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Prescribing Curvature Problems on the Bakry-Emery Ricci Tensor of a Compact Manifold with Boundary*

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Abstract The authors consider the problem of conformally deforming a metric such that the k-curvature defined by an elementary symmetric function of the eigenvalues of the Bakry-Emery Ricci tensor on a compact manifold with boundary to a prescribed function. A consequence of our main result is that there exists a complete metric such that the Monge-Ampère type equation with respect to its Bakry-Emery Ricci tensor is solvable, provided that the initial Bakry-Emery Ricci tensor belongs to a negative convex cone.

Keywords k-Curvature, Bakry-Emery Ricci tensor, Complete metric, Dirichlet problem
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1 Introduction

Let $(M^n, g), n \geq 3$ be a connected Riemannian manifold, and f a smooth function on M. The Ricci tensor on M is denoted by Ric (or Ric_g). In order to study a log Sobolev inequality of the diffusion operator, Bakry and Emery [1] introduced the following Bakry-Emery Ricci tensor:

$$\mathrm{Ric}_f = \mathrm{Ric} + \mathrm{Hess}(f).$$

In fact, the Bakry-Emery Ricci tensor also occurs naturally in many different subjects (see [12–14]). It has been widely studied recently. Many important geometric results of this tensor have been obtained, such as the measured Gromov-Hausdorff convergence theorem, volume comparison theorems, the splitting theorem, the rigidity theorem, etc., see [2, 12, 17, 20] and the references therein. Moreover, the Bakry-Emery Ricci tensor has a closed relation with Ricci flow (see [14]). There are some other interesting results (see [2, 8, 13, 16]).

In this paper, we consider the prescribing problems for this tensor. Let $\sigma_k : \mathbb{R}^n \to \mathbb{R}$ be the k-th elementary symmetric function, namely,

$$\sigma_k(\lambda) = \sum_{i_1<\cdots<i_k} \lambda_{i_1} \cdots \lambda_{i_k}, \quad \forall \lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n.$$

$\Gamma_k^+ = \{\lambda \in \mathbb{R}^n \mid \sigma_j(\lambda) > 0, \ 1 \leq j \leq k\}$ is an open convex cone. Let $\Gamma \subset \mathbb{R}^n$ be an open convex symmetric cone with the vertex at the origin satisfying $\Gamma_n^+ \subset \Gamma \subset \Gamma_1^+$. We call a metric g a

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Γ^--metric if it satisfies

$$-\lambda(g^{-1}\mathrm{Ric}_f) \in \Gamma,$$

where $\lambda(g^{-1}\mathrm{Ric}_f)$ is an n-vector composed of the eigenvalues of $g^{-1}\mathrm{Ric}_f$. Let $\Gamma^-[g]$ denote the set of all Γ^--metrics that are conformal to g.

Suppose $F : \mathbb{R}^n \to \mathbb{R}$ to be a general smooth symmetric homogeneous function of degree one with $F = 0$ on $\partial\Gamma$ satisfying the following structure conditions in Γ:

(C1) F is positive;

(C2) F is concave $\big($i.e., $\frac{\partial^2 F}{\partial\lambda_i \partial\lambda_j}$ is negative semi-definite$\big)$;

(C3) F is monotone $\big($i.e., $\frac{\partial F}{\partial \lambda_i}$ is positive$\big)$.

It follows from (C2) and $F(0) = 0$ that there exists some uniform constant $\Theta > 0$ such that

$$F(\lambda) \le \Theta \sum_i \lambda_i \quad \text{in } \Gamma. \tag{S1}$$

Since F is homogeneous and of degree one, by (C2), we have

$$\sum_i \frac{\partial F}{\partial \lambda_i}(\lambda) \ge F(e) > 0 \quad \text{in } \Gamma, \tag{S2}$$

where e is the identity of $\mathbb{R}^n$ (see [19]).

Let $(\overline{M}^n, g)$, $n \ge 3$ be a smooth compact Riemannian manifold with the boundary ∂M, and $f \in C^\infty(\overline{M})$. Given a positive function $\psi \in C^\infty(\overline{M})$, we study the problem of finding a smooth complete metric $\widetilde{g} \in \Gamma^-[g]$ such that

$$F(-\lambda(\widetilde{g}^{-1}\widetilde{\mathrm{Ric}_f})) = \psi(x) \quad \text{in } M, \tag{1.1}$$

where $\widetilde{\mathrm{Ric}_f} = \widetilde{\mathrm{Ric}} + \widetilde{\mathrm{Hess}}(f)$ and $\widetilde{\mathrm{Ric}}$ (resp. $\widetilde{\mathrm{Hess}}$) is the Ricci tensor (resp. Hessian) with respect to $\widetilde{g}$. Note that when $f =$ const. and $F = \sigma_k^{\frac{1}{k}}$ on Γ_k^+, (1.1) reduces to the following prescribed k-curvature equation:

$$\sigma_k^{\frac{1}{k}}(-\lambda(\widetilde{g}^{-1}\widetilde{\mathrm{Ric}})) = \psi(x) \quad \text{in } M. \tag{1.2}$$

In fact, the equation (1.2) has been extensively studied. Guan [4] and Gursky [5] proved that if $\mathrm{Ric}_g < 0$, there exists a complete conformal metric of the negative Ricci curvature satisfying (1.2). By a theorem of Lohkamp in [11], there always exist compact smooth metrics on $\overline{M}$ with negative Ricci curvature. The results in [4–5] imply that $\overline{M}$ admits a complete metric g such that the k-curvature defined by the negative eigenvalues of the Ricci tensor equals any given positive function. Note that in the case of $k = 1$ and $\psi =$ const., the equation (1.2) reduces to the Yamabe equation. If $k = n$, the equation (1.2) becomes the following Monge-Ampère type equation:

$$\det(-\lambda(\widetilde{g}^{-1}\widetilde{\mathrm{Ric}})) = \psi^n(x). \tag{1.3}$$

He and Sheng [7] solved the following equation:

$$\det(\lambda(\widetilde{g}^{-1}\widetilde{\mathrm{Ric}})) = \psi^n(x), \tag{1.4}$$

provided that (M, g) has the semi-positive Ricci curvature with a totally geodesic boundary, and is not conformal equivalent to a hemisphere. In the case of $\partial M = \varnothing$, Gursky and Viaclovsky in

[6] found the solution metric $\widetilde{g}$ satisfying (1.2) with $\widetilde{\mathrm{Ric}} < 0$. Li and Sheng obtained the same result in [9] by using a parabolic argument. Trudinger and Wang [18] solved (1.4) by requiring that (M, g) is not conformally equivalent to the unit sphere and has positive Ricci curvature.

In [21], we solved the equation (1.1) on a closed manifold. In this paper we study the prescribing curvature problem (1.1) on a compact manifold with boundary. The method we used here is inspired by [4] and a recent work of Li and Sheng [10], in which they considered the prescribing problem on the modified Schouten tensor A_g^τ for $\tau > n-1$.

Before stating our results, we first write out the corresponding partial differential equation of the equation (1.1). Let $\widetilde{g} = \mathrm{e}^{2u} g$, where u is defined on M. Under this conformal change, we have

$$\begin{aligned}\widetilde{\mathrm{Ric}}_f = \mathrm{Ric}_f &- (n-2)\nabla^2 u - \triangle u \cdot g + (n-2)(\mathrm{d}u \otimes \mathrm{d}u - |\nabla u|^2 g) \\ &- \mathrm{d}u \otimes \mathrm{d}f - \mathrm{d}f \otimes \mathrm{d}u + \langle \nabla u, \nabla f \rangle g,\end{aligned}$$

where the covariant derivative is taken with respect to the background metric g.

Set

$$\begin{aligned}\widehat{W}[u] = \nabla^2 u &+ \frac{1}{n-2}\triangle u g - (\mathrm{d}u \otimes \mathrm{d}u - |\nabla u|^2 g) - \frac{1}{n-2}\langle \nabla u, \nabla f \rangle g \\ &+ \frac{1}{n-2}(\mathrm{d}u \otimes \mathrm{d}f + \mathrm{d}f \otimes \mathrm{d}u) - \frac{1}{n-2}\mathrm{Ric}_f.\end{aligned}$$

For simplicity, we also use the notation $F(A)$ to denote $F(\lambda(g^{-1}A))$ for any smooth symmetric (0,2)-tensor A. Then the equation (1.1) becomes

$$F(\widehat{W}[u]) = \frac{\psi(x)}{n-2}\mathrm{e}^{2u}. \tag{1.5}$$

In order to find a complete metric satisfying (1.1), we only need to solve the following Dirichlet problem with infinite boundary value, i.e.,

$$\begin{cases} F(\widehat{W}[u]) = \dfrac{\psi(x)}{n-2}\mathrm{e}^{2u} & \text{in } M, \\ u = +\infty & \text{on } \partial M. \end{cases} \tag{1.6}$$

More generally, given a positive function $\Psi(x, z) \in C^\infty(\overline{M}^n \times \mathbb{R})$ and a function $\varphi \in C^\infty(\partial M)$, we consider the following equation:

$$\begin{cases} F(W[u]) = \Psi(x, u) & \text{in } M, \\ u = \varphi & \text{on } \partial M, \end{cases} \tag{1.7}$$

where

$$\begin{aligned}W[u] = \nabla^2 u &+ \gamma \triangle u g + \Big(s \mathrm{d}u \otimes \mathrm{d}u - \frac{t}{2}|\nabla u|^2 g\Big) + a(x)\langle \nabla u, \nabla f \rangle g \\ &+ b(x)(\mathrm{d}u \otimes \mathrm{d}f + \mathrm{d}f \otimes \mathrm{d}u) + T\end{aligned} \tag{1.8}$$

for $\gamma, s, t \in \mathbb{R}$, $\gamma > 0$, T is a smooth symmetric (0,2)-tensor, and $a(x)$, $b(x)$ are two smooth functions defined on M. Clearly, (1.7) is fully nonlinear and elliptic for the solutions u with $\lambda(g^{-1}W[u]) \in \Gamma$ (see [21]). Accordingly, we call a function $v \in C^2(M)$ admissible if $\lambda(g^{-1}W[v]) \in \Gamma$.

Our main results can be stated as follows.

Theorem 1.1 *Let $(\overline{M}^n, g)$, $n \geq 3$ be a smooth compact Riemannian manifold with the boundary ∂M and $f \in C^\infty(\overline{M})$. If $T \in \Gamma$, φ is a smooth function defined on a neighborhood of ∂M, and $\Psi(x,z) \in C^\infty(\overline{M}^n \times \mathbb{R})$ satisfies*

$$\Psi(x,z) > 0, \quad \partial_z \Psi > 0, \quad \lim_{z\to+\infty} \Psi(x,z) \to +\infty, \quad \lim_{z\to-\infty} \Psi(x,z) \to 0, \tag{1.9}$$

then there exists a unique admissible solution $u \in C^\infty(\overline{M})$ of the equation (1.7).

Remark 1.1 Different from the results of [10, 15], in this theorem, we need not add any restriction on $a(x)$, $b(x)$ and the coefficients $\gamma, s, t \in \mathbb{R}$, and just require $\gamma > 0$.

Applying Theorem 1.1 to the quotient of the elementary symmetric functions, i.e., $F = \big(\frac{\sigma_k}{\sigma_l}\big)^{\frac{1}{k-l}}$ on Γ_k^+, $0 \leq l < k \leq n$, and $\sigma_0 = 1$, we have the following corollary.

Corollary 1.1 *Let $(\overline{M}^n, g)$, $n \geq 3$ be a smooth compact Riemannian manifold with the boundary ∂M and $f \in C^\infty(\overline{M})$. If $-\lambda(g^{-1}\mathrm{Ric}_f) \in \Gamma_k^+$, then for each function $\varphi \in C^\infty(\partial M)$ and a positive function $\mathcal{K} \in C^\infty(\overline{M})$, there exists a unique smooth metric $\widetilde{g}$ satisfying*

$$\begin{cases} -\lambda(\widetilde{g}^{-1}\widetilde{\mathrm{Ric}_f}) \in \Gamma_k^+, \\ \Big(\dfrac{\sigma_k}{\sigma_l}\Big)^{\frac{1}{k-l}}(-\lambda(\widetilde{g}^{-1}\widetilde{\mathrm{Ric}_f})) = \mathcal{K} \quad \text{in } M, \\ \widetilde{g}|_{\partial M} = \mathrm{e}^{2\varphi} g|_{\partial M}. \end{cases} \tag{1.10}$$

By solving the Dirichlet problem (1.7) with infinite boundary data, we can obtain a complete metric satisfying (1.6) (compare with [4, 10]).

Theorem 1.2 *Let $(\overline{M}^n, g)$, $n \geq 3$ be a smooth compact Riemannian manifold with the boundary ∂M and $f \in C^\infty(\overline{M})$. Given any smooth positive function $\psi(x) \in C^\infty(\overline{M})$, if $T \in \Gamma$, then there exists a unique admissible solution $u \in C^\infty(\overline{M})$ of the equation*

$$\begin{cases} F(W[u]) = \dfrac{\psi(x)}{n-2}\mathrm{e}^{2u} & \text{in } M, \\ u = +\infty & \text{on } \partial M. \end{cases} \tag{1.10}$$

Moreover, there exist positive constants C and $0 < \theta \leq 1$, depending only on g, γ, s, t, $\|a\|_{L^\infty(\overline{M})}$, $\|b\|_{L^\infty(\overline{M})}$, $\|T\|_{g(\overline{M})}$, $\|\psi\|_{C^2(\overline{M})}$ and $\|f\|_{C^2(\overline{M})}$ such that

$$-C - \theta \log \rho(x) \leq u(x) \leq -\log \rho(x) + C,$$

where $\rho(x)$ is the distance function from x to ∂M with respect to the background metric g.

Then we have the following corollary.

Corollary 1.2 *Let $(\overline{M}^n, g)$, $n \geq 3$ be a smooth compact Riemannian manifold with the boundary ∂M and $f \in C^\infty(\overline{M})$. If $-\lambda(g^{-1}\mathrm{Ric}_f) \in \Gamma_k^+$, then there exists a unique smooth complete metric satisfying $-\lambda(\widetilde{g}^{-1}\widetilde{\mathrm{Ric}_f}) \in \Gamma_k^+$ and*

$$\Big(\frac{\sigma_k}{\sigma_l}\Big)^{\frac{1}{k-l}}(-\widetilde{g}^{-1}\widetilde{\mathrm{Ric}_f}) = \mathrm{const.} > 0 \quad \text{for all } 0 \leq l < k \leq n.$$

In particular, for $k = n$ and $l = 0$, we have

$$\det(-\lambda(\widetilde{g}^{-1}\widetilde{\mathrm{Ric}_f})) = \mathrm{const.}$$

Remark 1.2 Let (M,g) be a Riemannian manifold and f be a smooth function on M. By [2, 13], the N-Bakry-Emery Ricci tensor Ric_f^N and the N-Ricci tensor Ric_N are defined respectively by

$$\mathrm{Ric}_f^N = \mathrm{Ric} - \frac{1}{N}\mathrm{d}f\otimes\mathrm{d}f \quad \text{for } N>0,$$

$$\mathrm{Ric}_N = \begin{cases} \mathrm{Ric}+\mathrm{Hess}(f), & \text{if } N=\infty, \\ \mathrm{Ric}+\mathrm{Hess}(f)-\dfrac{1}{N-n}\mathrm{d}f\otimes\mathrm{d}f, & \text{if } n<N<\infty, \\ \mathrm{Ric}+\mathrm{Hess}(f)-\infty(\mathrm{d}f\otimes\mathrm{d}f), & \text{if } N=n, \\ -\infty, & \text{if } N<n. \end{cases}$$

Note that $\mathrm{d}f\otimes\mathrm{d}f$ is invariant under conformal changes. If M has a boundary, all the conclusions above are also valid for its N-Bakry-Emery Ricci tensor and the N-Ricci tensor.

This paper is organized as follows. In Sections 2–3, we establish a priori boundary and interior estimates of admissible solutions of (1.7). Then we prove Theorem 1.1 in Section 4 by using the a priori estimates and the standard continuity method. In Section 5, we solve the Dirichlet equation (1.10) by constructing two suitable barrier functions, and then prove Theorem 1.2.

2 Boundary Estimates

In this section, we establish a priori boundary estimates for the first and second derivatives of admissible solutions of (1.7) with a smooth Dirichlet data φ. We always assume that $\varphi\in C^\infty(\overline{M})$ throughout this paper.

2.1 Boundary C^1 estimates

For convenience, set

$$\nabla^2_{\mathrm{con}\,f}u := \nabla^2 u+\gamma\triangle u+\Big(s\mathrm{d}u\otimes\mathrm{d}u-\frac{t}{2}|\nabla u|^2 g\Big)$$

and

$$\nabla^u_{\mathrm{con}\,f}f := a(x)\langle\nabla u,\nabla f\rangle g+b(x)(\mathrm{d}u\otimes\mathrm{d}f+\mathrm{d}f\otimes\mathrm{d}u).$$

Then

$$W[u]=\nabla^2_{\mathrm{con}\,f}u+\nabla^u_{\mathrm{con}\,f}f+T.$$

A function w is said to be a subsolution of (1.7) if it satisfies the following equation:

$$\begin{cases} F(W[w])\ge\Psi(x,w) & \text{in } M, \\ w\le\varphi & \text{on } \partial M. \end{cases} \tag{2.1}$$

Changing the direction of the inequalities, one gets the definition of the supsolution of (1.7).

To estimate the gradient on the boundary, we need the following maximum principle for (1.7).

Proposition 2.1 (Maximum Principle) *Suppose that w and v are smooth sub- and supsolutions of the equation* (1.7) *with $w|_{\partial M}\le v|_{\partial M}$, respectively. If $\partial_z\Psi>0$ in $M\times\mathbb{R}$, then $w\le v$ on $\overline{M}$.*

One may prove this proposition by a contradictory argument. We omit its proof, and see [10, 21] for details.

By the maximum principle and the boundary distance function $\rho(x) := \mathrm{dist}_g(x, \partial M)$, we can construct two barrier functions later to control the gradient derivatives. Given any small positive constant δ, we set

$$M_\delta = \{x \in M \mid \rho(x) < \delta\}.$$

Since ∂M is smooth and $|\nabla \rho| = 1$ on ∂M, we may assume that $\rho(x)$ is smooth and $\frac{1}{2} \leq |\nabla \rho| \leq 2$ in M_δ for δ sufficiently small.

For any fixed point $x_0 \in \partial M$, we choose a local orthonormal coordinate system $\{x^i\}_{i=1,\cdots,n}$ in M_δ, such that ∂M is the plane $x^n = 0$. Let $\{e_1, \cdots, e_{n-1}, e_n\}$ be the corresponding coordinate vector fields, where e_n is the interior normal vector and e_α is the tangential direction vector, $\alpha = 1, 2, \cdots, n-1$.

Lemma 2.1 *Let $u \in C^2(\overline{M})$ be an admissible solution of* (1.7). *If $\min\limits_{\overline{M}} u \geq -\mu$ for some constant $\mu > 0$, and $\partial_z \Psi > 0$ on $\overline{M} \times \mathbb{R}$, then we have*

$$\partial_n u|_{\partial M} > -C,$$

where the constant C depends on μ, g, γ, s, t, $\|a\|_{L^\infty(\overline{M})}$, $\|b\|_{L^\infty(\overline{M})}$, $\|f\|_{C^1(\overline{M})}$, $\|T\|_{g(\overline{M})}$ and $\|\varphi\|_{C^2(\overline{M})}$.

Proof If there exists a local subsolution $u^- \in C^3(\overline{M}_\delta)$ of the equation (1.7), i.e.,

$$\begin{cases} F(W[u^-]) \geq \Psi(x, u^-), & x \in M_\delta, \\ u^-(x) \leq \varphi, & x \in \partial M_\delta, \end{cases} \tag{2.2}$$

then we complete the lemma. In fact, the maximum principle implies that $u^-(x) \leq u(x)$ on $\overline{M}_\delta$. Consequently, for any $x_0 \in \partial M$, we have

$$\frac{u(x) - u(x_0)}{\rho(x, x_0)} \geq \frac{u^-(x) - u^-(x_0)}{\rho(x, x_0)},$$

which implies that $\partial_n u|_{\partial M} \geq \partial_n u^-|_{\partial M}$.

Now, we construct a local subsolution u^- of (1.7) by using the method which is similar to that of [4, 10]. Set

$$u^- = \varphi + \theta \log \frac{\delta^2}{\rho + \delta^2}, \tag{2.3}$$

where θ is a positive constant to be fixed. Then $u^-|_{\partial M} = \varphi = u$, and

$$\varphi + \log \frac{\delta}{2} \leq u^-|_{\rho(x)=\delta} \leq \varphi + \log \delta.$$

Choosing $\delta < \mathrm{e}^{-\mu - \max\limits_{M} |\varphi|}$, we get $u^-(x) \leq u(x)$ on $\partial M_\delta \backslash \partial M$. Thus, $u^-(x) \leq u(x)$ on ∂M_δ.

It remains to verify $F(W[u^-]) \geq \Psi(x, u^-)$ in M_δ. Since

$$u_i^- = \varphi_i - \frac{\theta \rho_i}{\rho + \delta^2}, \quad u_{ij}^- = \varphi_{ij} - \frac{\theta \rho_{ij}}{\rho + \delta^2} + \frac{\theta \rho_i \rho_j}{(\rho + \delta^2)^2},$$

we have

$$\begin{aligned}
W[u^-]_{ij} = & \frac{\theta}{(\rho+\delta^2)^2}\Big(\gamma - \frac{t\theta}{2}\Big)|\nabla\rho|^2\delta_{ij} + \frac{\theta(1+s\theta)}{(\rho+\delta^2)^2}\rho_i\rho_j - \frac{\theta}{\rho+\delta^2}(\rho_{ij}+\gamma\triangle\rho\delta_{ij}) \\
& - \frac{\theta s}{\rho+\delta^2}\rho_i\varphi_j - \frac{\theta s}{\rho+\delta^2}\rho_j\varphi_i + \frac{\theta t\varphi_l\rho_l}{\rho+\delta^2}\delta_{ij} - \frac{\theta a}{\rho+\delta^2}\rho_l f_l\delta_{ij} - \frac{\theta b\rho_i f_j}{\rho+\delta^2} - \frac{\theta b\rho_j f_i}{\rho+\delta^2} \\
& + (\varphi_{ij}+\gamma\triangle\varphi\delta_{ij}) + s\varphi_i\varphi_j - \frac{t}{2}\varphi_l^2\delta_{ij} + a\varphi_l f_l\delta_{ij} + b(x)(\varphi_i f_j + \varphi_j f_i) + T_{ij} \\
\geq & \frac{\theta}{(\rho+\delta^2)^2}\Big(\gamma - \frac{t\theta}{2}\Big)|\nabla\rho|^2\delta_{ij} + \frac{\theta(1+s\theta)}{(\rho+\delta^2)^2}\rho_i\rho_j - C'\frac{\theta}{\rho+\delta^2}\delta_{ij} - C''\delta_{ij},
\end{aligned} \tag{2.4}$$

where the constants C' and C'' depend on $\|a\|_{L^\infty(\overline{M})}, \|b\|_{L^\infty(\overline{M})}, \|f\|_{C^1(\overline{M})}, \|T\|_{g(\overline{M})}$, and $\|\varphi\|_{C^2(\overline{M})}$.

Choose $\theta^{-1} \geq \max\big\{1, -s, \frac{t}{\gamma}\big\}$. Then $\gamma - t\theta > 0$ and $1 + s\theta > 0$. By $|\nabla\rho| \geq \frac{1}{2}$ in M_δ and (2.4), we have

$$W[u^-]_{ij} \geq \frac{\gamma\theta}{8(\rho+\delta^2)^2}\delta_{ij} + \frac{\theta(1+s\theta)}{(\rho+\delta^2)^2}\rho_i\rho_j - C'\frac{\theta}{\rho+\delta^2}\delta_{ij} - C''\delta_{ij}.$$

Note that $\rho+\delta^2 \leq 2\delta$ and $(\rho+\delta^2)^2 \leq 4\delta^2 < 4\delta$. By choosing $\delta < \min\big\{1, \frac{\gamma}{64C'}\frac{\gamma\theta}{64C''}\big\}$, we have

$$\frac{\gamma\theta}{32(\rho+\delta^2)^2} \geq \frac{C'\theta}{\rho+\delta^2}, \quad \frac{\gamma\theta}{16(\rho+\delta^2)^2} > C'',$$

which implies that

$$W[u^-]_{ij} \geq \frac{\gamma\theta}{32(\rho+\delta^2)^2}\delta_{ij} + \frac{\theta(1+s\theta)}{(\rho+\delta^2)^2}\rho_i\rho_j.$$

Hence, $W[u^-] \in \Gamma$ and u^- is admissible. Denote $F^{ij} = \frac{\partial F}{\partial W_{ij}}(W[u^-])$. We know that $\{F^{ij}\}$ is positive definite (see [3]), and $\sum\limits_i F^{ii} \geq F(e)$ (by (S2)). Note that F is homogeneous and of degree one. Then $F(W[u^-]) = F^{ij}W[u^-]_{ij}$. Thus, by (2.5), we have

$$F(W[u^-]) \geq \frac{\gamma\theta}{32(\rho+\delta^2)^2}\sum_i F^{ii} \geq \frac{\gamma\theta}{32(\rho+\delta^2)^2}F(e).$$

Since $\partial_z\Psi > 0$ for δ sufficiently small, we obtain

$$F(W[u^-]) \geq \frac{\gamma\theta}{32\delta^2}F(e) \geq \max_{\overline{M}\times\big[\min\limits_{\overline{M}}\varphi, \max\limits_{\overline{M}}\varphi\big]}\Psi(x,z) \geq \Psi(x,u^-).$$

Similarly, we can get the upper bound of $\partial_n u$ on ∂M.

Lemma 2.2 *Let $u \in C^2(\overline{M})$ be an admissible solution of* (1.7). *If $\max\limits_{\overline{M}} u \leq \mu$ for some constant $\mu > 0$, and $\partial_z\Psi > 0$ on $M \times \mathbb{R}$, then we have*

$$\partial_n u|_{\partial M} \leq C,$$

where the constant C depends on μ, g, γ, s, t, $\|a\|_{L^\infty(\overline{M})}$, $\|b\|_{L^\infty(\overline{M})}$, $\|f\|_{C^1(\overline{M})}$, $\|T\|_{g(\overline{M})}$ and $\|\varphi\|_{C^3(\overline{M})}$.

Proof Since u is admissible and $\Gamma\subset\Gamma_1^+$, we have

$$0<\operatorname{tr}(W[u])=(n\gamma+1)\triangle u+\Big(s-\frac{tn}{2}\Big)|\nabla u|^2+(an+2b)\langle\nabla u,\nabla f\rangle+\operatorname{tr}T. \tag{2.6}$$

Now, we construct a local supsolution $u^+\in C^3(\overline{M}_\delta)$ of (2.6), that is, the function u^+ satisfies

$$\begin{cases}\operatorname{tr}(W[u^+])\le 0, & x\in M_\delta,\\ u^+(x)\ge u, & x\in\partial\overline{M}_\delta.\end{cases} \tag{2.7}$$

Let τ be a small positive constant to be decided. Define

$$u^+=\varphi+\tau\log\frac{\rho+\delta^2}{\delta^2}.$$

A direct calculation shows that

$$\begin{aligned}\operatorname{tr}W[u^+]&=\frac{\tau}{(\rho+\delta^2)^2}\Big(\tau\Big(s-\frac{tn}{2}\Big)-(n\gamma+1)\Big)|\nabla\rho|^2\\&\quad+\frac{\tau}{\rho+\delta^2}\Big\{(n\gamma+1)\triangle\rho+2\Big(s-\frac{tn}{2}\Big)\langle\nabla\varphi,\nabla\rho\rangle+(an+2b)\langle\nabla\rho,\nabla f\rangle\Big\}\\&\quad+(n\gamma+1)\triangle\varphi+\Big(s-\frac{tn}{2}\Big)|\nabla\varphi|^2+(an+2b)\langle\nabla\varphi,\nabla f\rangle+\operatorname{tr}T.\end{aligned}$$

Choose τ, such that $\tau\big(s-\frac{tn}{2}\big)<1$. By $|\nabla\rho|\ge\frac12$, we have

$$\operatorname{tr}W[u^+]\le-\frac{\tau n\gamma}{4(\rho+\delta^2)^2}+\frac{\tau}{\rho+\delta^2}C'+C'',$$

where the constants C' and C'' depend on the $\|a\|_{L^\infty(\overline{M})}$, $\|b\|_{L^\infty(\overline{M})}$, $\|f\|_{C^1(\overline{M})}$, $\|T\|_{g(\overline{M})}$, $\|\varphi\|_{C^2(\overline{M})}$ and other known data. Then for δ sufficiently small, we have

$$\operatorname{tr}W[u^+]\le 0\quad\text{on }\overline{M}_\delta.$$

Note that $u^+|_{\partial M}=\varphi=u$, and $\max\limits_{\overline{M}}u\le\mu$. For a small δ, we have

$$u^+|_{\rho=\delta}=\varphi+\tau\log\frac{\delta+\delta^2}{\delta^2}\ge\min_{\overline{M}}\varphi+\tau\log\frac{1}{\delta}\ge\mu.$$

Hence, u^+ satisfies (2.7). By the maximum principle, we have $u^+|_{\overline{M}_\delta}\ge u|_{\overline{M}_\delta}$. Therefore, for any $x_0\in\partial M$,

$$\frac{u(x)-u(x_0)}{\rho(x,x_0)}\le\frac{u^+(x)-u^+(x_0)}{\rho(x,x_0)},$$

which implies that

$$\partial_n u|_{\partial M}\le\partial_n u^+|_{\partial M}.$$

Combining the above two lemmas, we obtain the following proposition.

Proposition 2.2 *Suppose that $u\in C^2(\overline{M})$ is an admissible solution of* (1.7). *If $\sup\limits_{\overline{M}}|u|\le\mu$ for some constant $\mu>0$, and $\partial_z\Psi>0$, then we have*

$$\sup_{\partial M}|\nabla_n u|\le C, \tag{2.8}$$

where the constant C depends on g,γ,s,t,μ, $\|a\|_{L^\infty(\overline{M})}$, $\|b\|_{L^\infty(\overline{M})}$, $\|f\|_{C^1(\overline{M})}$, $\|T\|_{g(\overline{M})}$ and $\|\varphi\|_{C^2(\overline{M})}$.

2.2 Boundary C^2 estimates

The method we use here to derive the second derivative estimates is similar to that of [4–5, 10]. For any fixed point $x_0 \in \partial M$, define a half ball B_δ^+ centered at x_0 of radius δ by

$$B_\delta^+ = \{x \in M : \ r(x) = \mathrm{dist}_g(x, x_0) < \delta\}.$$

We may assume that $r(x)$ is smooth in B_δ^+ for small δ. Then $|\nabla r| = 1$ in B_δ^+. Choose a local orthonormal frame $e_1, \cdots, e_n$ at x_0, where e_n is the inward unit normal vector. Since $\nabla_{ij} r^2(x_0) = 2\delta_{ij}$, we also assume that $\{\delta_{ij}\} \le \{\nabla_{ij} r^2\} \le 3\{\delta_{ij}\}$ in B_δ^+.

Let $u \in C^3(\overline{M})$ be an admissible solution of (1.7). Define a linearized operator L by

$$Lv = F^{ij} v_{ij} + \gamma \triangle v \mathcal{F} + 2s F^{ij} v_i u_j - t v_l u_l \mathcal{F} + a v_l f_l \mathcal{F} + 2b F^{ij} v_i f_j, \quad \forall v \in C^2(M), \tag{2.9}$$

where $F^{ij} = \frac{\partial F}{\partial W_{ij}}(W[u])$ and $\mathcal{F} = \mathrm{tr}(F^{ij}) = g_{ij} F^{ij}$.

In order to get the estimates of the normal and tangential derivatives of mixed type, we need the following two lemmas.

Lemma 2.3 *For any constant $\beta > 0$, there exist positive constants δ sufficiently small and N sufficiently large such that the barrier function*

$$w(x) = \rho - \frac{N}{2}\rho^2$$

satisfies

$$Lw \le -\beta \mathcal{F} \quad \text{in } M_\delta, \quad w \ge 0 \quad \text{on } \overline{M_\delta},$$

where δ and N depend on β, $\sup\limits_{\overline{M}} |\nabla u|$ and other known data.

Proof It is easy to check that

$$|L\rho| \ = \ |F^{ij}\rho_{ij} + \gamma \triangle \rho \mathcal{F} + 2s F^{ij} \rho_i u_j - t \rho_l u_l \mathcal{F} + a \rho_l f_l \mathcal{F} + 2b F^{ij} \rho_i f_j| \le C^* \mathcal{F},$$

where the constant C^* depends on $\gamma, |s|, |t|, n, g, \sup\limits_{\overline{M}} |\nabla u|, \|a\|_{L^\infty(\overline{M})}, \|b\|_{L^\infty(\overline{M})}$ and $\|f\|_{C^1(\overline{M})}$. Since $\{F^{ij}\}$ is positive definite and $|\nabla \rho| \ge \frac{1}{2}$ in $\overline{M_\delta}$, we have

$$\begin{aligned} Lw(x) &= L\rho - N\rho L\rho - N F^{ij}\rho_i\rho_j - N\gamma |\nabla\rho|^2 \mathcal{F} \\ &\le (1 + N\rho) C^* \mathcal{F} - \frac{1}{4} N \gamma \mathcal{F}. \end{aligned}$$

By choosing $\delta < \frac{\gamma}{8C^*} - \frac{1}{N}$ and $N > \max\left\{\frac{8C^*}{\gamma}, \frac{8\beta}{\gamma}\right\}$, we obtain

$$Lw(x) \le -\frac{1}{8} N \gamma \mathcal{F} \le -\beta \mathcal{F}.$$

Finally, for the fixed N, we can choose $\delta \le \frac{2}{N}$ to ensure that $w \ge 0$ on $\overline{M_\delta}$.

Lemma 2.4 *Let $h \in C^2(\overline{M}_\delta)$. If $h \le 0$ on ∂M, $h(x_0) = 0$ and*

$$-Lh \le D(1 + \mathcal{F}) \quad \text{in } M_\delta$$

for some positive constant D, then we have

$$\nabla_n h(x_0) \leq C,$$

where the constant C depends on D, β, $\|h\|_{C^0(\overline{M}_\delta)}$, $\sup_{\overline{M}} |\nabla u|$ and other known data, and β is the same constant as in Lemma 2.3*.*

Proof By Lemma 2.3, we can choose $A \gg \beta \gg 1$ such that $Aw(x) + \beta r^2(x) - h(x) \geq 0$ on ∂B_δ^+. It is clear that $|Lr^2| = |2rLr + 2F^{ij} r_i r_j + 2\gamma |\nabla r|^2 \mathcal{F}| \leq C'\mathcal{F}$, where C' depends on $\sup_{\overline{M}} |\nabla u|$, γ and δ. Note that $\mathcal{F} \geq F(e)$. By choosing $A \geq \max\left\{2C', \frac{4D}{\beta}, \frac{4D}{\beta F(e)}\right\}$, we have

$$L(Aw + \beta r^2 - h) \leq -A\beta\mathcal{F} + \beta C'\mathcal{F} + D + D\mathcal{F} \leq 0 \quad \text{in } M_\delta.$$

It follows from the maximum principle that $Aw + \beta r^2 - h \geq 0$ in B_δ^+. Since

$$\begin{aligned} &(Aw + \beta r^2 - h)(x_0) = 0, \\ &\nabla_n (Aw + \beta r^2 - h)(x_0) \geq 0, \end{aligned}$$

which implies that $\nabla_n h(x_0) \leq C$.

Now, we can get the following boundary estimates for the second derivatives.

Proposition 2.3 *Suppose that $u(x) \in C^3(\overline{M})$ is an admissible solution of* (1.7)*. Then*

$$\sup_{\partial M} |\nabla^2 u| \leq C, \tag{2.10}$$

where the constant C depends on $\|u\|_{C^1(\overline{M})}$, $\|\Psi\|_{C^1(\overline{M}\times[-\mu,\mu])}$, $\|\varphi\|_{C^3(\overline{M})}$, the geometric quantities of $(\overline{M}, g)$ and other known data.

Proof Since $u - \varphi = 0$ on ∂M, for any point $x_0 \in \partial M$, we have

$$\nabla_{\alpha\beta}(u - \varphi)(x_0) = -\nabla_n (u - \varphi)(x_0) \Pi(e_\alpha, e_\beta),$$

where $1 \leq \alpha, \beta \leq n - 1$ and Π denotes the second fundamental form of ∂M. Then one can get

$$|\nabla_{\alpha\beta} u(x_0)| \leq C, \tag{2.11}$$

where C depends on $\sup_{\partial M} |\nabla^2 \varphi|$, $\sup_{\partial M} |\nabla u|$ and the geometric quantities of $(\overline{M}, g)$.

To get the estimates of the normal and tangential derivatives of mixed type $\nabla_{\alpha n} u$, we differentiate the equation (1.7) with respect to e_k, that is,

$$\begin{aligned} \nabla_k \Psi = {} & F^{ij}(u_{ijk} + \gamma u_{llk}\delta_{ij}) + 2sF^{ij} u_{ik} u_j - t u_l u_{lk}\mathcal{F} + a_k u_l f_l \mathcal{F} + a(u_{lk} f_l + u_l f_{lk})\mathcal{F} \\ & + 2b_k F^{ij} u_i f_i + 2bF^{ij}(u_{ik} f_j + u_i f_{jk}) + F^{ij} T_{ij,k}. \end{aligned} \tag{2.12}$$

By the Ricci identities $u_{kij} = u_{ijk} + R_{kij}^p u_p$, (2.9) and (2.12), we have

$$\begin{aligned} L(u - \varphi)_k = {} & \nabla_k \Psi - (a_k u_l f_l + a u_l f_{lk})\mathcal{F} - 2F^{ij}(b_k u_i f_i + b u_i f_{jk}) \\ & + F^{ij} R_{kij}^p u_p + \gamma R_{kll}^p u_p \mathcal{F} - F^{ij} T_{ij,k} - F^{ij} \varphi_{kij} \\ & - \gamma \triangle \varphi_k \mathcal{F} - 2sF^{ij} \varphi_{ki} u_j + t\varphi_{kl} u_l \mathcal{F} - a\varphi_{kl} f_l \mathcal{F} - 2bF^{ij} \varphi_{ki} f_j. \end{aligned}$$

Then, for each $k = 1, \cdots, n$, we obtain that

$$|L(u-\varphi)_k| \le \widehat{C}(1+\mathcal{F}),$$

where $\widehat{C}$ depends on $n, g, \gamma, |s|, |t|, \|u\|_{C^1(\overline{M})}$, $\|\varphi\|_{C^3(\overline{M})}$, $\|\Psi\|_{C^1(\overline{M}\times[-\mu,\mu])}$, $\|T\|_{C^1(\overline{M})}$, $\|a\|_{C^1(\overline{M})}$, $\|b\|_{C^1(\overline{M})}$ and $\|f\|_{C^2(\overline{M})}$.

Applying Lemma 2.4 to $h = \pm\nabla_\alpha(u-\varphi)$, we immediately get the estimates

$$|\nabla_{\alpha n} u(x_0)| \le C. \tag{2.13}$$

It remains to estimate the bound of $\nabla_{nn}u$. Since u is admissible, by (2.6), we have $\triangle u(x_0) \ge -C$. Thus, $\nabla_{nn}u(x_0)$ has a lower bound by (2.11). Without loss of generality, we can assume $u_{nn}(x_0) \ge 0$ (otherwise we have done). Orthogonally decomposing the matrix $W[u]$ at $x_0 \in \partial M$ in terms of e_α and e_n, and using the known bounds in (2.11) and (2.13), we have

$$\begin{aligned} W[u]_{ij}(x_0) &= u_{ij} + \gamma\triangle u g_{ij} + \Big(su_iu_j - \frac{t}{2}|\nabla u|^2 g_{ij}\Big) + a(x)\langle\nabla u, \nabla f\rangle g_{ij} \\ &\quad + b(x)(u_if_j + f_iu_j) + T_{ij} \\ &\ge \begin{pmatrix} \gamma u_{nn} I_{n-1} & 0 \\ 0 & (1+\gamma)u_{nn} \end{pmatrix}(x_0) - C\delta_{ij} \\ &\ge (\gamma u_{nn}(x_0) - C)\delta_{ij}, \end{aligned}$$

where the constant C also depends only on $|s|, |t|, \sup\limits_{\partial M}|\nabla u|$, $\|a\|_{L^\infty(\overline{M})}$, $\|b\|_{L^\infty(\overline{M})}$ and $\|T\|_{g(\overline{M})}$. Then, we have

$$\begin{aligned} (\gamma u_{nn}(x_0) - C)\mathcal{F} &\le F(W[u])(x_0) \\ &= \Psi(x_0, u(x_0)) \\ &\le \max_{\overline{M}\times\left[\min\limits_{\overline{M}}\varphi, \max\limits_{\overline{M}}\varphi\right]} \Psi(x,u). \end{aligned}$$

Since $\mathcal{F} \ge F(e)$, we then obtain the upper bound of $\nabla_{nn}u(x_0)$.

3 Global Estimates

In this section, we first calculate the local interior estimates for admissible solutions of (1.7). By combining the local interior estimates and the boundary estimates in Section 2, we will derive a priori global C^2 estimates. Now, we divide the procedure into three steps.

3.1 Global C^0 estimate

Since the manifold is compact, we can get a global C^0 estimate easily.

Proposition 3.1 *Let $T \in \Gamma$, $\varphi \in C^\infty(\partial M)$ and $\Psi(x,z) \in C^\infty(\overline{M}^n \times \mathbb{R})$. If $\Psi(x,z)$ satisfies (1.9), then for any admissible solution $u \in C^2(\overline{M})$ of (1.7), we have*

$$\sup_{\overline{M}}|u| \le C_0, \tag{3.1}$$

where the constant C_0 only depends on $\|\Psi\|_{L^\infty(\overline{M})}$, $\|\varphi\|_{L^\infty(\partial M)}$ and $\|T\|_{g(\overline{M})}$.

Proof Let $x_{\min}$ be the minimum point of the function u on $\overline{M}$. If $x_{\min} \in \partial M$, then the lower bound of u can be obtained by

$$u(x) \geq u(x_{\min}) = \varphi(x_{\min}) \geq \min_{\partial M} \varphi(x).$$

If $x_{\min}$ is an interior point of M, then at $x_{\min}$, we have $\nabla u(x_{\min}) = 0$, $\nabla^2 u(x_{\min}) \geq 0$. Note that $\gamma > 0$ and $T \in \Gamma$. Then

$$\begin{aligned}\Psi(x_{\min}, u_{\min}) &= F(\nabla^2 u(x_{\min}) + \gamma \triangle u(x_{\min}) g + T(x_{\min})) \\ &\geq F(T)(x_{\min}) \geq \min_{\overline{M}} F(T) > 0.\end{aligned}$$

Hence, we can get the lower bound of u by the condition $\lim\limits_{z\to-\infty} \Psi(x,z) \to 0$.

Similarly, we can get the upper bound of u by considering its maximum and using the fact that $\lim\limits_{z\to+\infty} \Psi(x,z) \to +\infty$.

3.2 Global C^1 estimate

Let $B_r \subset M$ be a geodesic ball of radius $r > 0$. There exists a cutoff function $\zeta(x) \in C_0^\infty(B_r)$ such that $\zeta|_{B_{\frac{r}{2}}} = 1$, $\zeta|_{M\setminus B_r} = 0$, and

$$0 \leq \zeta \leq 1, \quad |\nabla \zeta| \leq b_0 \zeta^{\frac{1}{2}}, \quad |\nabla^2 \zeta| \leq b_0 \tag{3.2}$$

for some constant $b_0 > 0$.

Lemma 3.1 *Let $T \in \Gamma$, $\Psi(x,z) \in C^\infty(\overline{M} \times \mathbb{R})$, and $u \in C^3(B_r)$ be an admissible solution of* (1.7). *Then there exists a constant C depending only on $g, b_0, \gamma, r^{-1}, |s|, |t|$, $\|T\|_{C^1(\overline{M})}$, $\|a\|_{C^1(\overline{M})}$, $\|b\|_{C^1(\overline{M})}$, $\|f\|_{C^2(\overline{M})}$, $\|\Psi\|_{C^1(\overline{M})}$ and C_0 such that*

$$\sup_{B_{\frac{r}{2}}} |\nabla u| \leq C. \tag{3.3}$$

Proof Consider the following auxiliary function:

$$G = \zeta \omega \mathrm{e}^{\eta(u)},$$

where $\omega = \frac{1}{2}|\nabla u|^2$ and η is a function to be chosen later. Suppose that G attains its maximum at an interior point $x_0 \in B_r$. Choose a local normal coordinate frame e_i, $i = 1, \cdots, n$ at x_0 with respect to g such that $W[u](x_0)$ is diagonal. Then at x_0 we have

$$0 = (\log G)_i = \frac{\omega_i}{\omega} + \eta' u_i + \frac{\zeta_i}{\zeta}, \tag{3.4}$$

that is

$$\omega_i = -\omega\Big(\eta' u_i + \frac{\zeta_i}{\zeta}\Big) \tag{3.5}$$

and

$$0 \geq F^{ij}(\log G)_{ij} = F^{ij}\Big(\frac{\omega_{ij}}{\omega} - \frac{\omega_i \omega_j}{\omega^2}\Big) + F^{ij}(\eta'' u_i u_j + \eta' u_{ij}) + F^{ij}\Big(\frac{\zeta_{ij}}{\zeta} - \frac{\zeta_i \zeta_j}{\zeta^2}\Big). \tag{3.6}$$

By (3.5) and the Schwarz inequality, we have

$$F^{ij}\frac{\omega_i\omega_j}{\omega^2} \le (1+\alpha)\eta'^2 F^{ij}u_iu_j + \Big(1+\frac{1}{\alpha}\Big)F^{ij}\frac{\zeta_i\zeta_j}{\zeta^2}, \tag{3.7}$$

where α is any positive constant and we will choose a suitable one later. Note that

$$\omega_i = u_ku_{ki}, \quad \omega_{ij} = u_{ki}u_{kj} + u_ku_{kij}. \tag{3.8}$$

Substituting (3.7)–(3.8) into (3.6), we have

$$\begin{aligned}&\frac{1}{\omega}F^{ij}\Big(\delta_{kl} - \frac{u_ku_l}{2\omega}\Big)u_{ki}u_{lj} + \frac{1}{\omega}F^{ij}u_ku_{kij} + \eta' F^{ij}u_{ij}\\ &+ F^{ij}\Big(\eta'' - \frac{1+\alpha}{2}\eta'^2\Big)u_iu_j + F^{ij}\Big(\frac{\zeta_{ij}}{\zeta} - \frac{(1+3\alpha)\zeta_i\zeta_j}{2\alpha\zeta^2}\Big) \le 0.\end{aligned}$$

Note that the first term of the inequality above is non-negative, then by (3.2) and the Ricci identities, we have

$$\frac{1}{\omega}F^{ij}u_ku_{ijk} + \eta' F^{ij}u_{ij} + F^{ij}\Big(\eta'' - \frac{1+\alpha}{2}\eta'^2\Big)u_iu_j \le C\Big(1+\frac{1}{\zeta}\Big)\mathcal{F}, \tag{3.9}$$

where the constant C depends on n and b_0. Similarly, we can get

$$\frac{1}{\omega}\sum_{k,i}u_ku_{iik} + \eta'\triangle u + \Big(\eta'' - \frac{1+\alpha}{2}\eta'^2\Big)|\nabla u|^2 \le C\Big(1+\frac{1}{\zeta}\Big). \tag{3.10}$$

By (3.9)–(3.10), we have

$$\begin{aligned}&\frac{1}{\omega}u_kF^{ij}(u_{ijk} + \gamma u_{llk}\delta_{ij}) + \eta' F^{ij}(u_{ij} + \gamma\triangle u\delta_{ij})\\ &+ \Big(\eta'' - \frac{1+\alpha}{2}\eta'^2\Big)F^{ij}u_iu_j + \gamma\Big(\eta'' - \frac{1+\alpha}{2}\eta'^2\Big)|\nabla u|^2\mathcal{F} \le C\Big(1+\frac{1}{\zeta}\Big)\mathcal{F}.\end{aligned} \tag{3.11}$$

It follows from (3.5), (3.8) and (2.12) that

$$\begin{aligned}u_k(F^{ij}u_{ijk} + \gamma u_{llk}\delta_{ij}) = {}& u_k\nabla_k\Psi + 2\omega s\eta' F^{ij}u_iu_j - 2t\omega^2\eta'\mathcal{F} + 2b\omega\eta' F^{ij}u_if_j\\ &+ a\omega\eta'\langle\nabla u, \nabla f\rangle\mathcal{F} - au_ku_lf_{lk}\mathcal{F} - 2bF^{ij}u_iu_kf_{jk}\\ &- a_ku_k\langle\nabla u, \nabla f\rangle\mathcal{F} - 2b_ku_kF^{ij}u_if_j - u_kF^{ij}T_{ij,k}\\ &+ \frac{\omega}{\zeta}(2sF^{ij}u_j\zeta_i - t\langle\nabla\zeta, \nabla u\rangle\mathcal{F} + a\langle\nabla\zeta, \nabla f\rangle\mathcal{F} + 2bF^{ij}\zeta_if_j).\end{aligned}$$

By (3.2), the equality above implies

$$\begin{aligned}u_k(F^{ij}u_{ijk} + \gamma u_{llk}\delta_{ij}) \ge {}& u_k\nabla_k\Psi + 2\omega s\eta' F^{ij}u_iu_j - 2t\omega^2\eta'\mathcal{F}\\ &+ 2b\omega\eta' F^{ij}u_if_j + a\omega\eta'\langle\nabla u, \nabla f\rangle\mathcal{F} - C\omega\mathcal{F}\\ &- C|\nabla u|\mathcal{F} - C\frac{1}{\zeta}\omega(1+|\nabla u|)\mathcal{F},\end{aligned} \tag{3.12}$$

where the constant C depends only on $g, b_0, |s|, |t|, \|a\|_{C^1(\overline{M})}, \|b\|_{C^1(\overline{M})}, \|T\|_{C^1(\overline{M})}$ and $\|f\|_{C^2(\overline{M})}$.

Since F is homogeneous and of degree one, then

$$F^{ij}(u_{ij} + \gamma\triangle u\delta_{ij}) = \Psi - F^{ij}\Big(su_iu_j - \frac{t}{2}|\nabla u|^2\delta_{ij} + a\langle\nabla u, \nabla f\rangle\delta_{ij} + 2bu_if_j + T_{ij}\Big). \tag{3.13}$$

Note that $\nabla_k \Psi = \Psi_k + \Psi_u u_k$. It follows from (3.11)–(3.13) that

$$\begin{aligned}&\Big(\eta'' - \frac{1+\alpha}{2}\eta'^2 + s\eta'\Big)F^{ij}u_iu_j + \Big[2\gamma\Big(\eta'' - \frac{1+\alpha}{2}\eta'^2\Big) - t\eta'\Big]\omega\mathcal{F}\\ &\le C\Big(1+\frac{1}{|\nabla u|}\Big) + C\Big(\frac{1}{|\nabla u|} + \frac{1}{\zeta} + \frac{1}{\zeta}|\nabla u|\Big)\mathcal{F},\end{aligned} \tag{3.14}$$

where C also depends on $\gamma, g, \|T\|_{g(\overline{M})}$ and $\|\Psi\|_{C^1(\overline{M}\times[-C_0,C_0])}$.

Let $\eta(u) = v^{-N}$, where N is a large positive constant to be determined later and

$$v = 1 + u - \inf_{\{\zeta>0\}} u.$$

Since $0<\eta<1$, $\eta' = -N\frac{\eta}{v} < 0$ and $\eta'' = \frac{N\eta}{v^2}(N+1) > 0$, then

$$\eta'' - \frac{1+\alpha}{2}\eta'^2 = \frac{N\eta}{v^2}\Big(N+1-\frac{1+\alpha}{2}N\eta\Big) > \frac{N\eta}{v^2}\Big(1+\frac{1-\alpha}{2}N\Big).$$

Then for any fixed $\alpha \in (0,1)$, we have

$$\eta'' - \frac{1+\alpha}{2}\eta'^2 > \frac{(1-\alpha)N^2\eta}{2v^2} = \frac{(1-\alpha)N}{2v}|\eta'|.$$

Note that $1 \le v < 2C_0$. We may choose $\alpha = \frac{1}{2}$ and N large enough such that

$$\begin{cases}\eta'' - \dfrac{1+\alpha}{2}\eta'^2 + s\eta' > \Big(\dfrac{1}{4v}N - |s|\Big)|\eta'| > 0,\\ 2\gamma\Big(\eta'' - \dfrac{1+\alpha}{2}\eta'^2\Big) - t\eta' > \Big(\dfrac{\gamma N}{2v} - |t|\Big)|\eta'| > \dfrac{N}{4v}|\eta'|.\end{cases} \tag{3.15}$$

Therefore, by (3.14)–(3.15),

$$\frac{N}{4v}|\eta'|\omega\mathcal{F} \le C\Big(1+\frac{1}{|\nabla u|}\Big) + C\Big(\frac{1}{|\nabla u|} + \frac{1}{\zeta} + \frac{1}{\zeta}|\nabla u|\Big)\mathcal{F}.$$

We can assume $|\nabla u(x_0)| \ge 1$ (otherwise we have done). Then

$$\Big(\frac{N|\eta'|}{4v}|\nabla u|^2 - C\frac{1}{\zeta}(|\nabla u|+1)\Big)\mathcal{F} \le C, \tag{3.16}$$

which implies that

$$\sqrt{\zeta}|\nabla u|(x_0) \le C,$$

where the constant C also depends on N and C_0, from which it is easy to derive (3.3).

Choosing $\zeta \equiv 1$, we obtain the following global gradient estimates by (2.8) and (3.3).

Proposition 3.2 *Let $T \in \Gamma$ and $\Psi(x,z) \in C^\infty(\overline{M}\times\mathbb{R})$. Then for any admissible solution $u \in C^3(\overline{M})$ of (1.7), we have*

$$\sup_{\overline{M}} |\nabla u| \le C_1, \tag{3.17}$$

where the constant C_1 depends on $g, \gamma, |s|, |t|, \|T\|_{C^1(\overline{M})}, \|a\|_{C^1(\overline{M})}, \|b\|_{C^1(\overline{M})}, \|f\|_{C^2(\overline{M})}, \|\Psi\|_{C^1(\overline{M})}$ and $\max_{\overline{M}} |u|$.

3.3 Global C^2 estimate

As in Section 3.2, we first establish the interior second derivative estimates.

Lemma 3.2 *Let $T \in \Gamma$, $\Psi(x,z) \in C^\infty(M \times \mathbb{R})$, and $u \in C^4(B_r)$ be an admissible solution of* (1.7)*. Then there exists a constant C depending only on $g, b_0, \gamma, r^{-1}, |s|, |t|$, $\|T\|_{C^2(\overline{M})}$, $\|a\|_{C^2(\overline{M})}$, $\|b\|_{C^2(\overline{M})}$, $\|f\|_{C^3(\overline{M})}$, $\|\Psi\|_{C^2(\overline{M})}$ and $\|u\|_{C^1(B_r)}$ such that*

$$\sup_{B_{\frac{r}{2}}} |\nabla^2 u| \leq C. \tag{3.18}$$

Proof Consider the following auxiliary function:

$$H = \zeta(x) \mathrm{e}^{\eta(\omega)} (\nabla_{\xi\xi} u + s|\nabla_\xi u|^2),$$

where $\xi \in T_x M$ is a unit vector, $\omega = \frac{1}{2}|\nabla u|^2$, $\zeta \in C_0^\infty(B_r)$ satisfies $\zeta|_{B_{\frac{r}{2}}} = 1$, $\zeta|_{M \setminus B_r} = 0$ and (3.2), and the function η is chosen later. Suppose that H attains its maximum at a point $x_0 \in B_r$ and $\xi \in T_{x_0} M$. Choose a local orthnormal frame $\{e_i, i = 1, \cdots, n\}$ at x_0 with respect to g such that $e_1(x_0) = \xi$ and $\{W[u]_{ij}\}(x_0)$ is diagonal. Denote $K = \nabla_{11} u + s|\nabla_1 u|^2$. Without loss of generality, we can assume $K(x_0) > 1$. Then at x_0, we have

$$0 = (\log H)_i = \frac{K_i}{K} + \eta_i + \frac{\zeta_i}{\zeta}, \tag{3.19}$$

that is

$$K_i = -K\eta_i - \frac{\zeta_i}{\zeta} K \tag{3.20}$$

and

$$0 \geq F^{ii} (\log H)_{ii} = F^{ii} \Big(\frac{K_{ii}}{K} - \frac{K_i^2}{K} + \eta_{ii} + \frac{\zeta_{ii}}{\zeta} - \frac{\zeta_i^2}{\zeta} \Big). \tag{3.21}$$

By (3.20)–(3.21), we have

$$0 \geq F^{ii} K_{ii} + K F^{ii} (\eta_{ii} - 2\eta_i^2) + K F^{ii} \Big(\frac{\zeta_{ii}}{\zeta} - 3\frac{\zeta_i^2}{\zeta} \Big). \tag{3.22}$$

Note that

$$K_i = u_{11i} + 2su_1 u_{1i}, \quad K_{ii} = u_{11ii} + 2su_{1i}^2 + 2su_1 u_{1ii}. \tag{3.23}$$

By (3.22)–(3.23) and the Ricci identities, we have

$$\begin{aligned} 0 \geq{} & F^{ii} u_{ii11} + 2sF^{ii} u_{1i}^2 + 2su_1 F^{ii} u_{ii1} + KF^{ii}(\eta_{ii} - 2\eta_i^2) \\ & - \frac{C}{\zeta} K\mathcal{F} - C(1 + |\nabla^2 u|)\mathcal{F}, \end{aligned} \tag{3.24}$$

where the constant C depends only on g, b_0 and $\max\limits_{B_r} |\nabla u|$. Similarly, we have

$$\begin{aligned} 0 \geq{} & (\triangle u)_{11} + 2s \sum_i u_{1i}^2 + 2su_1 (\triangle u)_1 + K(\triangle \eta - 2|\nabla \eta|^2) \\ & - \frac{C}{\zeta} K - C(1 + |\nabla^2 u|). \end{aligned} \tag{3.25}$$

By (3.24)–(3.25), we have

$$0 \geq F^{ii}(u_{ii11} + \gamma(\triangle u)_{11}\delta_{ii}) + 2sF^{ii}u_{1i}^2 + 2s\gamma u_{1i}^2\mathcal{F} + 2su_1F^{ii}(u_{ii1} + \gamma(\triangle u)_1\delta_{ii}) \\ + KF^{ii}(\eta_{ii} - 2\eta_i^2) + \gamma K(\triangle\eta - 2|\nabla\eta|^2)\mathcal{F} - \frac{C}{\zeta}K\mathcal{F} - C(1 + |\nabla^2 u|)\mathcal{F}. \tag{3.26}$$

By (2.12), we have

$$\begin{aligned} &2su_1F^{ii}(u_{ii1} + \gamma(\triangle u)_1\delta_{ii}) \\ \geq\ &2su_1\nabla_1\Psi - 4s^2u_1F^{ii}u_{i1}u_i + 2stu_1u_lu_{l1}\mathcal{F} \\ &- 2su_1au_{l1}f_l\mathcal{F} - 4sbu_1F^{ii}u_{i1}f_i - C\mathcal{F}, \end{aligned} \tag{3.27}$$

where the constant C depends on $g, \gamma, |s|, |t|, \max_{B_r}|\nabla u|, \|T\|_{C^1(M)}, \|a\|_{C^1(M)}, \|b\|_{C^1(M)}$ and $\|f\|_{C^2(M)}$.

Differentiating the equation (1.7) twice, by the concavity of F, we obtain

$$\nabla_{11}\Psi \leq F^{ii}(u_{ii11} + \gamma(\triangle u)_{11}\delta_{ii}) + 2sF^{ii}(u_{i11}u_i + u_{i1}^2) - (tu_lu_{l11} + tu_{l1}^2)\mathcal{F} \\ + au_{l11}f_l\mathcal{F} + 2bF^{ii}u_{i11}f_i + C(1 + |\nabla^2 u|)\mathcal{F}, \tag{3.28}$$

where the constant C depends only on $g, \gamma, |s|, |t|$, $\|T\|_{C^2(M)}, \|a\|_{C^2(M)}$, $\|b\|_{C^2(M)}$, $\|f\|_{C^3(M)}$ and $\max_{B_r}|\nabla u|$.

Substituting (3.27)–(3.28) into (3.26) , we have

$$0 \geq \nabla_{11}\Psi + 2su_1\nabla_1\Psi + (2s\gamma + t)u_{1i}^2\mathcal{F} - 2sF^{ii}u_{i11}u_i + tu_lu_{l11}\mathcal{F} - au_{l11}f_l\mathcal{F} \\ - 2bF^{ii}u_{i11}f_i - 4s^2u_1F^{ii}u_{i1}u_i + 2stu_1u_lu_{l1}\mathcal{F} - 2asu_1u_{1l}f_l\mathcal{F} - 4sbu_1u_{i1}F^{ii}f_i \\ + KF^{ii}(\eta_{ii} - 2\eta_i^2) + \gamma K(\triangle\eta - 2|\nabla\eta|^2)\mathcal{F} - \frac{C}{\zeta}K\mathcal{F} - C(1 + |\nabla^2 u|)\mathcal{F}. \tag{3.29}$$

By (3.20), (3.23) and the Ricci identities, we have

$$u_{i11} = -2su_1u_{1i} - K\eta' u_lu_{1l} - \frac{\zeta_i}{\zeta}K + u_pR^p_{11i}. \tag{3.30}$$

Note that

$$\begin{cases} \nabla_1\Psi = \Psi_1 + \Psi_z u_1 \geq -C, \\ \nabla_{11}\Psi = \Psi_{11} + 2\Psi_{1z}u_1 + \Psi_{zz}u_1^2 + \Psi_z u_{11} \geq -C(K+1), \end{cases} \tag{3.31}$$

where the constant C depends only on $|s|, \|\Psi\|_{C^2(\overline{M}\times[-C_0,C_0])}$ and $\max_{B_r}|\nabla u|$.

Substituting (3.30)–(3.31) into (3.29), we have

$$0 \geq -C(K+1) + (2s\gamma + t)u_{1i}^2\mathcal{F} + KF^{ii}(\eta_{ii} - 2\eta_i^2) + \gamma K(\triangle\eta - 2|\nabla\eta|^2)\mathcal{F} \\ - C\Big(|\nabla^2 u| + \frac{1}{\zeta}\Big)K\mathcal{F} - C(1 + |\nabla^2 u|)\mathcal{F}. \tag{3.32}$$

Since $\{W[u]_{ij}(x_0)\}$ is diagonal, $|u_{ij}| = |-su_iu_j - 2bu_if_j - T_{ij}| \leq C$ for $i \neq j$. Hence,

$$\sum_i u_{1i}^2 = u_{11}^2 + \sum_{i\neq 1} u_{1i}^2 \leq K^2 + CK + C. \tag{3.33}$$

Note that $1 < K < C(|\nabla^2 u| + 1)$. Then by (3.32)–(3.33), we have

$$F^{ii}(\eta_{ii} - 2\eta_i^2) + \gamma(\triangle\eta - 2|\nabla\eta|^2)\mathcal{F} \le C + C\Big(1 + |\nabla^2 u| + \frac{1}{\zeta}\Big)\mathcal{F}. \tag{3.34}$$

Since

$$\eta_i = \eta'\omega_i = \eta' u_k u_{ki}, \quad \eta_{ii} = \eta'' u_k u_{ki} u_l u_{li} + \eta' u_{ki}^2 + \eta' u_k u_{kii},$$

we have

$$\begin{aligned} &F^{ii}(\eta_{ii} - 2\eta_i^2) + \gamma(\triangle\eta - 2|\nabla\eta|^2)\mathcal{F} \\ =& (\eta'' - 2\eta'^2)(F^{ii} + \gamma\delta_{ii}\mathcal{F})u_1 u_{ki} u_l u_{li} \\ &+ \eta' F^{ii} u_{ki}^2 + \eta'\gamma|\nabla^2 u|^2\mathcal{F} + \eta' u_k F^{ii}(u_{kii} + \gamma u_{kll}\delta_{ii}). \end{aligned} \tag{3.35}$$

Let

$$\eta(\omega) = \Big(1 - \frac{\omega}{M}\Big)^{-\alpha}, \quad \omega \in \Big[0, \frac{1}{2}M\Big],$$

where $M = 2\sup_M \omega$ and α is a small positive constant to be chosen later. Then we have

$$1 \le \eta \le 2^\alpha, \quad \eta' = \frac{\alpha\eta}{M\Big(1 - \dfrac{\omega}{M}\Big)} > 0.$$

If we choose $\alpha \le \frac{1}{2}$, then

$$\eta'' - 2\eta'^2 = \frac{\alpha(\alpha + 1 - 2\alpha\eta)\eta}{M^2\Big(1 - \dfrac{\omega}{M}\Big)^2} > 0.$$

Then by (2.12) and (3.35), we have

$$F^{ii}(\eta_{ii} - 2\eta_i^2) + \gamma(\triangle\eta - 2|\nabla\eta|^2)\mathcal{F} \ge \eta'\gamma|\nabla^2 u|^2\mathcal{F} - C - C(1 + |\nabla^2 u|)\mathcal{F}. \tag{3.36}$$

Combining (3.34) and (3.36), we have

$$\eta'\gamma|\nabla^2 u|^2\mathcal{F} \le C + C\Big(1 + \frac{1}{\zeta} + |\nabla^2 u|\Big)\mathcal{F}. \tag{3.37}$$

Multiplying ζ on both sides of (3.37), we obtain

$$\sqrt{\zeta}|\nabla^2 u|(x_0) \le C. \tag{3.38}$$

Hence, $\sqrt{\zeta}K(x_0) \le C$. This implies that

$$\sqrt{\zeta}\nabla_{\xi\xi}u \le C, \quad \forall x \in \{\zeta > 0\},\ \forall \xi \in T_x M,\ |\xi| = 1. \tag{3.39}$$

Since $\lambda(g^{-1}W[u]) \in \Gamma \subset \Gamma_1^+$, $\triangle u$ has a lower bound by Lemma 3.2. Then by (3.39), we get

$$\sqrt{\zeta}\nabla_{\xi\xi}u \ge -C, \quad \forall x \in \{\zeta > 0\},\ \forall \xi \in T_x M,\ |\xi| = 1. \tag{3.40}$$

Thus, (3.18) follows from (3.39)–(3.40).

Let $\zeta \equiv 1$. By (3.18) and (2.10), we derive the following global estimate for the second derivatives.

Proposition 3.3 *Let $T \in \Gamma$ and $\Psi(x,z) \in C^\infty(\overline{M} \times \mathbb{R})$. Then for any admissible solution $u \in C^4(\overline{M})$ of* (1.7)*, we have*

$$\sup_{\overline{M}} |\nabla^2 u| \le C_2, \tag{3.41}$$

where the constant C_2 depends only on $g, \gamma, |s|, |t|$, $\|T\|_{C^2(\overline{M})}$, $\|a\|_{C^2(\overline{M})}$, $\|b\|_{C^2(\overline{M})}$, $\|f\|_{C^3(\overline{M})}$, $\|\Psi\|_{C^2(\overline{M})}$ and $\|u\|_{C^1(\overline{M})}$.

4 Proof of Theorem 1.1

For any function h on M, define

$$\mathcal{P}[h] := F(W[h]) - \Psi(x,h).$$

Then any solution u of (1.7) satisfies $\mathcal{P}[u] = 0$. Let $u_p = u + pv$ for $p \in \mathbb{R}$. The linearized operator of equation (1.7) is

$$\begin{aligned}\mathcal{L}v :&= \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{P}[u_s]|_{s=0} \\ &= (F^{ij} + \delta_{ij}\gamma\mathcal{F})v_{ij} + 2sF^{ij}v_iu_j - (tv_lu_l - av_lf_l)\mathcal{F} + 2bF^{ij}v_if_j - \partial_z\Psi(x,u)v.\end{aligned} \tag{4.1}$$

Lemma 4.1 *Let $u \in C^2(\overline{M})$ be an admissible solution of equation* (1.7)*. If $\partial_z\Psi$ is positive on $M \times \mathbb{R}$, then $\mathcal{L} : C^{2,\alpha}(\overline{M}) \to C^{2,\alpha}(\overline{M})$ $(0 < \alpha < 1)$ is invertible.*

Proof Since $\partial_z\Psi$ is positive on $\overline{M} \times \mathbb{R}$, the coefficient of the zero order term in (4.1) is strictly negative. Hence, $\mathcal{L}$ is invertible in the Hölder space $C^{2,\alpha}(\overline{M})$.

Proof of Theorem 1.1 Note that the maximum principle in Proposition 2.1 ensures the uniqueness of solutions of (1.7). Now, we complete the proof by using the continuity method. Consider the following equation:

$$F(\nabla^2_{\mathrm{con}\,f}u + \nabla^u_{\mathrm{con}\,f}f + T^\beta) = \Psi_\beta(x,u), \quad \beta \in [0,1], \tag{4.2}$$

where

$$T_\beta = \beta T + \frac{1-\beta}{F(e)}g, \quad \Psi_\beta(x,u) = (1-\beta)\mathrm{e}^{2u} + \beta\Psi(x,u).$$

Clearly, T_β and Ψ_β satisfy the following conditions:

(1) $T_\beta \in \Gamma$ and $\|T_\beta\|_{C^4(M)} \le C$, where the constant C is independent of β;

(2) $\Psi_\beta(x,u) > 0$, $\partial_z\Psi_\beta > 0$, $\lim\limits_{z\to+\infty}\Psi_\beta(x,z) \to +\infty$ and $\lim\limits_{z\to-\infty}\Psi_\beta(x,z) \to 0$;

(3) $\|\Psi_\beta\|_{C^2(M\times[-C,C])} \le C$, where C is independent of β.

It follows from Section 2 and Section 3 that for each β, the admissible solution of (4.2) has uniform a priori C^2 estimates (independent of β). Then we obtain the uniform $C^{2,\alpha}$ estimates by the Evans-Krylov's theory. Define

$$I = \{\beta \in [0,1] \mid (4.2) \text{ has an admissible solution}\}.$$

Clearly, $u \equiv 0$ is the unique admissible solution of (4.2). Hence, $I \ne \varnothing$. Then by Proposition 2.1, $I \subset [0,1]$ is open. By the uniform a priori $C^{2,\alpha}$ estimates and the standard degree theory, we conclude that I is also closed. For $\beta = 1$, the equation (1.7) is solvable.

5 Proof of Theorem 1.2

The argument of the proof is similar to the one in [4, 10]. To solve the infinite boundary data Dirichlet problem (1.10), we consider a family of equations below

$$\begin{cases} F(W[u]) = \dfrac{\psi(x)}{n-2}\mathrm{e}^{2u} & \text{in } M, \\ u = \theta \log m & \text{on } \partial M, \end{cases} \tag{5.1}$$

where m is any positive integer and θ is a positive constant which will be chosen later.

For any fixed m, it follows from Theorem 1.1 that (5.1) has a unique admissible solution $u_m \in C^\infty(\overline{M})$. The maximum principle implies that

$$u_m \le u_{m+1}, \quad m = 1, 2, \cdots. \tag{5.2}$$

Next, for any m and a small $\delta > 0$, define a local barrier function by

$$u_m^- = \theta \log \frac{m\delta^2}{m\rho + \delta^2},$$

where $\rho(x) = \mathrm{dist}_g(x, \partial M),\ x \in \overline{M}$. Then $u_m^-|_{\partial M} = u_m|_{\partial M}$, and

$$\theta \log \frac{\delta}{2} \le u_m^-|_{\{\rho(x)=\delta\}} \le \theta \log \delta.$$

Therefore, we can choose δ small enough such that $u_m^- \le \min\limits_{\overline{M}} u_m$ on the boundary of M_δ in M. By a direct calculation, we have

$$W[u_m^-]_{ij} \ge \frac{\theta m^2}{(m\rho+\delta^2)^2}\Big(\gamma - \frac{t\theta}{2}\Big)|\nabla\rho|^2\delta_{ij} + \frac{\theta m^2(1+s\theta)}{(m\rho+\delta^2)^2}\rho_i\rho_j - C'\frac{\theta m}{m\rho+\delta^2}\delta_{ij} - C''(T)\delta_{ij},$$

where the constants C' and C'' depend only on $\|a\|_{L^\infty(\overline{M})}$, $\|b\|_{L^\infty(\overline{M})}$, $\|f\|_{C^1(\overline{M})}$, $\|T\|_{g(\overline{M})}$ and other known data.

Choose $\theta^{-1} \ge \max\big\{1, -s, \frac{t}{\gamma}\big\}$ and $\delta < \min\big\{1, \frac{\gamma}{64C'}\frac{\gamma\theta}{64C''}\big\}$. By $|\nabla\rho| \ge \frac{1}{2}$ in M_δ and (5.3), we have

$$W[u_m^-]_{ij} \ge \frac{\gamma\theta m^2}{32(m\rho+\delta^2)^2}g_{ij} + \frac{\theta m^2(1+s\theta)}{(m\rho+\delta^2)^2}\rho_i\rho_j.$$

If we require $\delta < \frac{(n-2)\gamma\theta F(e)}{32\max\limits_{\overline{M}}\psi(x)}$, then

$$F(W[u_m^-]) \ge \frac{\gamma\theta m^2}{32(m\rho+\delta^2)^2}F(e) = \frac{\gamma\theta}{32\delta^2}F(e)\cdot \mathrm{e}^{\frac{2u_m^-}{\theta}} \ge \frac{\max\psi(x)}{n-2}\mathrm{e}^{2u_m^-} \quad \text{in } M_\delta.$$

Therefore, the maximum principle implies that

$$u_m \ge u_m^- \quad \text{on } M_\delta. \tag{5.4}$$

In order to control the upper bound of u_m, we consider the following equation:

$$\Lambda(v) := A\triangle v + B|\nabla v|^2 + C|\nabla v| + D = \mathrm{e}^{2v}, \tag{5.5}$$

where

$$A = \frac{\Theta(n\gamma+1)(n-2)}{\min\limits_{\overline{M}}\psi}, \quad B = \frac{\Theta(2s-tn)(n-2)}{2\min\limits_{\overline{M}}\psi},$$
$$C = \frac{\Theta(n\|a\|_{L^\infty}+2\|b\|_{L^\infty})(n-2)}{\|\nabla f\|_{L^\infty}^{-1}\min\limits_{\overline{M}}\psi}, \quad D = \frac{\Theta(n-2)\|\mathrm{tr}\, T\|_{L^\infty}}{\min\limits_{\overline{M}}\psi}.$$

By (5.1), every u_m is a subsolution of (5.5). Thus, we only need to construct a local supsolution of (5.5). For any fixed point $y_0 \in M_{\frac{\delta}{2}}$, let x_0 be the nearest point of y_0 on ∂M. Choose the geodesic from x_0 to y_0, passing through y_0 and going out a small distance to another point z_0. Denote $\rho_0 = \mathrm{dist}_g(x_0, y_0)$, $R = \mathrm{dist}_g(z_0, x_0)$ and $r(x) = \mathrm{dist}_g(z_0, x)$ for any point $x \in \overline{M}$. When ρ_0 and δ are small enough, we can assume that $r(x)$ is smooth in the ball $B_{z_0}(R)$. Choose a local orthonormal frame $\{e_i, i = 1, \cdots, n\}$ at z_0. Note that A, C and D are positive constants.

Case (a) If $2s - nt > 0$, then $B > 0$. Consider the function $\widehat{v} \in B_{z_0}(R)$ defined by

$$\begin{aligned}\overline{v} = &-\log(R^2 - r^2(x)) + \tau\log\frac{R^2 - r^2(x)+\varepsilon}{\varepsilon} + \log 2\\ &+ \frac{1}{2}\log((n+1)A + B + RC) + \log R,\end{aligned}$$

where τ and ε are two positive constants to be chosen later. By a direct calculation, we have

$$\nabla\overline{v} = \frac{2r\nabla r}{R^2-r^2} - \frac{\tau 2r\nabla r}{R^2-r^2+\varepsilon} = 2r\nabla r\Big(\frac{1}{R^2-r^2} - \frac{\tau}{R^2-r^2+\varepsilon}\Big)$$

and

$$\triangle\overline{v} = \frac{\triangle r^2}{R^2-r^2} - \frac{\tau\triangle r^2}{R^2-r^2+\varepsilon} + \frac{4r^2|\nabla r|^2}{(R^2-r^2)^2} - \frac{4\tau r^2|\nabla r|^2}{(R^2-r^2+\varepsilon)^2}.$$

Note that $|\nabla r| = 1$. Then we have

$$\begin{aligned}\Lambda(\overline{v}) = &\frac{A\triangle r^2 + 2rC}{R^2-r^2} - \frac{\tau A\triangle r^2 + 2\tau rC}{R^2-r^2+\varepsilon} + \frac{4(A+B)r^2}{(R^2-r^2)^2} + \frac{4\tau r^2(B\tau - A)}{(R^2-r^2+\varepsilon)^2}\\ &- \frac{8\tau Br^2}{(R^2-r^2+\varepsilon)(R^2-r^2)} + D.\end{aligned} \tag{5.6}$$

Since $\triangle r^2(z_0) = 2n$, we can assume $n \le \triangle r^2 \le 3n$ in $B_{z_0}(R)$. Also note that $B > 0$, and then by (5.6) we obtain

$$\begin{aligned}\Lambda(\overline{v}) &\le \frac{3nA + 2rC}{R^2-r^2} - \frac{\tau An + 2\tau rC}{R^2-r^2+\varepsilon} + \frac{4(A+B)r^2}{(R^2-r^2)^2} + \frac{4\tau r^2(B\tau - A)}{(R^2-r^2+\varepsilon)^2} + D\\ &= \frac{(3nA+2rC)(R^2-r^2) + 4(A+B)r^2}{(R^2-r^2)^2} - \frac{\tau(An+2rC)}{R^2-r^2+\varepsilon} + \frac{4\tau r^2(B\tau - A)}{(R^2-r^2+\varepsilon)^2} + D\\ &\le \frac{[3nA + 2rC + 4(A+B)]R^2}{(R^2-r^2)^2} - \frac{\tau An}{R^2-r^2+\varepsilon} + \frac{4\tau r^2(B\tau - A)}{(R^2-r^2+\varepsilon)^2} + D.\end{aligned} \tag{5.7}$$

Choose $\tau \le \frac{A}{B}$, $\varepsilon < R^2$ and $R \le \sqrt{\frac{\tau nA}{2D}}$. It follows from (5.7) that

$$\begin{aligned}\Lambda(\overline{v}) &\le \frac{R^2}{(R^2-r^2)^2}[3nA + 2rC + 4(A+B)]\\ &< \frac{4R^2}{(R^2-r^2)^2}[(n+1)A + B + RC]\\ &\le \mathrm{e}^{2\overline{v}}.\end{aligned}$$

Note that $\overline{v}$ is infinite on $\partial B_{z_0}(R)$. For any $m \geq 1$, applying the maximum principle on this ball, we conclude that $u_m \leq \overline{v}$ on $\overline{B_{z_0}(R)}$. Thus,

$$\begin{aligned}
u_m(y_0) \leq \overline{v}(y_0) &= -\log \rho_0(2R-\rho_0) + \tau \log \frac{\rho_0(2R-\rho_0)+\varepsilon}{\varepsilon} + \log 2 \\
&\quad + \frac{1}{2}\log((n+1)A + B + RC) + \log R \\
&= -\log \rho_0 - \log \frac{(2R-\rho_0)}{2R} + \tau \log \frac{\rho_0(2R-\rho_0)+\varepsilon}{\varepsilon} \\
&\quad + \frac{1}{2}\log((n+1)A + B + RC).
\end{aligned}$$

So

$$u_m \leq -\log \rho + C. \tag{5.8}$$

By (5.2), (5.4) and (5.8), we have

$$u(x) := \lim_{m\to\infty} u_m(x) \quad \text{for all } x \in M$$

and

$$-C - \theta \log \rho \leq u(x) \leq -\log \rho + C \tag{5.9}$$

near the boundary ∂M.

For any compact subset $K \subset \overline{M}$, by the boundary control (5.9) and the a priori estimates in Section 2 and Section 3, we obtain

$$\|u_m\|_{C^{2,\alpha}(K)} \leq C,$$

where $0 < \alpha < 1$, and $C = C(K)$ is independent of m. Hence, the standard compactness argument and the Schauder regularity theory imply that $u \in C^\infty(\overline{M})$ is an admissible solution of (1.10).

Case (b) If $2s - nt \leq 0$, then $B \leq 0$. This case is much simpler than Case (a). Set $\widehat{v} \in B_{z_0}(R)$ defined by

$$\widehat{v} = -\log(R^2 - r^2(x)) + \log \frac{R^2 - r^2(x) + \varepsilon}{\varepsilon} + \log 2 + \frac{1}{2}\log((n+1)A + RC) + \log R,$$

where ε is a positive constant to be decided. Then

$$\begin{aligned}
\Lambda(\widehat{v}) &\leq A\triangle\widehat{v} + C|\nabla\widehat{v}| + D \\
&= \frac{A\triangle r^2 + 2rC}{R^2 - r^2} - \frac{A\triangle r^2 + 2rC}{R^2 - r^2 + \varepsilon} + \frac{4Ar^2}{(R^2-r^2)^2} - \frac{4Ar^2}{(R^2-r^2+\varepsilon)^2} + D \\
&\leq \frac{R^2}{(R^2-r^2)^2}(3nA + 2rC + 4A) - \frac{An}{R^2 - r^2 + \varepsilon} + D.
\end{aligned} \tag{5.10}$$

Choose ε and R as above, i.e., $\varepsilon < R^2$ and $R \leq \sqrt{\frac{nA}{2D}}$. Thus, we obtain

$$\Lambda(\widehat{v}) \leq \frac{R^2}{(R^2-r^2)^2}(3nA + 2rC + 4A) < \frac{4R^2}{(R^2-r^2)^2}((n+1)A + RC) \leq \mathrm{e}^{2\widehat{v}}.$$

The remaining argument is similar to the part in Case (a), and we omit it here.

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