Chin. Ann. Math.
35B(1), 2014, 125–138
DOI: 10.1007/s11401-013-0810-z

Chinese Annals of Mathematics, Series B

Almost Sure Asymptotics for Extremes of Non-stationary Gaussian Random Fields*

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Abstract In this paper, the authors prove an almost sure limit theorem for the maxima of non-stationary Gaussian random fields under some mild conditions related to the covariance functions of the Gaussian fields. As the by-products, the authors also obtain several weak convergence results which extended the existing results.

Keywords Almost sure limit theorem, Extremes, Gaussian random fields, Non-stationary
2000 MR Subject Classification 60F05, 62G70

1 Introduction

The almost sure central limit theorem (ASCLT for short) was first introduced independently by [3] and [17] for the partial sum, and then the concept was started to have applications in many areas. For example, [4–5] showed applications of ASCLTs for occupation measures of the Brownian motion on a compact Riemannian manifold and for diffusions and its application to path energy and eigenvalues of the Laplacian. His work was also followed up in many other applied areas, including condensed matter physics, statistical mechanics, ergodic theory, dynamical systems, occupational health psychology, control and information sciences and rehabilitation counseling and so on.

In its simplest form the ASCLT states that if $X_1, X_2, \cdots$ is an independent and identically distributed (i.i.d. for short) sequence of random variables with mean 0 and variance 1, then

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{t=1}^{n} \frac{1}{t} I(t^{-\frac{1}{2}} S_t \le x) = \Phi(x) \quad \text{a.e. } x \in \mathbb{R},$$

where $S_n = \sum\limits_{t=1}^{n} X_t$, I is an indicator function and $\Phi(x)$ stands for the standard normal distribution function.

Later on, [11] and independently [7] extended this principle by establishing the ASCLT for

Manuscript received January 4, 2012. Revised January 10, 2013.
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*Project supported by the National Natural Science Foundation of China (No. 11071182).

the maxima $M_t = \max_{k \le t} X_k$ of i.i.d. random variables. They proved that for any $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{t=1}^{n} \frac{1}{t} I(a_t(M_t - b_t) \le x) = G(x) \quad \text{a.e.} \tag{1.1}$$

with real sequences $a_t > 0$, $b_t \in \mathbb{R}$, $t \ge 1$ and a non-degenerate distribution $G(x)$. [10] and [6] extended (1.1) for weakly dependent stationary Gaussian sequences. We refer to [13] for the non-stationary Gaussian case, [19] for the more general dependent case and [9] for stationary Gaussian fields. The recent extension is the result of [20].

In this paper, we are interested in the similar problems for extremes of non-stationary Gaussian random fields. It is well-known that Gaussian random fields play a very important role in many applied sciences, such as image analysis, atmospheric sciences and geostatistics, among others. Firstly, we introduce some notations and notions of Gaussian random fields.

Denote the set of all positive integers and the set of all non-negative integers by $\mathbb{Z}$ and $\mathbb{N}$, respectively. Let $\mathbb{Z}^d$ and $\mathbb{N}^d$ be d-dimensional product spaces of $\mathbb{Z}$ and $\mathbb{N}$, respectively, where $d \ge 2$. In this paper, we only consider the case of $d = 2$ since it is notationally the simplest and the results for higher dimensions follow analogous arguments. For $\mathbf{i} = (i_1, i_2)$ and $\mathbf{j} = (j_1, j_2)$, $\mathbf{i} \le \mathbf{j}$ and $\mathbf{i} - \mathbf{j}$ mean $i_k \le j_k$, $k = 1, 2$ and $(i_1 - j_1, i_2 - j_2)$, respectively. $|\mathbf{i}|$ and $\mathbf{n} \to \infty$ mean $(|i_1|, |i_2|)$ and $n_k \to \infty$, $k = 1, 2$, respectively. Let $\mathbf{I_n} = \{\mathbf{j} \in \mathbb{Z}^2\colon 1 \le j_i \le n_i,\ i = 1, 2\}$ and $\chi_{\mathbf{E}}$ be the number of elements in $\mathbf{E}$ for any subset $\mathbf{E}$ of $\mathbb{Z}^2$. Let $\chi_{\mathbf{k}} = \prod_{i:k_i \ne 0} |k_i|$ for $\mathbf{k} = (k_1, k_2)$ and $\chi_{\mathbf{0}} = 1$. Note that $\chi_{\mathbf{k}} = \chi_{\mathbf{I_k}}$ when $\mathbf{k} \in \mathbb{Z}^2$. Also, let $\log \mathbf{n}$ and $\log\log \mathbf{n}$ denote $(\log n_1, \log n_2)$ and $(\log\log n_1, \log\log n_2)$, respectively. Let $\Phi(\cdot)$ and $\phi(\cdot)$ denote the standard Gaussian distribution function and its density function, respectively.

Let $\mathbf{X} = \{X_{\mathbf{n}}\}_{\mathbf{n} \ge \mathbf{1}}$ be a non-stationary standardized Gaussian random field on $\mathbb{R}^2$. Let $r_{\mathbf{ij}} = \mathrm{Cov}(X_{\mathbf{i}}, X_{\mathbf{j}})$ be the covariance functions of the Gaussian random field $\mathbf{X} = \{X_{\mathbf{n}}\}_{\mathbf{n} \ge \mathbf{1}}$.

[14] studied the extremes for non-stationary Gaussian random fields and obtained the following weak convergence result.

Theorem 1.1 *Let $\mathbf{X} = \{X_{\mathbf{n}}\}_{\mathbf{n} \ge \mathbf{1}}$ be a non-stationary standardized Gaussian random field. Assume that the covariance functions $r_{\mathbf{ij}}$ satisfy $|r_{\mathbf{ij}}| < \rho_{|\mathbf{i}-\mathbf{j}|}$ for some sequence $\{\rho_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{N}^2 - \{\mathbf{0}\}}$ such that*

$$\lim_{n_1\to\infty} \rho_{(n_1,0)} \log n_1 = 0, \qquad \lim_{n_2\to\infty} \rho_{(0,n_2)} \log n_2 = 0, \tag{1.2}$$

$$\lim_{\mathbf{n}\to\infty} \rho_{\mathbf{n}} \log \chi_{\mathbf{n}} = 0 \tag{1.3}$$

and $\sup_{\mathbf{n} \in \mathbb{N}^2 - \{\mathbf{0}\}} |\rho_{\mathbf{n}}| < 1$. Let the constants $\{u_{\mathbf{n},\mathbf{i}}, \mathbf{i} \le \mathbf{n}\}_{\mathbf{n} \ge \mathbf{1}}$ be such that $\lambda_{\mathbf{n}} = \min_{\mathbf{i} \in \mathbf{I_n}} u_{\mathbf{n},\mathbf{i}} \ge c(\log \chi_{\mathbf{n}})^{\frac{1}{2}}$ for some constant $c > 0$ and $\lim_{\mathbf{n}\to\infty} \sum_{\mathbf{i} \in \mathbf{I_n}} (1 - \Phi(u_{\mathbf{n},\mathbf{i}})) = \tau \in [0, \infty)$. Then

$$\lim_{\mathbf{n}\to\infty} P\Big(\bigcap_{\mathbf{i} \in \mathbf{I_n}} \{X_{\mathbf{i}} \le u_{\mathbf{n},\mathbf{i}}\} \Big) = \exp(-\tau). \tag{1.4}$$

For more detailed limit properties of the extremes and their applications for Gaussian random fields, we refer to [8–9, 14–15, 18]. For further results concerning the extremes in Gaussian random fields we refer the readers to [1–2, 8–9, 14–16].

In this paper, we concentrate on the almost sure limit theorem on extremes of non-stationary Gaussian random fields. We will extend (1.4) to the almost sure version. As a by-product, we find that (1.4) still holds under weaker conditions.

2 Main Results

Now, we state our main results.

Theorem 2.1 *Let* $\mathbf{X} = \{X_{\mathbf{n}}\}_{\mathbf{n}\geq \mathbf{1}}$ *be a non-stationary standardized Gaussian random field. Assume that the covariance functions* $r_{\mathbf{ij}}$ *satisfy* $|r_{\mathbf{ij}}| < \rho_{|\mathbf{i}-\mathbf{j}|}$ *for some sequence* $\{\rho_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{N}^2-\{\mathbf{0}\}}$ *such that for some* $\varepsilon > 0$,

$$\rho_{(n_1,0)} \log n_1 = O((\log\log n_1)^{-(1+\varepsilon)}), \quad \rho_{(0,n_2)} \log n_2 = O((\log\log n_2)^{-(1+\varepsilon)}), \tag{2.1}$$

$$\rho_{\mathbf{n}} \log \chi_{\mathbf{n}} = O((\chi_{\log\log \mathbf{n}})^{-(1+\varepsilon)}) \tag{2.2}$$

and $\sup\limits_{\mathbf{n}\in\mathbb{N}^2-\{\mathbf{0}\}} |\rho_{\mathbf{n}}| < 1$ *hold. Let the constants* $\{u_{\mathbf{n},\mathbf{i}}, \mathbf{i}\leq \mathbf{n}\}_{\mathbf{n}\geq\mathbf{1}}$ *be such that* $\chi_{\mathbf{n}}(1-\Phi(\lambda_{\mathbf{n}}))$ *is bounded, where* $\lambda_{\mathbf{n}} = \min\limits_{\mathbf{i}\in\mathbf{I_n}} u_{\mathbf{n},\mathbf{i}}$. *Suppose that* $\lim\limits_{\mathbf{n}\to\infty} \sum\limits_{\mathbf{i}\in\mathbf{I_n}} (1-\Phi(u_{\mathbf{n},\mathbf{i}})) = \tau \in [0,\infty)$ *holds. Then*

$$\lim_{\mathbf{n}\to\infty} \frac{1}{\chi_{\log \mathbf{n}}} \sum_{\mathbf{k}\in\mathbf{I_n}} \frac{1}{\chi_{\mathbf{k}}} I\Big(\bigcap_{\mathbf{i}\in\mathbf{I_k}} \{X_{\mathbf{i}} \leq u_{\mathbf{k},\mathbf{i}}\}\Big) = \exp(-\tau) \quad a.e. \tag{2.3}$$

As a special case, we have the following corollary.

Corollary 2.1 *Let* $\mathbf{X} = \{X_{\mathbf{n}}\}_{\mathbf{n}\geq \mathbf{1}}$ *be a non-stationary standardized Gaussian random field. Assume that the covariance functions* $r_{\mathbf{ij}}$ *satisfy* $|r_{\mathbf{ij}}| < \rho_{|\mathbf{i}-\mathbf{j}|}$ *for some sequence* $\{\rho_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{N}^2-\{\mathbf{0}\}}$ *such that* (2.1)–(2.2) *and* $\sup\limits_{\mathbf{n}\in\mathbb{N}^2-\{\mathbf{0}\}} |\rho_{\mathbf{n}}| < 1$ *hold. Let the constants* $\{u_{\mathbf{n}}\}_{\mathbf{n}\geq\mathbf{1}}$ *be such that* $\lim\limits_{\mathbf{n}\to\infty} \chi_{\mathbf{n}}(1-\Phi(u_{\mathbf{n}})) = \tau \in [0,\infty)$. *Then*

$$\lim_{\mathbf{n}\to\infty} \frac{1}{\chi_{\log \mathbf{n}}} \sum_{\mathbf{k}\in\mathbf{I_n}} \frac{1}{\chi_{\mathbf{k}}} I(M_{\mathbf{k}}(X) \leq u_{\mathbf{k}}) = \exp(-\tau) \quad a.s., \tag{2.4}$$

where $M_{\mathbf{k}}(X) = \max\limits_{\mathbf{i}\in\mathbf{I_k}} X_{\mathbf{i}}$.

Further, let $a_{\mathbf{n}} = \sqrt{2\log\chi_{\mathbf{n}}}$ *and* $b_{\mathbf{n}} = a_{\mathbf{n}} - \frac{\log\log\chi_{\mathbf{n}} + \log(4\pi)}{2a_{\mathbf{n}}}$, *and then*

$$\lim_{\mathbf{n}\to\infty} \frac{1}{\chi_{\log \mathbf{n}}} \sum_{\mathbf{k}\in\mathbf{I_n}} \frac{1}{\chi_{\mathbf{k}}} I(a_{\mathbf{k}}(M_{\mathbf{k}}(X) - b_{\mathbf{k}}) \leq x) = \exp(-\mathrm{e}^{-x}) \quad a.e.\ x \in \mathbb{R}. \tag{2.5}$$

Next, we give a weak convergence result which is an extension of Theorem A.

Theorem 2.2 *Let* $\mathbf{X} = \{X_{\mathbf{n}}\}_{\mathbf{n}\geq \mathbf{1}}$ *be a non-stationary standardized Gaussian random field. Assume that the covariance functions* $r_{\mathbf{ij}}$ *satisfy* $|r_{\mathbf{ij}}| < \rho_{|\mathbf{i}-\mathbf{j}|}$ *for some sequence* $\{\rho_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{N}^2-\{\mathbf{0}\}}$ *such that*

$$\rho_{(n_1,0)} \log n_1 \text{ and } \rho_{(0,n_2)} \log n_2 \text{ are bounded.} \tag{2.6}$$

In addition, assume that (1.3) *and* $\sup\limits_{\mathbf{n}\in\mathbb{N}^2-\{\mathbf{0}\}} |\rho_{\mathbf{n}}| < 1$ *hold. Let the constants* $\{u_{\mathbf{n},\mathbf{i}}, \mathbf{i}\leq \mathbf{n}\}_{\mathbf{n}\geq\mathbf{1}}$ *be such that* $\chi_{\mathbf{n}}(1-\Phi(\lambda_{\mathbf{n}}))$ *is bounded, where* $\lambda_{\mathbf{n}} = \min\limits_{\mathbf{i}\in\mathbf{I_n}} u_{\mathbf{n},\mathbf{i}}$. *Suppose that* $\lim\limits_{\mathbf{n}\to\infty} \sum\limits_{\mathbf{i}\in\mathbf{I_n}} (1-\Phi(u_{\mathbf{n},\mathbf{i}})) = \tau \in [0,\infty)$ *holds. Then* (1.4) *holds.*

Remark 2.1 The assertions of Theorems 2.1–2.2 still hold for stationary Gaussian fields with the similar conditions on the correlation functions. Note that even for the stationary case, Theorems 2.1–2.2 are still new results.

Example 2.1 (1) The assertions of Theorems 2.1–2.2 still hold for independent Gaussian random fields, and m-dependent Gaussian random fields.

(2) Let $Z_{\mathbf{1}}$ be a Gaussian field with mean 0, variance 1 and $Z_{\mathbf{n}} = Z_{(n_1,n_2)} = \frac{1}{n_1 n_2} Z_{(1,1)}$, and then $\mathbf{X} = \{Z_{\mathbf{n}}\}_{\mathbf{n}\geq\mathbf{1}}$ is a non-stationary Gaussian random field which satisfies the conditions of Theorems 2.1–2.2, where $\rho_{\mathbf{n}}$ can be chosen as follows:

$$\rho_{\mathbf{n}} = \rho_{(n_1,n_2)} = \frac{1}{n_1+1}\frac{1}{n_2+1}.$$

Using Theorem 2.2, we extend Theorem 6.2.1 of [12] to Gaussian random fields. The obtained result also tells us how to construct a non-stationary Gaussian random field by a stationary Gaussian field.

Corollary 2.2 *Let* $\mathbf{Y} = \{X_{\mathbf{n}} + m_{\mathbf{n}}\}_{\mathbf{n}\geq\mathbf{1}}$*, where* $\{X_{\mathbf{n}}\}_{\mathbf{n}\geq\mathbf{1}}$ *is the Gaussian random field satisfying the conditions of Theorem* 2.2 *and* $\{m_{\mathbf{n}}\}_{\mathbf{n}\geq\mathbf{1}}$ *satisfies*

$$\beta_{\mathbf{n}} = \max_{\mathbf{k}\in\mathbf{I_n}} |m_{\mathbf{k}}| = o(\sqrt{\chi_{\mathbf{n}}}), \tag{2.7}$$

and let $m^*_{\mathbf{n}}$ *be such that*

$$|m^*_{\mathbf{n}}| \leq \beta_{\mathbf{n}} \tag{2.8}$$

and

$$\frac{1}{\chi_{\mathbf{n}}} \sum_{\mathbf{i}\in\mathbf{I_n}} \exp\Big(a^*_{\mathbf{n}}(m_{\mathbf{i}} - m^*_{\mathbf{n}}) - \frac{1}{2}(m_{\mathbf{i}} - m^*_{\mathbf{n}})^2\Big) \to 1 \tag{2.9}$$

as $\mathbf{n}\to\infty$*, where* $a^*_{\mathbf{n}} = a_{\mathbf{n}} - \log\log\frac{\chi_{\mathbf{n}}}{2a_{\mathbf{n}}}$*. Then*

$$\lim_{\mathbf{n}\to\infty} P(a_{\mathbf{n}}(M_{\mathbf{n}}(Y) - b_{\mathbf{n}} - m^*_{\mathbf{n}}) \leq x) = \exp(-\mathrm{e}^{-x}), \tag{2.10}$$

where $M_{\mathbf{n}}(Y) = \max_{\mathbf{i}\in\mathbf{I_n}} Y_{\mathbf{i}}$*, and* $a_{\mathbf{n}}$ *and* $b_{\mathbf{n}}$ *are defined as in Corollary* 2.1.

Using Theorem 2.1, Corollary 2.2 can be extended to the almost sure version.

Corollary 2.3 *Let* $\mathbf{Y} = \{X_{\mathbf{n}} + m_{\mathbf{n}}\}_{\mathbf{n}\geq\mathbf{1}}$*, where* $\{X_{\mathbf{n}}\}_{\mathbf{n}\geq\mathbf{1}}$ *is the Gaussian random field satisfying the conditions of Theorem* 2.1 *and* $\{m_{\mathbf{n}}\}_{\mathbf{n}\geq\mathbf{1}}$ *satisfies the conditions of Corollary* 2.2*. Let* $m^*_{\mathbf{n}}$ *satisfy* (2.8)–(2.9) *and*

$$a_{\mathbf{n}}\Big(\max_{\mathbf{i}\in\mathbf{I_n}} m_{\mathbf{i}} - m^*_{\mathbf{n}}\Big) \ \textit{is bounded}. \tag{2.11}$$

Then

$$\lim_{\mathbf{n}\to\infty} \frac{1}{\chi_{\log\mathbf{n}}} \sum_{\mathbf{k}\in\mathbf{I_n}} \frac{1}{\chi_{\mathbf{k}}} I(a_{\mathbf{k}}(M_{\mathbf{k}}(Y) - b_{\mathbf{k}} - m^*_{\mathbf{k}}) \leq x) = \exp(-\mathrm{e}^{-x}) \quad a.e., \tag{2.12}$$

where $M_{\mathbf{n}}(Y) = \max_{\mathbf{i}\in\mathbf{I_n}} Y_{\mathbf{i}}$*, and* $a_{\mathbf{n}}$ *and* $b_{\mathbf{n}}$ *are defined as in Corollary* 2.1.

3 Auxiliary Results

In this section, we state and prove several lemmas which will be used in the proofs of our main results. As usual, $a_{\mathbf{n}} \ll b_{\mathbf{n}}$ means $a_{\mathbf{n}} = O(b_{\mathbf{n}})$. Let K denote positive constants whose values may vary from place to place.

The first lemma is the so-called normal comparison lemma which can be found in [12]. A simple special form of this theorem is given here.

Lemma 3.1 (cf. [12]) *Let* $\mathbf{X} = \{X_{\mathbf{n}}\}_{\mathbf{n}\geq\mathbf{1}}$ *and* $\mathbf{Y} = \{Y_{\mathbf{n}}\}_{\mathbf{n}\geq\mathbf{1}}$ *be standardized Gaussian random fields with covariance functions* $\Lambda^1_{\mathbf{ij}}$ *and* $\Lambda^2_{\mathbf{ij}}$*, respectively. Let* $\max_{\mathbf{i}\neq\mathbf{j}} |\gamma_{\mathbf{ij}}| = \gamma < 1$*, where* $\gamma_{\mathbf{ij}} = \max\{\Lambda^1_{\mathbf{ij}}, \Lambda^2_{\mathbf{ij}}\}$*. Then, for constants* $\{u_{\mathbf{n},\mathbf{i}}, \mathbf{i} \leq \mathbf{n}\}_{\mathbf{n}\geq\mathbf{1}}$*, we have*

$$\Big|P\Big(\bigcap_{\mathbf{i}\in\mathbf{I_n}}\{X_{\mathbf{i}} \leq u_{\mathbf{n},\mathbf{i}}\}\Big) - P\Big(\bigcap_{\mathbf{i}\in\mathbf{I_n}}\{Y_{\mathbf{i}} \leq u_{\mathbf{n},\mathbf{i}}\}\Big)\Big|$$
$$\leq K \sum_{\mathbf{i},\mathbf{j}\in\mathbf{I_n},\mathbf{i}\leq\mathbf{j},\mathbf{i}\neq\mathbf{j}} |\Lambda^1_{\mathbf{ij}} - \Lambda^2_{\mathbf{ij}}| \exp\Big(-\frac{u^2_{\mathbf{n},\mathbf{i}} + u^2_{\mathbf{n},\mathbf{j}}}{2(1+|\gamma_{\mathbf{ij}}|)}\Big),$$

where K *is some constant, depending only on* γ*.*

The second lemma is an extension of Lemma 3.1 of [10] from random sequences to random fields, which will play a crucial role in the proof of Theorem 2.1.

Lemma 3.2 *Let* $\{\xi_{\mathbf{k}}\}_{\mathbf{k}\geq\mathbf{1}}$*,* $\mathbf{k} \in \mathbb{Z}^d$*,* $d \geq 2$ *be a sequence of uniformly bounded random variables, i.e., there exists some* $M \in (0,\infty)$*, such that* $|\xi_{\mathbf{k}}| \leq M$ *a.s. for all* $\mathbf{k} \in \mathbb{Z}^d$*. If*

$$\mathrm{Var}\Big(\frac{1}{\chi_{\log\mathbf{n}}}\sum_{\mathbf{k}\in\mathbf{I_n}}\frac{1}{\chi_{\mathbf{k}}}\xi_{\mathbf{k}}\Big) \ll (\chi_{\log\log\mathbf{n}})^{-(1+\varepsilon)}$$

for some $\varepsilon > 0$*, then*

$$\lim_{\mathbf{n}\to\infty}\frac{1}{\chi_{\log\mathbf{n}}}\sum_{\mathbf{k}\in\mathbf{I_n}}\frac{1}{\chi_{\mathbf{k}}}(\xi_{\mathbf{k}} - E\xi_{\mathbf{k}}) = 0 \quad \textit{a.e.}$$

Proof We only prove the case of $d = 2$. Setting

$$\mu_{\mathbf{n}} = \frac{1}{\chi_{\log\mathbf{n}}}\sum_{\mathbf{k}\in\mathbf{I_n}}\frac{1}{\chi_{\mathbf{k}}}(\xi_{\mathbf{k}} - E\xi_{\mathbf{k}})$$

and $\mathbf{n_k} = (n_{k_1}, n_{k_2}) = (\exp(\exp(k_1^\nu)), \exp(\exp(k_2^\nu)))$ for some $\frac{1}{1+\varepsilon} < \nu < 1$, we have

$$\sum_{\mathbf{k}\geq\mathbf{3}} E\mu^2_{\mathbf{n_k}} \ll \sum_{k_2\geq 3}\sum_{k_1\geq 3}(k_1k_2)^{-\nu(1+\varepsilon)} < \infty.$$

Thus, by applying the Borel-Cantelli lemma, $\mu_{\mathbf{n_k}} \to 0$ a.s. Since for $\nu < 1$, $(\mathbf{k}+\mathbf{1})^\nu - \mathbf{k}^\nu \to \mathbf{0}$ as $\mathbf{k} \to \infty$ if $\nu < 1$, we have for $i = 1, 2$,

$$\frac{\log(n_{k_i+1})}{\log(n_{k_i})} = \frac{\exp((k_i+1)^\nu)}{\exp(k_i^\nu)} = \exp((k_i+1)^\nu - k_i^\nu) \to 1$$

as $k_i \to \infty$. Obviously for any given $\mathbf{n} \in \mathbb{Z}^2$, there exists an integer $\mathbf{k} \in \mathbb{Z}^2$, such that $\mathbf{n_k} < \mathbf{k} \leq \mathbf{n_{k+1}}$. Therefore

$$\begin{aligned}
|\mu_{\mathbf{n}}| &\leq \frac{1}{\chi_{\log\mathbf{n}}}\Big|\sum_{\mathbf{j}\in\mathbf{I_n}}\frac{1}{\chi_{\mathbf{j}}}(\xi_{\mathbf{j}} - E\xi_{\mathbf{j}})\Big| \\
&\leq \frac{1}{\chi_{\log\mathbf{n_k}}}\Big|\sum_{\mathbf{j}\in\mathbf{I_{n_k}}}\frac{1}{\chi_{\mathbf{j}}}(\xi_{\mathbf{j}} - E\xi_{\mathbf{j}})\Big| + \frac{1}{\chi_{\log\mathbf{n_k}}}\sum_{\mathbf{j}\in\mathbf{I_{n_{k+1}}}-\mathbf{I_{n_k}}}\frac{1}{\chi_{\mathbf{j}}}|(\xi_{\mathbf{j}} - E\xi_{\mathbf{j}})| \\
&\ll |\mu_{\mathbf{n_k}}| + \frac{1}{\chi_{\log\mathbf{n_k}}}\sum_{\mathbf{j}\in\mathbf{I_{n_{k+1}}}-\mathbf{I_{n_k}}}\frac{1}{\chi_{\mathbf{j}}}
\end{aligned}$$

$$\leq |\mu_{\mathbf{n_k}}| + \Big(\frac{\log n_{k_1+1}}{\log n_{k_1}} - 1\Big)\frac{\log n_{k_2+1}}{\log n_{k_2}} + \Big(\frac{\log n_{k_2+1}}{\log n_{k_2}} - 1\Big),$$

and thus

$$\lim_{\mathbf{n}\to\infty} \mu_{\mathbf{n}} = 0 \quad \text{a.s.}$$

The proof is complete.

In the following lemmas, we will intensively use the following notations and facts. By the assumption on $\lambda_{\mathbf{n}}$, we have $\chi_{\mathbf{n}}(1-\Phi(\lambda_{\mathbf{n}})) < K$, for a constant K and $\lambda_{\mathbf{n}} \to \infty$ as $\mathbf{n} \to \infty$. Since $\Phi(\cdot)$ is continuous, there exists $\omega_{\mathbf{n}}$, such that $\chi_{\mathbf{n}}(1-\Phi(\omega_{\mathbf{n}})) = K$, which combining with the fact that $1-\Phi(x) \sim \frac{\phi(x)}{x}$ as $x \to \infty$ implies

$$\omega_{\mathbf{n}} \leq \lambda_{\mathbf{n}}, \quad \exp\Big(-\frac{\omega_{\mathbf{n}}^2}{2}\Big) \backsim \frac{K\omega_{\mathbf{n}}}{\chi_{\mathbf{n}}}, \quad \omega_{\mathbf{n}} \backsim \sqrt{2\log\chi_{\mathbf{n}}} \tag{3.1}$$

for large $\mathbf{n}$. Let $\delta = \max_{\mathbf{i}\neq\mathbf{j}} |r_{\mathbf{ij}}| < 1$ and $\theta_{\mathbf{n}} = \exp(\eta\omega_{\mathbf{n}}^2)$, where η is a positive constant satisfying

$$\eta < \frac{1-\delta}{4(1+\delta)}.$$

Lemma 3.3 *Let $\mathbf{X} = \{X_{\mathbf{n}}\}_{\mathbf{n}\geq\mathbf{1}}$ be a non-stationary standardized Gaussian random field with covariance functions $r_{\mathbf{ij}}$ satisfying $\delta = \max_{\mathbf{i}\neq\mathbf{j}} |r_{\mathbf{ij}}| < 1$. Let the constants $\{u_{\mathbf{n},\mathbf{i}}, \mathbf{i} \leq \mathbf{n}\}_{\mathbf{n}\geq\mathbf{1}}$ be such that $\chi_{\mathbf{n}}(1-\Phi(\lambda_{\mathbf{n}}))$ is bounded, where $\lambda_{\mathbf{n}} = \min_{\mathbf{i}\in\mathbf{I_n}} u_{\mathbf{n},\mathbf{i}}$. Then, we have*

$$\sum_{\substack{\mathbf{i},\mathbf{j}\in\mathbf{I_n},\mathbf{i}\leq\mathbf{j},\mathbf{i}\neq\mathbf{j}\\ \chi_{|\mathbf{i}-\mathbf{j}|}\leq\theta_{\mathbf{n}}}} |r_{\mathbf{ij}}| \exp\Big(-\frac{u_{\mathbf{n},\mathbf{i}}^2 + u_{\mathbf{n},\mathbf{j}}^2}{2(1+|r_{\mathbf{ij}}|)}\Big) \ll (\chi_{\mathbf{n}})^{-\sigma_1} \tag{3.2}$$

with the constant $\sigma_1 > 0$.

Proof Using the facts in (3.1), it is easy to see that

$$\begin{aligned}
&\sum_{\substack{\mathbf{i},\mathbf{j}\in\mathbf{I_n},\mathbf{i}\leq\mathbf{j},\mathbf{i}\neq\mathbf{j}\\ \chi_{|\mathbf{i}-\mathbf{j}|}\leq\theta_{\mathbf{n}}}} |r_{\mathbf{ij}}| \exp\Big(-\frac{u_{\mathbf{n},\mathbf{i}}^2 + u_{\mathbf{n},\mathbf{j}}^2}{2(1+|r_{\mathbf{ij}}|)}\Big)\\
&\leq \delta \sum_{\substack{\mathbf{i},\mathbf{j}\in\mathbf{I_n},\mathbf{i}\leq\mathbf{j},\mathbf{i}\neq\mathbf{j}\\ \chi_{|\mathbf{i}-\mathbf{j}|}\leq\theta_{\mathbf{n}}}} \exp\Big(-\frac{\omega_{\mathbf{n}}^2}{1+\delta}\Big) \leq \delta\theta_{\mathbf{n}}^2\chi_{\mathbf{n}} \exp\Big(-\frac{\omega_{\mathbf{n}}^2}{1+\delta}\Big)\\
&\leq \delta\chi_{\mathbf{n}}\Big(\exp\Big(-\frac{\omega_{\mathbf{n}}^2}{2}\Big)\Big)^{\frac{2}{1+\delta}-2\eta} \ll \delta\chi_{\mathbf{n}}\Big(\frac{\omega_{\mathbf{n}}}{n_1 n_2}\Big)^{\frac{2}{1+\delta}-2\eta}\\
&\ll \delta(\chi_{\mathbf{n}})^{1-\frac{2}{1+\delta}+2\eta}\Big(\log\chi_{\mathbf{n}}\Big)^{\frac{1}{1+\delta}-\eta}.
\end{aligned}$$

Since $\eta < \frac{1-\delta}{4(1+\delta)}$ and $0 < \delta < 1$, we have $1 - \frac{2}{1+\delta} + 2\eta < 0$. Hence, there exists a constant $\sigma_1 > 0$, such that (3.2) holds.

Lemma 3.4 *Under the conditions of Theorem 2.2, we have*

$$\sum_{\substack{\mathbf{i},\mathbf{j}\in\mathbf{I_n},\mathbf{i}\leq\mathbf{j},\mathbf{i}\neq\mathbf{j}\\ \chi_{|\mathbf{i}-\mathbf{j}|}>\theta_{\mathbf{n}}}} |r_{\mathbf{ij}}| \exp\Big(-\frac{u_{\mathbf{n},\mathbf{i}}^2 + u_{\mathbf{n},\mathbf{j}}^2}{2(1+|r_{\mathbf{ij}}|)}\Big) = o(1) \tag{3.3}$$

as $\mathbf{n} \to \infty$. *Under the conditions of Theorem* 2.1, *we have*

$$\sum_{\substack{\mathbf{i},\mathbf{j} \in \mathbf{I_n}, \mathbf{i} \leq \mathbf{j}, \mathbf{i} \neq \mathbf{j} \\ \chi_{|\mathbf{i}-\mathbf{j}|} > \theta_{\mathbf{n}}}} |r_{\mathbf{ij}}| \exp\Big(- \frac{u_{\mathbf{n},\mathbf{i}}^2 + u_{\mathbf{n},\mathbf{j}}^2}{2(1 + |r_{\mathbf{ij}}|)}\Big) \ll (\chi_{\log\log \mathbf{n}})^{-(1+\varepsilon)}. \tag{3.4}$$

Proof Denote the sum in (3.3) and (3.4) by $S_{\mathbf{n}}$ and split it into three parts, the first for $\mathbf{i} < \mathbf{j}$, the second for $i_1 = j_1 \wedge i_2 < j_2$, and the third for $i_2 = j_2 \wedge i_1 < j_1$. We will denote them by $S_{\mathbf{n},i}$, $i = 1, 2, 3$, respectively. By (3.1), for large $\mathbf{n}$, there exists a constant $c > 0$, such that $\omega_{\mathbf{n}} > c\sqrt{\log(\chi_{\mathbf{n}})}$, and hence

$$\theta_{\mathbf{n}} = \exp(\eta\omega_{\mathbf{n}}^2) > \exp(c^2\eta\log(\chi_{\mathbf{n}})) = \chi_{\mathbf{n}}^{\alpha},$$

where $\alpha = c^2\eta$, and

$$\sum_{\substack{\mathbf{i},\mathbf{j} \in \mathbf{I_n}, \mathbf{i} \leq \mathbf{j}, \mathbf{i} \neq \mathbf{j} \\ \chi_{|\mathbf{i}-\mathbf{j}|} > \theta_{\mathbf{n}}}} |r_{\mathbf{ij}}| \leq \sum_{\substack{\mathbf{i},\mathbf{j} \in \mathbf{I_n}, \mathbf{i} \leq \mathbf{j}, \mathbf{i} \neq \mathbf{j} \\ \chi_{|\mathbf{i}-\mathbf{j}|} > \chi_{\mathbf{n}}^{\alpha}}} |r_{\mathbf{ij}}| \leq \sup_{\substack{\mathbf{i},\mathbf{j} \in \mathbf{I_n}, \mathbf{i} < \mathbf{j} \\ \chi_{|\mathbf{i}-\mathbf{j}|} > \chi_{\mathbf{n}}^{\alpha}}} \rho_{|\mathbf{i}-\mathbf{j}|} := \delta_{\theta_{\mathbf{n}}}^{(1)}.$$

For the first term $S_{\mathbf{n},1}$, applying the facts in (3.1), we get

$$\begin{aligned}
S_{\mathbf{n},1} &\leq \delta_{\theta_{\mathbf{n}}}^{(1)} \sum_{\substack{\mathbf{i},\mathbf{j} \in \mathbf{I_n}, \mathbf{i} < \mathbf{j} \\ \chi_{|\mathbf{i}-\mathbf{j}|} > \theta_{\mathbf{n}}}} \exp\Big(- \frac{\omega_{\mathbf{n}}^2}{1 + \delta_{\theta_{\mathbf{n}}}^{(1)}}\Big) \\
&\leq \delta_{\theta_{\mathbf{n}}}^{(1)} \chi_{\mathbf{n}}^2 \exp\Big(- \frac{\omega_{\mathbf{n}}^2}{1 + \delta_{\theta_{\mathbf{n}}}^{(1)}}\Big) \\
&\leq \delta_{\theta_{\mathbf{n}}}^{(1)} \chi_{\mathbf{n}}^2 \exp(-\omega_{\mathbf{n}}^2) \exp(\delta_{\theta_{\mathbf{n}}}^{(1)}\omega_{\mathbf{n}}^2) \\
&\ll (\chi_{\mathbf{n}}(1 - \Phi(\omega_{\mathbf{n}}))\omega_{\mathbf{n}})^2 \delta_{\theta_{\mathbf{n}}}^{(1)} \exp(\delta_{\theta_{\mathbf{n}}}^{(1)}\omega_{\mathbf{n}}^2) \\
&\ll \log(\chi_{\mathbf{n}}) \delta_{\theta_{\mathbf{n}}}^{(1)} \exp(\delta_{\theta_{\mathbf{n}}}^{(1)}\omega_{\mathbf{n}}^2) \\
&\leq \sup_{\substack{\mathbf{i},\mathbf{j} \in \mathbf{I_n}, \mathbf{i} < \mathbf{j} \\ \chi_{|\mathbf{i}-\mathbf{j}|} > \chi_{\mathbf{n}}^{\alpha}}} \rho_{|\mathbf{i}-\mathbf{j}|} \log(\chi_{|\mathbf{i}-\mathbf{j}|}) \exp(\delta_{\theta_{\mathbf{n}}}^{(1)}\omega_{\mathbf{n}}^2).
\end{aligned}$$

Now, using the conditions (1.3) and (2.2), we obtain the desired bounds on the right-hand sides of (3.3) and (3.4), respectively. For the second term, note that

$$\sup_{\substack{\mathbf{i},\mathbf{j} \in \mathbf{I_n}, i_1 = j_1 \\ |i_2 - j_2| > \theta_{\mathbf{n}}}} |r_{\mathbf{ij}}| \leq \sup_{\substack{\mathbf{i},\mathbf{j} \in \mathbf{I_n}, i_1 = j_1 \\ |i_2 - j_2| > \chi_{\mathbf{n}}^{\alpha}}} |r_{\mathbf{ij}}| \leq \sup_{\substack{\mathbf{i},\mathbf{j} \in \mathbf{I_n} \\ |i_2 - j_2| > \chi_{\mathbf{n}}^{\alpha}}} \rho_{(0,|i_2 - j_2|)} := \delta_{\theta_{\mathbf{n}}}^{(2)}.$$

Similarly, applying the facts in (3.1) again, we have

$$\begin{aligned}
S_{\mathbf{n},2} &\leq \delta_{\theta_{\mathbf{n}}}^{(2)} \sum_{\substack{\mathbf{i},\mathbf{j} \in \mathbf{I_n}, i_1 = j_1 \\ |i_2 - j_2| > \theta_{\mathbf{n}}}} \exp\Big(- \frac{\omega_{\mathbf{n}}^2}{1 + \delta_{\theta_{\mathbf{n}}}^{(2)}}\Big) \\
&\leq \delta_{\theta_{\mathbf{n}}}^{(2)} \chi_{\mathbf{n}} n_2 \exp\Big(- \frac{\omega_{\mathbf{n}}^2}{1 + \delta_{\theta_{\mathbf{n}}}^{(2)}}\Big) \\
&\leq n_1^{-1} \delta_{\theta_{\mathbf{n}}}^{(2)} \chi_{\mathbf{n}}^2 \exp(-\omega_{\mathbf{n}}^2) \exp(\delta_{\theta_{\mathbf{n}}}^{(2)}\omega_{\mathbf{n}}^2) \\
&\ll n_1^{-1} \log(\chi_{\mathbf{n}}) \delta_{\theta_{\mathbf{n}}}^{(2)} \exp(\delta_{\theta_{\mathbf{n}}}^{(2)}\omega_{\mathbf{n}}^2) \\
&\leq n_1^{-1} \sup_{\substack{\mathbf{i},\mathbf{j} \in \mathbf{I_n} \\ |i_2 - j_2| > \chi_{\mathbf{n}}^{\alpha}}} \rho_{(0,|i_2 - j_2|)} \log(|i_2 - j_2|) \exp(\delta_{\theta_{\mathbf{n}}}^{(2)}\omega_{\mathbf{n}}^2).
\end{aligned}$$

Now, using the condition (2.6) we get $S_{\mathbf{n},2} = o(1)$ as $\mathbf{n} \to \infty$. Using the condition (2.1), we get

$$S_{\mathbf{n},2} \ll n_1^{-1} (\log\log n_2)^{-(1+\varepsilon)} \ll (\chi_{\log\log \mathbf{n}})^{-(1+\varepsilon)}.$$

Likewise we can bound the third term.

Lemma 3.5 *Under the conditions of Theorem* 2.1, *for* $\mathbf{k}, \mathbf{n} \in \mathbb{N}^2$ *such that* $\mathbf{k} \neq \mathbf{n}$ *and* $u_{\mathbf{k},\mathbf{i}} \le u_{\mathbf{n},\mathbf{j}}$, *we have*

$$\sum_{\substack{\mathbf{i}\in \mathbf{I_k}, \mathbf{j} \in \mathbf{I_n} - \mathbf{I_k} \\ \mathbf{i} \le \mathbf{j}}} |r_{\mathbf{ij}}| \exp\Big(- \frac{u_{\mathbf{k},\mathbf{i}}^2 + u_{\mathbf{n},\mathbf{j}}^2}{2(1+|r_{\mathbf{ij}}|)}\Big) \ll (\chi_{\log\log \mathbf{n}})^{-(1+\varepsilon)}. \tag{3.5}$$

Proof Split the sum into two parts:

$$\sum_{\substack{\mathbf{i}\in \mathbf{I_k}, \mathbf{j} \in \mathbf{I_n} - \mathbf{I_k}, \mathbf{i} \le \mathbf{j} \\ \chi_{|\mathbf{i}-\mathbf{j}|} \le \theta_{\mathbf{n}}}} + \sum_{\substack{\mathbf{i}\in \mathbf{I_k}, \mathbf{j} \in \mathbf{I_n} - \mathbf{I_k}, \mathbf{i} \le \mathbf{j} \\ \chi_{|\mathbf{i}-\mathbf{j}|} > \theta_{\mathbf{n}}}} =: T_{\mathbf{n}}^{(1)} + T_{\mathbf{n}}^{(2)}.$$

For the first term, it follows from the facts in (3.1) that

$$\begin{aligned}
T_{\mathbf{n}}^{(1)} &\le \delta \sum_{\substack{\mathbf{i}\in \mathbf{I_k}, \mathbf{j} \in \mathbf{I_n} - \mathbf{I_k}, \mathbf{i} \le \mathbf{j} \\ \chi_{|\mathbf{i}-\mathbf{j}|} \le \theta_{\mathbf{n}}}} \exp\Big(- \frac{\omega_{\mathbf{k}}^2}{2(1+\delta)}\Big) \exp\Big(- \frac{\omega_{\mathbf{n}}^2}{2(1+\delta)}\Big) \\
&\le \delta \chi_{\mathbf{k}} \theta_{\mathbf{n}}^2 \exp\Big(- \frac{\omega_{\mathbf{k}}^2}{2(1+\delta)}\Big) \exp\Big(- \frac{\omega_{\mathbf{n}}^2}{2(1+\delta)}\Big) \\
&\le \delta \chi_{\mathbf{k}} \exp\Big(- \frac{\omega_{\mathbf{k}}^2}{2(1+\delta)}\Big) \exp\Big(- \Big(\frac{1}{2(1+\delta)} - 2\eta\Big)\omega_{\mathbf{n}}^2\Big) \\
&\ll \delta \chi_{\mathbf{k}} \Big(\frac{\omega_{\mathbf{k}}}{\chi_{\mathbf{k}}}\Big)^{\frac{1}{1+\delta}} \Big(\frac{\omega_{\mathbf{n}}}{\chi_{\mathbf{n}}}\Big)^{\frac{1}{1+\delta} - 4\eta} \\
&\ll \delta \chi_{\mathbf{k}}^{1-\frac{1}{1+\delta}} (\log \chi_{\mathbf{k}})^{\frac{1}{2(1+\delta)}} \chi_{\mathbf{n}}^{4\eta - \frac{1}{1+\delta}} (\log \chi_{\mathbf{n}})^{\frac{1}{2(1+\delta)} - 2\eta} \\
&\le \chi_{\mathbf{n}}^{4\eta - \frac{1}{1+\delta} + 1 - \frac{1}{1+\delta}} (\log \chi_{\mathbf{k}})^{\frac{1}{2(1+\delta)}} (\log \chi_{\mathbf{n}})^{\frac{1}{2(1+\delta)} - 2\eta}.
\end{aligned}$$

Since $\eta < \frac{1-\delta}{4(1+\delta)}$ and $0 < \delta < 1$, we have $4\eta - \frac{1}{1+\delta} + 1 - \frac{1}{1+\delta} < 0$. Hence, there exists a constant $\sigma_2 > 0$, such that $T_{\mathbf{n}}^{(1)} \ll \chi_{\mathbf{n}}^{-\sigma_2}$.

As in the proof of Lemma 3.4, split $T_{\mathbf{n}}^{(2)}$ into three parts, the first for $\mathbf{i} < \mathbf{j}$, the second for $i_1 = j_1 \wedge i_2 < j_2$, and the third for $i_2 = j_2 \wedge i_1 < j_1$ and denote them by $T_{\mathbf{n},i}^{(2)}$, $i = 1, 2, 3$, respectively.

For the first term $T_{\mathbf{n},1}^{(2)}$, in view of the facts (3.1), we have

$$\begin{aligned}
T_{\mathbf{n},1}^{(2)} &\le \delta_{\theta_{\mathbf{n}}}^{(1)} \sum_{\substack{\mathbf{i}\in \mathbf{I_n}, \mathbf{j} \in \mathbf{I_n} - \mathbf{I_k}, \mathbf{i} < \mathbf{j} \\ \chi_{|\mathbf{i}-\mathbf{j}|} > \theta_{\mathbf{n}}}} \exp\Big(- \frac{\omega_{\mathbf{k}}^2}{2(1+\delta_{\theta_{\mathbf{n}}}^{(1)})}\Big) \exp\Big(- \frac{\omega_{\mathbf{n}}^2}{2(1+\delta_{\theta_{\mathbf{n}}}^{(1)})}\Big) \\
&\le \delta_{\theta_{\mathbf{n}}}^{(1)} \chi_{\mathbf{k}} \chi_{\mathbf{n}} \exp\Big(- \frac{\omega_{\mathbf{k}}^2}{2(1+\delta_{\theta_{\mathbf{n}}}^{(1)})}\Big) \exp\Big(- \frac{\omega_{\mathbf{n}}^2}{2(1+\delta_{\theta_{\mathbf{n}}}^{(1)})}\Big) \\
&\le \delta_{\theta_{\mathbf{n}}}^{(1)} \chi_{\mathbf{k}} \exp\Big(- \frac{1}{2}\omega_{\mathbf{k}}^2\Big) \exp\Big(\frac{1}{2}\delta_{\theta_{\mathbf{n}}}^{(1)}\omega_{\mathbf{k}}^2\Big) \chi_{\mathbf{n}} \exp\Big(- \frac{1}{2}\omega_{\mathbf{n}}^2\Big) \exp\Big(\frac{1}{2}\delta_{\theta_{\mathbf{n}}}^{(1)}\omega_{\mathbf{n}}^2\Big) \\
&\ll \omega_{\mathbf{k}} \omega_{\mathbf{n}} \delta_{\theta_{\mathbf{n}}}^{(1)} \exp\Big(\frac{1}{2}\delta_{\theta_{\mathbf{n}}}^{(1)}\omega_{\mathbf{k}}^2\Big) \exp\Big(\frac{1}{2}\delta_{\theta_{\mathbf{n}}}^{(1)}\omega_{\mathbf{n}}^2\Big)
\end{aligned}$$

$$
\begin{aligned}
&\le \log(\chi_{\mathbf{n}}) \sup_{\substack{\mathbf{i},\mathbf{j}\in \mathbf{I_n}, \mathbf{i}<\mathbf{j} \\ \chi_{|\mathbf{i}-\mathbf{j}|}>\chi_{\mathbf{n}}^{\alpha}}} \rho_{|\mathbf{i}-\mathbf{j}|} \exp\Big(\frac{1}{2}\delta_{\theta_{\mathbf{n}}}^{(1)}\omega_{\mathbf{k}}^2\Big) \exp\Big(\frac{1}{2}\delta_{\theta_{\mathbf{n}}}^{(1)}\omega_{\mathbf{n}}^2\Big) \\
&\ll \sup_{\substack{\mathbf{i},\mathbf{j}\in \mathbf{I_n}, \mathbf{i}<\mathbf{j} \\ \chi_{|\mathbf{i}-\mathbf{j}|}>\chi_{\mathbf{n}}^{\alpha}}} \rho_{|\mathbf{i}-\mathbf{j}|} \log(\chi_{|\mathbf{i}-\mathbf{j}|}) \\
&\ll (\chi_{\log\log \mathbf{n}})^{-(1+\varepsilon)},
\end{aligned}
$$

where we have used the condition (2.2) in the last step.

Similarly, taking into account the facts in (3.1), we get

$$
\begin{aligned}
T_{\mathbf{n},2}^{(2)} &\le \delta_{\theta_{\mathbf{n}}}^{(2)} \sum_{\substack{\mathbf{i}\in \mathbf{I_k}, \mathbf{j}\in \mathbf{I_n}-\mathbf{I_k}, i_1=j_1 \\ |i_2-j_2|>\theta_{\mathbf{n}}}} \exp\Big(-\frac{\omega_{\mathbf{k}}^2}{2(1+\delta_{\theta_{\mathbf{n}}}^{(2)})}\Big) \exp\Big(-\frac{\omega_{\mathbf{n}}^2}{2(1+\delta_{\theta_{\mathbf{n}}}^{(2)})}\Big) \\
&\le \delta_{\theta_{\mathbf{n}}}^{(2)} \chi_{\mathbf{k}} n_2 \exp\Big(-\frac{\omega_{\mathbf{k}}^2}{2(1+\delta_{\theta_{\mathbf{n}}}^{(2)})}\Big) \exp\Big(-\frac{\omega_{\mathbf{n}}^2}{2(1+\delta_{\theta_{\mathbf{n}}}^{(2)})}\Big) \\
&\le n_1^{-1} \delta_{\theta_{\mathbf{n}}}^{(2)} \chi_{\mathbf{k}} \chi_{\mathbf{n}} \exp\Big(-\frac{1}{2}\omega_{\mathbf{k}}^2\Big) \exp\Big(\frac{1}{2}\delta_{\theta_{\mathbf{n}}}^{(2)}\omega_{\mathbf{k}}^2\Big) \exp\Big(-\frac{1}{2}\omega_{\mathbf{n}}^2\Big) \exp\Big(\frac{1}{2}\delta_{\theta_{\mathbf{n}}}^{(2)}\omega_{\mathbf{n}}^2\Big) \\
&\le n_1^{-1} \Big(\chi_{\mathbf{k}} \exp\Big(-\frac{1}{2}\omega_{\mathbf{k}}^2\Big)\Big)\Big(\chi_{\mathbf{n}} \exp\Big(-\frac{1}{2}\omega_{\mathbf{n}}^2\Big)\Big) \delta_{\theta_{\mathbf{n}}}^{(2)} \exp\Big(\frac{1}{2}\delta_{\theta_{\mathbf{n}}}^{(2)}\omega_{\mathbf{k}}^2\Big) \exp\Big(\frac{1}{2}\delta_{\theta_{\mathbf{n}}}^{(2)}\omega_{\mathbf{n}}^2\Big) \\
&\ll n_1^{-1} \omega_{\mathbf{k}}\omega_{\mathbf{n}} \delta_{\theta_{\mathbf{n}}}^{(2)} \exp\Big(\frac{1}{2}\delta_{\theta_{\mathbf{n}}}^{(2)}\omega_{\mathbf{k}}^2\Big) \exp\Big(\frac{1}{2}\delta_{\theta_{\mathbf{n}}}^{(2)}\omega_{\mathbf{n}}^2\Big) \\
&\ll n_1^{-1} \log(\chi_{\mathbf{n}}) \sup_{\substack{\mathbf{i},\mathbf{j}\in \mathbf{I_n} \\ |i_2-j_2|>\chi_{\mathbf{n}}^{\alpha}}} \rho_{(0,|i_2-j_2|)} \exp\Big(\frac{1}{2}\delta_{\theta_{\mathbf{n}}}^{(2)}\omega_{\mathbf{k}}^2\Big) \exp\Big(\frac{1}{2}\delta_{\theta_{\mathbf{n}}}^{(2)}\omega_{\mathbf{n}}^2\Big) \\
&\le n_1^{-1} \sup_{\substack{\mathbf{i},\mathbf{j}\in \mathbf{I_n} \\ |i_2-j_2|>\chi_{\mathbf{n}}^{\alpha}}} \rho_{(0,|i_2-j_2|)} \log(|i_2-j_2|) \\
&\ll (\chi_{\log\log \mathbf{n}})^{-(1+\varepsilon)},
\end{aligned}
$$

where we have used the condition (2.1) in the last step. Likewise, we can bound the third term.

Lemma 3.6 *Under the conditions of Theorem* 2.1, *for* $\mathbf{k},\mathbf{n} \in \mathbb{N}^2$ *such that* $\mathbf{k} \ne \mathbf{n}$ *and* $u_{\mathbf{k},\mathbf{i}} \le u_{\mathbf{n},\mathbf{j}}$, *we have*

$$
\begin{aligned}
\text{(i)}\quad &\Big|P\Big(\bigcap_{\mathbf{i}\in \mathbf{I_k}}\{X_{\mathbf{i}} \le u_{\mathbf{k},\mathbf{i}}\}, \bigcap_{\mathbf{j}\in \mathbf{I_n}-\mathbf{I_k}}\{X_{\mathbf{j}} \le u_{\mathbf{n},\mathbf{j}}\}\Big) \\
&\quad - P\Big(\bigcap_{\mathbf{i}\in \mathbf{I_k}}\{X_{\mathbf{i}} \le u_{\mathbf{k},\mathbf{i}}\}\Big) P\Big(\bigcap_{\mathbf{j}\in \mathbf{I_n}-\mathbf{I_k}}\{X_{\mathbf{j}} \le u_{\mathbf{n},\mathbf{j}}\}\Big)\Big| \ll (\chi_{\log\log \mathbf{n}})^{-(1+\varepsilon)}, \\
\text{(ii)}\quad &E\Big|I\Big(\bigcap_{\mathbf{i}\in \mathbf{I_n}}\{X_{\mathbf{i}} \le u_{\mathbf{n},\mathbf{i}}\}\Big) - I\Big(\bigcap_{\mathbf{i}\in \mathbf{I_n}-\mathbf{I_k}}\{X_{\mathbf{i}} \le u_{\mathbf{n},\mathbf{i}}\}\Big)\Big| \\
&\ll \frac{\chi_{\mathbf{n}} - \chi_{\mathbf{I_n}-\mathbf{I_k}}}{\chi_{\mathbf{n}}} + (\chi_{\log\log \mathbf{n}})^{-(1+\varepsilon)}.
\end{aligned}
$$

Proof For part (i), using Lemmas 3.1 and 3.5, we have

$$
\begin{aligned}
&\Big|P\Big(\bigcap_{\mathbf{i}\in \mathbf{I_k}}\{X_{\mathbf{i}} \le u_{\mathbf{k},\mathbf{i}}\}, \bigcap_{\mathbf{j}\in \mathbf{I_n}-\mathbf{I_k}}\{X_{\mathbf{j}} \le u_{\mathbf{n},\mathbf{j}}\}\Big) - P\Big(\bigcap_{\mathbf{i}\in \mathbf{I_k}}\{X_{\mathbf{i}} \le u_{\mathbf{k},\mathbf{i}}\}\Big) P\Big(\bigcap_{\mathbf{j}\in \mathbf{I_n}-\mathbf{I_k}}\{X_{\mathbf{j}} \le u_{\mathbf{n},\mathbf{j}}\}\Big)\Big| \\
&\le K \sum_{\substack{\mathbf{i}\in \mathbf{I_k}, \mathbf{j}\in \mathbf{I_n}-\mathbf{I_k} \\ \mathbf{i}\le \mathbf{j}}} |r_{\mathbf{ij}}| \exp\Big(-\frac{u_{\mathbf{k},\mathbf{i}}^2 + u_{\mathbf{n},\mathbf{j}}^2}{2(1+|r_{\mathbf{ij}}|)}\Big) \ll (\chi_{\log\log \mathbf{n}})^{-(1+\varepsilon)}.
\end{aligned}
$$

For part (ii), we have

$$\begin{aligned}
&E\Big|I\Big(\bigcap_{\mathbf{i}\in\mathbf{I_n}}\{X_{\mathbf{i}}\le u_{\mathbf{n},\mathbf{i}}\}\Big)-I\Big(\bigcap_{\mathbf{i}\in\mathbf{I_n}-\mathbf{I_k}}\{X_{\mathbf{i}}\le u_{\mathbf{n},\mathbf{i}}\}\Big)\Big|\\
&=P\Big(\bigcap_{\mathbf{i}\in\mathbf{I_n}-\mathbf{I_k}}\{X_{\mathbf{i}}\le u_{\mathbf{n},\mathbf{i}}\}\Big)-P\Big(\bigcap_{\mathbf{i}\in\mathbf{I_n}}\{X_{\mathbf{i}}\le u_{\mathbf{n},\mathbf{i}}\}\Big)\\
&\le\Big|P\Big(\bigcap_{\mathbf{i}\in\mathbf{I_n}-\mathbf{I_k}}\{X_{\mathbf{i}}\le u_{\mathbf{n},\mathbf{i}}\}\Big)-\prod_{\mathbf{i}\in\mathbf{I_n}-\mathbf{I_k}}\Phi(u_{\mathbf{n},\mathbf{i}})\Big|+\Big|P\Big(\bigcap_{\mathbf{i}\in\mathbf{I_n}}\{X_{\mathbf{i}}\le u_{\mathbf{n},\mathbf{i}}\}\Big)-\prod_{\mathbf{i}\in\mathbf{I_n}}\Phi(u_{\mathbf{n},\mathbf{i}})\Big|\\
&\quad+\Big|\prod_{\mathbf{i}\in\mathbf{I_n}-\mathbf{I_k}}\Phi(u_{\mathbf{n},\mathbf{i}})-\prod_{\mathbf{i}\in\mathbf{I_n}}\Phi(u_{\mathbf{n},\mathbf{i}})\Big|\\
&=:R_{\mathbf{n},1}+R_{\mathbf{n},2}+R_{\mathbf{n},3}.
\end{aligned}$$

Using Lemmas 3.1 and 3.3–3.4, we know that $R_{\mathbf{n},1}$ and $R_{\mathbf{n},2}$ are both bounded by

$$K\sum_{\mathbf{i},\mathbf{j}\in\mathbf{I_n},\mathbf{i}\le\mathbf{j},\mathbf{i}\ne\mathbf{j}}|r_{\mathbf{ij}}|\exp\Big(-\frac{u_{\mathbf{n},\mathbf{i}}^2+u_{\mathbf{n},\mathbf{j}}^2}{2(1+|r_{\mathbf{ij}}|)}\Big)\ll(\chi_{\log\log\mathbf{n}})^{-(1+\varepsilon)}.$$

Using the fact that $\chi_{\mathbf{n}}(1-\Phi(\lambda_{\mathbf{n}}))$ is bounded, we have

$$\begin{aligned}
R_{\mathbf{n},3}&=\prod_{\mathbf{i}\in\mathbf{I_n}-\mathbf{I_k}}\Phi(u_{\mathbf{n},\mathbf{i}})-\prod_{\mathbf{i}\in\mathbf{I_n}}\Phi(u_{\mathbf{n},\mathbf{i}})\\
&\le1-\prod_{\mathbf{i}\in\mathbf{I_n}\setminus(\mathbf{I_n}-\mathbf{I_k})}\Phi(u_{\mathbf{n},\mathbf{i}})\\
&\le\sum_{\mathbf{i}\in\mathbf{I_n}\setminus(\mathbf{I_n}-\mathbf{I_k})}(1-\Phi(u_{\mathbf{n},\mathbf{i}}))\\
&\le(\chi_{\mathbf{n}}-\chi_{\mathbf{I_n}-\mathbf{I_k}})(1-\Phi(\lambda_{\mathbf{n}}))\\
&=\frac{\chi_{\mathbf{n}}-\chi_{\mathbf{I_n}-\mathbf{I_k}}}{\chi_{\mathbf{n}}}\chi_{\mathbf{n}}(1-\Phi(\lambda_{\mathbf{n}}))\\
&\ll\frac{\chi_{\mathbf{n}}-\chi_{\mathbf{I_n}-\mathbf{I_k}}}{\chi_{\mathbf{n}}}.
\end{aligned}$$

This completes the proof of the lemma.

4 Proof of Main Results

In this section, we give the proofs of our main results.

Proof of Theorem 2.1 First, note that conditions (2.1) and (2.2) imply (2.6) and (1.3), respectively, and hence (1.4) holds under the conditions of Theorem 2.1. Then we have

$$\lim_{\mathbf{n}\to\infty}\frac{1}{\chi_{\log\mathbf{n}}}\sum_{\mathbf{k}\in\mathbf{I_n}}\frac{1}{\chi_{\mathbf{k}}}P\Big(\bigcap_{\mathbf{i}\in\mathbf{I_k}}\{X_{\mathbf{i}}\le u_{\mathbf{k},\mathbf{i}}\}\Big)=\exp(-\tau)\quad\text{a.s.}$$

Therefore, it suffices to prove that

$$\lim_{\mathbf{n}\to\infty}\frac{1}{\chi_{\log\mathbf{n}}}\sum_{\mathbf{k}\in\mathbf{I_n}}\frac{1}{\chi_{\mathbf{k}}}\Big\{I\Big(\bigcap_{\mathbf{i}\in\mathbf{I_k}}\{X_{\mathbf{i}}\le u_{\mathbf{k},\mathbf{i}}\}\Big)-P\Big(\bigcap_{\mathbf{i}\in\mathbf{I_k}}\{X_{\mathbf{i}}\le u_{\mathbf{k},\mathbf{i}}\}\Big)\Big\}=0\quad\text{a.s.}$$

Let

$$\xi_{\mathbf{k}}=I\Big(\bigcap_{\mathbf{i}\in\mathbf{I_k}}\{X_{\mathbf{i}}\le u_{\mathbf{k},\mathbf{i}}\}\Big)-P\Big(\bigcap_{\mathbf{i}\in\mathbf{I_k}}\{X_{\mathbf{i}}\le u_{\mathbf{k},\mathbf{i}}\}\Big).$$

Note that $|\xi_{\mathbf{k}}| \leq 1$ for all $\mathbf{k} \in \mathbb{Z}^2$. By Lemma 3.2, we only need to show that

$$\mathrm{Var}\Big(\frac{1}{\chi_{\log \mathbf{n}}} \sum_{\mathbf{k}\in\mathbf{I_n}} \frac{1}{\chi_{\mathbf{k}}} I\Big(\bigcap_{\mathbf{i}\in\mathbf{I_k}} \{X_{\mathbf{i}} \leq u_{\mathbf{k},\mathbf{i}}\}\Big)\Big) \ll (\chi_{\log\log \mathbf{n}})^{-(1+\varepsilon)}. \tag{4.1}$$

Now, we have

$$\begin{aligned}\mathrm{Var}\Big(\frac{1}{\chi_{\log \mathbf{n}}} \sum_{\mathbf{k}\in\mathbf{I_n}} \frac{1}{\chi_{\mathbf{k}}} I\Big(\bigcap_{\mathbf{i}\in\mathbf{I_k}} \{X_{\mathbf{i}} \leq u_{\mathbf{k},\mathbf{i}}\}\Big)\Big) &= \frac{1}{(\chi_{\log \mathbf{n}})^2}\Big\{\sum_{\mathbf{k}\in\mathbf{I_n}} \frac{1}{\chi_{\mathbf{k}}^2} E\xi_{\mathbf{k}}^2 + \sum_{\mathbf{k}\neq\mathbf{l}} \frac{1}{\chi_{\mathbf{k}}\chi_{\mathbf{l}}} E\xi_{\mathbf{k}}\xi_{\mathbf{l}}\Big\}\\ &= \mathrm{A}_1 + \mathrm{A}_2.\end{aligned}$$

Since $|\xi_{\mathbf{k}}| \leq 1$, it follows that

$$\mathrm{A}_1 \leq \frac{1}{(\chi_{\log \mathbf{n}})^2} \sum_{\mathbf{k}\in\mathbf{I_n}} \frac{1}{\chi_{\mathbf{k}}^2} \leq \frac{K}{(\chi_{\log \mathbf{n}})^2}. \tag{4.2}$$

Note that for $\mathbf{k} \neq \mathbf{l}$ such that $u_{\mathbf{k},\mathbf{i}} \leq u_{\mathbf{l},\mathbf{j}}$,

$$\begin{aligned}|E\xi_{\mathbf{k}}\xi_{\mathbf{l}}| &= \Big|\mathrm{Cov}\Big(I\Big(\bigcap_{\mathbf{i}\in\mathbf{I_k}} \{X_{\mathbf{i}} \leq u_{\mathbf{k},\mathbf{i}}\}\Big), I\Big(\bigcap_{\mathbf{j}\in\mathbf{I_l}} \{X_{\mathbf{i}} \leq u_{\mathbf{l},\mathbf{j}}\}\Big)\Big)\Big|\\ &\leq \Big|\mathrm{Cov}\Big(I\Big(\bigcap_{\mathbf{i}\in\mathbf{I_k}} \{X_{\mathbf{i}} \leq u_{\mathbf{k},\mathbf{i}}\}\Big), I\Big(\bigcap_{\mathbf{j}\in\mathbf{I_l}} \{X_{\mathbf{i}} \leq u_{\mathbf{l},\mathbf{j}}\}\Big) - I\Big(\bigcap_{\mathbf{j}\in\mathbf{I_l}-\mathbf{I_k}} \{X_{\mathbf{i}} \leq u_{\mathbf{l},\mathbf{j}}\}\Big)\Big)\Big|\\ &\quad + \Big|\mathrm{Cov}\Big(I\Big(\bigcap_{\mathbf{i}\in\mathbf{I_k}} \{X_{\mathbf{i}} \leq u_{\mathbf{k},\mathbf{i}}\}\Big), I\Big(\bigcap_{\mathbf{j}\in\mathbf{I_l}-\mathbf{I_k}} \{X_{\mathbf{i}} \leq u_{\mathbf{l},\mathbf{j}}\}\Big)\Big)\Big|\\ &\leq E\Big|I\Big(\bigcap_{\mathbf{j}\in\mathbf{I_l}} \{X_{\mathbf{i}} \leq u_{\mathbf{l},\mathbf{j}}\}\Big) - I\Big(\bigcap_{\mathbf{j}\in\mathbf{I_l}-\mathbf{I_k}} \{X_{\mathbf{i}} \leq u_{\mathbf{l},\mathbf{j}}\}\Big)\Big|\\ &\quad + \Big|\mathrm{Cov}\Big(I\Big(\bigcap_{\mathbf{i}\in\mathbf{I_k}} \{X_{\mathbf{i}} \leq u_{\mathbf{k},\mathbf{i}}\}\Big), I\Big(\bigcap_{\mathbf{j}\in\mathbf{I_l}-\mathbf{I_k}} \{X_{\mathbf{i}} \leq u_{\mathbf{l},\mathbf{j}}\}\Big)\Big)\Big|\\ &\ll \frac{\chi_{\mathbf{l}} - \chi_{\mathbf{I_l}-\mathbf{I_k}}}{\chi_{\mathbf{l}}} + \frac{1}{(\chi_{\log\log \mathbf{l}})^{(1+\varepsilon)}},\end{aligned}$$

where we have used Lemma 3.6 in the last step. Now, we have

$$\begin{aligned}\mathrm{A}_2 &\leq 2\frac{1}{(\chi_{\log \mathbf{n}})^2} \sum_{\mathbf{k}\neq\mathbf{l}, u_{\mathbf{k},\mathbf{i}}\leq u_{\mathbf{l},\mathbf{j}}} \frac{1}{\chi_{\mathbf{k}}\chi_{\mathbf{l}}} E\xi_{\mathbf{k}}\xi_{\mathbf{l}}\\ &\ll \frac{1}{(\chi_{\log \mathbf{n}})^2} \sum_{\mathbf{k}\neq\mathbf{l}} \frac{1}{\chi_{\mathbf{k}}\chi_{\mathbf{l}}}\Big\{\frac{\chi_{\mathbf{l}} - \chi_{\mathbf{I_l}-\mathbf{I_k}}}{\chi_{\mathbf{l}}} + \frac{1}{(\chi_{\log\log \mathbf{l}})^{(1+\varepsilon)}}\Big\}\\ &= \mathrm{A}_{21} + \mathrm{A}_{22}.\end{aligned}$$

In order to estimate A_{21}, we define $\mathbb{A}_{\mathbf{m}} = \{(\mathbf{k},\mathbf{l}) \in \mathbf{I_n} \times \mathbf{I_n} \colon (2m_j - 1)(k_j - l_j) \geq 0,\ \mathbf{k} \neq \mathbf{l}\}$ for $\mathbf{m} \in \Lambda \equiv \{(m_1, m_2) \colon m_j = 0, 1,\ j = 1, 2,\ \mathbf{m} \neq \mathbf{1}\}$. Let $\mathbf{a}^{\mathbf{m}}$ denote $(a_1^{m_1}, a_2^{m_2})$ for $\mathbf{a} \in \mathbb{R}^2$ and $\mathbf{m} \in \Lambda$. Then, we have

$$\mathrm{A}_{21} = \frac{1}{(\chi_{\log \mathbf{n}})^2} \sum_{\mathbf{m}\in\Lambda} \sum_{(\mathbf{k},\mathbf{l})\in\mathbb{A}_{\mathbf{m}}} \frac{1}{\chi_{\mathbf{k}}\chi_{\mathbf{l}}} \frac{\chi_{\mathbf{l}} - \chi_{\mathbf{I_l}-\mathbf{I_k}}}{\chi_{\mathbf{l}}}.$$

Since $\frac{\chi_{\mathbf{l}} - \chi_{\mathbf{I_l}-\mathbf{I_k}}}{\chi_{\mathbf{l}}}$ becomes $\frac{\chi_{\mathbf{k}^{\mathbf{1}-\mathbf{m}}}}{\chi_{\mathbf{l}^{\mathbf{1}-\mathbf{m}}}}$ for $(\mathbf{k},\mathbf{l}) \in \mathbb{A}_{\mathbf{m}}$, it follows that

$$\mathrm{A}_{21} = \frac{1}{(\chi_{\log \mathbf{n}})^2} \sum_{\mathbf{m}\in\Lambda} \sum_{(\mathbf{k},\mathbf{l})\in\mathbb{A}_{\mathbf{m}}} \frac{\chi_{\mathbf{k}^{\mathbf{1}-\mathbf{m}}}}{\chi_{\mathbf{k}}\chi_{\mathbf{l}}\chi_{\mathbf{l}^{\mathbf{1}-\mathbf{m}}}}$$

$$\ll \frac{1}{(\chi_{\log \mathbf{n}})^2} \sum_{\mathbf{m}\in\Lambda} \chi_{\log \mathbf{n}} \chi_{(\log \mathbf{n})^{\mathbf{m}}}$$
$$\ll (\chi_{\log \mathbf{n}})^{-\nu}$$

for some $\nu > 0$. For A_{22}, we have

$$\begin{aligned} \mathrm{A}_{22} &= \frac{1}{(\chi_{\log \mathbf{n}})^2} \sum_{\mathbf{k}\neq\mathbf{l}} \frac{1}{\chi_{\mathbf{k}}\chi_{\mathbf{l}}} \frac{1}{(\chi_{\log\log \mathbf{l}})^{(1+\varepsilon)}} \\ &\ll \frac{1}{(\chi_{\log \mathbf{n}})^2} \frac{1}{(\chi_{\log\log \mathbf{n}})^{(1+\varepsilon)}} \sum_{\mathbf{k}\neq\mathbf{l}} \frac{1}{\chi_{\mathbf{k}}\chi_{\mathbf{l}}} \\ &\ll \frac{1}{(\chi_{\log\log \mathbf{n}})^{(1+\varepsilon)}}. \end{aligned}$$

Therefore,

$$\mathrm{A}_2 \ll \frac{1}{(\chi_{\log\log \mathbf{n}})^{(1+\varepsilon)}}.$$

This and (4.2) together establish (4.1).

Proof of Theorem 2.2 Let $\mathbf{Y} = \{Y_{\mathbf{n}}\}_{\mathbf{n}\geq\mathbf{1}}$ be an independent standardized Gaussian random field. It is easy to see that

$$\begin{aligned} \Big|P\Big(\bigcap_{\mathbf{i}\in\mathbf{I_n}}\{X_{\mathbf{i}} \leq u_{\mathbf{n},\mathbf{i}}\}\Big) - \exp(-\tau)\Big| &\leq \Big|P\Big(\bigcap_{\mathbf{i}\in\mathbf{I_n}}\{X_{\mathbf{i}} \leq u_{\mathbf{n},\mathbf{i}}\}\Big) - \prod_{\mathbf{i}\in\mathbf{I_n}} P\Big(\{Y_{\mathbf{i}} \leq u_{\mathbf{n},\mathbf{i}}\}\Big)\Big| \\ &\quad + \Big|\prod_{\mathbf{i}\in\mathbf{I_n}} P\Big(\{Y_{\mathbf{i}} \leq u_{\mathbf{n},\mathbf{i}}\}\Big) - \exp(-\tau)\Big|. \end{aligned}$$

By Lemmas 3.1 and 3.3–3.4, we have

$$\begin{aligned} &\Big|P\Big(\bigcap_{\mathbf{i}\in\mathbf{I_n}}\{X_{\mathbf{i}} \leq u_{\mathbf{n},\mathbf{i}}\}\Big) - \prod_{\mathbf{i}\in\mathbf{I_n}} P(\{Y_{\mathbf{i}} \leq u_{\mathbf{n},\mathbf{i}}\})\Big| \\ &\leq K \sum_{\mathbf{i},\mathbf{j}\in\mathbf{I_n},\mathbf{i}\leq\mathbf{j},\mathbf{i}\neq\mathbf{j}} |r_{\mathbf{ij}}| \exp\Big(-\frac{u_{\mathbf{n},\mathbf{i}}^2 + u_{\mathbf{n},\mathbf{j}}^2}{2(1+|r_{\mathbf{ij}}|)}\Big) = o(1). \end{aligned}$$

By Lemma 6.1.1 in [12] and the condition that $\lim_{\mathbf{n}\to\infty} \sum_{\mathbf{i}\in\mathbf{I_n}} (1-\Phi(u_{\mathbf{n},\mathbf{i}})) = \tau \in [0,\infty)$, the second sum is also $o(1)$. The proof is complete.

Proof of Corollary 2.2 Let $u_{\mathbf{n},\mathbf{i}} = u_{\mathbf{n}} + m_{\mathbf{n}}^* - m_{\mathbf{i}}$, where $u_{\mathbf{n}} = \frac{x}{a_{\mathbf{n}}} + b_{\mathbf{n}}$. Then the probability on the left-hand side of (2.10) can be written as

$$P\Big(\bigcap_{\mathbf{i}\in\mathbf{I_n}}\{Y_{\mathbf{i}} \leq u_{\mathbf{n}} + m_{\mathbf{n}}^*\}\Big) = P\Big(\bigcap_{\mathbf{i}\in\mathbf{I_n}}\{X_{\mathbf{i}} \leq u_{\mathbf{n},\mathbf{i}}\}\Big).$$

Since $|m_{\mathbf{n}}^*| < \beta_{\mathbf{n}}$ for sufficiently large $\mathbf{n}$, and $u_{\mathbf{n}} \sim \sqrt{2\log\chi_{\mathbf{n}}}$, it follows that

$$\min_{\mathbf{i}\in\mathbf{I_n}} u_{\mathbf{n},\mathbf{i}} = \sqrt{2\log\chi_{\mathbf{n}}}(1+o(1)).$$

Thus if it is shown that

$$\lim_{\mathbf{n}\to\infty} \sum_{\mathbf{i}\in\mathbf{I_n}} (1-\Phi(u_{\mathbf{n},\mathbf{i}})) = \mathrm{e}^{-x}, \tag{4.3}$$

the result will follow from Theorem 2.2. To see that (4.3) holds, we note that $\lambda_{\mathbf{n}} = \min_{\mathbf{i}\in\mathbf{I_n}} u_{\mathbf{n},\mathbf{i}} \to \infty$ as $\mathbf{n} \to \infty$,

$$\sum_{\mathbf{i}\in\mathbf{I_n}}(1-\Phi(u_{\mathbf{n},\mathbf{i}})) \sim \frac{1}{\sqrt{2\pi}}\sum_{\mathbf{i}\in\mathbf{I_n}}\frac{\exp(-\frac{u_{\mathbf{n},\mathbf{i}}^2}{2})}{u_{\mathbf{n},\mathbf{i}}}$$

$$\sim \frac{1}{\sqrt{2\pi}}\frac{\exp(-\frac{u_{\mathbf{n}}^2}{2})}{u_{\mathbf{n}}}\sum_{\mathbf{i}\in\mathbf{I_n}}\exp\left(u_{\mathbf{n}}(m_{\mathbf{i}}-m_{\mathbf{n}}^*)-\frac{1}{2}(m_{\mathbf{i}}-m_{\mathbf{n}}^*)^2\right) \tag{4.4}$$

and

$$\left|\frac{u_{\mathbf{n},\mathbf{i}}}{u_{\mathbf{n}}}-1\right| = \left|\frac{m_{\mathbf{i}}-m_{\mathbf{n}}^*}{u_{\mathbf{n}}}\right| \le \frac{K\beta_{\mathbf{n}}}{\sqrt{\log\chi_{\mathbf{n}}}} \to 0$$

uniformly in $\mathbf{i} \le \mathbf{n}$. Clearly, we also have

$$\left|u_{\mathbf{n}}(m_{\mathbf{i}}-m_{\mathbf{n}}^*)-\frac{1}{2}(m_{\mathbf{i}}-m_{\mathbf{n}}^*)^2\right| \le 2|u_{\mathbf{n}}-a_{\mathbf{n}}^*|\beta_{\mathbf{n}} \le \frac{K\beta_{\mathbf{n}}}{\sqrt{\log\chi_{\mathbf{n}}}}.$$

Therefore, according to (2.7), (2.9) and (4.4), we have

$$\sum_{\mathbf{i}\in\mathbf{I_n}}(1-\Phi(u_{\mathbf{n},\mathbf{i}})) \sim \chi_{\mathbf{n}}(1-\Phi(u_{\mathbf{n}}))\frac{1}{\chi_{\mathbf{n}}}\sum_{\mathbf{i}\in\mathbf{I_n}}\exp\left(a_{\mathbf{n}}^*(m_{\mathbf{i}}-m_{\mathbf{n}}^*)-\frac{1}{2}(m_{\mathbf{i}}-m_{\mathbf{n}}^*)^2\right)$$

$$\to \mathrm{e}^{-x}$$

since $\chi_{\mathbf{n}}(1-\Phi(u_{\mathbf{n}})) \to \mathrm{e}^{-x}$ by a direct calculation. Hence (4.3) holds and the proof of the corollary is complete.

Proof of Corollary 2.3 As in the proof of Corollary 2.2, let $u_{\mathbf{n},\mathbf{i}} = u_{\mathbf{n}} + m_{\mathbf{n}}^* - m_{\mathbf{i}}$, where $u_{\mathbf{n}} = \frac{x}{a_{\mathbf{n}}} + b_{\mathbf{n}}$. Then, by Corollary 2.2 we have

$$P(a_{\mathbf{n}}(M_{\mathbf{n}}(Y)-b_{\mathbf{n}}-m_{\mathbf{n}}^*) \le x) = P\Big(\bigcap_{\mathbf{i}\in\mathbf{I_n}}\{X_{\mathbf{i}} \le u_{\mathbf{n},\mathbf{i}}\}\Big) \to \exp(-\mathrm{e}^{-x})$$

as $\mathbf{n} \to \infty$. Hence, it suffices to prove that

$$\frac{1}{\chi_{\log\mathbf{n}}}\sum_{\mathbf{k}\in\mathbf{I_n}}\frac{1}{\chi_{\mathbf{k}}}I\Big(\bigcap_{\mathbf{i}\in\mathbf{I_k}}\{X_{\mathbf{i}} \le u_{\mathbf{k},\mathbf{i}}\}\Big) \to \exp(-\mathrm{e}^{-x}) \quad \text{a.s.}$$

as $\mathbf{n} \to \infty$, which will be done by showing that

$$\chi_n(1-\Phi(\lambda_n)) \text{ is bounded,} \tag{4.5}$$

due to Theorem 2.1, where $\lambda_{\mathbf{n}} = \min_{\mathbf{i}\in\mathbf{I_n}} u_{\mathbf{n},\mathbf{i}} = u_{\mathbf{n}} + m_{\mathbf{n}}^* - \max_{\mathbf{i}\in\mathbf{I_n}} m_{\mathbf{i}}$. By the definitions of $\beta_{\mathbf{n}}$ and $m_{\mathbf{n}}^*$, we have

$$\begin{aligned}
&u_{\mathbf{n}}\Big(\max_{\mathbf{i}\in\mathbf{I_n}} m_{\mathbf{i}} - m_{\mathbf{n}}^*\Big) - \frac{1}{2}\Big(\max_{\mathbf{i}\in\mathbf{I_n}} m_{\mathbf{i}} - m_{\mathbf{n}}^*\Big)^2\\
&= o(1) + \sqrt{2\log\chi_{\mathbf{n}}}\Big(\max_{\mathbf{i}\in\mathbf{I_n}} m_{\mathbf{i}} - m_{\mathbf{n}}^*\Big)\Big(1 - \frac{\log\log\chi_{\mathbf{n}} + \log 4\pi}{4\log\chi_{\mathbf{n}}} - \frac{\max_{\mathbf{i}\in\mathbf{I_n}} m_{\mathbf{i}} - m_{\mathbf{n}}^*}{2\sqrt{2\log\chi_{\mathbf{n}}}}\Big)\\
&\le o(1) + K
\end{aligned}$$

for large $\mathbf{n}$. Hence

$$\begin{aligned}\chi_{\mathbf{n}}(1-\Phi(\lambda_{\mathbf{n}})) &\sim \chi_{\mathbf{n}}\lambda_{\mathbf{n}}^{-1}\exp\Big(-\frac{1}{2}\Big(u_{\mathbf{n}}+m_{\mathbf{n}}^{*}-\max_{\mathbf{i}\in\mathbf{I_n}}m_{\mathbf{i}}\Big)^{2}\Big)\\ &\sim \chi_{\mathbf{n}}(1-\Phi(u_{\mathbf{n}}))\exp\Big(-u_{\mathbf{n}}\Big(m_{\mathbf{n}}^{*}-\max_{\mathbf{i}\in\mathbf{I_n}}m_{\mathbf{i}}\Big)-\frac{1}{2}\Big(m_{\mathbf{n}}^{*}-\max_{\mathbf{i}\in\mathbf{I_n}}m_{\mathbf{i}}\Big)^{2}\Big)\\ &\ll \chi_{\mathbf{n}}(1-\Phi(u_{\mathbf{n}}))\end{aligned}$$

for large $\mathbf{n}$. Hence (4.5) holds and the proof of the corollary is complete.

Acknowledgement The authors would like to thank the referees for their careful reading and helpful comments that have helped to improve the quality of the paper.

References

[1] Adler, R. and Taylor, J. E., Random Fields and Geometry, Springer-Verlag, New York, 2007.

[2] Berman, S., Sojourns and Extremes of Stochastic Processes, Wadsworth and Brooks/Cole, Boston, 1992.

[3] Brosamler, G. A., An almost everywhere central limit theorem, *Math. Proc. Cambridge Philos. Soc.*, **104**, 1988, 561–574.

[4] Brosamler, G. A., An almost everywhere central limit-theorem for the occupation measures of Brownian motion on a compact Riemannian manifold, *C. R. Acad. Sci. Paris Sér. I Math.*, **307**, 1988, 919–922.

[5] Brosamler, G. A., A simultaneous almost everywhere centrallimit-theorem for diffusions and its application to path energy and eigenvalues of the Laplacian, *Illinois Journal of Mathematics*, **34**, 1990, 526–556.

[6] Chen, S. and Lin, Z., Almost sure limit theorems for a stationary normal sequence, *Appl. Math. Lett.*, **20**, 2006, 316–322.

[7] Cheng, S., Peng, L. and Qi, Y., Almost sure convergence in extreme value theoy, *Math. Nachr.*, **190**, 1998, 43–50.

[8] Choi, H., Central limit theory and extemes of random fields, PhD Dissertation in Univ. of North Carolina at Chapel Hill, 2002.

[9] Choi, H., Almost sure limit theorem for stationary Gaussian random fields, *Journal of the Korean Statistical Society*, **39**, 2010, 449–454.

[10] Csáki, E. and Gonchigdanzan, K., Almost sure limit theorem for the maximum of stationary Gaussian sequences, *Statist. Probab. Lett.*, **58**, 2002, 195–203.

[11] Fahrner, I. and Stadtmüller, U., On almost sure max-limit theorems, *Statist. Probab. Lett.*, **37**, 1998, 229–236.

[12] Leadbetter, M. R., Lindgren, G. and Rootzén, H., Extremes and Related Properties of Random Sequences and Processes, Springer-Verlag, New York, 1983.

[13] Peng, Z. and Nadarajah, S., Almost sure limit theorem for Gaussian squences, *Theory Probab. Appl.*, **55**, 2011, 361–367.

[14] Pereira, L., On the extremal behavior of a nonstationary normal random field, *Journal of Statistical Planning and Inference*, **140**, 2010, 3567–3576.

[15] Pereira, L., Asymptotic location of largest values of a stationary random field, *Communications in Statistics-Theory and Methods*, 2011, DOI: 10.1080/03610926.2011.650270.

[16] Piterbarg, V. I., Asymptotic Methods in the Theory of Gaussian Processes and Fields, AMS, Providence, 1996.

[17] Schatte, P., On strong versions of the central limit theorem, *Math. Nachr.*, **137**, 1988, 249–256.

[18] Tan, Z., The limit theorems on extremes for Gaussian random fields, *Statist. Probab. Lett.*, **83**, 2013, 436–444.

[19] Tan, Z. and Peng, Z., Almost sure convergence for non-stationary random sequences, *Statist. Probab. Lett.*, **79**, 2009, 857–863.

[20] Zhuang, G., Peng., Z. and Tan, Z., Almost sure central limit theorem for partial sums of Markov chain, *Chin. Ann. Math. Ser. B*, 2012, **33**, 2012, 73–82.