $R$ of $\mathbf{S}^1\backslash\{A, B\}$ and furthermore, $z_b$ is closer to $A$ than $z_a$ is. In what follows, we denote by $(P, Q)$ the minor on $\mathbf{S}^1$ connecting two non-antipodal labeling points $P$ and $Q$ on $\mathbf{S}^1$. There are several cases to be considered.

**Case 1** Both $z_a$ and $z_b$ lie in the arc $(UW)$ (see Figure 8). In this case, if $T_a(\Delta_D)$ covers $z_b$, then $T_bT_a(\Delta_D^*)$ is not disjoint from $\Delta_D^*$. But we know that $\tau_c^{-1}\tau_D(\Delta_D^*)$ is disjoint from $\Delta_D^*$. This is a contradiction. If $T_a(\Delta_D) \cap \mathbf{S}^1 \subset (Uz_b)$, then $T_bT_a(\Delta_D) \subset \Delta_D^*$. Again this contradicts that $\tau_c^{-1}\tau_D(\Delta_D^*)$ is disjoint from $\Delta_D^*$. If $T_a(\Delta_D)\cap\mathbf{S}^1 \subset (z_az_b)$, then one easily sees that $T_bT_a(\Delta_D) \subset \Delta_D$ or $T_bT_a(\Delta_D) \subset \Delta_D^*$, both of which would imply that $T_bT_a(\Delta_D^*)$ is not disjoint from $\Delta_D^*$.

**Case 2** $z_b \in (AU)$ and $z_a \in (UW)$ (see Figure 8). Then $T_a(\Delta_D) \cap \mathbf{S}^1 \subset (Uz_a)$. Hence $T_bT_a(\Delta_D) \subset \Delta_D$. This implies that $T_bT_a(\Delta_D^*)$ is not disjoint from $\Delta_D^*$. If $z_b \in (UW)$ and $z_a \in (BW)$, by considering the inverse $T_a^{-1}T_b^{-1}$ of $T_bT_a$ and by the same argument as above, we see that this case does not occur.

**Case 3** $z_a, z_b \in (AU)$ (see Figure 9). Noting that $b$ is preperipheral, so $b$ bounds a twice punctured disk $\mathscr{E}$ containing $x$. This implies that $\dot{S}\backslash\mathscr{E}$ is not of type $(0, 3)$. We can choose a non-trivial simple closed geodesic $\gamma$ on $\dot{S}\backslash\mathscr{E}$, which can also be viewed as a geodesic on $\dot{S}$ that satisfies the conditions: (i) $\gamma$ is not preperipheral, (ii) $\gamma$ is disjoint from $b$, and (iii) $\gamma$ intersects $D$ and $a$. Let $(\tau_\gamma, \Omega_\gamma, \mathscr{U}_\gamma)$ be the configuration corresponding to $\gamma$. By Lemma 2.2 of [19], $z_b \in \Omega_\gamma \cap \mathbf{S}^1$, and there exists a maximal element $\Delta \in \mathscr{U}_\gamma$ so that $\partial\Delta$ crosses $\overline{UW}$. It could be the case that $\Delta$ covers $z_a$, as shown in Figure 9. But it could also be the case that $\Delta$ does not cover $z_a$. Since $z_b \in \Omega_\gamma \cap \mathbf{S}^1$, there exists a maximal element, and call it $\Delta$ too, such that $z_a \in \Delta \cap \mathbf{S}^1$. In any case, $z_b$ is not contained in $\overline{\Delta} \cap \mathbf{S}^1$ and $\Delta$ is disjoint from $\overline{AB}$.

Now we have $T_a(\mathbf{H}\backslash\Delta) \subset \Delta$, and thus $T_bT_a(\Delta) \cap \Delta \neq \varnothing$. But $\tau_c^{-1}\tau_D(\Delta) \cap \Delta = \varnothing$. We conclude that

$$T_bT_a(\Delta) \neq \tau_c^{-1}\tau_D(\Delta).$$

So $T_bT_a \neq \tau_c^{-1}\tau_D$.

**Case 4** $z_a \in (BW)$ and $z_b \in (AU)$ (see Figure 10). Assume without loss of generality that both $a\cup b$ and $c\cup D$ fill $\dot{S}$. Since $\widetilde{c} = \widetilde{D}$, there exists $h \in G$ sending $\overline{U_0W_0} = \partial\Delta_c$ to $\overline{UW} = \partial\Delta_D$. Hence $h$ is hyperbolic and its axis $\mathrm{axis}(h)$ separates $z_a$ from $z_b$. By assumption, $c \cup D$ fills $\dot{S}$. From Lemma 2.2 of [21], $\mathrm{axis}(h)$ intersects at least one geodesic $\overline{U_1W_1}$ in $\{\varrho^{-1}(\widetilde{c})\} = \{\varrho^{-1}(\widetilde{D})\}$ between $\overline{U_0W_0}$ and $\overline{UW}$. In general, we let $\overline{U_1W_1}, \cdots, \overline{U_kW_k} \in \{\varrho^{-1}(\widetilde{c})\}$ be the geodesics between $\overline{U_0W_0}$ and $\overline{UW}$, where $k \geq 1$. We redraw Figure 10 as Figure 11 and Figure 12.

Let $\mathscr{R}_{\widetilde{c}}$ denote the collection of components of $\mathbf{H}\backslash\{\varrho^{-1}(\widetilde{c})\}$. Then there exists a bijection $\chi$ between $\mathscr{R}_{\widetilde{c}}$ and the set of geodesics $c_0 \subset \dot{S}$ with $\widetilde{c}_0 = \widetilde{c}$.

Let $\Omega_0 \in \mathscr{R}_{\widetilde{c}}$ be contained in the region bounded by $\overline{U_0W_0}$ and be disjoint from $\overline{AB}$. For $1 \leq j \leq k$, we let $\Omega_j \in \mathscr{R}_{\widetilde{c}}$ be contained in the region bounded by $\overline{U_{j-1}W_{j-1}}$ and $\overline{U_jW_j}$. Finally, denote by $\Omega_{k+1} \in \mathscr{R}_{\widetilde{c}}$ the component contained in the region bounded by $\overline{UW}$ and $\overline{U_kW_k}$.

It is clear that $A \in (U_0U_1)$ and $B \in (W_kW)$. From Figure 4, we have $z_a \in (BW)$ and $z_b \in (AU)$. If $z_b \in (AU_1)$ (see Figure 11), we consider the component $\Delta_1$ of $\mathbf{H}\backslash\overline{U_1W_1}$ containing $\overline{UW}$. Then $T_b^{-1}(\Delta_1) \cap \mathbf{S}^1 \subset (z_bU_1)$. Write $\Delta^* = T_a^{-1}T_b^{-1}(\Delta_1)$. If $\Delta^* \neq \tau_D^{-1}\tau_c(\Delta_1)$, we are done. So we assume that $\Delta^* = \tau_D^{-1}\tau_c(\Delta_1)$.

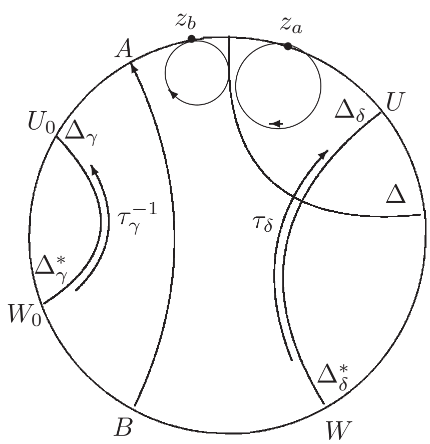


Figure 9 Both $z_a, z_b$ are inside of $\Delta_c \cap \Delta_D$

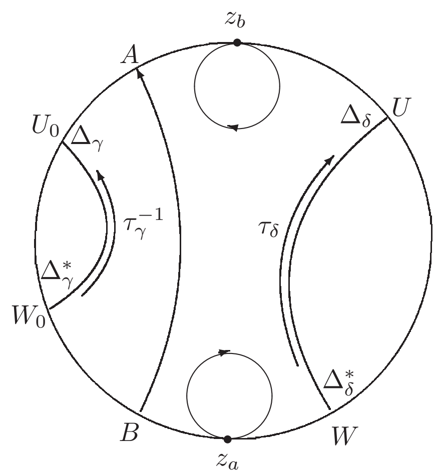


Figure 10 $z_a, z_b \in \Delta_c$ are separated by $\Delta_D^*$

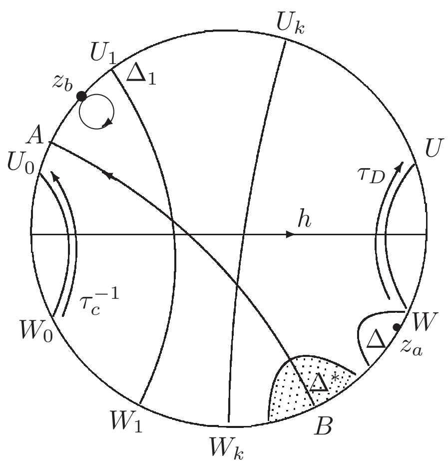


Figure 11 $z_b \in (AU_1)$ and $z_a \in (BW)$

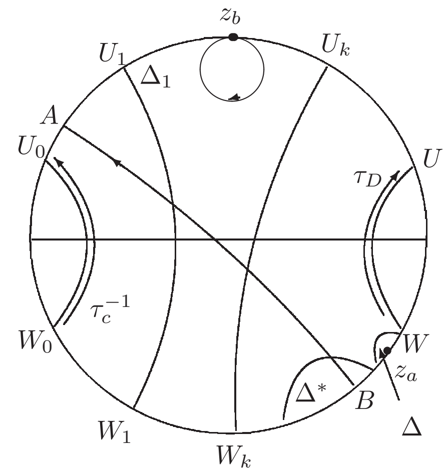


Figure 12 $z_b \in (U_1U)$ and $z_a \in (BW)$

By construction $z_a \notin \Delta^* \cap \mathbf{S}^1$ and $\Delta^*$ covers the repelling fixed point $B$ (otherwise, we immediately see that $T_bT_a \neq \tau_c^{-1}\tau_D$). $\Delta^*$ is shown as a shaded region in Figure 11. Let $(\tau, \Omega, \mathscr{U})$ be the configuration defined by $\Delta^*$.

Let $c' \subset S$ denote the simple closed geodesic corresponding to $\Omega_1$. By Lemma 2.1 of [20], $c'$ is disjoint from $c$. Since $z_a \in \Delta_1 \cap \mathbf{S}^1$, by Lemma 2.2 of [19], $c'$ intersects $a$. Since $b \in \mathscr{T}(x)$, by construction we know that $c'$ intersects $t_b(a)$, i.e., $t_b^{-1}(c')$ intersects $a$. It follows that $t_a^{-1}t_b^{-1}(c')$ intersects $a$. This tells us that $\mathscr{U}$ contains a maximal element $\Delta$ which covers $z_a$ (Lemma 2.2 of [19]). But since $z_a$ is disjoint from $\Delta^*$, $\Delta$ is disjoint from $\Delta^*$.

We claim that $\Delta$ does not cross axis($h$). Otherwise, we note that $\partial\Delta \in \{\varrho^{-1}(\widetilde{c})\}$. This implies that $\overline{U_kW_k}$ would not be the last geodesic in $\{\varrho^{-1}(\widetilde{c})\}$ that lies in between $\overline{U_0W_0}$ and $\overline{UW}$, and crosses axis($h$). Therefore, $\Delta$ is disjoint from both $\Delta^*$ and axis($h$). $\Delta$ is shown in Figure 11 too. Since $\tau_c(\Delta_1) \cap \mathbf{S}^1 \subset (W_0W_1)$, it is disjoint from $(U_0W_0)$. But we know that $\tau_D^{-1}\tau_c(\Delta_1) = \Delta^*$. We conclude that $\tau_D(\Delta) \cap \mathbf{S}^1$ is disjoint from $(U_0W_0)$. It follows that $T_bT_a(\Delta) \neq \tau_c^{-1}\tau_D(\Delta)$, which in turn implies that $T_bT_a(\Delta) \neq \tau_c^{-1}\tau_D$.

If $z_b \in (U_1U)$ (see Figure 12), again, we let $\Delta_1$ be the component of $\mathbf{H}\backslash\overline{U_1W_1}$ that contains $\overline{UW}$ (as shown in Figure 12). Let $\Delta^* = \tau_D^{-1}\tau_c(\Delta_1)$. We may also assume that

$\Delta^* = T_a^{-1}T_b^{-1}(\Delta_1)$. Let $\Delta$ be shown as in Figure 12. Then by the same argument as above, we conclude that $T_a^{-1}T_b^{-1}(\Delta) \neq \tau_D^{-1}\tau_c(\Delta)$, which implies that $T_bT_a \neq \tau_c^{-1}\tau_D$.

This completes the proof of Lemma 6.1.

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