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The Φ-Group for n Submodules Systems as a Generalization of the K-Theory*

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Abstract In this paper, the authors construct a Φ-group for n submodules, which generalizes the classical K-theory and gives more information than the classical ones. This theory is related to the classification theory for indecomposable systems of n subspaces.

Keywords System of n subspaces, System of n submodules, Φ-Group, K-Theory, Indecomposable system
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1 Introduction

The description of systems $\mathcal{S} = (H; E_1, E_2, \cdots, E_n)$ of n subspaces H_i $(i = 1, \cdots, n)$, of a Hilbert space H, which can be finite or infinite dimensional, up to an isomorphism or the unitary equivalence, is famous as the multi-space theory, and the classification of these systems is a subject which attracts many mathematicians' attention. In a finite dimensional space, the classification of indecomposable systems of n subspaces for $n = 1, 2$ and 3 is simple. Jordan blocks give indecomposable systems of 4 subspaces. But there exist many other kinds of indecomposable systems of 4 subspaces. Therefore, it was surprising that Gelfand and Ponomarev [1] gave a complete classification of indecomposable systems of four subspaces in a finite dimensional space.

In this paper, we generalize this theory to the case of A-modules, where A is an involutive algebra, and we construct a group, called Φ-group, which is a generalization of the K-group and gives more information of the algebra A than the K-theory. This group, which can be regarded as the multi-operator edition of the K-group, has essential relations with the problem of classification of systems of n subspaces when $A = \mathbb{C}$.

2 Preliminaries

We first recall the basic notations of systems of n subspaces.

Let H be a Hilbert space and $E_1, \cdots, E_n$ be n subspaces in H. Then we say that $\mathcal{S} = (H; E_1, \cdots, E_n)$ is a system of n subspaces in H or an n subspace system in H. Let $\mathcal{T} = (K; F_1, \cdots, F_n)$ be another system of n subspaces in a Hilbert space K. Then $\varphi : \mathcal{S} \to \mathcal{T}$ is

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a homomorphism if $\varphi : H \to K$ is a bounded linear operator satisfying that $\varphi(E_i) \subseteq F_i$ for $i = 1, \cdots, n$. Moreover, φ is an isomorphism if it is an invertible linear operator and $\varphi(E_i) = F_i$ for $i = 1, \cdots, n$. We say that systems $\mathcal{S}$ and $\mathcal{T}$ are isomorphic if there exists an isomorphism $\varphi : \mathcal{S} \to \mathcal{T}$. And if φ is moreover a unitary operator, we say that the two systems are unitarily equivalent.

There are notations about direct sum and indecomposable systems (see [2]), and the main work on multi-subspace systems is about the classification. Many problems of linear algebra can be reduced to the classification of the systems of subspaces in a finite dimensional vector space. In a finite dimensional space, the classification of indecomposable systems of n subspaces for $n = 1, 2$ and 3 is simple. Gelfand and Ponomarev [1] gave a complete classification of indecomposable systems of four subspaces in a finite dimensional space.

Proposition 2.1 (see [3]) *Let H be a Hilbert space and $\mathcal{S} = (H; E)$ be a system of one subspace. Then $\mathcal{S} = (H; E)$ is indecomposable if and only if $\mathcal{S} \cong (\mathbb{C}; 0)$ or $\mathcal{S} \cong (\mathbb{C}; \mathbb{C})$.*

Let $\mathcal{S} = (H; E_1, E_2)$ be a system of two subspace in a Hilbert space H. Then $\mathcal{S}$ is indecomposable if and only if $\mathcal{S}$ is isomorphic to one of the following four commutative systems:

$$\mathcal{S}_1 = (\mathbb{C}; \mathbb{C}, 0), \quad \mathcal{S}_2 = (\mathbb{C}; 0, \mathbb{C}), \quad \mathcal{S}_3(\mathbb{C}; \mathbb{C}, \mathbb{C}), \quad \mathcal{S}_4 = (\mathbb{C}; 0, 0).$$

Gelfand and Ponomarev [1] claimed that there exist only nine finite-dimensional indecomposable systems of three subspaces. But we do not know whether there exists an infinite-dimensional transitive system of three subspaces.

Proposition 2.2 (see [1]) *Let $\mathcal{S} = (H; E_1, E_2, E_3)$ be an indecomposable system of three subspaces. If H is finite-dimensional, then $\mathcal{S}$ is isomorphic to one of the following nine systems:*

$$\mathcal{S}_1 = (\mathbb{C}; 0, 0, 0), \quad \mathcal{S}_2 = (\mathbb{C}; \mathbb{C}, 0, 0), \quad \mathcal{S}_3 = (\mathbb{C}; 0, \mathbb{C}, 0), \quad \mathcal{S}_4 = (\mathbb{C}; 0, 0, \mathbb{C}), \quad \mathcal{S}_5 = (\mathbb{C}; \mathbb{C}, \mathbb{C}, 0),$$
$$\mathcal{S}_6 = (\mathbb{C}; \mathbb{C}, 0, \mathbb{C}), \quad \mathcal{S}_7 = (\mathbb{C}; 0, \mathbb{C}, \mathbb{C}), \quad \mathcal{S}_8 = (\mathbb{C}; \mathbb{C}, \mathbb{C}, \mathbb{C}), \quad \mathcal{S}_9 = (\mathbb{C}^2; \mathbb{C}(1, 0), \mathbb{C}(0, 1), \mathbb{C}(1, 1)).$$

One of the main problems to tackle is the classification of indecomposable systems $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ of four subspaces in a Hilbert space H. In the case when H is finite-dimensional, Gelfand and Ponomarev completely classified indecomposable systems and gave a complete list of them in [1].

Now we generalize the former definition to the term of (right) modules.

Definition 2.1 *Given an involutive algebra A, let $\mathcal{H}$ be a finitely generated free A-module and $E_1, \cdots, E_n$ be n finitely generated projective submodules of $\mathcal{H}$. Then we say that $\mathcal{S} = (\mathcal{H}; E_1, \cdots, E_n)$ is a system of n-submodules in $\mathcal{H}$.*

Let $\mathcal{T} = (\mathcal{H}'; F_1, \cdots, F_n)$ be another system of n-submodules. Then $\varphi : \mathcal{S} \to \mathcal{T}$ is called a homomorphism if $\varphi : \mathcal{H} \to \mathcal{H}'$ is a module map satisfying that $\varphi(E_i) \subseteq F_i$ for $i = 1, \cdots, n$. And $\varphi : \mathcal{S} \to \mathcal{T}$ is called an isomorphism if $\varphi : \mathcal{H} \to \mathcal{H}'$ is an isomorphism satisfying that $\varphi(E_i) = F_i$ for $i = 1, \cdots, n$. We say that system $\mathcal{S}$ and $\mathcal{T}$ are isomorphic if there exists an isomorphism $\varphi : \mathcal{S} \to \mathcal{T}$.

Let $\mathcal{S} = (\mathcal{H}; E_1, \cdots, E_n)$ and $\mathcal{T} = (\mathcal{H}'; F_1, \cdots, F_n)$ be two systems of n submodules in the module $\mathcal{H}$. Then their direct sum $S \oplus T$ is defined by

$$\mathcal{S} \oplus \mathcal{T} := (\mathcal{H} \oplus \mathcal{H}' : E_1 \oplus F_1, \cdots, E_n \oplus F_n).$$

Similar to the typical systems of n subspaces, we also have the notation of indecomposability and irreducibility.

Let us introduce an important kind of A-module, and thus A^n, which denotes the direct sum $A \oplus A \cdots \oplus A$ of n copies of A, A^n becomes a module over A with the module action defined by

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} a = \begin{pmatrix} a_1 a \\ \vdots \\ a_n a \end{pmatrix}.$$

We are mainly interested in the system of n-submodules in $\mathcal{H}$, which is of this type, denoted by $(A^m; E_1, E_2, \cdots, E_n)$, where each E_k is a finitely generated projectively submodule of A^m. Then each E_k is isomorphic to $p_k A^m$ for some projection p_k on A^m. Then we can write the system of n submodules in the form: $(p_1, p_2, \cdots, p_n)$, where each p_i is a projection, namely

$$p_i^* = p_i, \quad p_i^2 = p_i.$$

In this paper, we construct an Abelian group, namely Φ-group, for systems of n operators $(p_1, p_2, \cdots, p_n)$ to generalize the classical K-group. The Φ-group for systems of one operator is just the classical K-group, and when $n \geq 2$, the Φ-group contains the K-group as a direct summand, and hence we can see that $\Phi(A)$ contains more information of A than that of $K(A)$.

We mainly describe the Φ-group for systems of n operators when $n = 2$, and it only has some of the propositions of the classical K-theory. We compute the Φ-group for multi-operators when the operators have some relations. In fact, to compute the Φ-group is the process to describe the structure of multi-operators up to unitary equivalence.

Finally, we remark that the Φ-group has a relationship with the problem of the classification of systems of n subspaces. The Φ-group can be regarded as a classification theory for systems of n subspaces up to unitary equivalence when $A = \mathbb{C}$.

We firstly discuss the case when the involutive algebra A is unital. Since every finite size projection on A^n is in fact a matrix $x \in M_n(A)$ such that $x^2 = x$ and $x^* = x$, then every system of one submodule corresponds to such a fixed matrix x. Therefore, the Grothendiek group of stable isomorphism classes of the systems of one submodule is nothing but the K-group $K_0(A)$.

In what follows, we will mainly discuss the systems of two submodules, and thus pairs of projections.

3 Systems of two Submodules and the Group $\Phi_0(A)$

In this section, we define the Φ-group for systems of two submodules, and thus pairs of projections on A^n, $n \geq 1$, for a given involutive algebra A. We begin the procedure from the unital case.

Definition 3.1 *Given a unital involutive algebra A, let X be the set of all the pairs of projections (p_1, p_2) and $\sim$ be the smallest equivalence relation on X, such that*

(1) $(p_1, p_2) \sim (p_1 \oplus 0_m, p_2 \oplus 0_n)$ *for any m, n in $\mathbb{N}$;*

(2) $(p_1, p_2) \sim u(p_1, p_2)u^*$ *for any unitary u;*

(3) $(p_1, p_2) \sim (q_1, q_2)$ *if there exists a pair (r_1, r_2) in X and a unitary u such that*

$$(p_1 \oplus r_1, p_2 \oplus r_2) = u(q_1 \oplus r_1, q_2 \oplus r_2)u^*.$$

Remark 3.1 By (1) of this definition, we can assume that any pair of projections has the same size. The check of (2) is similar to the check of (3), and the former is much easier. Hence, from now on, we assume that any pair of projections has the same size, and we use (3) to check if two pairs of projections are equivalent.

Let $[(p_1, p_2)]$ denote the equivalence class of (p_1, p_2), and we denote the set of the class by

$$\Omega_2 := \{[(p_1, p_2)] : (p_1, p_2) \in X\}.$$

We define an addition on Ω_2 by

$$[(p_1, p_2)] + [(q_1, q_2)] := [(p_1 \oplus q_1, p_2 \oplus q_2)].$$

The addition is well-defined as follows.

We firstly fix (q_1, q_2). If $(p_1, p_2) \sim (p_1', p_2')$, by Remark 3.1, we only check (3) in Definition 3.1. Thus there exist a pair $(v_1, v_2) \in X$ and a unitary u such that

$$u(p_1 \oplus v_1, p_2 \oplus v_2)u^* = (p_1' \oplus v_1, p_2' \oplus v_2).$$

Then

$$\begin{aligned} &(u_1 \oplus 1)(p_1 \oplus v_1 \oplus q_1, p_2 \oplus v_2 \oplus q_2)(u_1 \oplus 1)^* \\ = &(p_1' \oplus v_1 \oplus q_1, p_2' \oplus v_2 \oplus q_2). \end{aligned}$$

Using the procedure of changing orders, we have two unitary x and y such that

$$\begin{aligned} &(u_1 \oplus 1)x(p_1 \oplus q_1 \oplus v_1, p_2 \oplus q_2 \oplus v_2)x^*(u_1 \oplus 1)^* \\ = &(u_1 \oplus 1)(p_1 \oplus v_1 \oplus q_1, p_2 \oplus v_2 \oplus q_2)(u_1 \oplus 1)^* \\ = &(p_1' \oplus v_1 \oplus q_1, p_2' \oplus v_2 \oplus q_2) \\ = &y(p_1' \oplus q_1 \oplus v_1, p_2' \oplus q_2 \oplus v_2)y^*. \end{aligned}$$

Hence

$$(p_1 \oplus q_1, p_2 \oplus q_2) \sim (p_1' \oplus q_1, p_2' \oplus q_2),$$

and thus

$$[(p_1 \oplus q_1, p_2 \oplus q_2)] = [(p_1' \oplus q_1, p_2' \oplus q_2)].$$

Next, we fix (p_1', p_2'), and supposing that $(q_1', q_2') \sim (q_1, q_2)$, with the same procedure, we can prove that

$$[(p_1 \oplus q_1, p_2 \oplus q_2)] = [(p_1' \oplus q_1', p_2' \oplus q_2')].$$

Next we give an important proposition as follows.

Proposition 3.1 *Ω_2 is an Abelian semi-group with cancellation.*

Proof Since the propositions of association and commutation are obvious, we only need to check the cancellation. If

$$[(p_1, p_2)] + [(r_1, r_2)] = [(q_1, q_2)] + [(r_1, r_2)],$$

then

$$(p_1 \oplus r_1, p_2 \oplus r_2) \sim (q_1 \oplus r_1, q_2 \oplus r_2).$$

By Remark 3.1, there exists a pair (s_1, s_2) and a unitary u such that

$$u(p_1 \oplus r_1 \oplus s_1, p_2 \oplus r_2 \oplus s_2)u^* = (q_1 \oplus r_1 \oplus s_1, q_2 \oplus r_2 \oplus s_2).$$

Hence $[(p_1, p_2)] = [(q_1, q_2)]$.

In the end, we note that this simi-group has zero element. Since $(p_1, p_2) \sim (p_1 \oplus 0_n, p_2 \oplus 0_n)$, we can see directly that

$$[(p_1, p_2)] + [(0_n, 0_n)] = [(p_1, p_2)]$$

for any system (p_1, p_2) and any natural number n, and therefore $[(0_n, 0_n)]$ is the zero element for any natural number n.

Then it is convenient to give our main definition.

Definition 3.2 $\Phi(A)$*:= the Grothendieck of* Ω_2.

Remark 3.2 We can only consider the commutative systems of two submodules, and using the same procedure we get the Φ-group, denoted by $\Phi_c(A)$ which is a subgroup of $\Phi(A)$. In Section 5, we will compute some examples of $\Phi_c(A)$ for different A.

The functor Φ for unital involutive algebras Let A and B be unital involutive algebras, and let $\varphi : A \to B$ be a *-homomorphism. Associate to φ a group homomorphism $\Phi(\varphi) : \Phi(A) \to \Phi(B)$ as follows. φ extends to a *-homomorphism $\varphi : M_n(A) \to M_n(B)$ for each n. A unital *-homomorphism maps projections to projections and unitaries to unitaries. Then we can define $\Phi(\varphi) : \Phi(A) \to \Phi(B)$ by $\Phi(\varphi)[(p_1, p_2)] = [(\varphi(p_1), \varphi(p_2))]$. It is easy to check that it is a group homomorphism, and therefore we get the following proposition.

Proposition 3.2 (Functoriality of Φ for Unital Involutive Algebras)
(i) *For each unital involutive algebra A,* $\Phi(\mathrm{id_A}) = \mathrm{id}_{\Phi(\mathrm{A})}$.
(ii) *If* A, B *and* C *are unital involutive algebras, and if* $\varphi : A \to B$ *and* $\psi : B \to C$ *are* **-homomorphisms, then* $\Phi(\psi \circ \varphi) = \Phi(\psi)\Phi(\varphi)$.

The non-unital case If A is an involutive algebra, unital or non-unital, $\widetilde{A}$ being its unitalization, then

$$0 \to A \to \widetilde{A} \to \mathbb{C} \to 0$$

is a short exact sequence, and we define

$$\Phi_0(A) := \ker(\Phi(\widetilde{A}) \to \Phi(\mathbb{C})).$$

Remark 3.3 In Section 4 of this paper, we will compute $\Phi(\mathbb{C})$ and we will see that it is not trivial.

Proposition 3.3 Φ_0 *is a covariant functor from the category of involutive algebras to the category of Abelian groups.*

Proof The proof is similar to the case of the usual K-theory. Let $\varphi : A \to B$ be a homomorphism between involutive algebras A and B, and define $\widetilde{\varphi} : \widetilde{A} \to \widetilde{B}$ by

$$\widetilde{\varphi}(a + \alpha I_A) = \varphi(a) + \alpha I_B.$$

Then there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{l_A} & \widetilde{A} & \xrightarrow{\pi_A} & \mathbb{C} & \longrightarrow & 0 \\ & & \big\downarrow \varphi & & \big\downarrow \widetilde{\varphi} & & \big\| & & \\ 0 & \longrightarrow & B & \xrightarrow[l_B]{} & \widetilde{B} & \xrightarrow[\pi_B]{} & \mathbb{C} & \longrightarrow & 0 \end{array}$$

We get the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Phi_0(A) & \longrightarrow & \Phi(\widetilde{A}) & \xrightarrow{\Phi(\pi_A)} & \Phi(\mathbb{C}) & \longrightarrow & 0 \\ & & & & \big\downarrow \Phi(\widetilde{\varphi}) & & \big\| & & \\ 0 & \longrightarrow & \Phi_0(B) & \longrightarrow & \Phi(\widetilde{B}) & \xrightarrow[\Phi(\pi_B)]{} & \Phi(\mathbb{C}) & \longrightarrow & 0 \end{array}$$

Then $\Phi_0(\varphi) = \Phi(\widetilde{\varphi})\,|_{\Phi_0(A)}$, and by the commutativity of the above diagram, we see that $\Phi(\widetilde{\varphi})\Phi_0(A) \subset \Phi_0(B)$, since Φ is a covariant functor from the category of involutive algebras to the category of Abelian groups, so is Φ_0.

4 Propositions of the Group $\Phi_0(A)$

The standard picture of the group $\Phi_0(A)$ If A is an involutive algebra, unital or non-unital, $\widetilde{A}$ being its unitalization, then

$$0 \to A \to \widetilde{A} \xrightarrow{\pi} \mathbb{C} \to 0$$

is a short exact sequence. Then

$$\Phi_0(A) := \{[(p_1,p_2)] - [(q_1,q_2)] \in \Phi(\widetilde{A}) : \pi_*([(p_1,p_2)] - [(q_1,q_2)]) = 0\}.$$

$[\pi(p_1,p_2)] = [\pi(q_1,q_2)]$ implies that there exists a pair of projections (x,y) and a unitary u such that

$$u^*(\pi(p_1) \oplus x, \pi(p_2) \oplus y)u = (\pi(q_1) \oplus x, \pi(q_2) \oplus y).$$

Since u, x and y are scalar matrices, we have

$$\pi(u^*(p_1 \oplus x, p_2 \oplus y)u) = \pi(q_1 \oplus x, q_2 \oplus y).$$

As $[(p_1 \oplus x, p_2 \oplus y)] - [(q_1 \oplus x, q_2 \oplus y)] = [(p_1,p_2)] - [(q_1,q_2)]$, we replace p_1 with $p_1 \oplus x$, p_2 with $p_2 \oplus y$, q_1 with $q_1 \oplus x$ and q_2 with $q_2 \oplus y$, then we have

$$\pi(u^*(p_1,p_2)u) = \pi(q_1,q_2).$$

As $[u^*(p_1,p_2)u] = [(p_1,p_2)]$, we replace p_1 with u^*p_1u and p_2 with u^*p_2u, and then we have

$$\pi(p_1,p_2) = \pi(q_1,q_2).$$

Conversely, given $[(p_1,p_2)] - [(q_1,q_2)] \in \Phi(\widetilde{A})$ such that $\pi(p_1,p_2) = \pi(q_1,q_2)$, and then obviously we get $\pi_*([(p_1,p_2)] - [(q_1,q_2)]) = 0$, and hence

$$[(p_1,p_2)] - [(q_1,q_2)] \in \Phi_0(A).$$

Then we get the standard picture of $\Phi_0(A)$:

$$\Phi_0(A) := \{[(p_1,p_2)] - [(q_1,q_2)] \in \Phi(\widetilde{A}) : \pi(p_1,p_2) = \pi(q_1,q_2)\}.$$

By the standard picture of $\Phi_0(A)$, we can demonstrate that $\Phi_0(A) \cong \Phi(A)$ for the unital involutive algebra A. This is an important fact since it ensures that we can denote by $\Phi_0(A)$ whether A is unital or not.

Proposition 4.1 (Direct Sums) *For every pair of involutive algebras A and B, we have*

$$\Phi_0(A \oplus B) \cong \Phi_0(A) \oplus \Phi_0(B).$$

Proof Let $\iota_A : A \to A \oplus B$ and $\iota_B : B \to A \oplus B$ be the canonical inclusion maps, and they induce a homomorphism

$$\Phi_0(\iota_A) \oplus \Phi_0(\iota_B) : \Phi_0(A) \oplus \Phi_0(B) \to \Phi_0(A \oplus B),$$

which maps (g,h) in $\Phi_0(A) \oplus \Phi_0(B)$ to $\Phi_0(\iota_A)(g) + \Phi_0(\iota_B)(h)$. We have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Phi_0(A)\oplus\Phi_0(B) & \longrightarrow & \Phi(\widetilde{A})\oplus\Phi(\widetilde{B}) & \longrightarrow & \Phi(\mathbb{C})\oplus\Phi(\mathbb{C}) & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle \Phi_0(\iota_A)\oplus\Phi_0(\iota_B)} & & \Big\downarrow {\scriptstyle \Phi(\iota_{\widetilde{A}})\oplus\Phi(\iota_{\widetilde{B}})} & & \Big\downarrow {\scriptstyle \Phi(\iota_{\mathbb{C}})\oplus\Phi(\iota_{\mathbb{C}})} & & \\
0 & \longrightarrow & \Phi_0(A\oplus B) & \longrightarrow & \Phi(\widetilde{A}\oplus\widetilde{B}) & \longrightarrow & \Phi(\mathbb{C}\oplus\mathbb{C}) & \longrightarrow & 0
\end{array}$$

By 5-lemma, we only have to show the case when A and B are both unital. This is obvious, since every element in the matrix $M_n(A \oplus B)$ is of the form (a,b) for $a \in A$ and $b \in B$, and the product and addition of matrices happen on each component independently.

In the K-theory, the Morita invariance $K_0(A) \cong K_0(M_n(A))$ is a well-known result. The key point of the proof is to show the unital case, and thus $K(A) \cong K(M_n(A))$ when A is a unital involutive algebra, and the general case is got by 5-lemma (see [4]). Unfortunately in the case of Φ_0-group, since the functor Φ_0 does not preserve the split exact sequence, we can not use 5-lemma. But we can still show the unital case in a direct way.

Proposition 4.2 (Morita Invariance) *Let A be a unital involutive algebra and let n be a natural number. Then $\Phi(A)$ is isomorphic to $\Phi(M_n(A))$.*

Proof We will show that the $*$-homomorphism

$$\lambda_A : A \to M_n(A), \quad a \mapsto \begin{pmatrix} a & \\ & 0_{n-1} \end{pmatrix}$$

induces an isomorphism $\alpha : \Phi(A) \to \Phi(M_n(A))$ with $\alpha[(p_1,p_2)] = [((\lambda_A)_m(p_1), (\lambda_A)_m(p_2))]$, where p_1, p_2 are the sizes of m and $(\lambda_A)_m$ is the *-homomorphism $M_m(A) \to M_{mn}(A)$ induced by λ_A.

We should check that this definition is well given. If $[(q_1,q_2)] = [(p_1,p_2)]$, then by Remark 3.2, there exists a pair (r_1,r_2) and a unitary u such that $(q_1 \oplus r_1, q_2 \oplus r_2) = u(p_1 \oplus r_1, p_2 \oplus r_2)u^*$. Without lost of generalization, we assume that p_i, q_i, r_i $(i = 1,2)$ are all of size m.

Let $\{e_1, e_2, \cdots, e_{2mn}\}$ be the standard basis for $\mathbb{C}^{2mn}$, and let v be a permutation unitary in $M_{2mn}(\mathbb{C})$ that fulfills

$$ve_i = e_n(i-1)+1, \quad i = 1,2,\cdots,2m.$$

Then

$$(q_1 \oplus r_1 \oplus 0_{2m(n-1)}, q_2 \oplus r_2 \oplus 0_{2m(n-1)}) = v^*((\lambda_A)_{2m}(q_1 \oplus r_1, q_2 \oplus r_2))v$$

and

$$(p_1 \oplus r_1 \oplus 0_{2m(n-1)}, p_2 \oplus r_2 \oplus 0_{2m(n-1)}) = v^*((\lambda_A)_{2m}(p_1 \oplus r_1, p_2 \oplus r_2))v.$$

Since

$$\begin{aligned} &(q_1 \oplus r_1 \oplus 0_{2m(n-1)}, q_2 \oplus r_2 \oplus 0_{2m(n-1)}) \\ = &(u \oplus 1_{2m(n-1)})(p_1 \oplus r_1 \oplus 0_{2m(n-1)}, p_2 \oplus r_2 \oplus 0_{2m(n-1)})(u \oplus 1_{2m(n-1)})^*, \end{aligned}$$

then

$$\begin{aligned} &v^*((\lambda_A)_m q_1 \oplus (\lambda_A)_m r_1, (\lambda_A)_m q_2 \oplus (\lambda_A)_m r_2)v \\ = &(u \oplus 1_{2m(n-1)})v^*((\lambda_A)_m p_1 \oplus (\lambda_A)_m r_1, (\lambda_A)_m p_2 \oplus (\lambda_A)_m r_2)v(u \oplus 1_{2m(n-1)})^*. \end{aligned}$$

Therefore,

$$[((\lambda_A)_m q_1, (\lambda_A)_m q_2)] = [((\lambda_A)_m p_1, (\lambda_A)_m p_2)].$$

For each natural number k, let $\gamma_k : M_k(M_n(A)) \to M_{kn}(A)$ be the *-isomorphism given by viewing each element of $M_k(M_n(A))$ as one big matrix in $M_{kn}(A)$. Define $\beta : \Phi(M_n(A)) \to \Phi(A)$ by $\beta[(p_1, p_2)] = [\gamma_k(p_1, p_2)]$ for $p_1, p_2 \in M_k(M_n(A))$.

We should show that this definition is well given. In fact, given a pair (q_1, q_2) such that $(q_1, q_2) \sim (p_1, p_2)$, by Remark 3.1, there exists a pair (x, y) and a unitary u such that $u^*(p_1 \oplus x, p_2 \oplus y)u = (q_1 \oplus x, q_2 \oplus y)$. Suppose that p_i, q_i are in $M_k(M_n(A))$ and x, y are in $M_l(M_n(A))$, so u is in $M_{k+l}(M_n(A))$. Then

$$\begin{aligned} \gamma_{k+l}(q_1 \oplus x, q_2 \oplus y) &= \gamma_{k+l}(u^*(p_1 \oplus x, p_2 \oplus y)u), \\ (\gamma_k(q_1) \oplus \gamma_l(x), \gamma_k(q_2) \oplus \gamma_l(y)) &= (\gamma_{k+l}u)^*(\gamma_k(p_1) \oplus \gamma_l(x), \gamma_k(p_2) \oplus \gamma_l(y))(\gamma_{k+l}u). \end{aligned}$$

Since $\gamma_{k+l}u$ is also a unitary element, we have

$$\gamma_k(q_1, q_2) \sim \gamma_k(p_1, p_2),$$

and thus

$$[\gamma_k(q_1, q_2)] = [\gamma_k(p_1, p_2)].$$

We claim that β is the inverse to α. To prove this claim it suffices to show that

$$\begin{aligned} (\lambda_A)_{kn}(\gamma_k(p_1, p_2)) &\sim (p_1, p_2), \quad p_1, p_2 \in M_k(M_n(A)), \\ \gamma_k((\lambda_A)_k(q_1, q_2)) &\sim (q_1, q_2), \quad q_1, q_2 \in M_k(A), \end{aligned}$$

where $(\lambda_A)_m$ is the *-homomorphism $M_m(A) \to M_m(M_n(A))$ induced by λ_A. We prove the second claim, and the proof of the first claim is similar.

Let $\{e_1, e_2, \cdots, e_{kn}\}$ be the standard basis for $\mathbb{C}^{kn}$, and let u be a permutation unitary in $M_{kn}(\mathbb{C})$ that fulfills

$$ue_i = e_{n(i-1)+1}, \quad i = 1, 2, \cdots, k.$$

Then

$$(q_1, q_2) \sim (q_1 \oplus 0_{(n-1)k}, q_2 \oplus 0_{(n-1)k}) = u^*(\gamma_k((\lambda_A)_k(q_1, q_2)))u.$$

Therefore

$$\gamma_k((\lambda_A)_k(q_1,q_2)) \sim (q_1,q_2).$$

The direct system and the direct limit

Recall that the direct limit (A,φ_i) of the direct system of involutive algebras

$$\{A_i \xrightarrow{\varphi_{ji}} A_j : i \le j; i,j \in J\}$$

is characterized by (i) $A = \bigcup_{i\in J} \varphi_i(A_i)$ and (ii) $\ker(\varphi_i) = \bigcup_{j\ge i} \ker(\varphi_{ji})$.

Theorem 4.1 *Suppose that (A,φ_i) is the direct limit of the direct system of involutive algebras $\{A_i \xrightarrow{\varphi_{ji}} A_j \mid i \le j; i,j \in J\}$, and then $\{\Phi_0(A_i) \xrightarrow{\varphi_{ji}^*} \Phi_0(A_j) \mid i \le j\}$ is a direct system of Abelian groups with a direct limit $\{\Phi_0(A_i) \xrightarrow{\varphi_i^*} \Phi_0(A) \mid i \in J\}$.*

Proof We have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Phi_0(A_i) & \longrightarrow & \Phi(\widetilde{A_i}) & \longrightarrow & \Phi(\mathbb{C}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_i^*} & & \downarrow{\scriptstyle \widetilde{\varphi_i}^*} & & \| & & \\ 0 & \longrightarrow & \Phi_0(A) & \longrightarrow & \Phi(\widetilde{A}) & \longrightarrow & \Phi(\mathbb{C}) & \longrightarrow & 0 \end{array}$$

Since the direct limit preserves exactness, by 5-lemma, we may assume that all A_i and A are unital and that φ_i and φ_{ji} preserve units. It suffices to show that $\Phi(A) = \varinjlim \Phi(A_i)$. We prove it by two steps.

(1) $\Phi(A) = \bigcup_{i\in J} \varphi_i^*(\Phi(A_i))$.

For any projections $p,q \in M_n(A)$, there are $i,j \in J$ and $p_i \in M_n(A_i)$, $q_j \in M_n(A_j)$ such that $\varphi_i(p_i) = p$, $\varphi_j(q_j) = q$, so $\varphi_i(p_i^2 - p_i) = 0$, $\varphi_i(p_i)^* = \varphi_i(p_i^*) = \varphi_i(p_i)$. Thus $\varphi_{ki}(p_i^2 - p_i) = 0$, $\varphi_{ki}(p_i)^* = \varphi_{ki}(p_i)$ for some $k \ge i$, and then φ_{ki} is a projection in $M_n(A_k)$. Similarly, there is some $l \ge j$ such that $\varphi_{lj}(q_j)$ is a projection in $M_n(A_l)$. Let $t \ge k,l$. Then $\varphi_{ti}(p_i)$ and $\varphi_{tj}(q_j)$ are projections in $M_n(A_t)$ such that

$$\varphi_t^*[(\varphi_{ti}(p_i),\varphi_{tj}(q_j))] = [(\varphi_t\varphi_{ki}(p_i),\varphi_t\varphi_{tj}(q_j))] = [(\varphi_i(p_i),\varphi_j(q_j))] = [(p,q)].$$

(2) $\ker\varphi_i^* = \bigcup_{j\ge i} \ker\varphi_{ji}^*$.

Letting $\varphi_i^*([(p_i,q_i)] - [(r_i,s_i)]) = 0$, thus $[(\varphi_i(p_i),\varphi_i(q_i))] = [(\varphi_i(r_i),\varphi_i(s_i))]$. Then by Remark 3.1 there are some unitary u in $M_n(A)$ and some projections $x,y \in M_n(A)$ such that

$$u(\varphi_i(p_i)\oplus x, \varphi_i(q_i)\oplus y)u^* = (\varphi_i(r_i)\oplus x, \varphi_i(s_i)\oplus y).$$

There exists a j such that $u = \varphi_j(u_j)$, $x = \varphi_j(x_j)$, $y = \varphi_j(y_j)$, where u_j is a unitary, and thus

$$\varphi_j(u_j)(\varphi_i(p_i)\oplus\varphi_j(x_j), \varphi_i(q_i)\oplus\varphi_j(y_j))\varphi_j(u_j)^* = (\varphi_i(r_i)\oplus\varphi_j(x_j), \varphi_i(s_i)\oplus\varphi_j(y_j)).$$

Let $k \ge i,j$, and then

$$\begin{aligned} &\varphi_k(\varphi_{kj}(u_j)(\varphi_{ki}(p_i)\oplus\varphi_{kj}(x_j), \varphi_{ki}(q_i)\oplus\varphi_{kj}(y_j))\varphi_{kj}(u_j)^*) \\ =\ &\varphi_k\varphi_{kj}(u_j)(\varphi_k\varphi_{ki}(p_i)\oplus\varphi_k\varphi_{kj}(x_j), \varphi_k\varphi_{ki}(q_i)\oplus\varphi_k\varphi_{kj}(y_j))\varphi_k\varphi_{kj}(u_j)^* \end{aligned}$$

$$\begin{aligned}&= \varphi_j(u_j)(\varphi_i(p_i)\oplus\varphi_j(x_j),\varphi_i(q_i)\oplus\varphi_j(y_j))\varphi_j(u_j)^*\\&= (\varphi_i(r_i)\oplus\varphi_j(x_j),\varphi_i(s_i)\oplus\varphi_j(y_j))\\&= (\varphi_k\varphi_{ki}(r_i)\oplus\varphi_k\varphi_{kj}(x_j),\varphi_k\varphi_{ki}(s_i)\oplus\varphi_k\varphi_{kj}(y_j))\\&= \varphi_k(\varphi_{ki}(r_i)\oplus\varphi_{kj}(x_j),\varphi_{ki}(s_i)\oplus\varphi_{kj}(y_j)).\end{aligned}$$

Enlarge k if necessary, and we can get

$$\begin{aligned}&\varphi_{kj}(u_j)(\varphi_{ki}(p_i)\oplus\varphi_{kj}(x_j),\varphi_{ki}(q_i)\oplus\varphi_{kj}(y_j))\varphi_{kj}(u_j)^*\\&= (\varphi_{ki}(r_i)\oplus\varphi_{kj}(x_j),\varphi_{ki}(s_i)\oplus\varphi_{kj}(y_j)).\end{aligned}$$

Therefore,

$$[(\varphi_{ki}(p_i),\varphi_{ki}(q_i))]=[(\varphi_{ki}(r_i),\varphi_{ki}(s_i))],$$

and thus

$$\varphi_{ki}^*([(p_i,q_i)]-[(r_i,s_i)])=0.$$

The relationship between Φ-groups and K-groups We firstly study the unital case. Suppose that A is a unital involutive algebra, and X is the corresponding set in Definition 3.1. If we only consider the subset of X consisting of pairs with the form $(p,0)$, we get a direct summand of $\Phi(A)$ and thus $\{[(p,0)]-[(q,0)]:(p,0),(q,0)\in X\}$, which obviously is isomorphic to the typical $K(A)$. Similarly, we have another direct summand $\{[(0,p)]-[(0,q)]:(0,p),(0,q)\in X\}$ which is also isomorphic to $K(A)$. Hence, we get that

$$K(A)\oplus K(A)\subseteq\Phi(A),$$

and $\Phi(A)/(K(A)\oplus K(A))$ is an Abelian group.

For the general involutive algebra A, unital or not unital, by the standard picture of $\Phi_0(A)$ and $K_0(A)$, we also have that

$$K_0(A)\oplus K_0(A)\subseteq\Phi_0(A).$$

5 The Computation of Φ-Groups

The computation of Φ-groups is in fact the description of the structure theory of pairs up to unitary equivalence, and it is a subproblem to study the pairs of self-adjoint operators. Even for a pair of projections acting on Hilbert spaces, the problem of describing, up to unitary equivalence, irreducible (undecomposible) pairs without any relation is very difficult. For example, let p,q be projections acting on $\mathbb{C}^2$. To simplify the problem, we fix p as $\begin{pmatrix}1&\\&0\end{pmatrix}$ and q as any projection on $\mathbb{C}^2$. By the equations $q^*=q$ and $q^2=q$, we see that q has the forms of $\begin{pmatrix}1&\\&1\end{pmatrix}$ or $\begin{pmatrix}\lambda&\sqrt{\lambda(1-\lambda)}\mathrm{e}^{\mathrm{i}\theta}\\\sqrt{\lambda(1-\lambda)}\mathrm{e}^{-\mathrm{i}\theta}&1-\lambda\end{pmatrix}$, where $0\le\lambda\le1,\ 0\le\theta<2\pi$. Consider the unitary matrix u which is commutative with p, and thus u is of the form $\begin{pmatrix}\mathrm{e}^{\mathrm{i}\alpha}&\\&\mathrm{e}^{\mathrm{i}\beta}\end{pmatrix}$, where $0\le\alpha,\beta<2\pi$. Acting on (p,q) by this kind of u, we get

$$\begin{aligned}u(p,q)u^*&=u\Big(\begin{pmatrix}1&\\&0\end{pmatrix},\begin{pmatrix}\lambda&\sqrt{\lambda(1-\lambda)}\mathrm{e}^{\mathrm{i}\theta}\\\sqrt{\lambda(1-\lambda)}\mathrm{e}^{-\mathrm{i}\theta}&1-\lambda\end{pmatrix}\Big)u^*\\&=\Big(\begin{pmatrix}1&\\&0\end{pmatrix},\begin{pmatrix}\lambda&\sqrt{\lambda(1-\lambda)}\mathrm{e}^{\mathrm{i}(\theta+\alpha-\beta)}\\\sqrt{\lambda(1-\lambda)}\mathrm{e}^{-\mathrm{i}(\theta+\alpha-\beta)}&1-\lambda\end{pmatrix}\Big),\end{aligned}$$

where $0 \leq \lambda \leq 1,\ 0 \leq \theta, \alpha, \beta < 2\pi$ and q is of the nontrivial kind. Then for different λ_1 and λ_2 in $[0,1]$,

$$\left(\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \begin{pmatrix} \lambda_1 & \sqrt{\lambda_1(1-\lambda_1)}\mathrm{e}^{\mathrm{i}(\theta+\alpha-\beta)} \\ \sqrt{\lambda_1(1-\lambda_1)}\mathrm{e}^{-\mathrm{i}(\theta+\alpha-\beta)} & 1-\lambda_1 \end{pmatrix}\right)$$

and

$$\left(\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \begin{pmatrix} \lambda_2 & \sqrt{\lambda_2(1-\lambda_2)}\mathrm{e}^{\mathrm{i}(\theta+\alpha-\beta)} \\ \sqrt{\lambda_2(1-\lambda_2)}\mathrm{e}^{-\mathrm{i}(\theta+\alpha-\beta)} & 1-\lambda_2 \end{pmatrix}\right)$$

can not be unitarily isomorphic. Therefore, the unitarily isomorphism classes for the kind of pairs (p, q), where

$$p = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}$$

is of the card $\aleph$. So we only consider the pairs of projections that satisfy an algebraic relation.

Next, we give the general theory of self-adjiont operators by [5].

Let H be a separable complex (finite or infinite-dimensional) Hilbert space. We consider the pairs A and B of self-adjiont operators which are solutions of the equation

$$P_2(A,B) = \alpha A^2 + \beta_1 AB + \beta_2 BA + \gamma B^2 + \delta A + \varepsilon B + \chi I = 0,$$

where $\alpha, \beta_1, \beta_2, \gamma, \delta, \varepsilon, \chi \in \mathbb{C}$. Suppose that

$$P_2^*(A,B) = \overline{\alpha} A^2 + \overline{\beta_1} BA + \overline{\beta_2} AB + \overline{\gamma} B^2 + \overline{\delta} A + \overline{\varepsilon} B + \overline{\chi} I = P_2(A,B).$$

So we can write this equation as

$$P_2 = \alpha A^2 + \beta\{A,B\} + \mathrm{i}\eta[A,B] + \gamma B^2 + \delta A + \varepsilon B + \chi I = 0, \tag{5.1}$$

where $\alpha, \beta, \eta, \gamma, \delta, \varepsilon, \chi \in \mathbb{R}$, $[A,B] = AB - BA$ is the commutator, and $\{A,B\} = AB + BA$ is the anticommutator. We also have that $\beta = \frac{1}{2}(\beta_1 + \beta_2)$ and $\eta = \frac{1}{2\mathrm{i}}(\beta_1 - \beta_2)$.

By using an affine change of variables, (5.1) can be divided into four groups:

(a) Wild relations: $0 = 0$ or $A^2 = I$.

(b) Binormal relations: $A^2 + B^2 = I$ or $A^2 = B^2$ or $A^2 - B^2 = I$.

(c) Lie algebras and their non-linear transformations: $[A,B] = 0$ or $\frac{1}{\mathrm{i}}[A,B] = I$ or $\frac{1}{\mathrm{i}}[A,B] = A$ or $\frac{1}{\mathrm{i}}[A,B] = A^2$ or $\frac{1}{\mathrm{i}}[A,B] = A^2 + I$ or $\frac{1}{\mathrm{i}}[A,B] = A^2 - I$ or $\frac{1}{\mathrm{i}}[A,B] = A^2 + B$.

(d) Quantum relations: $\frac{1}{\mathrm{i}}[A,B] = q(A^2+B^2)$ $(q > 0)$ or $\frac{1}{\mathrm{i}}[A,B] = q(A^2+B^2) + I (q \in \mathbb{R}, q \neq 0)$ or $\frac{1}{\mathrm{i}}[A,B] = q(A^2-B^2)$ $(q > 0)$ or $\frac{1}{\mathrm{i}}[A,B] = q(A^2-B^2) + I (q \in \mathbb{R}, q \neq 0)$.

In what follows, we study each of these groups of the relations for projections A and B.

(a) Wild relations. The relation $0 = 0$ means that A and B do not satisfy any relation. For projection A, $A^2 = I$ means $A = I$, and therefore $AB = BA$.

(b) Binormal relations. For projections A and B, the relation $A^2 + B^2 = I$ implies that $A = I - B$, and hence $AB = BA$. For the relation $A^2 = B^2$, $A = B$ for projections A and B, and therefore $AB = BA$. The third relation $A^2 - B^2 = I$ holds only when $A = I$ and $B = 0$, which also implies that $AB = BA$.

(c) Lie algebras and their non-linear transformation. In this group of relations, the first six relations are partial cases of the relation

$$[A,B] = \mathrm{i}P_2(A), \tag{5.2}$$

where $P_2(A)$ is a real quadratic polynomial. Even for bounded self-adjoint pairs, (5.2) implies that $[A,B]=0$ by Proposition 1.19 in [5]. For the last relation $\frac{1}{\mathrm{i}}[A,B]=A^2+B$, we can transform it into $[A,(A^2+B)]=\mathrm{i}(A^2+B)$, and then it is converted to the form in (5.2), so we get $[A,(A^2+B^2)]=0$, and thus $[A,B]=0$.

(d) Quantum relations. By Proposition 1.13 in [5], the pair of bounded self-adjoint operators A,B satisfies $\frac{1}{\mathrm{i}}[A,B]=q(A^2+B^2)$ $(q>0)$ and then $A=B=0$, so $AB=BA$. For the second relation $\frac{1}{\mathrm{i}}[A,B]=q(A^2+B^2)+I$ $(q\in\mathbb{R}, q\neq 0)$, we can transform it into the relation $[A,A+B]=\mathrm{i}q(A+B)+\mathrm{i}I$ for a pair of projections A and B, and then it becomes the form of (5.2) in case (c), so we have $[A,B]=0$. For the next two relations, $[A,B]=0$ also holds for bounded self-adjoint operators A and B.

In summary, given a pair of projections which satisfy the relation (5.1), they either have no relation, or satisfy

$$[A,B]=0.$$

We consider the case where the pairs of projections have commutative relations. Thus we consider the commutative systems and give some examples of the computations of $\Phi_c(A)$. It is not hard to see that $\Phi_c(A)$, as a subgroup of $\Phi(A)$, inherits all the propositions of $\Phi(A)$.

Example 5.1 $\Phi_c(A)$ for $A=\mathbb{C}, M_n(\mathbb{C}), C(S^1)$.

For commutative projections p_1,p_2 in $M_n(\mathbb{C})$, they can be simultaneously diagonalized, and thus there is a unitary u in $M_n(\mathbb{C})$ such that up_1u^* and up_2u^* are diagonal matrices and the elements in the diagonal are 0 or 1 since both up_1u^* and up_2u^* are projections. Then we get a couple $(\mathrm{diag}(i_1,i_2,\cdots,i_n), \mathrm{diag}(j_1,j_2,\cdots,j_n))$ with i_s and j_t being 0 or 1 for $1\le s,t\le n$. Therefore, each couple of (i_s,j_s) is an element of the set $\{(1,0),(0,1),(1,1)\}$. Although the relative position of the couples (i_s,j_s) may change for different unitary matrices, the number of times each element in the set $\{(1,0),(0,1),(1,1)\}$ appears will be unchanged, and thus if $(1,0)$, $(0,1)$ and $(1,1)$ appear n_1,n_2,n_3 times respectively in the set $\{(i_1,j_1),(i_2,j_2),\cdots(i_n,j_n)\}$, then n_1,n_2,n_3 are constant for different unitary transformations.

In fact, if there are two unitary matrices u_1 and u_2 such that $u_1(p_1,p_2)u_1^*$ and $u_2(p_1,p_2)u_2^*$ correspond to (n_1,n_2,n_3) and (m_1,m_2,m_3) respectively, and that $(n_1,n_2,n_3)\neq(m_1,m_2,m_3)$, then $(u_1p_1u_1^*)(u_1p_2u_1^*)$ and $(u_2p_1u_2^*)(u_2p_2u_2^*)$ will have different ranks, so $u_1(p_1p_2)u_1^*$ and $u_2(p_1p_2)u_2^*$ have different ranks, which is impossible since unitary transformation dose not change a matix's rank. Then we can give a definition of the rank for commutative pairs of projections, which is a generalization of the rank of one matrix.

Definition 5.1 (Rank for Commutative Pairs of Projections) *Given two projections $p,q\in M_n(\mathbb{C})$, suppose that $u(p,q)u^*$ is the simultaneously diagonalized form. Let r_1,r_2,r_3 be the times $(1,0),(0,1),(1,1)$ appearing in the diagonalized form respectively, and then we call the triple (r_1,r_2,r_3) the rank of the pair (p,q).*

In what follows, we claim that the rank (r_1,r_2,r_3) is invariant under the equivalent relation $\sim$.

Proposition 5.1 *Suppose that (r_1,r_2,r_3) is the rank of a pair of projections $p_1,p_2\in M_n(\mathbb{C})$, and then (r_1,r_2,r_3) is invariant under the equivalence relation $\sim$.*

Proof We show the case (3) in Definition 3.1.

Give another pair of projections $q_1, q_2 \in M_n(\mathbb{C})$ for which there exists a unitary u and a pair of projections (x, y) such that

$$u(p_1 \oplus x, p_2 \oplus y)u^* = (q_1 \oplus x, q_2 \oplus y).$$

Hence the pair $(p_1 \oplus x, p_2 \oplus y)$ and the pair $(q_1 \oplus x, q_2 \oplus y)$ have the same rank (k_1, k_2, k_3). Suppose that $(p_1, p_2), (q_1, q_2), (x, y)$ can be diagonalized by unitary matrices u_1, u_2, u_3 respectively, and then $(p_1 \oplus x, p_2 \oplus y)$ can be diagonalized by the unitary $(u_1 \oplus u_3)$ and $(q_1 \oplus x, q_2 \oplus y)$ can be diagonalized by the unitary $(u_2 \oplus u_3)$. If the rank of (q_1, q_2) is (s_1, s_2, s_3) and the rank of (x, y) is (l_1, l_2, l_3), then we have

$$(k_1, k_2, k_3) = (r_1, r_2, r_3) + (l_1, l_2, l_3) = (s_1, s_2, s_3) + (l_1, l_2, l_3).$$

Since the addition is the canonical one for vectors, we have that

$$(r_1, r_2, r_3) = (s_1, s_2, s_3).$$

By Proposition 5.1, we have

$$\Omega_2 = \mathbb{N} \oplus \mathbb{N} \oplus \mathbb{N},$$

and hence

$$\Phi_c(\mathbb{C}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

By Proposition 4.2 for the commutative case, we have $\Phi_c(M_k(\mathbb{C})) = \Phi_c(\mathbb{C}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

Since two commutative matrices in $M_n(C(S^1))$ can also be diagonalized to the constant matrices simultaneously, we have

$$\Phi_c(C(S^1)) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

Remark 5.1 By the same procedure, we can also construct the Φ-theory for n submodules, denoted by $\Phi^n(A)$, and we can also compute the Φ-group for the commutative systems for Hilbert subspaces, which in general, is $\Phi_c^n(\mathbb{C}) = \bigoplus_{i=1}^{2^n-1} \mathbb{Z}$.

The relationship between the Φ-group and the problem of classification of systems of *n*-subspaces

In the computation of Φ-groups, we see that it is the process to describe the unitary equivalence class for multi-operator, and we should find the irreducible form of the operators as a base for the Φ-group. When $A = \mathbb{C}$, it is the problem of classification of systems of n-subspaces in a finite dimensional Hilbert space which we have introduced in the second section of this paper. In fact $\Phi(\mathbb{C})$ describes the stable unitary equivalence class for the systems of n subspaces in a finite dimensional Hilbert space, but in the case of finite dimensions, the stable unitary equivalence class is almost the unitary equivalence class.

When $n = 1$, $\Phi^1(\mathbb{C}) = \mathbb{Z}$, there is only one direct summand that corresponds to the non-trivial indecomposable systems of one subspaces up to unitary equivalence, namely $(\mathbb{C}; \mathbb{C})$.

When $n = 2$, $\Phi_c^2(\mathbb{C}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, there are three direct summands that correspond to the non-trivial indecomposable commutative systems of two subspaces up to unitary equivalence, namely

$$\mathcal{S}_1 = (\mathbb{C}; \mathbb{C}, 0), \quad \mathcal{S}_2 = (\mathbb{C}; 0, \mathbb{C}), \quad \mathcal{S}_3 = (\mathbb{C}; \mathbb{C}, \mathbb{C}).$$

When $n = 3$, $\Phi_c^3(\mathbb{C}) = \bigoplus_{i=1}^{7} \mathbb{Z}$, there are seven direct summands that correspond to the non-trivial indecomposable commutative systems of three subspaces up to unitary equivalence, namely

$$\mathcal{S}_1 = (\mathbb{C}; \mathbb{C}, 0, 0), \quad \mathcal{S}_2 = (\mathbb{C}; 0, \mathbb{C}, 0), \quad \mathcal{S}_3 = (\mathbb{C}; 0, 0, \mathbb{C}),$$
$$\mathcal{S}_4 = (\mathbb{C}; \mathbb{C}, \mathbb{C}, 0), \quad \mathcal{S}_5 = (\mathbb{C}; \mathbb{C}, 0, \mathbb{C}), \quad \mathcal{S}_6 = (\mathbb{C}; 0, \mathbb{C}, \mathbb{C}), \quad \mathcal{S}_7 = (\mathbb{C}; \mathbb{C}, \mathbb{C}, \mathbb{C}).$$

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