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$\mathcal{T}_C$-Gorenstein Projective, $\mathcal{L}_C$-Gorenstein Injective and $\mathcal{H}_C$-Gorenstein Flat Modules*

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Abstract The authors introduce and investigate the $\mathcal{T}_C$-Gorenstein projective, $\mathcal{L}_C$-Gorenstein injective and $\mathcal{H}_C$-Gorenstein flat modules with respect to a semidualizing module C which shares the common properties with the Gorenstein projective, injective and flat modules, respectively. The authors prove that the classes of all the $\mathcal{T}_C$-Gorenstein projective or the $\mathcal{H}_C$-Gorenstein flat modules are exactly those Gorenstein projective or flat modules which are in the Auslander class with respect to C, respectively, and the classes of all the $\mathcal{L}_C$-Gorenstein injective modules are exactly those Gorenstein injective modules which are in the Bass class, so the authors get the relations between the Gorenstein projective, injective or flat modules and the C-Gorenstein projective, injective or flat modules. Moreover, the authors consider the $\mathcal{T}_C(R)$-projective and $\mathcal{L}_C(R)$-injective dimensions and $\mathcal{T}_C(R)$-precovers and $\mathcal{L}_C(R)$-preenvelopes. Finally, the authors study the $\mathcal{H}_C$-Gorenstein flat modules and extend the Foxby equivalences.

Keywords $\mathcal{T}_C$-Gorenstein projective module, C-Gorenstein projective module, Semidualizing module, Foxby equivalence, Precover, Preenvelope
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1 Introduction

Semidualizing modules are the common generalizations of dualizing modules and free modules of rank one. Foxby [4], Vasconcelos [12] and Golod [5] initiated the study of semidualizing modules under different names. A semidualizing module C induces some interesting classes of modules, such as the Auslander class $\mathcal{A}_C(R)$, the Bass class $\mathcal{B}_C(R)$, the C-projective modules $\mathcal{P}_C(R)$, the C-injective modules $\mathcal{I}_C(R)$ and the C-flat modules $\mathcal{F}_C(R)$, etc. These classes of modules were investigated by many authors and the Foxby equivalence between the Auslander class $\mathcal{A}_{\mathcal{D}}(R)$ and the Bass class $\mathcal{B}_{\mathcal{D}}(R)$ with respect to a dualizing module $\mathcal{D}$ was also extended to the semidualizing case (more details can be found in [7, 10]).

Recall that Enochs and Jenda introduced and studied Gorenstein projective R-modules as a generalization of Auslander's G-modules to the non-finitely modules. An R-module M is called Gorenstein projective if there exists an exact sequence

$$\mathbb{P} = \cdots \xrightarrow{\partial_2^{\mathbb{P}}} P_1 \xrightarrow{\partial_1^{\mathbb{P}}} P_0 \xrightarrow{\partial_0^{\mathbb{P}}} \cdots ,$$

such that the complex $\mathrm{Hom}(\mathbb{P}, Q)$ is exact for each projective module Q and $M \cong \mathrm{Coker}(\partial_1^{\mathbb{P}})$. The class of all Gorenstein projective R-modules, denoted by $\mathcal{GP}(R)$ and the class of all Goren-

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stein injective R-modules, denoted by $\mathcal{GI}(R)$, are defined dually, while an R-module M is called Gorenstein flat if there is an exact sequence of flat R-modules,

$$\mathbb{F} = \cdots \xrightarrow{\partial_2^{\mathbb{F}}} F_1 \xrightarrow{\partial_1^{\mathbb{F}}} F_0 \xrightarrow{\partial_0^{\mathbb{F}}} \cdots ,$$

such that $\mathbb{F} \otimes E$ is exact for any injective R-module E and $M \cong \mathrm{Coker}(\partial_1^{\mathbb{F}})$. The class of all Gorenstein flat R-modules is denoted by $\mathcal{GF}(R)$.

Recently, Holm, Jørgensen, Sather-Wagstaff, and White extended the Gorenstein projective (injective, flat) modules to C-Gorenstein projective (injective, flat) modules via the complete $\mathcal{PP}_C$-resolution ($\mathcal{FF}_C$-resolution, $\mathcal{I}_C\mathcal{I}$-resolution). Recall that a complete $\mathcal{PP}_C$-resolution is an exact sequence of R-modules

$$\mathbb{P} = \cdots \to P_1 \to P_0 \to C \otimes_R P^0 \to C \otimes_R P^1 \to \cdots ,$$

where P_i and P^i for $i \in \mathbb{Z}$ are projective, and the complex $\mathrm{Hom}(\mathbb{P}, C \otimes_R Q)$ is exact for each projective R-module Q. And the complete $\mathcal{I}_C\mathcal{I}$-resolution is defined similarly. Note that the complete $\mathcal{FF}_C$-resolution is an exact sequence of R-modules

$$\mathbb{F} = \cdots \to F_1 \to F_0 \to C \otimes_R F^0 \to C \otimes_R F^1 \to \cdots ,$$

where F_i and F^i for $i \in \mathbb{Z}$ are flat, and the complex $\mathrm{Hom}(C, E) \otimes \mathbb{F}$ is exact for all injective modules E. The class of all C-Gorenstein projective (injective or flat) modules is denoted by $\mathcal{GP}_C(R)$ ($\mathcal{GI}_C(R)$ or $\mathcal{GF}_C(R)$). Note that if the semidualizing module C is the regular module R, then C-Gorenstein projective (injective or flat) modules are just Gorenstein projective (injective or flat).

Note that the two functors $\mathrm{Hom}(C, -)$ and $C \otimes -$ provide equivalences between the class of projective modules and C-projective modules, injective modules and C-injective modules, and flat modules and C-flat modules. A natural question arises: Do the functors $\mathrm{Hom}(C, -)$ and $C \otimes -$ provide the equivalence between the classes $\mathcal{GP}(R)$ and $\mathcal{GP}_C(R)$ ($\mathcal{GI}(R)$ and $\mathcal{GI}_C(R)$ or $\mathcal{GF}(R)$ and $\mathcal{GF}_C(R)$)?

The authors noticed that Sather-Wagstaff, Sharif and White [8] defined the class $\mathcal{G}(\mathcal{P}_C(R))$, which is consists of the modules that are built by a complete resolution of $\mathcal{P}_C(R)$-modules and they [10] also proved that the functors $C \otimes -$ and $\mathrm{Hom}(C, -)$ provide natural equivalence between the classes $\mathcal{GP}(R) \cap \mathcal{A}_C(R)$ and $\mathcal{G}(\mathcal{P}_C(R))$, which helps us to answer the above question significantly.

In this paper, in order to study the relations between the classes of Gorenstein projective modules and C-Gorenstein projective modules, we define the $\mathcal{T}_C$-Gorenstein projective modules. Similarly, we define the $\mathcal{L}_C$-Gorenstein injective and $\mathcal{H}_C$-Gorenstein flat modules. And we get the following Foxby equivalences, in which the first two can be deduced from Theorem 3.1 and the work of [10] (see Corollary 3.1 and Theorem 5.2).

Corollary A *Letting C be a semidualizing R-module, we have the following equivalent classes provided by the functors $C \otimes -$ and* $\mathrm{Hom}(C, -)$:

(1) $\mathcal{T}_C(R) = \mathcal{GP}(R) \cap \mathcal{A}_C(R) \underset{\mathrm{Hom}_R(C,-)}{\overset{C\otimes -}{\rightleftarrows}} \mathcal{GP}_C(R) \cap \mathcal{B}_C(R)$;

(2) $\mathcal{GI}_C(R) \cap \mathcal{A}_C(R) \underset{\mathrm{Hom}_R(C,-)}{\overset{C\otimes -}{\rightleftarrows}} \mathcal{L}_C(R) = \mathcal{GI}(R) \cap \mathcal{B}_C(R)$;

(3) $\mathcal{H}_C(R) = \mathcal{GF}(R) \cap \mathcal{A}_C(R) \underset{\mathrm{Hom}_R(C,-)}{\overset{C\otimes -}{\rightleftarrows}} \mathcal{GF}_C(R) \cap \mathcal{B}_C(R)$,

where (3) *holds when R is coherent.*

In Section 4, we study the $\mathcal{T}_C(R)$-projective and $\mathcal{L}_C(R)$-injective dimensions, and the $\mathcal{T}_C(R)$-precovers or $\mathcal{L}_C(R)$-preenvelopes, which extends the results of Holm [6]. Particularly, we have (see Theorem 4.1) the following result.

Theorem A *Let M be an R-module and n a nonnegative integer. Denote $\mathcal{P}^{<\infty}$ by the class of R-modules with finite projective dimensions. The following are equivalent*:

(1) $\mathcal{T}_C(R)$-$pd(M) = n < \infty$.
(2) *M admits a special $\mathcal{T}_C(R)$-precover: $0 \to K \to T \to M \to 0$ with $T \in \mathcal{T}_C(R)$ and $pd(K) = n - 1$.*
(3) *M admits a special $\mathcal{P}^{<\infty}$-preenvelope: $0 \to M \to L \to T' \to 0$ with $pd(L) = n$ and $T' \in \mathcal{T}_C(R)$.*

As an application, we prove that the classical finitistic projective dimension and the injective dimension are equal to the finitistic $\mathcal{T}_C(R)$-projective dimension and the $\mathcal{L}_C(R)$-injective dimension, respectively (see Proposition 4.1). Moreover, we get the following result (see Theorem 4.3).

Theorem B *Let R be a Gorenstein ring and C be a semidualizing module. Then the Auslander class $\mathcal{A}_C(R) = \widehat{\mathcal{T}_C(R)}$ and the Bass class $\mathcal{B}_C(R) = \widetilde{\mathcal{L}_C(R)}$.*

In Section 5, we define and study the $\mathcal{H}_C$-Gorenstein flat modules over a commutative coherent ring R. We have the following results (see Theorem 5.1 and Proposition 5.1).

Proposition A *Let R be coherent and M an R-module. Then $M \in \mathcal{H}_C(R) \Leftrightarrow M^+ \in \mathcal{L}_C(R)$.*

Hence, many properties of the $\mathcal{H}_C$-Gorenstein flat modules can be obtained from the $\mathcal{L}_C$-Gorenstein injective modules. Particularly, we extend the Foxby equivalence

$$\begin{tikzcd}
\mathcal{P}(R) \arrow[r, shift left, "C\otimes-"] \arrow[d, hook] & \mathcal{P}_C(R) \arrow[l, shift left, "\mathrm{Hom}_R(C,-)", "\sim"'] \arrow[d, hook] \\
\mathcal{H}_C(R) \arrow[r, shift left, "C\otimes-"] \arrow[d, hook] & \mathcal{GF}_C(R)\cap\mathcal{B}_C \arrow[l, shift left, "\mathrm{Hom}_R(C,-)", "\sim"'] \arrow[d, hook] \\
\widehat{\mathcal{H}_C(R)} \arrow[r, shift left, "C\otimes-"] \arrow[d, hook] & \widehat{\mathcal{GF}_C(R)}\cap\mathcal{B}_C \arrow[l, shift left, "\mathrm{Hom}_R(C,-)", "\sim"'] \arrow[d, hook] \\
\mathcal{A}_C \arrow[r, shift left, "C\otimes-"] & \mathcal{B}_C \arrow[l, shift left, "\mathrm{Hom}_R(C,-)", "\sim"']
\end{tikzcd}.$$

Notation A Throughout this paper, R is a commutative ring with an identity, all the modules are unitary, and C is a semidualizing R-module. The class of all the projective, injective or flat R-modules is denoted by $\mathcal{P}(R)$, $\mathcal{I}(R)$ or $\mathcal{F}(R)$, respectively. For an R-module M, let $pd(M)$, $id(M)$, $Gpd(M)$ and $Gid(M)$ denote the projective, injective, Gorenstein projective and Gorenstein injective dimensions of M, respectively. For unexplained concepts and notations, we refer the readers to [8–10].

2 Preliminaries

In this section, we introduce a number of definitions, notions and facts which will be used throughout this paper.

Definition 2.1 (cf. [13, 1.8]) *An R-module C is called semidualizing if*
(1) *C admits a degreewise finite generated projective resolution,*
(2) *the natural homothety map $R \to \mathrm{Hom}_R(C, C)$ is an isomorphism, and*
(3) *$\mathrm{Ext}^i_R(C, C) = 0$ for any $i \geq 1$.*

Definition 2.2 (cf. [11]) *Let C be a semidualizing R-module. The Auslander class with respect to C, denoted by $\mathcal{A}_C(R)$, consists of all the R-modules M satisfying*

(1) $\mathrm{Tor}_i^R(C,M) = \mathrm{Ext}_R^i(C, C\otimes M) = 0$ *for any* $i \geq 1$,

(2) *the natural map* $M \to \mathrm{Hom}(C, C\otimes M)$ *is an isomorphism.*

Dually, the Bass class with respect to C*, denoted by* $\mathcal{B}_C(R)$*, consists of all the* R*-modules* M *satisfying*

(1) $\mathrm{Ext}_R^i(C,M) = \mathrm{Tor}_i^R(C, \mathrm{Hom}(C,M)) = 0$ *for any* $i \geq 1$,

(2) *the natural evaluation map* $C \otimes \mathrm{Hom}(C,M) \to M$ *is an isomorphism.*

Fact 2.1 Let C be a semidualizing R-module. The classes $\mathcal{A}_C(R)$ and $\mathcal{B}_C(R)$ are closed under extensions, kernels of epimorphisms and Cokernels of monomorphisms (cf. [7, Corollary 3.6]). The class $\mathcal{A}_C(R)$ contains all the R-modules of finite flat dimensions and those of finite $\mathcal{I}_C$-injective dimensions, and the category $\mathcal{B}_C(R)$ contains all the R-modules of finite injective dimensions by [7, Corollaries 6.1–6.2].

Let $\mathscr{X}$ be a class of R-modules. We denote by $\mathscr{X}^\perp$ the subcategory of R-modules M such that $\mathrm{Ext}_R^1(X,M) = 0$ for all $X \in \mathscr{X}$. Similarly, ${}^\perp\mathscr{X}$ denotes the subcategory of modules M such that $\mathrm{Ext}_R^1(M,X) = 0$ for all $X \in \mathscr{X}$.

Definition 2.3 (cf. [2]) *Let* $\mathscr{X}$ *be a class of* R*-modules and* M *be any* R*-module. An* $\mathscr{X}$*-precover of* M *is called special if there is an exact sequence* $0 \to L \to X \to M \to 0$ *with* $X \in \mathscr{X}$ *and* $L \in \mathscr{X}^\perp$*. The special preenvelope is defined dually.*

3 $\mathcal{T}_C$-Gorenstein Projective and $\mathcal{L}_C$-Gorenstein Injective Modules

In this section, we give the definitions and some properties of the $\mathcal{T}_C$-Gorenstein projective and $\mathcal{L}_C$-Gorenstein injective modules.

Definition 3.1 *Let* C *be a semidualizing* R*-module. An* R*-module* M *is called* $\mathcal{T}_C$*-Gorenstein projective if there exists an exact complex of projective* R*-modules*

$$\mathbb{P} = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots ,$$

such that the following conditions hold:

(1) *The complex* $C \otimes \mathbb{P}$ *is exact.*

(2) *The complex* $\mathrm{Hom}(\mathbb{P}, Q)$ *is exact for all the projective* R*-modules* Q*.*

(3) *There is an isomorphism* $M \cong \mathrm{Coker}(P_1 \to P_0)$*.*

Denote the class of all $\mathcal{T}_C$*-Gorenstein projective modules by* $\mathcal{T}_C(R)$*.*

An R*-module* M *is called* $\mathcal{L}_C$*-Gorenstein injective if there exists an exact complex of injective* R*-modules*

$$\mathbb{I} = \cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots ,$$

such that the following conditions hold:

(1) *The complex* $\mathrm{Hom}(C, \mathbb{I})$ *is exact.*

(2) *The complex* $\mathrm{Hom}(E, \mathbb{I})$ *is exact for all the injective* R*-modules* E*.*

(3) *There exists an isomorphism* $M \cong \mathrm{Ker}(I_0 \to I^0)$*.*

Denote the class of all $\mathcal{L}_C$*-Gorenstein injective modules by* $\mathcal{L}_C(R)$*.*

Remark 3.1 Let C be a semidualizing R-module.

(1) When $C = R$, we have that

$$\mathcal{T}_C(R) = \mathcal{GP}(R), \quad \mathcal{L}_C(R) = \mathcal{GI}(R).$$

(2) By symmetry, every kernel or cokernel of the morphisms in the complex $\mathbb{P}$ is $\mathcal{T}_C$-Gorenstein projective and every kernel or cokernel of the morphisms in the complex $\mathbb{I}$ is $\mathcal{L}_C$-Gorenstein injective.

(3) By definition, we have

$$\mathcal{P}(R) \subseteq \mathcal{T}_C(R) \subseteq \mathcal{GP}(R), \quad \mathcal{I}(R) \subseteq \mathcal{L}_C(R) \subseteq \mathcal{GI}(R).$$

The following theorem implies that the class of $\mathcal{T}_C$-Gorenstein projective modules or the class of $\mathcal{L}_C$-Gorenstein injective modules shares many common properties with the class of Gorenstein projective or injective modules.

Theorem 3.1 *Let C be a semidualizing R-module. Then*
(1) $\mathcal{T}_C(R) = \mathcal{GP}(R) \cap \mathcal{A}_C(R)$;
(2) $\mathcal{L}_C(R) = \mathcal{GI}(R) \cap \mathcal{B}_C(R)$.

Proof We only prove (1). By Fact 2.1, the classes $\mathcal{A}_C(R)$ and $\mathcal{B}_C(R)$ are closed under extensions, kernels of epimorphisms and cokernels of monomorphisms, so $\mathcal{GP}(R) \cap \mathcal{A}_C(R) \subseteq \mathcal{T}_C(R)$ is straightforward to prove. On the other hand, by Remark 3.1(3), $\mathcal{T}_C(R) \subseteq \mathcal{GP}(R)$. We only need to show $\mathcal{T}_C(R) \subseteq \mathcal{A}_C(R)$. In fact, for any R-module M, if $M \in \mathcal{T}_C(R)$, then there exists an exact sequence of projective modules

$$\mathbb{P} = \cdots \to P_1 \to P_0 \to P_{-1} \to \cdots,$$

such that $C \otimes \mathbb{P}$ is exact and $M \cong \mathrm{Coker}(P_1 \to P_0)$. So $\mathrm{Tor}_i(C, M) = 0$ for $i \geq 1$ and $C \otimes \mathbb{P}$ is an exact complex of $\mathcal{P}_C(R)$ and $C \otimes M \cong \mathrm{Coker}(C \otimes P_1 \to C \otimes P_0)$. By Fact 2.1, $P_i \in \mathcal{A}_C(R)$ for $i \in \mathbb{Z}$, so $\mathrm{Hom}(C, C \otimes P_i) \cong P_i$. Thus $\mathrm{Hom}(C, C \otimes M) \cong M$. Clearly, $\mathrm{Ext}^i(C, C \otimes M) = 0$ by [9, Lemma 1.9(b)]. Hence $M \in \mathcal{A}_C(R)$ and the result follows.

Following from the well-known properties of the classes $\mathcal{GP}(R)$, $\mathcal{GI}(R)$, $\mathcal{A}_C(R)$ and $\mathcal{B}_C(R)$ (cf. [6–7]), by Theorem 3.1, we have the following proposition.

Proposition 3.1 *Let C be a semidualizing R-module. Then*
(1) *the class $\mathcal{T}_C(R)$ is closed under direct sums and the class $\mathcal{L}_C(R)$ is closed under direct products*;
(2) *the class $\mathcal{T}_C(R)$ is projectively resolving and $\mathcal{L}_C(R)$ is injectively resolving*;
(3) *both the classes $\mathcal{T}_C(R)$ and $\mathcal{L}_C(R)$ are closed under direct summands.*

Sather-Wagstaff, Sharif and White proved that

$$\mathcal{G}(\mathcal{P}_C(R)) = \mathcal{GP}_C(R) \cap \mathcal{B}_C(R) \text{ and } \mathcal{G}(\mathcal{I}_C(R)) = \mathcal{GI}_C(R) \cap \mathcal{A}_C(R).$$

And by Theorem 3.1, we can prove the following equivalence provided by the functors $C \otimes -$ and $\mathrm{Hom}(C, -)$, which answers partially the question put forward in the introduction. As the conclusion was also showed by Sather-Wagstaff, Sharif and White [10, Remark 2.11], we omit the proof.

Corollary 3.1

(1) $\mathcal{T}_C(R) = \mathcal{GP}(R) \cap \mathcal{A}_C(R) \underset{\mathrm{Hom}_R(C,-)}{\overset{C\otimes -}{\underset{\sim}{\rightleftarrows}}} \mathcal{GP}_C(R) \cap \mathcal{B}_C(R).$

(2) $\mathcal{GI}_C(R) \cap \mathcal{A}_C(R) \underset{\mathrm{Hom}_R(C,-)}{\overset{C\otimes -}{\underset{\sim}{\rightleftarrows}}} \mathcal{L}_C(R) = \mathcal{GI}(R) \cap \mathcal{B}_C(R).$

4 $\mathcal{T}_C(R)$-Precovers and $\mathcal{L}_C(R)$-Preenvelopes

In this section we want to study the existence of $\mathcal{T}_C(R)$-precovers and $\mathcal{L}_C(R)$-preenvelopes. Moreover, we also study the $\mathcal{T}_C(R)$ projective dimensions and $\mathcal{L}_C(R)$ injective dimensions and we get some good results which extend the results of Holm [6].

Let $\mathscr{X}$ be a class of R modules. We denote by $\widehat{\mathscr{X}}$ the class of R-modules with finite $\mathscr{X}$-projective dimensions and by $\widetilde{\mathscr{X}}$ the class of R-modules with finite $\mathscr{X}$-injective dimensions.

Firstly, we prove the following lemma.

Lemma 4.1 *Let M be an R-module. Denote by $\mathcal{T}_C(R)$-pd(M) and $\mathcal{L}_C(R)$-id(M), the $\mathcal{T}_C$-Gorenstein projective and $\mathcal{L}_C$-Gorenstein injective dimensions of M, respectively.*

(1) If $\mathcal{T}_C(R)$-pd$(M) < \infty$, then $\mathcal{T}_C(R)$-pd$(M) = Gpd(M)$.

(2) If $\mathcal{L}_C(R)$-id$(M) < \infty$, then $\mathcal{L}_C(R)$-id$(M) = Gid(M)$.

In particular, $\widehat{\mathcal{T}_C(R)} = \widehat{\mathcal{GP}(R)} \cap \mathcal{A}_C(R)$ and $\widetilde{\mathcal{L}_C(R)} = \widetilde{\mathcal{GI}(R)} \cap \mathcal{B}_C(R)$.

Proof We only prove (1) and the proof of (2) is similar. By Remark 3.1(3), we have an inequality $Gpd(M) \leq \mathcal{T}_C(R)$-pd(M). Next let $Gpd(M) = n < \infty$. Then there exists an exact sequence

$$0 \to G \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

with each $P_i \in \mathcal{P}(R)$ and $G \in \mathcal{GP}(R)$. By assumption, $\mathcal{T}_C(R)$-$pd(M) < \infty$, so $M \in \mathcal{A}_C(R)$ by Theorem 3.1 and Fact 2.1. So $G \in \mathcal{A}_C(R)$ also by Fact 2.1. Thus $G \in \mathcal{GP}(R) \cap \mathcal{A}_C(R)$. So $G \in \mathcal{T}_C(R)$ by Theorem 3.1. Hence $\mathcal{T}_C(R)$-$pd(M) \leq n = Gpd(M)$ and (1) follows.

Theorem 4.1 *Let M be an R-module and n a nonnegative integer. Denote by $\mathcal{P}^{<\infty}$ the class of R-modules with finite projective dimensions. The following are equivalent.*

(1) $\mathcal{T}_C(R)$-$pd(M) = n < \infty$.

(2) M admits a special $\mathcal{T}_C(R)$-precover: $0 \to K \to T \to M \to 0$ with $T \in \mathcal{T}_C(R)$ and $pd(K) = n - 1$.

(3) M admits a special $\mathcal{P}^{<\infty}$-preenvelope: $0 \to M \to L \to T' \to 0$ with $pd(L) = n$ and $T' \in \mathcal{T}_C(R)$.

Proof (1) $\Rightarrow$ (2). By Lemma 4.1, $Gpd_R(M) = \mathcal{T}_C(R)$-$pd(M) = n$. So M admits a surjective Gorenstein projective precover: $0 \to K \to G \xrightarrow{\Phi} M \to 0$ with G being Gorenstein projective and $pd(K) = n - 1$ by [6, Theorem 2.1]. Since $pd(K) = n - 1 < \infty$, $K \in \mathcal{GP}(R)^{\perp}$. So the Gorenstein projective precover Φ is special by Definition 2.3. We claim that $G \in \mathcal{T}_C(R)$. In fact, as $\mathcal{T}_C(R)$-$pd(M) = n$, $M \in \mathcal{A}_C(R)$ by Theorem 3.1 and Fact 2.1. Clearly, $K \in \mathcal{A}_C(R)$, and thus $G \in \mathcal{A}_C(R)$ also by Fact 2.1. So $G \in \mathcal{T}_C(R)$ by Theorem 3.1. Hence let $T = G$, and then $\Phi : T \to M \to 0$ is the desired special $\mathcal{T}_C(R)$-precover of M.

(2) $\Rightarrow$ (3). Consider the exact sequence $0 \to K \to T \to M \to 0$ with $T \in \mathcal{T}_C(R)$ and $pd(K) = n - 1$. Since $T \in \mathcal{T}_C(R)$, there is an exact sequence $0 \to T \to P \to T' \to 0$ with $P \in \mathcal{P}(R)$ and $T' \in \mathcal{T}_C(R)$. Thus we have the following pushout diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K \arrow[r] \arrow[d, equal] & T \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] & P \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\
 & & T' \arrow[r, equal] \arrow[d] & T' \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} .$$

Since $pd(K) = n - 1$, $pd(L) = n < \infty$ by the exact sequence in the middle row of the above pushout diagram. By Definition 3.1, $T' \in {}^{\perp}\mathcal{P}^{<\infty}$, so the exact sequence $0 \to M \to L \to T' \to 0$ with $T' \in \mathcal{T}_C(R)$ is a special $\mathcal{P}^{<\infty}$-preenvelope of M by Definition 2.3 and [6, Proposition 2.3].

(3) $\Rightarrow$ (1). Since $pd(L) = n$, there exists an exact sequence $0 \to L' \to P_0 \to L \to 0$ with P_0

projective and $pd(L') = n - 1$. Consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & T'' \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & T' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] & L \arrow[r] \arrow[d] & T' \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd} .$$

Since $\mathcal{T}_C(R)$ is projectively resolving, $T'' \in \mathcal{T}_C(R)$. Moreover, by the Snake lemma, we get an exact sequence

$$0 \to L' \to T'' \to M \to 0.$$

As $pd(L') = n - 1$, $\mathcal{T}_C(R)$-$pd(M) = n$. And the theorem follows.

Similarly we have the following result.

Theorem 4.2 *Let M be an R-module and n a nonegative integer. Denote by $\mathcal{I}^{<\infty}$ the class of R-modules with finite injective dimensions. The following are equivalent.*

(1) $\mathcal{L}_C(R)$-$id(M) = n < \infty$.

(2) *M admits a special $\mathcal{L}_C(R)$-preenvelope: $0 \to M \to L \to K \to 0$ with $L \in \mathcal{L}_C(R)$ and $id(K) = n - 1$.*

(3) *M admits a special $\mathcal{I}^{<\infty}$-precover: $0 \to L \to K \to M \to 0$ with $id(K) = n$ and $L \in \mathcal{L}_C(R)$.*

The next proposition is an application of Theorems 4.1–4.2.

Recall that the finitistic projective dimension $\mathrm{FPD}(R)$ is defined as $\mathrm{FPD}(R) = \sup\{pd(M) \mid pd(M) < \infty\}$ and the finitistic injective dimension $\mathrm{FID}(R) = \sup\{id(M) \mid id(M) < \infty\}$. Holm [6] defined the finitistic Gorenstein projective dimension $\mathrm{FGPD}(R)$ and the finitistic Gorenstein injective dimension $\mathrm{FGID}(R)$, and he proved the equalities $\mathrm{FGPD}(R) = \mathrm{FPD}(R)$ and $\mathrm{FGID}(R) = \mathrm{FID}(R)$ (cf. [6, Theorems 2.28 and 2.29]). Similarly, we prove the following equalities and note that we use a different way from Holm's.

Proposition 4.1 *Let $\mathrm{FTPD}(R) = \sup\{\mathcal{T}_C(R)\text{-}pd(M) \mid \mathcal{T}_C(R)\text{-}pd(M) < \infty\}$ and $\mathrm{FLID}(R) = \sup\{\mathcal{L}_C(R)\text{-}id(M) \mid \mathcal{L}_C\text{-}id(M) < \infty\}$ denote the finitistic $\mathcal{T}_C(R)$-Gorenstein projective and $\mathcal{L}_C(R)$-Gorenstein injective dimensions of the base ring R, respectively. Then $\mathrm{FPD}(R) = \mathrm{FTPD}(R)$ and $\mathrm{FID}(R) = \mathrm{FLID}(R)$.*

Proof We only prove $\mathrm{FPD}(R) = \mathrm{FTPD}(R)$. Clearly $\mathrm{FPD}(R) \leq \mathrm{FTPD}(R)$. On the other hand, if M is a module with $0 \leq \mathcal{T}_C(R)$-$pd(M) \leq n$, where n is a nonegative integer, then there exists a module L with $pd(L) = n$ by Theorem 4.1. Hence, if we assume that $0 \leq \mathrm{FTPD}(R) = n$, then we can find an R-module L with $pd(L) = n$, so $\mathrm{FPD}(R) \geq n$, and $\mathrm{FPD}(R) = \mathrm{FTPD}(R)$.

Enochs, Jenda and Xu [3, Corollaries 2.4 and 2.6] showed that when R is a local Cohen-Macaulay ring with a dualizing module $\mathcal{D}$, the Auslander classes with respect to $\mathcal{D}$ are exactly the R-modules with finite Gorenstein projective (flat) dimensions and the Bass classes with respect to $\mathcal{D}$ are exactly the R-modules with finite Gorenstein injective dimensions. While the Gorenstein ring is always a local Cohen-Macaulay ring, in this case, R is the only dualizing module (cf. [2, Remark 9.5.15]). Enochs and Jenda [2, Theorem 12.3.1] proved that every R-module has finite Gorenstein projective dimensions, if and only if every R-module has finite Gorenstein flat dimensions, if and only if every R-module has finite the Gorenstein injective dimensions over Gorenstein ring R. Hence we have the following result, noting that when $C = R$, the result is exactly the [3, Corollaries 2.4 and 2.6]:

Theorem 4.3 *Let R be a Gorenstein ring and C be a semidualizing module. Then the Auslander class $\mathcal{A}_C(R) = \widehat{\mathcal{T}_C(R)}$ and the Bass class $\mathcal{B}_C(R) = \widetilde{\mathcal{L}_C(R)}$.*

Proof By Lemma 4.1, we know that $\widehat{\mathcal{T}_C(R)} = \widehat{\mathcal{GP}(R)} \cap \mathcal{A}_C(R)$ and $\widetilde{\mathcal{L}_C(R)} = \widetilde{\mathcal{GI}(R)} \cap \mathcal{B}_C(R)$. Moreover, R is Gorenstein, so every R-module has a finite Gorenstein projective and Gorenstein injective dimension. Hence we have that $\mathcal{A}_C \subseteq \widehat{\mathcal{GP}(R)}$ and $\mathcal{B}_C \subseteq \widetilde{\mathcal{GI}(R)}$. So

$$\widehat{\mathcal{GP}(R)} \cap \mathcal{A}_C(R) = \mathcal{A}_C(R), \quad \widetilde{\mathcal{GI}(R)} \cap \mathcal{B}_C(R) = \mathcal{B}_C(R).$$

5 $\mathcal{H}_C(R)$-Gorenstein Flat Modules

In this section, we will give the definition of $\mathcal{H}_C(R)$-Gorenstein flat modules which share the common properties with the Gorenstein flat R-modules.

Definition 5.1 *Let C be a semidualizing R-module. An R-module M is called $\mathcal{H}_C$-Gorenstein flat if there is an exact complex of flat R-modules*

$$\mathbb{F} = \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots ,$$

such that the following conditions hold:

(1) *The complex $C \otimes \mathbb{F}$ is exact.*

(2) *The complex $I \otimes \mathbb{F}$ is exact for any injective R-module I.*

(3) *There exists an isomorphism $M \cong \mathrm{Coker}(F_1 \to F_0)$.*

Denote the class of all $\mathcal{H}_C$-Gorenstein flat modules by $\mathcal{H}_C(R)$.

Clearly, any flat module is $\mathcal{H}_C$-Gorenstein flat, and any $\mathcal{H}_C$-Gorenstein flat module is Gorenstein flat. Moreover, when $C = R$, $\mathcal{H}_C(R)$ is exactly the class of Gorenstein flat modules.

Theorem 5.1 $\mathcal{H}_C(R) = \mathcal{GF}(R) \cap \mathcal{A}_C(R)$. *Particularly, an R-module M is in $\mathcal{GF}(R) \cap \mathcal{A}_C(R)$, if and only if there exists an exact sequence*

$$\mathbb{F} = \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

such that both $C \otimes \mathbb{F}$ and $I \otimes \mathbb{F}$ are exact for any injective R-module I and $M \cong \mathrm{Coker}(F_1 \to F_0)$.

Proof Clearly $\mathcal{H}_C(R) \subseteq \mathcal{GF}(R)$. Assume $M \in \mathcal{H}_C(R)$, so there exists an exact complex of flat modules $\mathbb{F}$ such that $C \otimes \mathbb{F}$ is exact. Thus $\mathrm{Tor}_i(C, M) = 0$ for $i \geq 1$. As $F \in \mathcal{A}_C(R)$ for every flat module F, $\mathrm{Hom}(C, C \otimes \mathbb{F}) \cong \mathbb{F}$. Hence the exact complex $C \otimes \mathbb{F}$ is $\mathrm{Hom}(C, -)$-exact and $\mathrm{Hom}(C, C \otimes M) \cong M$. Moreover, by [9, Lemma 1.9], we get that $\mathrm{Ext}^i(C, C \otimes M) = 0$ for $i \geq 1$. Thus $M \in \mathcal{A}_C(R)$ by Definition 2.2 and $\mathcal{H}_C(R) \subseteq \mathcal{GF}(R) \cap \mathcal{A}_C(R)$. The converse containment follows from Fact 2.1 and Definition 5.1.

Hence, following the properties of the classes $\mathcal{GF}(R)$ and $\mathcal{A}_C(R)$ (cf. [6–7]), we know that the class of $\mathcal{H}_C(R)$ is projective resolving. Furthermore, $\mathcal{H}_C(R)$ is closed under direct sums and direct summands.

When R is coherent, Holm [6, Theorem 3.6] showed that M is a Gorenstein flat module if and only if the Pontryagin dual $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is Gorenstein injective, and Sather-Wagstaff, Sharif and White [9, Lemma 4.2] proved that M is $\mathcal{G}_C$-flat, if and only if the Pontryagin dual M^+ is $\mathcal{G}_C$-injective.

By Theorem 5.1, we have the extension result.

Proposition 5.1 *Let R be coherent and M an R-module. Then $M \in \mathcal{H}_C(R) \Leftrightarrow M^+ \in \mathcal{L}_C(R)$.*

Proof On one hand, $M \in \mathcal{A}_C(R) \Leftrightarrow M^+ \in \mathcal{B}_C(R)$ by [1, (3.2.9)]. On the other hand, $M \in \mathcal{GF}(R) \Leftrightarrow M^+ \in \mathcal{GI}(R)$ by [6, Theorem 3.6]. Hence the result follows from Theorem 5.1.

Based on Proposition 5.1, we can easily get the following result.

Theorem 5.2 *The functors* $C\otimes -$ *and* $\mathrm{Hom}(C,-)$ *provide the equivalence between the classes* $\mathcal{H}_C(R)$ *and* $\mathcal{GF}_C(R)\cap\mathcal{B}_C(R)$

$$\mathcal{H}_C(R) \underset{\mathrm{Hom}_R(C,-)}{\overset{C\otimes -}{\underset{\sim}{\rightleftarrows}}} \mathcal{GF}_C(R)\cap\mathcal{B}_C(R).$$

Proof On one hand, by Proposition 5.1, $M\in\mathcal{H}_C(R)\Leftrightarrow M^+\in\mathcal{L}_C(R)$. Moreover, by Corollary 3.1,

$$M^+\in\mathcal{L}_C(R)\Leftrightarrow \mathrm{Hom}(C,M^+)\in\mathcal{GI}_C(R)\cap\mathcal{A}_C(R).$$

But $\mathrm{Hom}(C,M^+)\cong(C\otimes M)^+$, so

$$(C\otimes M)^+\in\mathcal{GI}_C(R)\cap\mathcal{A}_C(R)\Leftrightarrow C\otimes M\in\mathcal{GF}_C(R)\cap\mathcal{B}_C(R)$$

by [9, Lemma 4.2] and [1, (3.2.9)]. On the other hand, by [9, Lemma 4.2] and [1, (3.2.9)], $M\in\mathcal{GF}_C(R)\cap\mathcal{B}_C(R)\Leftrightarrow M^+\in\mathcal{GI}_C(R)\cap\mathcal{A}_C(R)$. By Corollary 3.1, $M^+\in\mathcal{GI}_C(R)\cap\mathcal{A}_C(R)\Leftrightarrow C\otimes M^+\in\mathcal{L}_C(R)$. As $C\otimes M^+\cong(\mathrm{Hom}(C,M))^+$, by Proposition 5.1, we have that

$$M\in\mathcal{GF}_C(R)\cap\mathcal{B}_C(R)\Leftrightarrow \mathrm{Hom}(C,M))\in\mathcal{H}_C(R).$$

As any projective module is $\mathcal{H}_C$-Gorenstein flat, every R-module has an $\mathcal{H}_C(R)$-projective dimension. By Fact 2.1 and Theorem 5.1, we can easily get that

$$\widehat{\mathcal{H}_C(R)}=\widehat{\mathcal{GF}(R)}\cap\mathcal{A}_C(R).$$

Hence we have the extended Foxby equivalence

$$\begin{tikzcd}
\mathcal{P}(R) \arrow[r, shift left, "C\otimes -"] \arrow[d, hook] & \mathcal{P}_C(R) \arrow[l, shift left, "\mathrm{Hom}_R(C,-)"', "\sim"] \arrow[d, hook] \\
\mathcal{H}_C(R) \arrow[r, shift left, "C\otimes -"] \arrow[d, hook] & \mathcal{GF}_C(R)\cap\mathcal{B}_C \arrow[l, shift left, "\mathrm{Hom}_R(C,-)"', "\sim"] \arrow[d, hook] \\
\widehat{\mathcal{H}_C(R)} \arrow[r, shift left, "C\otimes -"] \arrow[d, hook] & \widehat{\mathcal{GF}_C(R)}\cap\mathcal{B}_C \arrow[l, shift left, "\mathrm{Hom}_R(C,-)"', "\sim"] \arrow[d, hook] \\
\mathcal{A}_C \arrow[r, shift left, "C\otimes -"] & \mathcal{B}_C \arrow[l, shift left, "\mathrm{Hom}_R(C,-)"', "\sim"]
\end{tikzcd}.$$

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References

[1] Christensen, L. W., Gorenstein Dimensions, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2000.

[2] Enochs, E. E. and Jenda, O. M. G., Relative homological algebra, De Gruyter, Berlin, New York, 2000.

[3] Enochs, E. E., Jenda, O. M. G. and Xu, Jinzhong, Foxby duality and Gorenstein injective and projective modules, *Trans. Amer. Math. Soc.*, **348**(8), 1996, 3223–3234.

[4] Foxby, H. B., Gorenstein modules and related modules, *Math. Scand*, **31**, 1972, 267–284.

[5] Golod, E. S., *G*-dimension and generalized perfect ideals, Algebraic geometry and its applications, *Trudy Mat. Inst. Steklov*, **165**, 1984, 62–66 (in Russian).

[6] Holm, H., Gorenstein homological dimensions, *J. Pure Appl. Algebra*, **189**, 2004, 167–193.

[7] Holm, H. and White, D., Foxby equivalence over associative rings, *J. Math. Kyoto Univ.*, **47**, 2008, 781–808.

[8] Sather-Wagstaff, S., Sharif, T. and White, D., Stability of Gorenstein categories, *J. London Math. Soc.*, **77**(2), 2008, 481–502.

[9] Sather-Wagstaff, S., Sharif, T. and White, D., AB-context and stabiity for Gorenstein flat modules with respect to semidualizing modules, *Algebra and Representation Theory*, **54**, 2010, 430–435.

[10] Sather-Wagstaff, S., Sharif, T. and White, D., Tate cohomology with respect to semidualizing modules, *J. Algebra*, **324**, 2010, 2336–2368.

[11] Takahashi, R. and White, D., Homological aspects of semidualizing modules, *Math. Scand*, **106**(1), 2010, 5–22.

[12] Vasconcelos, W. V., Divisor theory in module categories, North-Holland Publishing Co., Amsterdam, 1974.

[13] White, D., Gorenstein projective dimension with respect to a semidualizing module, *J. Commut. Algebra,* **2**(1), 2010, 111–137.