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Crossed Products over Weak Hopf Algebras Related to Cleft Extensions and Cohomology*

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Abstract The authors present the general theory of cleft extensions for a cocommutative weak Hopf algebra H. For a right H-comodule algebra, they obtain a bijective correspondence between the isomorphisms classes of H-cleft extensions $A_H \hookrightarrow A$, where A_H is the subalgebra of coinvariants, and the equivalence classes of crossed systems for H over A_H. Finally, they establish a bijection between the set of equivalence classes of crossed systems with a fixed weak H-module algebra structure and the second cohomology group $H^2_{\varphi_{Z(A_H)}}(H, Z(A_H))$, where $Z(A_H)$ is the center of A_H.

Keywords Monoidal category, Weak Hopf algebra, Cleft extension, Weak crossed product, Sweedler cohomology for weak Hopf algebras
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1 Introduction

Weak Hopf algebras (or quantum groupoids in the terminology of Nikshych and Vainerman [21]) were introduced by Böhm, Nill and Szlachányi [9] as a new generalization of Hopf algebras and groupoid algebras. The main differences with other Hopf algebraic constructions, such as quasi-Hopf algebras and rational Hopf algebras, are the following: Weak Hopf algebras are coassociative but the coproduct is not required to preserve the unity morphism or, equivalently, the counity is not an algebra morphism. Some motivations to study weak Hopf algebras come from their connection with the theory of algebra extensions, the important applications in the study of dynamical twists of Hopf algebras and their link with quantum field theories and operator algebras (see [21]). It is well-known that groupoid algebras of finite groupoids provide examples of weak Hopf algebras. If G is a finite groupoid (a category with a finite number of objects such that each morphism is invertible) then the groupoid algebra over a commutative ring R is an example of cocommutative weak Hopf algebra. There are more interesting examples of cocommutative weak Hopf algebras, for example recently Bulacu [11–12] proved that Cayley-

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Dickson and Clifford algebras provide examples of commutative and cocommutative weak Hopf algebras in some suitable symmetric monoidal categories of graded vector spaces.

As in the Hopf algebra setting, it is possible to define a theory of crossed products for weak Hopf algebras. The key to extend the crossed product constructions of the Hopf world to the weak setting is the use of idempotent morphisms combined with the ideas given in [10]. In [4, 16], the authors defined a product on $A\otimes V$, for an algebra A and an object V both living in a strict monoidal category $\mathcal{C}$, where every idempotent splits. In order to obtain that product, we must consider two morphisms $\psi_V^A : V\otimes A \to A\otimes V$ and $\sigma_V^A : V\otimes V \to A\otimes V$ which satisfy some twisted-like and cocycle-like conditions, respectively. Associated to these morphisms, it is possible to define an idempotent morphism $\nabla_{A\otimes V} : A\otimes V \to A\otimes V$, that becomes the identity in the classical case. The image of this idempotent inherits the associative product from $A\otimes V$. In order to define a unit for $\mathrm{Im}\,(\nabla_{A\otimes V})$, and hence to obtain an algebra structure, we require the existence of a preunit $\nu : K \to A\otimes V$, and, under these conditions, it is possible to characterize weak crossed products with a unit as products on $A\otimes V$ that are morphisms of left A-modules with preunit. Finally, it is convenient to observe that, if the preunit is a unit, the idempotent becomes the identity, and we recover the classical examples of the Hopf algebra setting. The theory presented in [4, 16] contains as a particular instance the one developed by Brzeziński in [10] as well as all the crossed products constructed in the weak setting, for example the ones defined in [13, 18, 22]. Recently, Böhm [8] showed that a monad in the weak version of the Lack and Street's 2-category of monads in a 2-category is identical to a crossed product system in the sense of [4], and also in [17] we can find that unified crossed products (see [1]) and partial crossed products (see [20]) are particular instances of weak crossed products. An interesting example of weak crossed product comes from the theory of weak cleft extensions associated to weak Hopf algebras. This notion was introduced in [2], and it provides examples of weak crossed products satisfying twisted and cocycle conditions (see [4]). These crossed products are deeply connected with Galois theory as we can see in the intrinsic characterization of weak cleftness in terms of weak Galois extensions with normal basis obtained in [19]. We want to point out that, when we particularize this weak cleft theory to the Hopf algebra setting, we obtain a more general notion than the usual one of cleft extension (see [19, Definition 7.2.1]) because in this case the uniqueness of the cleaving morphism is not guaranteed.

The theory of crossed products in the Hopf setting arises as a generalization of the classical smash products, and by the results obtained by Doi and Takeuchi [15], we know that every cleft extension $D \hookrightarrow A$ with cleaving morphism f, such that $f(1_H) = 1_A$ induces a crossed product $D\sharp_\sigma H$, where $\sigma : H\otimes H \to D$ is a suitable convolution invertible morphism (a normal 2-cocycle). Conversely, in [7] we can find the reverse result, that is, if $D\sharp_\sigma H$ is a crossed product, the extension $D \hookrightarrow D\sharp_\sigma H$ is cleft. On the other hand, Sweedler [23] introduced the cohomology of a cocommutative Hopf algebra H with coefficients in a commutative H-module algebra A. We will denote these cohomology groups as $H_{\varphi_A}(H^\bullet, A)$, where φ_A is a fixed action of H over A. In [23], we can find an interesting interpretation of the second cohomology group $H^2_{\varphi_A}(H, A)$ in terms of extensions: This group classifies the set of equivalence classes of cleft extensions, i.e., classes of equivalent crossed products determined by a 2-cocycle. This result was extended by Doi [14] proving that, in the noncommutative case, there exists a bijection between the isomorphism classes of H-cleft extensions D of A and equivalence classes of crossed systems for H over A with a fixed action. If H is cocommutative, the equivalence is described by $H^2_{\varphi_{\mathcal{Z}(A)}}(H, \mathcal{Z}(A))$, where $\mathcal{Z}(A)$ is the center of A.

The aim of this paper is to extend the preceding results to cocommutative weak Hopf algebras completing the program initiated in [5]. To do that, in the second section, we introduce the notion of H-cleft extension for a weak Hopf algebra H, and we prove that this kind of extensions are examples of weak cleft extensions as the ones introduced in [2] and satisfying that the classical notion used in the papers of Doi and Takeuchi is obtained when we particularize to the Hopf setting. We also prove that, under cocommutative conditions, we can assume without loss of generality that the associated cleaving morphism is a total integral. In the third section, assuming that H is cocommutative, we prove that it is possible to identify the set of crossed systems in a weak setting as the set of weak crossed products induced by a weak left action and a convolution invertible twisted normal 2-cocycle. As a consequence, we obtain the main result of this section that assures the following: If (A, ρ_A) is a right H-comodule algebra, there exists a bijective correspondence between the equivalence classes of H-cleft extensions $A_H \hookrightarrow A$ and the equivalence classes of crossed systems for H over A_H, where A_H denotes the subalgebra of coinvariants in the weak setting. Finally, in the fourth section, we generalize the result obtained by Doi and Takeuchi about the characterization of equivalence classes of crossed systems using the second Sweedler cohomology group. To obtain this generalization, we must use the cohomology theory of algebras over commutative weak Hopf algebras developed in [5]. The main result contained in [5, Theorem 3.11] asserts that if (A, φ_A) is a commutative left H-module algebra, there exists a bijection between the second cohomology group, denoted by $H^2_{\varphi_A}(H, A)$, and the equivalence classes of weak crossed products $A \otimes_\alpha H$, where $\alpha : H \otimes H \to A$ is a morphism satisfying normal and 2-cocycle conditions. Then, by this bijection and using the results of the previous sections, we obtain the description of the bijection between the isomorphism classes of H-cleft extensions $A_H \hookrightarrow B$ and the equivalence classes of crossed systems for H over A_H in terms of $H^2_{\varphi_{\mathcal{Z}(A_H)}}(H, \mathcal{Z}(A_H))$.

2 Integrals over Weak Hopf Algebras

From now on, $\mathcal{C}$ denotes a strict symmetric category with tensor product denoted by $\otimes$ and unit object K. With c, we will denote the natural isomorphism of symmetry, and we also assume that $\mathcal{C}$ has equalizers. Then, under these conditions, every idempotent morphism $q : Y \to Y$ splits, i.e., there exist an object Z and morphisms $i : Z \to Y$ and $p : Y \to Z$, such that $q = i \circ p$ and $p \circ i = \mathrm{id}_Z$. We denote the class of objects of $\mathcal{C}$ by $|\mathcal{C}|$ and for each object $M \in |\mathcal{C}|$, the identity morphism by $\mathrm{id}_M : M \to M$. For simplicity of notation, given objects M, N, P in $\mathcal{C}$ and a morphism $f : M \to N$, we write $P \otimes f$ for $\mathrm{id}_P \otimes f$ and $f \otimes P$ for $f \otimes \mathrm{id}_P$.

An algebra in $\mathcal{C}$ is a triple $A = (A, \eta_A, \mu_A)$, where A is an object in $\mathcal{C}$ and $\eta_A : K \to A$ (unit), $\mu_A : A \otimes A \to A$ (product) are morphisms in $\mathcal{C}$, such that $\mu_A \circ (A \otimes \eta_A) = \mathrm{id}_A = \mu_A \circ (\eta_A \otimes A)$, $\mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A)$. We will say that an algebra A is commutative if $\mu_A \circ c_{A,A} = \mu_A$. Given two algebras $A = (A, \eta_A, \mu_A)$ and $B = (B, \eta_B, \mu_B)$, $f : A \to B$ is an algebra morphism, if $\mu_B \circ (f \otimes f) = f \circ \mu_A$ and $f \circ \eta_A = \eta_B$. If A, B are algebras in $\mathcal{C}$, the object $A \otimes B$ is an algebra in $\mathcal{C}$, where $\eta_{A\otimes B} = \eta_A \otimes \eta_B$ and $\mu_{A\otimes B} = (\mu_A \otimes \mu_B) \circ (A \otimes c_{B,A} \otimes B)$.

For an algebra A, we define the center of A as a subalgebra $Z(A)$ of A with inclusion algebra morphism $i_{Z(A)} : Z(A) \to A$ satisfying $\mu_A \circ (A \otimes i_{Z(A)}) = \mu_A \circ c_{A,A} \circ (A \otimes i_{Z(A)})$, and such that if $f : B \to A$ is a morphism with $\mu_A \circ (A \otimes f) = \mu_A \circ c_{A,A} \circ (A \otimes f)$, there exists a unique morphism $f' : B \to Z(A)$ satisfying $i_{Z(A)} \circ f' = f$. Trivially $Z(A)$ is a commutative algebra.

A coalgebra in $\mathcal{C}$ is a triple $D = (D, \varepsilon_D, \delta_D)$, where D is an object in $\mathcal{C}$ and $\varepsilon_D : D \to K$ (counit), $\delta_D : D \to D \otimes D$ (coproduct) are morphisms in $\mathcal{C}$, such that $(\varepsilon_D \otimes D) \circ \delta_D = \mathrm{id}_D = (D \otimes \varepsilon_D) \circ \delta_D$, $(\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D$. We will say that D is cocommutative, if $c_{D,D} \circ \delta_D = \delta_D$ holds. If $D = (D, \varepsilon_D, \delta_D)$ and $E = (E, \varepsilon_E, \delta_E)$ are coalgebras, $f : D \to E$ is a coalgebra morphism, if $(f \otimes f) \circ \delta_D = \delta_E \circ f$ and $\varepsilon_E \circ f = \varepsilon_D$. When D, E are coalgebras in $\mathcal{C}$, $D \otimes E$ is a coalgebra in $\mathcal{C}$, where $\varepsilon_{D \otimes E} = \varepsilon_D \otimes \varepsilon_E$ and $\delta_{D \otimes E} = (D \otimes c_{D,E} \otimes E) \circ (\delta_D \otimes \delta_E)$.

If A is an algebra, B a coalgebra and $\alpha : B \to A$, $\beta : B \to A$ are morphisms, we define the convolution product by $\alpha \wedge \beta = \mu_A \circ (\alpha \otimes \beta) \circ \delta_B$.

Let A be an algebra. The pair (M, φ_M) is a left A-module, if M is an object in $\mathcal{C}$ and $\varphi_M : A \otimes M \to M$ is a morphism in $\mathcal{C}$ satisfying $\varphi_M \circ (\eta_A \otimes M) = \mathrm{id}_M$, $\varphi_M \circ (A \otimes \varphi_M) = \varphi_M \circ (\mu_A \otimes M)$. Given two right A-modules (M, φ_M) and (N, φ_N), $f : M \to N$ is a morphism of right A-modules if $\varphi_N \circ (A \otimes f) = f \circ \varphi_M$.

Let C be a coalgebra. The pair (M, ρ_M) is a right C-comodule, if M is an object in $\mathcal{C}$ and $\rho_M : M \to M \otimes C$ is a morphism in $\mathcal{C}$ satisfying $(M \otimes \varepsilon_C) \circ \rho_M = \mathrm{id}_M$, $(M \otimes \rho_M) \circ \rho_M = (M \otimes \delta_C) \circ \rho_M$. Given two right C-comodules (M, ρ_M) and (N, ρ_N), $f : M \to N$ is a morphism of right C-comodules, if $(f \otimes C) \circ \rho_M = \rho_N \circ f$.

By weak Hopf algebras, we understand the objects introduced in [9] as a generalization of ordinary Hopf algebras. Here we recall the definition of these objects in a monoidal symmetric setting.

Definition 2.1 *A weak Hopf algebra H is an object in $\mathcal{C}$ with an algebra structure (H, η_H, μ_H) and a coalgebra structure $(H, \varepsilon_H, \delta_H)$, such that the following axioms hold*:

(1)
$$\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ \delta_{H \otimes H}.$$

(2)
$$\begin{aligned} \varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) &= (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes \delta_H \otimes H) \\ &= (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes (c_{H,H} \circ \delta_H) \otimes H). \end{aligned}$$

(3)
$$\begin{aligned} (\delta_H \otimes H) \circ \delta_H \circ \eta_H &= (H \otimes \mu_H \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H) \\ &= (H \otimes (\mu_H \circ c_{H,H}) \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H). \end{aligned}$$

(4) *There exists a morphism $\lambda_H : H \to H$ in $\mathcal{C}$ (called the antipode of H) satisfying:*
(i) $\mathrm{id}_H \wedge \lambda_H = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)$,
(ii) $\lambda_H \wedge \mathrm{id}_H = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$,
(iii) $\lambda_H \wedge \mathrm{id}_H \wedge \lambda_H = \lambda_H$.

It is easy to see that a weak Hopf algebra is a Hopf algebra if and only if the morphism δ_H is unit-preserving or if and only if the counit is a homomorphism of algebras.

If H is a weak Hopf algebra in $\mathcal{C}$, the antipode is unique, antimultiplicative, anticomultiplicative and leaves the unit and the counit invariant as follows:

$$\lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H}, \quad \delta_H \circ \lambda_H = c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H, \tag{2.1}$$
$$\lambda_H \circ \eta_H = \eta_H, \quad \varepsilon_H \circ \lambda_H = \varepsilon_H. \tag{2.2}$$

If we define the morphisms Π_H^L (target), Π_H^R (source), $\overline{\Pi}_H^L$ and $\overline{\Pi}_H^R$ by

$$\Pi_H^L = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H),$$
$$\Pi_H^R = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)),$$
$$\overline{\Pi}_H^L = (H \otimes (\varepsilon_H \circ \mu_H)) \circ ((\delta_H \circ \eta_H) \otimes H), \quad \overline{\Pi}_H^R = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)),$$

respectively, it is straightforward to show that they are idempotent and Π_H^L, Π_H^R satisfy the equalities

$$\Pi_H^L = \mathrm{id}_H \wedge \lambda_H, \quad \Pi_H^R = \lambda_H \wedge \mathrm{id}_H, \tag{2.3}$$

respectively (see [9]). Moreover, we have

$$\Pi_H^L \circ \overline{\Pi}_H^L = \Pi_H^L, \quad \Pi_H^L \circ \overline{\Pi}_H^R = \overline{\Pi}_H^R, \quad \Pi_H^R \circ \overline{\Pi}_H^L = \overline{\Pi}_H^L, \quad \Pi_H^R \circ \overline{\Pi}_H^R = \Pi_H^R, \tag{2.4}$$
$$\overline{\Pi}_H^L \circ \Pi_H^L = \overline{\Pi}_H^L, \quad \overline{\Pi}_H^L \circ \Pi_H^R = \Pi_H^R, \quad \overline{\Pi}_H^R \circ \Pi_H^L = \Pi_H^L, \quad \overline{\Pi}_H^R \circ \Pi_H^R = \overline{\Pi}_H^R, \tag{2.5}$$
$$\Pi_H^L = \overline{\Pi}_H^R \circ \lambda_H = \lambda_H \circ \overline{\Pi}_H^L, \quad \Pi_H^R = \overline{\Pi}_H^L \circ \lambda_H = \lambda_H \circ \overline{\Pi}_H^R, \tag{2.6}$$
$$\Pi_H^L \circ \lambda_H = \Pi_H^L \circ \Pi_H^R = \lambda_H \circ \Pi_H^R, \quad \Pi_H^R \circ \lambda_H = \Pi_H^R \circ \Pi_H^L = \lambda_H \circ \Pi_H^L. \tag{2.7}$$

For the morphisms target, we have the following identities:

$$\mu_H \circ (H \otimes \Pi_H^L) = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H), \tag{2.8}$$
$$\mu_H \circ (\Pi_H^R \otimes H) = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes \delta_H), \tag{2.9}$$
$$\mu_H \circ (H \otimes \overline{\Pi}_H^L) = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (\delta_H \otimes H), \tag{2.10}$$
$$\mu_H \circ (\overline{\Pi}_H^R \otimes H) = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes \delta_H), \tag{2.11}$$
$$(H \otimes \Pi_H^L) \circ \delta_H = (\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H), \tag{2.12}$$
$$(\Pi_H^R \otimes H) \circ \delta_H = (H \otimes \mu_H) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)), \tag{2.13}$$
$$(\overline{\Pi}_H^L \otimes H) \circ \delta_H = (H \otimes \mu_H) \circ ((\delta_H \circ \eta_H) \otimes H), \tag{2.14}$$
$$(H \otimes \overline{\Pi}_H^R) \circ \delta_H = (\mu_H \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)), \tag{2.15}$$
$$\Pi_H^L \circ \mu_H \circ (H \otimes \Pi_H^L) = \Pi_H^L \circ \mu_H. \tag{2.16}$$

Definition 2.2 *Let H be a weak Hopf algebra. We will say that a right H-comodule (A, ρ_A) is a right H-comodule algebra, if A is an algebra such that $\rho_A \circ \mu_A = \mu_{A\otimes H} \circ (\rho_A \otimes \rho_A)$ and any one of the following equivalent conditions holds*:

(1) $(A \otimes \Pi_H^L) \circ \rho_A = (\mu_A \otimes H) \circ (A \otimes c_{H,A}) \circ ((\rho_A \circ \eta_A) \otimes A)$,
(2) $(A \otimes \overline{\Pi}_H^R) \circ \rho_A = (\mu_A \otimes H) \circ (A \otimes (\rho_A \circ \eta_A))$,
(3) $(A \otimes \Pi_H^L) \circ \rho_A \circ \eta_A = \rho_A \circ \eta_A$,
(4) $(A \otimes \overline{\Pi}_H^R) \circ \rho_A \circ \eta_A = \rho_A \circ \eta_A$,
(5) $(\rho_A \otimes H) \circ \rho_A \circ \eta_A = (A \otimes \mu_H \otimes H) \circ (\rho_A \otimes \delta_H) \circ (\eta_A \otimes \eta_H)$,
(6) $(\rho_A \otimes H) \circ \rho_A \circ \eta_A = (A \otimes (\mu_H \circ c_{H,H}) \otimes H) \circ (\rho_A \otimes \delta_H) \circ (\eta_A \otimes \eta_H)$.

If (A, ρ_A) is a right H-comodule algebra, the triple (A, H, Γ_A^H) is a right-right weak entwining structure, where

$$\Gamma_A^H = (A \otimes \mu_H) \circ (c_{H,A} \otimes H) \circ (H \otimes \rho_A) \tag{2.17}$$

(see [13]). Therefore the following identity holds:

$$(A \otimes \varepsilon_H) \circ \Gamma_A^H = \mu_A \circ (e_A \otimes A), \tag{2.18}$$

where

$$e_A = (A \otimes \varepsilon_H) \circ \Gamma_A^H \circ (H \otimes \eta_A). \tag{2.19}$$

We denote by $\mathcal{M}_A^H(\Gamma_A^H)$ the category of weak entwined modules, i.e., the objects M in $\mathcal{C}$ together with two morphisms $\phi_M : M \otimes A \to A$ and $\rho_M : M \to M \otimes H$, such that (M, ϕ_M) is a right A-module, (M, ρ_M) is a right H-comodule, and such that the following equality:

$$\rho_M \circ \phi_M = (\phi_M \otimes H) \circ (M \otimes \Gamma_A^H) \circ (\rho_M \otimes A) \tag{2.20}$$

holds. Then, if (A, ρ_A) is a right H-comodule algebra, (A, μ_A, ρ_A) is an object of $\mathcal{M}_A^H(\Gamma_A^H)$.

If (A, ρ_A) is a right H-comodule algebra, we define the subalgebra of coinvariants of A as the equalizer:

$$A_H \xrightarrow{i_A} A \overset{\rho_A}{\underset{\zeta_A}{\rightrightarrows}} A \otimes H,$$

where $\zeta_A = (\mu_A \otimes H) \circ (A \otimes c_{H,A}) \circ ((\rho_A \circ \eta_A) \otimes A)$. Note that, as a consequence of Definition 2.2(1), we have that $\zeta_A = (A \otimes \Pi_H^L) \circ \rho_A$. It is easy to see that $(A_H, \eta_{A_H}, \mu_{A_H})$ is an algebra, with η_{A_H} and μ_{A_H} to be the factorizations through the equalizer i_A of the morphisms η_A and $\mu_A \circ (i_A \otimes i_A)$, respectively. For example, the weak Hopf algebra H is a right H-comodule algebra with right H-comodule structure given by $\rho_H = \delta_H$, and subalgebra of coinvariants H_H is the image of the idempotent morphism Π_H^L. In what follows, we will denote this image by H_L.

Definition 2.3 *Let H be a weak Hopf algebra, and (A, ρ_A) be a right H-comodule algebra. We define an integral as a morphism of right H-comodules $f : H \to A$. Moreover, if $f \circ \eta_H = \eta_A$, we say that the integral is total.*

An integral $f : H \to A$ is convolution invertible, if there exists a morphism $f^{-1} : H \to A$ (called the convolution inverse of f), such that

(1) $f^{-1} \wedge f = e_A$,

(2) $f \wedge f^{-1} = (A \otimes (\varepsilon_H \circ \mu_H)) \circ ((\rho_A \circ \eta_A) \otimes H)$,

(3) $f^{-1} \wedge f \wedge f^{-1} = f^{-1}$,

where e_A is the morphism defined in (2.19).

Trivially, the inverse is unique. Moreover, using the condition Definition 2.3(1), if f is an integral convolution invertible, we get that $f \wedge f^{-1} \wedge f = f$. Finally, when f is a total integral, we can rewrite the equality Defnition 2.3(1) as $f^{-1} \wedge f = f \circ \Pi_H^R$ and Definition 2.3(2) as $f \wedge f^{-1} = f \circ \overline{\Pi}_H^L$.

Example 2.1 Let H be a weak Hopf algebra, such that $\Pi_H^L = \overline{\Pi}_H^L$ (equivalently, $\Pi_H^R = \overline{\Pi}_H^R$). Then the identity id_H is a convolution invertible total integral with inverse λ_H. Note that this equality is always true in the Hopf algebra setting. In our case, it holds, for example, if H is a cocommutative weak Hopf algebra.

Definition 2.4 *Let H be a weak Hopf algebra and (A, ρ_A) be a right H-comodule algebra. We say that $A_H \hookrightarrow A$ is an H-cleft extension, if there exists an integral $f : H \to A$ convolution invertible, such that the morphism $f \wedge f^{-1}$ factorizes through the equalizer i_A. In what follows, the morphism f will be called a cleaving morphism associated to the H-cleft extension $A_H \hookrightarrow A$.*

Proposition 2.1 *Let H be a weak Hopf algebra, and (A, ρ_A) be a right H-comodule algebra, such that $A_H \hookrightarrow A$ is an H-cleft extension with cleaving morphism f. Then the equality*

$$\rho_A \circ f^{-1} = (f^{-1} \otimes \lambda_H) \circ c_{H,H} \circ \delta_H \tag{2.21}$$

holds.

Proof We define the following morphisms: $r = \rho_A \circ f^{-1}$, $s = \rho_A \circ f$ and $t = (f^{-1} \otimes \lambda_H) \circ c_{H,H} \circ \delta_H$.

First of all, we show that $s \wedge r = s \wedge t$. Indeed,

$$\begin{aligned} s \wedge r &= \rho_A \circ (f \wedge f^{-1}) \\ &= (A \otimes \Pi_H^L) \circ \rho_A \circ (f \wedge f^{-1}) \\ &= (\mu_A \otimes (\Pi_H^L \circ \mu_H \circ (H \otimes \Pi_H^L))) \circ (A \otimes c_{H,A} \otimes H) \circ ((\rho_A \circ f) \otimes (\rho_A \circ f^{-1})) \circ \delta_H \\ &= (\mu_A \otimes \Pi_H^L) \circ (A \otimes c_{H,A}) \circ ((\rho_A \circ f) \otimes f^{-1}) \circ \delta_H \\ &= (\mu_A \otimes H) \circ (A \otimes c_{H,A}) \circ (((f \otimes (\mu_H \circ (H \otimes \lambda_H) \circ \delta_H)) \circ \delta_H) \otimes f^{-1}) \circ \delta_H \\ &= s \wedge t. \end{aligned}$$

In the foregoing calculations, the first equality follows by using that A is a right H-comodule algebra; the second one follows because $A_H \hookrightarrow A$ is H-cleft; in the third one, we use (2.16); the fourth one relies on the equality $(\mu_A \otimes \mu_H) \circ (A \otimes c_{H,A} \otimes \Pi_H^L) \circ (\rho_A \otimes \rho_A) = (\mu_A \otimes H) \circ (A \otimes c_{H,A}) \otimes (\rho_A \otimes A)$, the fifth one is a consequence of the definition of Π_H^L; finally, in the last one, we use that f is an integral.

Using similar techniques (2.6) and (2.10), we obtain that $t \wedge s = r \wedge s$.

On the other hand, using Definition 2.1(2) and that f is an integral we have that the equality $(f^{-1} \wedge f) \circ \mu_H = ((\varepsilon_H \circ \mu_H) \otimes (f^{-1} \wedge f)) \circ (H \otimes \delta_H)$ holds.

Now we use the previous equality and that f is a convolution invertible integral to get that $t \wedge s \wedge t = t$.

$$\begin{aligned} t \wedge s \wedge t &= (\mu_A \otimes H) \circ (A \otimes c_{H,A}) \circ (A \otimes \mu_H \otimes A) \circ (c_{H,A} \otimes H \otimes A) \\ &\quad \circ (\lambda_H \otimes (((f^{-1} \wedge f) \otimes \Pi_H^L) \circ \delta_H) \otimes A) \circ (\delta_H \otimes f^{-1}) \circ \delta_H \\ &= (\mu_A \otimes H) \circ (A \otimes c_{H,A}) \circ (A \otimes \mu_H \otimes A) \circ (c_{H,A} \otimes H \otimes A) \\ &\quad \circ (\lambda_H \otimes ((((f^{-1} \wedge f) \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \\ &\quad \circ ((\delta_H \circ \eta_H) \otimes H)) \otimes A) \circ (\delta_H \otimes f^{-1}) \circ \delta_H \\ &= (\mu_A \otimes H) \circ (A \otimes c_{H,A}) \circ (A \otimes \mu_H \otimes A) \circ (c_{H,A} \otimes H \otimes A) \\ &\quad \circ (\lambda_H \otimes (((((\varepsilon_H \circ \mu_H) \otimes (f^{-1} \wedge f)) \circ (H \otimes \delta_H)) \otimes H) \\ &\quad \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)) \otimes A) \\ &\quad \circ (\delta_H \otimes f^{-1}) \circ \delta_H \\ &= c_{H,A} \circ ((\lambda_H \wedge \Pi_H^L) \otimes (f^{-1} \wedge f \wedge f^{-1})) \circ \delta_H = t. \end{aligned}$$

Taking into account that $r \wedge s \wedge r = r$, the equalities $r = r \wedge s \wedge r = t \wedge s \wedge r = t \wedge s \wedge t = t$ hold and we conclude the proof.

Proposition 2.2 *Let H be a cocommutative weak Hopf algebra, and (A, ρ_A) be a right H-comodule algebra. If there exists a convolution invertible integral $f : H \to A$, then $A_H \hookrightarrow A$ is an H-cleft extension.*

Proof Let f^{-1} be the convolution inverse of f. We have to show that $f \wedge f^{-1}$ factorizes through the equalizer i_A. Indeed,

$$\begin{aligned}
&\zeta_A \circ (f \wedge f^{-1})\\
&= (A \otimes \Pi_H^L) \circ \rho_A \circ (A \otimes (\varepsilon_H \circ \mu_H)) \circ ((\rho_A \circ \eta_A) \otimes H)\\
&= (A \otimes H \otimes (\varepsilon_H \circ \mu_H)) \circ (A \otimes ((\Pi_H^L \otimes H) \circ \delta_H) \otimes H) \circ ((\rho_A \circ \eta_A) \otimes H)\\
&= (A \otimes H \otimes (\varepsilon_H \circ \mu_H)) \circ (A \otimes ((\Pi_H^L \otimes H) \circ c_{H,H} \circ \delta_H) \otimes H) \circ ((\rho_A \circ \eta_A) \otimes H)\\
&= (A \otimes (\varepsilon_H \circ \mu_H) \otimes H) \circ (\rho_A \otimes c_{H,H}) \circ (((A \otimes \Pi_H^L) \circ \rho_A \circ \eta_A) \otimes H)\\
&= (A \otimes (\varepsilon_H \circ \mu_H) \otimes H) \circ (\rho_A \otimes c_{H,H}) \circ ((\rho_A \circ \eta_A) \otimes H)\\
&= (A \otimes H \otimes (\varepsilon_H \circ \mu_H)) \circ (A \otimes (c_{H,H} \circ \delta_H) \otimes H) \circ ((\rho_A \circ \eta_A) \otimes H)\\
&= (A \otimes H \otimes (\varepsilon_H \circ \mu_H)) \circ (A \otimes \delta_H \otimes H) \circ ((\rho_A \circ \eta_A) \otimes H)\\
&= \rho_A \circ (f \wedge f^{-1}).
\end{aligned}$$

In the foregoing calculations, the first and the last equalities follow by Definition 2.3(2); the second, fourth and sixth ones use the condition of comodule for A; in the third and seventh ones we use that H is cocommutative; finally the fifth one follows by Definition 2.2(3).

Remark 2.1 Let H be a weak Hopf algebra, and (A, ρ_A) be a right H-comodule algebra. We want to point out the relation between the notion of H-cleft extension and the one of weak H-cleft extension given in [2]. In [2], we introduce the set $\mathrm{Reg}^{WR}(H, A)$ as the one whose elements are the morphisms $h : H \to A$, such that there exists a morphism $h^{-1} : H \to A$, called the left weak inverse of h, such that $h^{-1} \wedge h = e_A$, where e_A is the morphism defined in (2.19) for the right-right weak entwining structure Γ_A^H associated to (A, ρ_A) (see (2.17)).

Then, following [2, Definition 1.9], we say that $A_H \hookrightarrow A$ is a weak H-cleft extension if there exists a morphism $h : H \to A$ in $\mathrm{Reg}^{WR}(H, A)$ of right H-comodules, such that the equality $\Gamma_A^H \circ (H \otimes h^{-1}) \circ \delta_H = \zeta_A \circ (e_A \wedge h^{-1})$ holds. Moreover, we can assume without loss of generality that $e_A \wedge h^{-1} = h^{-1}$ and the previous equality can be expressed as

$$\Gamma_A^H \circ (H \otimes h^{-1}) \circ \delta_H = \zeta_A \circ h^{-1}, \tag{2.22}$$

and, as a consequence [2, Proposition 1.12], the morphism $q_A = \mu_A \circ (A \otimes h^{-1}) \circ \rho_A : A \to A$ factorizes through i_A. Therefore, there exists a unique morphism $p_A : A \to A_H$, such that $q_A = i_A \circ p_A$. Then, $h \wedge h^{-1} = q_A \circ h$, and, as a consequence, $h \wedge h^{-1}$ admits a factorization through i_A. Moreover, by [2, Remark 1.10], we know that if there exists an $h \in \mathrm{Reg}^{WR}(H, A)$ of right H-comodules,

$$\Gamma_A^H = (\mu_A \otimes H) \circ (A \otimes (\rho_A \circ \mu_A)) \circ (((h^{-1} \otimes h) \circ \delta_H) \otimes A). \tag{2.23}$$

Theorem 2.1 *Let H be a weak Hopf algebra, and (A, ρ_A) be a right H-comodule algebra. If there exists an $h \in \mathrm{Reg}^{WR}(H, A)$ of right H-comodules, such that $e_A \wedge h^{-1} = h^{-1}$, the following assertions are equivalent:*

(i) *The morphism $h \wedge h^{-1}$ factorizes through the equalizer i_A and h^{-1} satisfies* (2.21)*.*

(ii) *The equality* (2.22) *holds.*

Proof If (ii) holds, $A_H \hookrightarrow A$ is a weak H-cleft extension and then $h \wedge h^{-1}$ admits a factorization through i_A. The equality (2.21) follows in a similar way to the proof given in Proposition 2.1 by using that $e_A \wedge h^{-1} = h^{-1}$.

Conversely, assume that (i) holds. Then

$$
\begin{aligned}
& \Gamma_A^H \circ (H \otimes h^{-1}) \circ \delta_H \\
&= (\mu_A \otimes H) \circ (A \otimes (\rho_A \circ \mu_A)) \circ (((h^{-1} \otimes h) \circ \delta_H) \otimes h^{-1}) \circ \delta_H \\
&= (\mu_A \otimes \Pi_H^L) \circ (h^{-1} \otimes (\rho_A \circ (h \wedge h^{-1}))) \circ \delta_H \\
&= ((\mu_A \circ (A \otimes \mu_A)) \otimes (\Pi_H^L \circ \mu_H)) \circ (h^{-1} \otimes h \otimes c_{H,A} \otimes H) \\
&\quad \circ (H \otimes \delta_H \otimes (\rho_A \circ h^{-1})) \circ (H \otimes \delta_H) \circ \delta_H \\
&= (\mu_A \otimes \Pi_H^L) \circ (e_A \otimes c_{H,A}) \circ (\delta_H \otimes h^{-1}) \circ \delta_H \\
&= (((A \otimes \varepsilon_H) \circ \Gamma_A^H) \otimes \Pi_H^L) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes h^{-1}) \circ \delta_H \\
&= (A \otimes (\varepsilon_H \circ \mu_H) \otimes \Pi_H^L) \circ (c_{H,A} \otimes c_{H,H}) \circ (H \otimes c_{H,A} \otimes H) \\
&\quad \circ (\delta_H \otimes (\rho_A \circ h^{-1})) \circ \delta_H \\
&= (A \otimes (\Pi_H^L \circ \mu_H)) \circ (c_{H,A} \otimes H) \circ (H \otimes (\rho_A \circ h^{-1})) \circ \delta_H \\
&= (h^{-1} \otimes \Pi_H^L) \circ c_{H,H} \circ \delta_H,
\end{aligned}
$$

where the first equality follows by (2.23), the second one uses that $h \wedge h^{-1}$ factorizes through i_A and the third one follows because A is a weak entwined module and h is a morphism of right H-comodules. In the fourth equality, we use (2.21), and the fifth one is a consequence of the properties of Π_H^L. The sixth one follows by (2.18), and the seventh one relies on the definition of Γ_A^H. Using (2.8) and (2.16), we obtain the eighth equality. Finally, the last one follows by (2.21) and the properties of Π_H^L.

On the other hand,

$$
\zeta_A \circ h^{-1} = (A \otimes \overline{\Pi}_H^R) \circ \rho_A \circ h^{-1} = (h^{-1} \otimes (\overline{\Pi}_H^R \circ \lambda_H)) \circ c_{H,H} \circ \delta_H = (h^{-1} \otimes \Pi_H^L) \circ c_{H,H} \circ \delta_H.
$$

Then the proof is complete.

Corollary 2.1 *Let H be a weak Hopf algebra, and (A, ρ_A) be a right H-comodule algebra. If $A_H \hookrightarrow A$ is an H-cleft extension, then it is a weak H-cleft extension.*

Proof If $A_H \hookrightarrow A$ is an H-cleft extension, there exists an integral $f : H \to A$ convolution invertible, such that the morphism $f \wedge f^{-1}$ factorizes through the equalizer i_A, where f^{-1} is the convolution inverse of f. Then, $f \in \mathrm{Reg}^{WR}(H, A)$, $e_A \wedge f^{-1} = f^{-1}$, and by using Proposition 2.1, the equality (2.21) holds. Therefore, as a consequence of the previous theorem, we obtain that $A_H \hookrightarrow A$ is a weak H-cleft extension.

Remark 2.2 As a consequence of Corollary 2.1, the results proved in [2–3] for weak H-cleft extensions can be applied for H-cleft extensions. For example, if $A_H \hookrightarrow A$ is an H-cleft extension with cleaving morphism f, the morphism $q_A = \mu_A \circ (A \otimes f^{-1}) \circ \rho_A$ factorizes through the equalizer i_A, i.e., there exists a morphism $p_A : A \to A_H$ such that $i_A \circ p_A = q_A$. Also, using [4, Lemmas 3.9 and 3.11], we have the following equalities:

$$\mu_A \circ (q_A \otimes f) \circ \rho_A = \mathrm{id}_A, \tag{2.24}$$

$$p_A \circ \mu_A \circ (i_A \otimes A) = \mu_{A_H} \circ (A_H \otimes p_A). \tag{2.25}$$

Definition 2.5 *Let H be a weak Hopf algebra. Two H-cleft extensions $A_H \hookrightarrow A$ and $B_H \hookrightarrow B$ are equivalent (written by $A_H \hookrightarrow A \sim B_H \hookrightarrow B$), if $A_H = B_H$, and there exists a morphism of right H-comodule algebras $T : A \to B$, such that $T \circ i_A = i_B$.*

Note that, if the H-cleft extensions $A_H \hookrightarrow A$ and $A_H \hookrightarrow B$ are equivalent, and f is a cleaving morphism for $A_H \hookrightarrow A$, it is easy to show that $g = T \circ f$ is a cleaving morphism for $A_H \hookrightarrow B$ with $g^{-1} = T \circ f^{-1}$. Under these conditions, T is an isomorphism. If f is the cleaving morphism associated to $A_H \hookrightarrow A$, we define four morphisms as follows: $\gamma_A = (p_A \otimes H) \circ \rho_A : A \to A_H \otimes H$, $\chi_A = \mu_A \circ (i_A \otimes f) : A_H \otimes H \to A$ and $\gamma_B = (p_B \otimes H) \circ \rho_B : B \to A_H \otimes H$, $\chi_B = \mu_B \circ (i_B \otimes g) : A_H \otimes H \to B$, where p_A and p_B are the factorizations of $q_A = \mu_A \circ (A \otimes f^{-1}) \circ \rho_A$, $q_B = \mu_B \circ (A \otimes g^{-1}) \circ \rho_B$, respectively, and i_A, i_B are the corresponding equalizer morphisms. Then,

$$\begin{aligned}
\chi_B \circ \gamma_A &= \mu_B \circ ((i_B \circ p_A) \otimes g) \circ \rho_A = \mu_B \circ ((T \circ q_A) \otimes (T \circ f)) \circ \rho_A \\
&= T \circ \mu_A \circ (A \otimes (f^{-1} \wedge f)) \circ \rho_A \\
&= T \circ \mu_A \circ (A \otimes e_A) \circ \rho_A = T, \\
i_B \circ p_B \circ T &= \mu_B \circ (B \otimes (T \circ f^{-1})) \circ \rho_B \circ T = T \circ q_A = T \circ i_A \circ p_A = i_B \circ p_A.
\end{aligned}$$

If we define $T^{-1} : B \to A$ by $T^{-1} = \chi_A \circ \gamma_B$, we have

$$T \circ T^{-1} = \mu_B \circ ((T \circ i_A \circ p_B) \otimes (T \circ f)) \otimes \rho_B = \mu_B \circ (B \otimes e_B) \circ \rho_B = \mathrm{id}_B,$$

and in a similar way, $T^{-1} \circ T = \mathrm{id}_A$. Therefore, T is an isomorphism.

Obviously, " $\sim$ " is an equivalence relation, and we denote by $[B_H \hookrightarrow B]$ the isomorphisms class of the H-cleft extension $B_H \hookrightarrow B$.

It is a well-known fact in the Hopf algebra setting that, if $A_H \hookrightarrow A$ is an H-cleft extension with convolution invertible integral f, the morphism $h = \mu_A \circ (f \otimes (f^{-1} \circ \eta_H))$ is a total integral which is convolution invertible. There is a similar result for weak Hopf algebras, although we want to point out that the proof is very different, because in our case $\delta_H \circ \eta_H \neq \eta_H \otimes \eta_H$. Actually, in order to give the convolution inverse of this morphism, we assume that the weak Hopf algebra is cocommutative. This hypothesis can be removed in the classical case, because for Hopf algebras the morphisms Π_H^L, Π_H^R $\overline{\Pi}_H^L$ and $\overline{\Pi}_H^R$ trivialize.

Proposition 2.3 *Let H be a weak Hopf algebra with invertible antipode. If $A_H \hookrightarrow A$ is an H-cleft extension with cleaving morphism f, then $h = \mu_A \circ (f \otimes (f^{-1} \circ \eta_H))$ is a total integral. Moreover, if H is cocommutative, h is convolution invertible.*

Proof The morphism $h = \mu_A \circ (f \otimes (f^{-1} \circ \eta_H))$ is an integral, where f^{-1} is the convolution inverse of f. Indeed,

$$\rho_A \circ h = \mu_{A \otimes H} \circ ((\rho_A \circ f) \otimes (\rho_A \circ f^{-1} \circ \eta_H))$$

$$
\begin{aligned}
&= (A \otimes (\lambda_H \circ \lambda_H^{-1})) \circ \mu_{A\otimes H} \circ (((f \otimes H) \circ \delta_H) \otimes ((f^{-1} \otimes \lambda_H) \circ c_{H,H} \circ \delta_H \circ \eta_H))\\
&= (\mu_A \otimes H) \circ (f \otimes (c_{H,A} \circ (\lambda_H \otimes f^{-1}) \circ (\mu_H \otimes H)\\
&\quad \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes \lambda_H^{-1}))) \circ \delta_H\\
&= (\mu_A \otimes H) \circ (f \otimes (c_{H,A} \circ (\lambda_H \otimes f^{-1}) \circ (H \otimes \Pi_H^L) \circ \delta_H \circ \lambda_H^{-1})) \circ \delta_H\\
&= (\mu_A \otimes H) \circ (f \otimes ((f^{-1} \circ \Pi_H^L \circ \lambda_H^{-1}) \otimes H) \circ \delta_H) \circ \delta_H\\
&= (h \otimes H) \circ \delta_H.
\end{aligned}
$$

The first equality follows because A is a right H-comodule algebra, the second one uses that λ_H is an isomorphism, the third and the fifth ones are the consequences of the antimultiplicative property for λ_H, λ_H^{-1} and the naturality of c, the fourth one follows by (2.12), and finally, the last one uses (2.6).

Now, using Proposition 2.1, the properties of the antipode and (2.6), we have

$$
\begin{aligned}
&(A \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,A} \otimes H) \circ (H \otimes (\rho_A \circ f^{-1} \circ \eta_H))\\
&= (A \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,A} \otimes H) \circ (H \otimes ((f^{-1} \otimes \lambda_H) \circ c_{H,H} \circ \delta_H \circ \eta_H))\\
&= ((\varepsilon_H \circ \mu_H \circ c_{H,H}) \otimes f^{-1}) \circ (\lambda_H^{-1} \otimes (\delta_H \circ \eta_H))\\
&= f^{-1} \circ \Pi_H^L \circ \lambda_H^{-1} = f^{-1} \circ \overline{\Pi}_H^R.
\end{aligned}
$$

As a consequence, $h = f \wedge (f^{-1} \circ \overline{\Pi}_H^R)$, and then we get that h is total because, by the previous equality and (2.15), $h \circ \eta_H = (f \wedge (f^{-1} \circ \overline{\Pi}_H^R)) \circ \eta_H = (f \wedge f^{-1}) \circ \eta_H = \eta_A$.

Now we assume that H is cocommutative. We define $h^{-1} = \mu_A \circ ((f \circ \eta_H) \otimes f^{-1})$. Following a similar way to the one developed for h, it is easy to prove the equalities $h^{-1} = (f \circ \Pi_H^R) \wedge f^{-1}$ and $\mu_A \circ (f^{-1} \otimes (f \circ \eta_H)) = \mu_A \circ (f^{-1} \otimes (f \circ \Pi_H^R \circ \lambda_H)) \circ c_{H,H} \circ \delta_H$. As a consequence of the last equation, taking into account that H is cocommutative, we obtain that

$$\mu_A \circ ((f^{-1} \circ \eta_H) \otimes (f \circ \eta_H)) = \eta_H. \tag{2.26}$$

We conclude the proof showing that h^{-1} is the convolution inverse of h. Condition (2) in Definition 2.3 follows because, as a consequence of (2.26), $h \wedge h^{-1} = f \wedge f^{-1} = (A \otimes (\varepsilon_H \circ \mu_H)) \circ ((\rho_A \circ \eta_A) \otimes H)$.

As far as Definition 2.3(1), using that f is a convolution invertible integral, A is an H-comodule algebra and (2.15), we get that $h^{-1} \wedge h = (A \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,A} \otimes H) \circ (H \otimes (\rho_A \circ \eta_A)) = e_A$.

The proof for the condition (3) in Definition 2.3 for h follows a similar pattern, and we leave the details to the reader. Then h is convolution invertible.

Remark 2.3 As a consequence of the previous proposition, in the cocommutative setting, we can assume that the integral is total.

In the following definition, we recall the notion of left weak H-module algebra introduced in [5].

Definition 2.6 *Let H be a weak Hopf algebra. We will say that A is a left weak H-module algebra if A is an algebra and there exists a morphism $\varphi_A : H \otimes A \to A$ satisfying*

(1) $\varphi_A \circ (\eta_H \otimes A) = \mathrm{id}_A$,

(2) $\varphi_A \circ (H \otimes \mu_A) = \mu_A \circ (\varphi_A \otimes \varphi_A) \circ (H \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes A \otimes A)$,

(3) $\varphi_A \circ (\mu_H \otimes \eta_A) = \varphi_A \circ (H \otimes (\varphi_A \circ (H \otimes \eta_A)))$,

and any one of the following equivalent conditions holds:

(4) $\varphi_A \circ (\Pi_H^L \otimes A) = \mu_A \circ ((\varphi_A \circ (H \otimes \eta_A) \otimes A)$,

(5) $\varphi_A \circ (\overline{\Pi}_H^L \otimes A) = \mu_A \circ c_{A,A} \circ ((\varphi_A \circ (H \otimes \eta_A) \otimes A)$,

(6) $\varphi_A \circ (\Pi_H^L \otimes \eta_A) = \varphi_A \circ (H \otimes \eta_A)$,

(7) $\varphi_A \circ (\overline{\Pi}_H^L \otimes \eta_A) = \varphi_A \circ (H \otimes \eta_A)$,

(8) $\varphi_A \circ (H \otimes (\varphi_A \circ (H \otimes \eta_A))) = ((\varphi_A \circ (H \otimes \eta_A)) \otimes (\varepsilon_H \circ \mu_H)) \circ (\delta_H \otimes H)$,

(9) $\varphi_A \circ (H \otimes (\varphi_A \circ (H \otimes \eta_A))) = ((\varepsilon_H \circ \mu_H) \otimes (\varphi_A \circ (H \otimes \eta_A))) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)$.

If we replace (3) *by*

(3)$'$ $\varphi_A \circ (\mu_H \otimes A) = \varphi_A \circ (H \otimes \varphi_A)$,

we will say that (A, φ_A) *is a left* H*-module algebra.*

Remark 2.4 Note that as a consequence of Definition 2.6(4)–(5) if the weak Hopf algebra is cocommutative the morphism $\varphi_A \circ (H \otimes \eta_A)$ factorizes through the center of A, that is, there exists a unique morphism $n_A : H \to Z(A)$, such that $i_{Z(A)} \circ n_A = \varphi_A \circ (H \otimes \eta_A)$. Moreover, if H is a Hopf algebra and (A, φ_A) is a left weak H-module algebra, conditions (4)–(9) in Definition 2.6 imply that $\varepsilon_H \otimes \eta_A = \varphi_A \circ (H \otimes \eta_A)$. As a consequence, the equality (3) in Definition 2.6 is always true and φ_A is a weak action of H on A (see [6]).

Proposition 2.4 *Let H be a cocommutative weak Hopf algebra. If $A_H \hookrightarrow A$ is an H-cleft extension with cleaving morphism f, the pair (A_H, φ_{A_H}) is a left weak H-module algebra, where φ_{A_H} is the factorization of the morphism $\varphi_A = \mu_A \circ (A \otimes (\mu_A \circ c_{A,A})) \circ (((f \otimes f^{-1}) \circ \delta_H) \otimes i_A)$ through the equalizer i_A.*

Proof If $A_H \hookrightarrow A$ is an H-cleft extension, as a consequence of Corollary 2.1, we have that $A_H \hookrightarrow A$ is a weak H-cleft extension, and then, using [2, Proposition 1.15], we know that φ_{A_H} factorizes through the equalizer i_A and satisfies Definition 2.2(2). Moreover it is easy to see that

$$\varphi_{A_H} = p_A \circ \mu_A \circ (f \otimes i_A), \tag{2.27}$$

and then Definition 2.2(1) holds.

As far as Definition 2.2(3),

$$\begin{aligned}
&\varphi_{A_H} \circ (H \otimes (\varphi_{A_H} \circ (H \otimes \eta_{A_H})))\\
&= p_A \circ \mu_A \circ (f \otimes (q_A \circ f))\\
&= p_A \circ \mu_A \circ (f \otimes (f \wedge f^{-1}))\\
&= ((p_A \circ \mu_A) \otimes (\varepsilon_H \circ \mu_H)) \circ (f \otimes (\rho_A \circ \eta_A) \circ H)\\
&= (p_A \otimes (\varepsilon_H \circ \mu_H)) \circ (A \otimes \overline{\Pi}_H^R \otimes H) \circ ((\rho_A \circ f) \otimes H)\\
&= ((p_A \circ q_A \circ f) \otimes (\varepsilon_H \circ \mu_H)) \circ (\delta_H \otimes H)\\
&= ((p_A \circ (f \wedge f^{-1})) \otimes (\varepsilon_H \circ \mu_H)) \circ (\delta_H \otimes H)\\
&= (p_A \otimes (\varepsilon_H \circ \mu_H)) \circ ((\rho_A \circ \eta_A) \otimes \mu_H)\\
&= (p_A \circ (f \wedge f^{-1})) \circ \mu_H\\
&= \varphi_{A_H} \circ (\mu_H \otimes \eta_{A_H}),
\end{aligned}$$

where the first equality follows by (2.27); the second one follows because $q_A \circ f = f \wedge f^{-1}$; the third, sixth and seventh ones are consequences of Definition 2.3(2); in the fourth one, we use that A is a right H-comodule algebra; in the fifth one, we use (2.11) and that f is an integral; finally, in the last one, we use that $p_A \circ (f \wedge f^{-1}) = \varphi_{A_H} \circ (H \otimes \eta_{A_H})$.

It only remains to show one of the equivalent conditions (4)–(9) in Definition 2.6. We get (6) in Definition 2.6:

$$\begin{aligned}\varphi_{A_H} \circ (\Pi_H^L \otimes \eta_{A_H}) &= p_A \circ (f \wedge f^{-1}) \circ \Pi_H^L = p_A \circ (A \otimes (\varepsilon_H \circ \mu_H)) \circ ((\rho_A \circ \eta_A) \otimes \Pi_H^L) \\ &= p_A \circ (f \wedge f^{-1}) = \varphi_{A_H} \circ (H \otimes \eta_{A_H}).\end{aligned}$$

3 Crossed Systems for Weak Hopf Algebras

Taking into account the theory developed in the previous section, in the remainder of this paper, we will assume that H is a cocommutative weak Hopf algebra. In this section, we generalize the theory of crossed systems over a Hopf algebra given by Doi [14] to the weak setting. Also we obtain a bijective correspondence between the isomorphisms classes of H-cleft extensions $[A_H \hookrightarrow A]$ and the equivalence classes of crossed systems for H over A_H. Following [5, Definition 1.18], we have the following definition.

Definition 3.1 *Let (A, φ_A) be a left weak H-module algebra. We define* $\mathrm{Reg}_{\varphi_A}(H, A)$, *as the set of morphisms $h : H \to A$, such that there exists a morphism $h^{-1} : H \to A$ (the convolution regular inverse of h) satisfying the following equalities:*

(1 $h \wedge h^{-1} = h^{-1} \wedge h = u_1$,

(2) $h \wedge h^{-1} \wedge h = h$,

(3) $h^{-1} \wedge h \wedge h^{-1} = h^{-1}$,

where $u_1 = \varphi_A \circ (H \otimes \eta_A)$.

In a similar way, $\mathrm{Reg}_{\varphi_A}(H \otimes H, A)$ *is the set of morphisms $\sigma : H \otimes H \to A$, such that there exists a morphism $\sigma^{-1} : H \otimes H \to A$ satisfying:*

(4) $\sigma \wedge \sigma^{-1} = \sigma^{-1} \wedge \sigma = u_2$,

(5) $\sigma \wedge \sigma^{-1} \wedge \sigma = \sigma$,

(6) $\sigma^{-1} \wedge \sigma \wedge \sigma^{-1} = \sigma^{-1}$,

where $u_2 = \varphi_A \circ (H \otimes u_1)$.

Note that h^{-1} is unique because, if there exists a morphism $g : H \to A$ satisfying Definition 3.1 (1)–(3) for h, we have

$$g = g \wedge h \wedge g = u_1 \wedge g = h^{-1} \wedge h \wedge g = h^{-1} \wedge u_1 = h^{-1} \wedge h \wedge h^{-1} = h^{-1}.$$

The proof for the unicity of σ^{-1} is similar and, of course, the sets $\mathrm{Reg}_{\varphi_A}(H, A)$, $\mathrm{Reg}_{\varphi_A}(H \otimes H, A)$ may be empty. Note that, using (3) of Definition 2.6, we have

$$u_2 = u_1 \circ \mu_H. \tag{3.1}$$

Also, as a consequence of [5, Proposition 1.19], we know that, if (A, φ_A) is a left weak H-module algebra, such that there exists an $h : H \to A$ satisfying that:

$$h \wedge h^{-1} = h^{-1} \wedge h = u_1, \quad h \wedge h^{-1} \wedge h = h, \quad h^{-1} \wedge h \wedge h^{-1} = h^{-1},$$

the following equalities are equivalent:

$$h \circ \eta_H = \eta_A, \tag{3.2}$$

$$h \circ \Pi_H^L = u_1, \tag{3.3}$$

$$h \circ \overline{\Pi}_H^L = u_1. \tag{3.4}$$

In a similar way, it is possible to see that, if $\sigma : H \otimes H \to A$ is a morphism, such that

$$\sigma \wedge \sigma^{-1} = \sigma^{-1} \wedge \sigma = u_2, \quad \sigma \wedge \sigma^{-1} \wedge \sigma = \sigma, \quad \sigma^{-1} \wedge \sigma \wedge \sigma^{-1} = \sigma^{-1},$$

the following equalities are equivalent:

$$\sigma \circ (\eta_H \otimes H) = u_1, \tag{3.5}$$

$$\sigma \circ (\Pi_H^L \otimes H) \circ \delta_H = u_1, \tag{3.6}$$

$$\sigma \circ c_{H,H} \circ (H \otimes \overline{\Pi}_H^L) \circ \delta_H = u_1. \tag{3.7}$$

Finally, the following assertions are equivalent:

$$\sigma \circ (H \otimes \eta_H) = u_1, \tag{3.8}$$

$$\sigma \circ (H \otimes \Pi_H^R) \circ \delta_H = u_1, \tag{3.9}$$

$$\sigma \circ c_{H,H} \circ (\overline{\Pi}_H^R \otimes H) \circ \delta_H = u_1. \tag{3.10}$$

Proposition 3.1 *Let (A, φ_A) be a left weak H-module algebra. If there exists an $h : H \to A$ satisfying the following equalities: $h \wedge h^{-1} = h^{-1} \wedge h = u_1$, $h \wedge h^{-1} \wedge h = h$, $h^{-1} \wedge h \wedge h^{-1} = h^{-1}$, we have that $h \circ \eta_H = \eta_A$ if and only if $h^{-1} \circ \eta_H = \eta_A$.*

Proof If $h \circ \eta_H = \eta_A$, using (2.12) and (3.3), we have

$$h^{-1} \circ \eta_H = (h^{-1} \wedge u_1) \circ \eta_H = (h^{-1} \wedge (h \circ \Pi_H^L)) \circ \eta_H = u_1 \circ \eta_H = \eta_A.$$

Conversely, if $h^{-1} \circ \eta_H = \eta_A$, $h \circ \eta_H = (h \wedge u_1) \circ \eta_H = (h \wedge (h^{-1} \circ \Pi_H^L)) \circ \eta_H = u_1 \circ \eta_H = \eta_A$.

Definition 3.2 *Let (A, φ_A) be a left weak H-module algebra, and $\sigma : H \otimes H \to A$ be a morphism satisfying Definition* 3.1(4)–(6)*. We say that (φ_A, σ) is a crossed system for H over A if the following conditions hold:*

(1)

$$\begin{aligned}&\mu_A \circ (A \otimes \varphi_A) \circ (\sigma \otimes \mu_H \otimes A) \circ (\delta_{H\otimes H} \otimes A)\\ &= \mu_A \circ ((\varphi_A \circ (H \otimes \varphi_A)) \otimes A) \circ (H \otimes H \otimes c_{A,A}) \circ (H \otimes H \otimes \sigma \otimes A) \circ (\delta_{H\otimes H} \otimes A).\end{aligned}$$

(2)

$$((\varphi_A \circ (H \otimes \sigma))) \wedge (\sigma \circ (H \otimes \mu_H)) = ((\sigma \otimes \varepsilon_H) \wedge (\sigma \circ (\mu_H \otimes H)).$$

(3)

$$\sigma \circ (H \otimes \eta_H) = \sigma \circ (\eta_H \otimes H) = \varphi_A \circ (H \otimes \eta_A).$$

It is clear that our condition (4) in Definition 3.1 over σ implies that it is left invertible in the sense of [22, Definition 4.1]. In any case, to obtain the main results of this paper and a good cohomological interpretation, we need the right invertibility, that is Definition 3.1(4)–(6). Moreover, the morphism σ is in $\mathrm{Reg}_{\varphi_A}(H\otimes H, A)$, and Definition 3.2(2) is equivalent to

$$\mu_A\circ(\sigma^{-1}\otimes(\varphi_A\circ(H\otimes\sigma)))\circ(\delta_{H\otimes H}\otimes H)=(\sigma\circ(\mu_H\otimes H))\wedge(\sigma^{-1}\circ(H\otimes\mu_H)). \quad (3.11)$$

Two crossed systems for H over A, (φ_A,σ) and (ϕ_A,τ) are said to be equivalent, denoted by $(\varphi_A,\sigma)\approx(\phi_A,\tau)$, if $\varphi_A\circ(H\otimes\eta_A)=\phi_A\circ(H\otimes\eta_A)$ and there exists an h in $\mathrm{Reg}_{\varphi_A}(H,A)\cap\mathrm{Reg}_{\phi_A}(H,A)$ with $h\circ\eta_H=\eta_A$, such that

$$\varphi_A=\mu_A\circ(\mu_A\otimes A)\circ(h\otimes\phi_A\otimes h^{-1})\circ(\delta_H\otimes c_{H,A})\circ(\delta_H\otimes A), \quad (3.12)$$

$$\begin{aligned}\sigma=\;&\mu_A\circ(\mu_A\otimes h^{-1})\circ(\mu_A\otimes\tau\otimes\mu_H)\circ(h\otimes\phi_A\otimes\delta_{H\otimes H})\\&\circ(\delta_H\otimes h\otimes H\otimes H)\circ\delta_{H\otimes H}.\end{aligned} \quad (3.13)$$

Proposition 3.2 *The relation $\approx$ is an equivalence relation.*

Proof Let (φ_A,σ) be a crossed system. The morphism u_1 is in $\mathrm{Reg}_{\varphi_A}(H,A)$ with inverse $u_1^{-1}=u_1$, and satisfies that $u_1\circ\eta_H=\eta_A$. Moreover, using that (A,φ_A) is a left weak H-module algebra, we have

$$\begin{aligned}&\mu_A\circ(\mu_A\otimes A)\circ(u_1\otimes\varphi_A\otimes u_1^{-1})\circ(\delta_H\otimes c_{H,A})\circ(\delta_H\otimes A)\\&=\mu_A\circ(\mu_A\otimes A)\circ((\varphi_A\circ(H\otimes\eta_A))\otimes\varphi_A\otimes(\varphi_A\circ(H\otimes\eta_A)))\circ(\delta_H\otimes c_{H,A})\circ(\delta_H\otimes A)\\&=\mu_A\circ(\varphi_A\otimes(\varphi_A\circ(H\otimes\eta_A)))\circ(\delta_H\otimes c_{H,A})\circ(\delta_H\otimes A)=\varphi_A,\end{aligned}$$

and get (3.12).

As far as (3.13), using that (A,φ_A) is a left weak H-module algebra and taking into account that σ is in $\mathrm{Reg}_{\varphi_A}(H\otimes H,A)$, we have that

$$\begin{aligned}&\mu_A\circ(\mu_A\otimes u_1^{-1})\circ(\mu_A\otimes\sigma\otimes\mu_H)\circ(u_1\otimes\varphi_A\otimes\delta_{H\otimes H})\circ(\delta_H\otimes u_1\otimes H\otimes H)\circ\delta_{H\otimes H}\\&=\mu_A\circ((\mu_A\circ(\varphi_A\circ(H\otimes\eta_A))\otimes\varphi_A)\otimes(\sigma\wedge(\varphi_A\circ(\mu_H\otimes\eta_A))))\\&\quad\circ(\delta_H\otimes(\varphi_A\circ(H\otimes\eta_A))\otimes H\otimes H\circ\delta_{H\otimes H})\\&=\mu_A\circ((\varphi_A\circ(\mu_H\otimes\eta_A))\otimes(\sigma\wedge\sigma^{-1}\wedge\sigma))\circ\delta_{H\otimes H})\\&=\sigma\wedge\sigma^{-1}\wedge\sigma=\sigma,\end{aligned}$$

and the relation is reflexive.

In order to get that $\approx$ is symmetrical, assume that $(\varphi_A,\sigma)\approx(\phi_A,\tau)$. Let h be the morphism in $\mathrm{Reg}_{\varphi_A}(H,A)\cap\mathrm{Reg}_{\phi_A}(H,A)$ satisfying (3.12)–(3.13), such that $h\circ\eta_H=\eta_A$. Then the inverse h^{-1} is in $\mathrm{Reg}_{\varphi_A}(H,A)\cap\mathrm{Reg}_{\phi_A}(H,A)$. As a consequence of Proposition 3.1, we obtain that $h^{-1}\circ\eta_H=\eta_A$. Moreover,

$$\begin{aligned}&\mu_A\circ(\mu_A\otimes A)\circ(h^{-1}\otimes\varphi_A\otimes h)\circ(\delta_H\otimes c_{H,A})\circ(\delta_H\otimes A)\\&=\mu_A\circ(\mu_A\otimes A)\circ((h^{-1}\wedge h)\otimes\phi_A\otimes(h^{-1}\wedge h))\circ(\delta_H\otimes c_{H,A})\circ(\delta_H\otimes A)\\&=\mu_A\circ(\mu_A\otimes A)\circ((\phi_A\circ(H\otimes\eta_A))\otimes\phi_A\otimes(\phi_A\circ(H\otimes\eta_A)))\circ(\delta_H\otimes c_{H,A})\circ(\delta_H\otimes A)\\&=\mu_A\circ(\phi_A\otimes(\phi_A\circ(H\otimes\eta_A)))\circ(H\otimes c_{H,A})\circ(\delta_H\otimes A)=\phi_A,\end{aligned}$$

by using that $(\varphi_A,\sigma)\approx(\psi_A,\tau)$, Definition 3.1(1) and that (A,ψ_A) is a left weak H-module algebra. In a similar way, we obtain (3.13) and the relation is symmetrical.

Finally we show the transitivity. Assume that $(\varphi_A, \sigma) \approx (\phi_A, \tau)$ and $(\phi_A, \tau) \approx (\chi_A, \gamma)$ with morphisms h in $\mathrm{Reg}_{\varphi_A}(H, A) \cap \mathrm{Reg}_{\phi_A}(H, A)$ and g in $\mathrm{Reg}_{\phi_A}(H, A) \cap \mathrm{Reg}_{\chi_A}(H, A)$, respectively. Then, the convolution product $h \wedge g$ is in $\mathrm{Reg}_{\varphi_A}(H, A) \cap \mathrm{Reg}_{\chi_A}(H, A)$. Using (2.12), (3.3), Defnition 3.1(2) and that $g^{-1} \wedge g = h^{-1} \wedge h$, we obtain that $(h \wedge g) \circ \eta_H = \eta_A$.

The proof for the conditions (3.12)–(3.13) follows a similar pattern to the well-known proof in the classical case, and we leave the details to the reader.

Remark 3.1 We have given the detailed calculus for the above proposition in order to illustrate the differences when working with weak Hopf algebras. Note that the proof is trivial in the classical case: If H is a Hopf algebra, the relation is reflexive using the morphism $h = \varepsilon_H \otimes \eta_A$, and it is easy to get that it is symmetrical because $h \wedge h^{-1} = h^{-1} \wedge h = \varepsilon_H \otimes \eta_A$. Obviously, these equalities are not true for weak Hopf algebras.

Proposition 3.3 *Let (A, φ_A) be a left weak H-module algebra and $\sigma \in \mathrm{Reg}_{\varphi_A}(H \otimes H, A)$. The following assertions hold:*

(i) $\sigma \circ (\eta_H \otimes H) = u_1 \Leftrightarrow \sigma^{-1} \circ (\eta_H \otimes H) = u_1$.

(ii) $\sigma \circ (H \otimes \eta_H) = u_1 \Leftrightarrow \sigma^{-1} \circ (H \otimes \eta_H) = u_1$.

Proof We prove (i). The proof of (ii) is similar, and we leave the details to the reader.

$$
\begin{aligned}
&\sigma^{-1} \circ (\eta_H \otimes H) \\
&= (\sigma^{-1} \wedge u_2) \circ (\eta_H \otimes H) \\
&= \mu_A \circ (\sigma^{-1} \otimes (u_1 \circ \mu_H)) \circ \delta_{H\otimes H} \circ (\eta_H \otimes H) \\
&= \mu_A \circ ((\sigma^{-1} \circ c_{H,H}) \otimes u_1)) \circ (((\overline{\Pi}_H^L \otimes H) \circ \delta_H) \otimes H) \circ \delta_H \\
&= \mu_A \circ ((\sigma^{-1} \circ c_{H,H}) \otimes \sigma) \circ (H \otimes (((\overline{\Pi}_H^L \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \\
&\quad \circ ((\delta_H \circ \eta_H) \otimes H)) \otimes H) \circ (H \otimes \delta_H) \circ \delta_H \\
&= \mu_A \circ ((\sigma^{-1} \circ c_{H,H}) \otimes \sigma) \circ (H \otimes ((H \otimes (((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H))) \\
&\quad \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H) \otimes H)) \otimes H) \circ (H \otimes \delta_H) \circ \delta_H \\
&= \mu_A \circ ((\sigma^{-1} \circ c_{H,H}) \otimes \sigma) \circ (H \otimes ((H \otimes (((\varepsilon_H \circ \mu_H) \otimes H) \\
&\quad \circ (H \otimes c_{H,H}) \circ ((c_{H,H} \circ \delta_H) \otimes H))) \\
&\quad \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H) \otimes H)) \otimes H) \circ (H \otimes (c_{H,H} \circ \delta_H)) \circ \delta_H \\
&= ((\mu_A \circ (\sigma^{-1} \otimes \sigma)) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes c_{H,H} \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes c_{H,H} \otimes H) \\
&\quad \circ ((\delta_H \circ \eta_H) \otimes ((\delta_H \otimes H) \circ \delta_H) \\
&= (u_1 \otimes \varepsilon_H) \circ \delta_H \circ \mu_H \circ (\eta_H \otimes H) = u_1,
\end{aligned}
$$

where the first and the eighth equalities follow by the properties of σ, the second one uses the definition of u_2, the third one follows by (2.14), in the fourth one, we use (3.6) and (2.12), the fifth one is a consequence of the definition of $\overline{\Pi}_H^L$, the sixth one follows by Definition 2.1(3), the seventh one uses the cocommutativity of H, and the last one uses the unit-counit properties.

The proof of the converse is the same changing σ by σ^{-1}.

Remark 3.2 The equalities (1)–(3) of Definition 3.2 have a clear meaning in the theory of weak crossed products introduced in [4, 8]. The full details can be also found in [5, Section 2]. In this point, we give a brief summary adapted to our setting, i.e., there are some changes in the notation.

Let (A, φ_A) be a left weak H-module algebra and $\sigma : H \otimes H \to A$ be a morphism. We define the morphisms $\psi_H^A : H \otimes A \to A \otimes H$ and $\sigma_H^A : H \otimes H \to A \otimes H$, by $\psi_H^A = (\varphi_A \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A)$ and $\sigma_H^A = (\sigma \otimes \mu_H) \circ \delta_{H\otimes H}$, respectively.

Then, the morphism $\nabla_{A\otimes H} = (\mu_A \otimes H) \circ (A \otimes \psi_H^A) \circ (A \otimes H \otimes \eta_A)$ is an idempotent, and we will denote by $A \times_\sigma H$, $i_{A\otimes H} : A \times_\sigma H \to A \otimes H$ and $p_{A\otimes H} : A \otimes H \to A \times_\sigma H$ the object, the injection and the projection associated to the factorization of $\nabla_{A\otimes H}$, respectively.

Considering the quadruple $\mathbb{A}_H = (A, H, \psi_H^A, \sigma_H^A)$, where (A, φ_A) is a left weak H-module algebra and $\sigma \in \mathrm{Reg}_{\varphi_A}(H \otimes H, A)$, we say that $\mathbb{A}_H$ satisfies the twisted condition if

$$(\mu_A \otimes H) \circ (A \otimes \psi_H^A) \circ (\sigma_H^A \otimes A) = (\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\psi_H^A \otimes H) \circ (H \otimes \psi_H^A), \tag{3.14}$$

and the cocycle condition holds if

$$(\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\sigma_H^A \otimes H) = (\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\psi_H^A \otimes H) \circ (H \otimes \sigma_H^A). \tag{3.15}$$

For the product defined by

$$\mu_{A\otimes_\sigma H} = (\mu_A \otimes H) \circ (\mu_A \otimes \sigma_H^A) \circ (A \otimes \psi_H^A \otimes H), \tag{3.16}$$

if the twisted and the cocycle conditions hold, we obtain that it is associative and normalized with respect to $\nabla_{A\otimes H}$ (i.e., $\nabla_{A\otimes H} \circ \mu_{A\otimes_\sigma H} = \mu_{A\otimes_\sigma H} = \mu_{A\otimes_\sigma H} \circ (\nabla_{A\otimes H} \otimes \nabla_{A\otimes H})$). We say that $A \otimes_\sigma H = (A \otimes H, \mu_{A\otimes_\sigma H})$ is a weak crossed product if $\mathbb{A}_H$ satisfies (3.14)–(3.15).

Due to the normality condition, the object $A \times_\sigma H$ is an algebra with product $\mu_{A\times_\sigma H} = p_{A\otimes H} \circ \mu_{A\otimes_\sigma H} \circ (i_{A\otimes H} \otimes i_{A\otimes H})$, and unit $\eta_{A\times_\sigma H} = p_{A\otimes H} \circ (\eta_A \otimes \eta_H)$ (see [16, Propositions 3.7–3.8]). Moreover, $\nu = \nabla_{A\otimes H} \circ (\eta_A \otimes \eta_H)$ is a preunit for $\mu_{A\otimes_\sigma H}$.

Therefore, if (φ_A, σ) is a crossed system for H over A, we have that $A \otimes_\sigma H = (A \otimes H, \mu_{A\otimes_\sigma H})$ is a weak crossed product with preunit $\nu = \nabla_{A\otimes H} \circ (\eta_A \otimes \eta_H)$. Conversely, if the pair (φ_A, σ) satisfies that $A \otimes_\sigma H = (A \otimes H, \mu_{A\otimes_\sigma H})$ is a weak crossed product with preunit $\nu = \nabla_{A\otimes H} \circ (\eta_A \otimes \eta_H)$ and normalized with respect to $\nabla_{A\otimes H}$, we obtain that (φ_A, σ) is a crossed system for H over A (see [5, Corollary 2.20]).

In the following result, we characterize crossed products with an H-module structure φ_A.

Theorem 3.1 *Let (A, φ_A) be a left weak H-module algebra and $\sigma \in \mathrm{Reg}_{\varphi_A}(H \otimes H, A)$ satisfying Definition* 3.2(1)*. The following assertions are equivalent:*

(i) (A, φ_A) *is a left H-module algebra.*

(ii) *The morphism σ factorizes through the center of A.*

Proof Let (A, φ_A) be a left H-module algebra. We define $\gamma_\sigma : A \otimes H \otimes H \to A$ as $\gamma_\sigma = \mu_A \circ ((\mu_A \circ c_{A,A}) \otimes A) \circ (A \otimes ((\sigma \otimes \sigma^{-1}) \circ \delta_{H\otimes H}))$. Then,

$$\gamma_\sigma = \mu_A \circ (A \otimes u_2) \tag{3.17}$$

because

$$\begin{aligned}
\gamma_\sigma &= \mu_A \circ ((\mu_A \circ c_{A,A}) \otimes A) \circ (A \otimes (((\sigma \wedge u_2) \otimes \sigma^{-1}) \circ \delta_{H\otimes H})) \\
&= \mu_A \circ ((\mu_A \circ (A \otimes \mu_A) \circ (A \otimes (c_{A,A} \circ (A \otimes u_1)))) \otimes A) \circ (c_{A,A} \otimes H \otimes A) \\
&\quad \circ (A \otimes \sigma_H^A \otimes \sigma^{-1}) \circ (A \otimes \delta_{H\otimes H})
\end{aligned}$$

$$
\begin{aligned}
&= \mu_A \circ ((\mu_A \circ (A \otimes (\varphi_A \circ (\Pi_H^L \otimes A) \circ c_{A,H})) \circ (c_{A,A} \otimes H)) \otimes A) \\
&\quad \circ (A \otimes \sigma_H^A \otimes \sigma^{-1}) \circ (A \otimes \delta_{H\otimes H}) \\
&= \mu_A \circ ((\mu_A \circ (A \otimes (A \otimes (\varphi_A \circ c_{A,H} \circ (A \otimes (\mu_H \circ (H \otimes \lambda_H) \circ (\mu_H \otimes \mu_H) \\
&\quad \circ \delta_{H\otimes H})))))) \otimes A) \circ (c_{A,A} \otimes H \otimes H \otimes A) \circ (A \otimes \sigma \otimes H \otimes H \otimes A) \\
&\quad \circ (A \otimes \delta_{H\otimes H} \otimes \sigma^{-1}) \circ (A \otimes \delta_{H\otimes H}) \\
&= \mu_A \circ ((\mu_A \circ (A \otimes \varphi_A) \circ (\sigma_H^A \otimes A) \circ (H \otimes H \otimes (\varphi_A \circ c_{A,H} \circ (A \otimes \lambda_H))) \\
&\quad \circ (H \otimes c_{A,H} \otimes \mu_H) \circ (c_{A,H} \otimes H \otimes H \otimes H) \circ (A \otimes \delta_{H\otimes H})) \otimes \sigma^{-1}) \circ (A \otimes \delta_{H\otimes H}) \\
&= \mu_A \circ ((\mu_A \circ ((\varphi_A \circ (H \otimes \varphi_A)) \otimes A) \circ (H \otimes H \otimes c_{A,A}) \\
&\quad \circ (((H \otimes H \otimes \sigma) \circ \delta_{H\otimes H}) \otimes A) \\
&\quad \circ (H \otimes H \otimes (\varphi_A \circ c_{A,H} \circ (A \otimes \lambda_H))) \circ (H \otimes c_{A,H} \otimes \mu_H) \\
&\quad \circ (c_{A,H} \otimes H \otimes H \otimes H) \circ (A \otimes \delta_{H\otimes H})) \otimes \sigma^{-1}) \circ (A \otimes \delta_{H\otimes H}) \\
&= \mu_A \circ (((\varphi_A \circ (\mu_H \otimes A)) \circ (H \otimes c_{A,H})) \otimes A) \circ (c_{A,H} \otimes \lambda_H \otimes A) \\
&\quad \circ (A \otimes ((\mu_H \otimes \mu_H) \circ \delta_{H\otimes H}) \otimes (\sigma \wedge \sigma^{-1})) \circ (A \otimes \delta_{H\otimes H}) \\
&= \mu_A \circ ((\varphi_A \circ c_{A,H} \circ (A \otimes (\Pi_H^L \circ \mu_H))) \otimes u_2) \circ (A \otimes \delta_{H\otimes H}) \\
&= \mu_A \circ ((\varphi_A \circ (\overline{\Pi}_H^L \otimes A) \circ c_{A,H}) \otimes u_1) \circ (A \otimes (\delta_H \circ \mu_H)) \\
&= \mu_A \circ ((\mu_A \circ c_{A,A} \circ (u_1 \otimes A) \circ c_{A,H}) \otimes u_1) \circ (A \otimes (\delta_H \circ \mu_H)) \\
&= \mu_A \circ (A \otimes u_2),
\end{aligned}
$$

where the first and the eighth equalities follow by Definition 3.1(4)–(5), the second one uses (3.1), the third one is a consequence of Definition 2.6(4), the fourth one follows by the definition of Π_H^L, the fifth and the seventh equalities are consequences of Definition 2.6(3)$'$. The sixth and the nineth equalities follow by the cocommutativity of H, the tenth one uses Definition 2.6(5), and the last one uses Definition 2.6(2).

Therefore, σ factorizes through the center of A because

$$
\begin{aligned}
&\mu_A \circ (A \otimes \sigma) \\
&= \mu_A \circ (A \otimes (u_2 \wedge \sigma)) \\
&= \mu_A \circ (\gamma_\sigma \otimes \sigma) \circ (A \otimes \delta_{H\otimes H}) \\
&= \mu_A \circ ((\mu_A \circ ((\mu_A \circ c_{A,A}) \otimes A) \circ (A \otimes ((\sigma \otimes \sigma^{-1}) \circ \delta_{H\otimes H}))) \otimes \sigma) \circ (A \otimes \delta_{H\otimes H}) \\
&= \mu_A \circ ((\mu_A \circ c_{A,A}) \otimes A) \circ (A \otimes ((\sigma \otimes u_2) \circ \delta_{H\otimes H})) \\
&= \mu_A \circ (A \otimes (\varphi_A \circ c_{A,H} \circ (A \otimes \overline{\Pi}_H^L))) \circ (c_{A,A} \otimes H) \circ (A \otimes \sigma_H^A) \\
&= \mu_A \circ (A \otimes (\varphi_A \circ (\Pi_H^L \otimes A) \circ c_{A,H})) \circ (c_{A,A} \otimes H) \circ (A \otimes \sigma_H^A) \\
&= \mu_A \circ (A \otimes (\mu_A \circ (u_1 \otimes A) \circ c_{A,H})) \circ (c_{A,A} \otimes H) \circ (A \otimes \sigma_H^A) \\
&= \mu_A \circ c_{A,A} \circ (A \otimes \sigma),
\end{aligned}
$$

where the first, the fourth and the last equalities follow by Definition 3.1(4)–(5), the second one uses (3.17), the third one is a consequence of the definition of γ_σ, the fifth and the seventh equalities follow by Definition 2.6(5), and the sixth one is a consequence of the cocommutativity of H.

Conversely, assume that the morphism σ factorizes through the center of A. Then, using

Definition 2.6(2)–(3), conditions (4)–(6) in Definition 3.1, the twisted condition and that H is cocommutative, we get that (A, φ_A) is a left H-module algebra.

Corollary 3.1 *Let (A, φ_A) be a left weak H-module algebra. The following assertions are equivalent:*

(i) (A, φ_A) *is a left H-module algebra.*

(ii) (φ_A, u_2) *is a crossed system for H over A.*

Proof It is straightforward.

Remark 3.3 In the conditions of Corollary 3.1, if (A, φ_A) is a left H-module algebra, we have that for the crossed system (φ_A, u_2) the equality $\sigma_H^A = (u_1 \otimes H) \circ \delta_H \circ \mu_H$ holds. Then, the associated crossed product defined is $\mu_{A \otimes_{u_2} H} = \nabla_{A \otimes H} \circ (\mu_A \otimes \mu_H) \circ (A \otimes \psi_H^A \otimes H)$ and therefore $\mu_{A \times_{u_2} H} = p_{A \otimes H} \circ (\mu_A \otimes \mu_H) \circ (A \otimes \psi_H^A \otimes H) \circ (i_{A \otimes H} \otimes i_{A \otimes H})$. In this case, we say that the weak crossed product is smash.

On the other hand, for a left weak H-module algebra, if the equality $\varphi_A = \varphi_A \circ (\Pi_H^L \otimes A)$ holds, using Definition 2.6(4), we obtain that

$$\mu_{A \times_\sigma H} = p_{A \otimes H} \circ (\mu_A \otimes H) \circ (\mu_A \otimes \sigma_H^A) \circ (A \otimes c_{H,A} \otimes H) \circ (i_{A \otimes H} \otimes i_{A \otimes H}).$$

In this case, the weak crossed product is called twisted.

Proposition 3.4 *Let (φ_A, σ) be a crossed system for H over A. Then, the algebra $A \times_\sigma H$ is a right H-comodule algebra for the coaction $\rho_{A \times_\sigma H} = (p_{A \otimes H} \otimes H) \circ (A \otimes \delta_H) \circ i_{A \otimes H}$. Moreover, $(A \times_\sigma H)_H = A$.*

Proof As a consequence of [5, Proposition 3.2], we obtain that $A \times_\sigma H$ is a right H-comodule algebra for the coaction $\rho_{A \times_\sigma H} = (p_{A \otimes H} \otimes H) \circ (A \otimes \delta_H) \circ i_{A \otimes H}$. Moreover, it is easy to prove that

$$A \xrightarrow{i_{A \times_\sigma H}} A \times H \underset{(A \times H \otimes \Pi_H^L) \circ \rho_{A \times_\sigma H}}{\overset{\rho_{A \times_\sigma H}}{\rightrightarrows}} A \times H \otimes H$$

is an equalizer diagram, where $i_{A \times_\sigma H} = p_{A \otimes H} \circ (A \otimes \eta_H)$.

In the following proposition, we establish the relation between crossed systems and H-cleft extensions.

Proposition 3.5 *Let (φ_A, σ) be a crossed system for H over A. Then $A \hookrightarrow A \times_\sigma H$ is an H-cleft extension.*

Proof The morphism $f = p_{A \otimes H} \circ (\eta_A \otimes H) : H \to A \times_\sigma H$ is a total integral. Obviously, $f \circ \eta_H = \eta_{A \times_\sigma H}$. Moreover, using that $\nabla_{A \otimes H}$ is a morphism of right H-comodules, we get that f is an integral. We define $f^{-1} = p_{A \otimes H} \circ (\sigma^{-1} \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \lambda_H) \otimes H) \circ \delta_H$. We will show that f^{-1} is the convolution inverse of f. First note that Definition 2.3(1) holds:

$$\begin{aligned}
&f^{-1} \wedge f \\
&= p_{A \otimes H} \circ (\mu_A \otimes \sigma_H^A) \circ (A \otimes ((u_1 \otimes H) \circ \delta_H) \otimes H) \circ (((\sigma^{-1} \otimes H) \circ (H \otimes c_{H,H}) \\
&\quad \circ ((\delta_H \circ \lambda_H) \otimes H) \circ \delta_H) \otimes H) \circ \delta_H \\
&= p_{A \otimes H} \circ ((\mu_A \circ (A \otimes (u_2 \wedge \sigma))) \otimes \mu_H) \circ (\sigma^{-1} \otimes \delta_{H \otimes H})
\end{aligned}$$

$$
\begin{aligned}
&\circ ((((H \otimes c_{H,H}) \circ ((\delta_H \circ \lambda_H) \otimes H)\delta_H) \otimes H) \circ \delta_H \\
&= p_{A\otimes H} \circ (\mu_A \otimes H) \circ (\sigma^{-1} \otimes \sigma_H^A) \circ ((((H \otimes c_{H,H}) \circ ((\delta_H \circ \lambda_H) \otimes H) \circ \delta_H) \otimes H) \circ \delta_H \\
&= p_{A\otimes H} \circ ((u_1 \circ \mu_H) \otimes \mu_H) \circ \delta_{H\otimes H} \circ (\lambda_H \otimes H) \circ \delta_H \\
&= p_{A\otimes H} \circ (u_1 \otimes H) \circ \delta_H \circ \Pi_H^R \\
&= p_{A\otimes H} \circ \nabla_{A\otimes H} \circ (\eta_A \otimes \Pi_H^R) \\
&= (A \times H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,A\times H} \otimes H) \circ (H \otimes (\rho_{A\times_\sigma H} \circ \eta_{A\times_\sigma H})).
\end{aligned}
$$

In the previous calculations, the first equality follows by the normalized condition for the product $\mu_{A\otimes_\sigma H}$; the second one uses that (A, φ_A) is a left weak H-module; the third and fourth ones hold because σ is in $\mathrm{Reg}_{\varphi_A}(H \otimes H, A)$; the fifth one follows by the definition of Π_H^R; the sixth one uses $\nabla_{A\otimes H} = ((\mu_A \circ (A \otimes u_1)) \otimes H) \circ (A \otimes \delta_H)$; finally, in the last one we use that f is a total integral.

The proof of Definition 2.3(2) follows a similar pattern, by the equality

$$((\sigma \circ (H \otimes \mu_H)) \wedge (\sigma^{-1} \circ (\mu_H \otimes H))) \circ (H \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H) \circ \delta_H = u_1,$$

which follows by (2.12)–(2.13) as well as $\sigma \in \mathrm{Reg}_{\varphi_A}(H \otimes H, A)$ and (3.11).

To finish the proof, we only need to show that $f^{-1} \wedge f \wedge f^{-1} = f^{-1}$. First of all, using that H is cocommutative, it is easy to see that $(f^{-1} \otimes \lambda_H) \circ c_{H,H} \circ \delta_H = \rho_{A\times_\sigma H} \circ f^{-1}$. Using this equality, the fact that $\lambda_H \circ \lambda_H = \mathrm{id}_H$ (which follows because H is cocommutative) and (2.6), we have

$$
\begin{aligned}
f^{-1} \wedge f \wedge f^{-1} &= \mu_{A\times_\sigma H} \circ (f^{-1} \otimes ((f \wedge f^{-1}) \circ \lambda_H \circ \lambda_H)) \circ \delta_H \\
&= \mu_{A\times_\sigma H} \circ (f^{-1} \otimes (f \circ \overline{\Pi}_H^L \circ \lambda_H \circ \lambda_H)) \circ c_{H,H} \circ \delta_H \\
&= \mu_{A\times_\sigma H} \circ (A \times H \otimes (f \circ \Pi_H^R)) \circ \rho_{A\times H} \circ f^{-1} \\
&= \mu_{A\times_\sigma H} \circ (A \times H \otimes (f^{-1} \wedge f)) \circ \rho_{A\times H} \circ f^{-1} = f^{-1}.
\end{aligned}
$$

Then the proof is completed.

Proposition 3.6 *Let $A_H \hookrightarrow A$ be an H-cleft extension. The morphism*

$$\sigma_A := (\mu_A \circ (f \otimes f)) \wedge (f^{-1} \circ \mu_H) : H \otimes H \to A,$$

where $f : H \to A$ is a convolution invertible total integral, factorizes through the equalizer i_A. Moreover, if $\varphi_{A_H} : H \otimes A_H \to A_H$ is the left weak H-module structure defined in Proposition 2.4*, the factorization of σ_A is a morphism in $\mathrm{Reg}_{\varphi_{A_H}}(H \otimes H, A_H)$ satisfying the condition* (3) *in Definition* 3.2 *and with convolution inverse the factorization through the equalizer i_A of the morphism $\sigma_A^{-1} := (f \circ \mu_H) \wedge (\mu_A \circ c_{A,A} \circ (f^{-1} \otimes f^{-1}))$.*

Proof If $A_H \hookrightarrow A$ is an H-cleft extension, Corollary 2.1 implies that $A_H \hookrightarrow A$ is a weak H-cleft extension. Then, using [2, Proposition 1.17], we obtain that σ_A factorizes through the equalizer i_A and, if σ_{A_H} is this factorization, the equality

$$\sigma_{A_H} = p_A \circ \mu_A \circ (f \otimes f) \tag{3.18}$$

holds.

On the other hand, the morphism σ_A^{-1} factorizes through the equalizer i_A. Indeed,

$$
\begin{aligned}
&\rho_A \circ \sigma_A^{-1} \\
&= \mu_{A\otimes H} \circ (f \otimes H \otimes \mu_{A\otimes H}) \circ (\mu_H \otimes \mu_H \otimes (\rho_A \circ f^{-1}) \otimes (\rho_A \circ f^{-1})) \circ (\delta_{H\otimes H} \otimes c_{H,H}) \circ \delta_{H\otimes H} \\
&= (\mu_A \otimes \mu_H) \circ (f \otimes \mu_{A\otimes H} \otimes H) \circ (H \otimes c_{H,A} \otimes c_{H,A} \otimes H) \\
&\quad \circ (\mu_H \otimes H \otimes [(A \otimes \overline{\Pi}_H^R) \circ \rho_A \circ f^{-1}] \otimes (\rho_A \circ f^{-1})) \circ (H \otimes c_{H,H} \otimes c_{H,H}) \\
&\quad \circ (\delta_H \otimes H \otimes H \otimes H) \circ \delta_{H\otimes H} \\
&= (\mu_A \otimes H) \circ (f \otimes [(A \otimes \mu_H) \circ (c_{H,A} \otimes (((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes \delta_H))) \circ (H \otimes \mu_A \otimes H \otimes H) \\
&\quad \circ (H \otimes A \otimes c_{H,A} \otimes H) \circ (H \otimes (\rho_A \circ f^{-1}) \otimes (\rho_A \circ f^{-1})) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)]) \\
&\quad \circ (\mu_H \otimes H \otimes H) \circ \delta_{H\otimes H} \\
&= (\mu_A \otimes H) \circ (f \otimes [(A \otimes \mu_H) \circ (c_{H,A} \otimes H) \circ (H \otimes \mu_A \otimes H) \circ (H \otimes f^{-1} \otimes (\rho_A \circ f^{-1})) \\
&\quad \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)]) \circ (\mu_H \otimes H \otimes H) \circ \delta_{H\otimes H} \\
&= (\mu_A \otimes H) \circ (f \otimes [((\mu_A \circ c_{A,A} \circ (f^{-1} \otimes f^{-1})) \otimes \Pi_H^L) \circ (H \otimes c_{H,H}) \circ ((c_{H,H} \otimes \delta_H) \otimes H)]) \\
&\quad \circ (\mu_H \otimes H \otimes H) \circ \delta_{H\otimes H} \\
&= (\mu_A \otimes H) \circ (f \otimes [(A \otimes \mu_H) \circ (c_{H,A} \otimes H) \circ (H \otimes \mu_A \otimes H) \circ (H \otimes f^{-1} \otimes (\rho_A \circ f^{-1})) \\
&\quad \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)]) \circ (\mu_H \otimes H \otimes H) \circ \delta_{H\otimes H} \\
&= (A \otimes \overline{\Pi}_H^R) \circ \rho_A \circ \sigma_A^{-1} \\
&= (A \otimes \Pi_H^L) \circ \rho_A \circ \sigma_A^{-1}.
\end{aligned}
$$

In the last computations, the first and the fourth equalities follows because A is a right H-comodule algebra, the second one relies on Theorem 2.1. In the third one, we use (2.11). In the fifth one, we apply the identity

$$
\begin{aligned}
&((\mu_A \circ c_{A,A} \circ (f^{-1} \otimes f^{-1})) \otimes \Pi_H^L) \circ (H \otimes c_{H,H}) \circ ((c_{H,H} \circ \delta_H) \otimes H) \\
&= (A \otimes \mu_H) \circ (c_{H,A} \otimes H) \circ (H \otimes \mu_A \otimes H) \circ (H \otimes f^{-1} \otimes (\rho_A \circ f^{-1})) \\
&\quad \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H). \qquad (3.19)
\end{aligned}
$$

Finally, the sixth one is obtained by (2.5) and the idempotent character of $\overline{\Pi}_H^R$, the seventh follows by repetition of the previous computations but in inverse order, and the last one relies on (2.4).

Let $\sigma_{A_H}^{-1}$ be the factorization of σ_A^{-1}. We will finish the proof by showing that σ_{A_H} is a morphism in $\mathrm{Reg}_{\varphi_{A_H}}(H \otimes H, A_H)$ with inverse $\sigma_{A_H}^{-1}$. First of all, note that $A_H \hookrightarrow A$ is an H-cleft extension and then, using Proposition 2.2, the morphism $f \wedge f^{-1}$ factorizes through the equalizer i_A. Now, using that H is a weak Hopf algebra, f is an integral, A is an H-comodule algebra, A_H is a weak H-module algebra and the equality (2.24) we obtain that $i_A \circ (\sigma_{A_H} \wedge \sigma_{A_H}^{-1}) = i_A \circ \varphi_{A_H} \circ (\mu_H \otimes \eta_{A_H})$. Then, using that i_A is a monomorphism, $\sigma_{A_H} \wedge \sigma_{A_H}^{-1} = \varphi_{A_H} \circ (\mu_H \otimes \eta_{A_H})$.

The proof of the equality $\sigma_{A_H}^{-1} \wedge \sigma_{A_H} = \varphi_{A_H} \circ (\mu_H \circ (H \otimes \eta_{A_H}))$ follows a similar pattern, using (2.9), (2.25), (3.18) and the equality $e_A \circ \mu_H \circ (\Pi_H^R \otimes H) = e_A \circ \mu_H$, where e_A is the morphism defined in (2.19).

To prove Definition 3.1(5), we compose with the equalizer i_A

$$
i_A \circ (\sigma_{A_H} \wedge \sigma_{A_H}^{-1} \wedge \sigma_{A_H}) = i_A \circ (\sigma_{A_H} \wedge (\varphi_{A_H} \circ (\mu_H \otimes \eta_{A_H}))
$$

$$= (\mu_A \circ (f \otimes f)) \wedge (f^{-1} \circ \mu_H) \wedge ((f \wedge f^{-1}) \circ \mu_H)$$
$$= (\mu_A \circ (f \otimes f)) \wedge ((f^{-1} \wedge f \wedge f^{-1}) \circ \mu_H) = i_A \circ \sigma_{A_H},$$

and then $\sigma_{A_H} \wedge \sigma_{A_H}^{-1} \wedge \sigma_{A_H} = \sigma_{A_H}$. In a similar way, using $f \wedge f^{-1} \wedge f = f$, we get Definition 3.1(6).

To finish the proof, we only need to show that σ_{A_H} satisfies the normal condition. Indeed, it is easy to see that $i_A \circ \sigma_{A_H} \circ (\eta_H \otimes H) = q_A \circ \mu_A \circ (\eta_A \otimes f) = q_A \circ f = f \wedge f^{-1} = i_A \circ \varphi_{A_H} \circ (H \otimes \eta_{A_H})$ and $i_A \circ \sigma_{A_H} \circ (H \otimes \eta_H) = q_A \circ f = f \wedge f^{-1} = i_A \circ \varphi_{A_H} \circ (H \otimes \eta_{A_H})$. Therefore, $\sigma_{A_H} \circ (\eta_H \otimes H) = \sigma_{A_H} \circ (H \otimes \eta_H) = \varphi_{A_H} \circ (H \otimes \eta_{A_H})$.

In the next theorem, we prove that each H-cleft extension determines a unique equivalence class of crossed systems for H over A. First we need a fundamental result in the study of H-cleft extensions that generalizes [15, Theorem 11].

Theorem 3.2 *Let $A_H \hookrightarrow A$ be an H-cleft extension with cleaving morphism f. Then, the pair $(\varphi_{A_H}, \sigma_{A_H})$ is a crossed system for H over A_H, where φ_{A_H} is the weak H-module structure defined in Proposition* 2.4 *and σ_{A_H} is the morphism obtained in Proposition* 3.6. *Moreover, the H-cleft extensions $A_H \hookrightarrow A$ and $A_H \hookrightarrow A_H \times_{\sigma_{A_H}} H$ are equivalent.*

Proof First note that in this case $\psi_H^{A_H} = (p_A \otimes H) \circ \rho_A \circ \mu_A \circ (f \otimes i_A)$ and $\sigma_H^{A_H} = (p_A \otimes H) \circ \rho_A \circ \mu_A \circ (f \otimes f)$. Then, as a consequence of [4, Proposition 3.13], we have that the quadruple $(A_H, H, \psi_H^{A_H}, \sigma_H^{A_H})$ satisfies the twisted and the cocycle conditions. Therefore, the theory exposed in Remark 3.2 leads to get that $(\varphi_{A_H}, \sigma_{A_H})$ is a crossed system for H over A_H. Moreover, by [4, Lemma 3.11], we obtain that $\nabla_{A_H \otimes H} = (p_A \otimes H) \circ \rho_A \circ \mu_A \circ (i_A \otimes f)$.

Taking into account Proposition 3.5, we know that $A_H \hookrightarrow A_H \times_{\sigma_{A_H}} H$ is an H-cleft extension. Also, using [4, 3.10], there exists a right H-comodule algebra isomorphism $T = p_{A_H \otimes H} \circ (p_A \otimes H) \circ \rho_A : A \to A_H \times_{\sigma_{A_H}} H$, such that $T^{-1} = \mu_A \circ (i_A \otimes f) \circ i_{A_H \otimes H}$ and $T^{-1} \circ i_{A_H \times_{\sigma_{A_H}} H} = \mu_A \circ (i_A \otimes f) \circ \nabla_{A_H \otimes H} \circ (A_H \otimes \eta_H) = \mu_A \circ (i_A \otimes (f \circ \eta_H)) = i_A$. Therefore, $A_H \hookrightarrow A$ and $A_H \hookrightarrow A_H \times_{\sigma_{A_H}} H$ are equivalent.

Proposition 3.7 *Let (φ_A, σ) be a crossed system for H over A. Let $A \hookrightarrow A \times_\sigma H$ be the H-cleft extension constructed in Proposition* 3.5. *Then, if (ϕ_A, τ) is the crossed system associated to the H-cleft extension $A \hookrightarrow A \times_\sigma H$, we have that $(\phi_A, \tau) = (\varphi_A, \sigma)$.*

Proof Using Proposition 3.4 and Theorem 3.2, the convolution invertible total integral $f = p_{A \otimes H} \circ (\eta_A \otimes H)$ determines a crossed system (ϕ_A, τ), where ϕ_A and τ are defined by $\phi_A = p_{A \times_\sigma H} \circ \mu_{A \times_\sigma H} \circ (f \otimes i_{A \times_\sigma H})$ and $\tau = p_{A \times_\sigma H} \circ \mu_{A \times_\sigma H} \circ (f \otimes f)$, where $p_{A \times_\sigma H}$ is the factorization through the equalizer $i_{A \times_\sigma H} = p_{A \otimes H} \circ (A \otimes \eta_H)$ of the morphism $q_{A \times_\sigma H} = \mu_{A \times_\sigma H} \circ (A \times_\sigma H \otimes f^{-1}) \circ \rho_{A \times_\sigma H}$.

We will show that the crossed systems (φ_A, σ) and (ϕ_A, τ) coincide. First of all note that, using the properties of the antipode, that H is a cocommutative weak Hopf algebra, as far as Definition 3.2(2), (3.6) and (3.9), the equality

$$\begin{aligned} u_1 = \mu_A \circ ((\varphi_A \circ (H \otimes \sigma^{-1})) \otimes \sigma) \circ (H \otimes H \otimes ((c_{H,H} \otimes H) \circ (H \otimes c_{H,H}))) \\ \circ ((\delta_{H \otimes H} \circ (H \otimes \lambda_H) \circ \delta_H) \otimes H) \circ \delta_H \end{aligned} \tag{3.20}$$

holds. Then we can obtain a simple expression for the morphism $q_{A \times_\sigma H}$. Indeed,

$$q_{A \times_\sigma H} = \mu_{A \times_\sigma H} \circ (p_{A \otimes H} \otimes f^{-1}) \circ (A \otimes \delta_H) \circ i_{A \otimes H}$$

$$
\begin{aligned}
&= p_{A\otimes H}\circ(\mu_A\otimes H)\circ(\mu_A\otimes\sigma\otimes\mu_H)\circ(A\otimes\varphi_A\otimes\delta_{H\otimes H})\circ(A\otimes H\otimes c_{H,A}\otimes H)\\
&\quad\circ(A\otimes\delta_H\otimes\sigma^{-1}\otimes H)\circ(A\otimes H\otimes H\otimes c_{H,H})\circ(A\otimes H\otimes(\delta_H\circ\lambda_H)\otimes H)\\
&\quad\circ(A\otimes H\otimes\delta_H)\circ(A\otimes\delta_H)\circ i_{A\otimes H}\\
&= p_{A\otimes H}\circ(\mu_A\otimes\Pi_H^L)\circ(A\otimes\mu_A\otimes H)\circ(A\otimes((H\otimes\varphi_A)\circ\sigma^{-1})\otimes\sigma\otimes H)\\
&\quad\circ(A\otimes H\otimes H\otimes c_{H,H}\otimes H\otimes H)\\
&\quad\circ(A\otimes H\otimes c_{H,H}\otimes c_{H,H}\otimes H)\circ(A\otimes\delta_H\otimes(\delta_H\circ\lambda_H)\otimes H)\circ(A\otimes H\otimes(c_{H,H}\circ\delta_H))\\
&\quad\circ(A\otimes\delta_H)\circ i_{A\otimes H}\\
&= p_{A\otimes H}\circ(\mu_A\otimes\Pi_H^L)\circ(A\otimes(\mu_A\circ((\varphi_A\circ(H\otimes\sigma^{-1}))\otimes\sigma)\circ(H\otimes H\otimes c_{H,H}\otimes H)\\
&\quad\circ(H\otimes c_{H,H}\otimes c_{H,H})\circ(\delta_H\otimes(\delta_H\circ\lambda_H)\otimes H)\circ(H\otimes\delta_H)\circ\delta_H))\circ i_{A\otimes H}\\
&= p_{A\otimes H}\circ(\mu_A\otimes\Pi_H^L)\circ(A\otimes u_1\otimes H)\circ(A\otimes\delta_H)\circ i_{A\otimes H}\\
&= p_{A\otimes H}\circ(A\otimes\Pi_H^L)\circ\nabla_{A\otimes H}\circ i_{A\otimes H}\\
&= p_{A\otimes H}\circ(A\otimes\Pi_H^L)\circ i_{A\otimes H}.
\end{aligned}
$$

In the following calculations, the first equality follows by the H-comodule structure for $A\times_\sigma H$; in the second one we use that $\mu_{A\otimes_\sigma H}\circ(\nabla_{A\otimes H}\otimes\nabla_{A\otimes H})=\mu_{A\otimes_\sigma H}$; the third one relies on the antimultiplicativity of the antipode; the fourth one relies on cocommutativity of H; the fifth one follows by (3.20); the seventh one is a consequence of the definition of $\nabla_{A\otimes H}$; finally the last one uses that $\nabla_{A\otimes H}\circ i_{A\otimes H}=i_{A\otimes H}$.

On the other hand, $q_{A\times_\sigma H}=i_{A\times_\sigma H}\circ p_{A\times_\sigma H}=p_{A\otimes H}\circ(p_{A\times_\sigma H}\otimes\eta_H)$ and then $(A\otimes\varepsilon_H)\circ i_{A\otimes H}\circ q_{A\times_\sigma H}=(A\otimes\varepsilon_H)\circ\nabla_{A\otimes H}\circ(p_{A\times_\sigma H}\otimes\eta_H)=p_{A\times_\sigma H}$. As a consequence, $p_{A\times_\sigma H}=(A\otimes\varepsilon_H)\circ i_{A\otimes H}$ because

$$
\begin{aligned}
p_{A\times_\sigma H}&=(A\otimes\varepsilon_H)\circ\nabla_{A\otimes H}\circ(A\otimes\Pi_H^L)\circ i_{A\otimes H}\\
&=(\mu_A\otimes\varepsilon_H)\otimes(A\otimes(\varphi_A\circ(\Pi_H^L\otimes\eta_A))\otimes H)\circ i_{A\otimes H}\\
&=(A\otimes\varepsilon_H)\circ\nabla_{A\otimes H}\circ i_{A\otimes H}=(A\otimes\varepsilon_H)\circ i_{A\otimes H}.
\end{aligned}
$$

Using this equality it is easy to see that $(\varphi_A,\sigma)=(\phi_A,\tau)$. Indeed,

$$
\begin{aligned}
\phi_A&=p_{A\times_\sigma H}\circ\mu_{A\times_\sigma H}\circ(f\otimes i_{A\times_\sigma H})\\
&=(\mu_A\otimes\varepsilon_H)\circ(\varphi_A\otimes\sigma\otimes\mu_H)\circ(H\otimes A\otimes\delta_{H\otimes H})\circ(H\otimes c_{H,A}\otimes H)\circ(\delta_H\otimes A\otimes\eta_H)\\
&=\mu_A\circ(\varphi_A\otimes(\sigma\circ(H\otimes\Pi_H^R)\circ\delta_H))\circ(H\otimes c_{H,A})\circ(\delta_H\otimes A)\\
&=\mu_A\circ(\varphi_A\otimes u_1)\circ(H\otimes c_{H,A})\circ(\delta_H\otimes A)=\varphi_A,
\end{aligned}
$$

and $\tau=p_{A\times_\sigma H}\circ\mu_{A\times_\sigma H}\circ(f\otimes f)=(A\otimes\varepsilon_H)\circ\nabla_{A\otimes H}\circ(\sigma\otimes\mu_H)\circ\delta_{H\otimes H}=\sigma\wedge u_2=\sigma$.

The following proposition is the weak version of [14, Lemma 2.1].

Proposition 3.8 *Let $A_H\hookrightarrow A$ be an H-cleft extension with cleaving morphism f. Assume that $g:H\to A$ is another convolution invertible total integral with associated crossed system (ϕ_{A_H},τ_{A_H}). Then the crossed systems $(\varphi_{A_H},\sigma_{A_H})$ and (ϕ_{A_H},τ_{A_H}) are equivalent.*

Proof The morphism $\widetilde{h}=f\wedge g^{-1}$ factorizes through the equalizer i_A. Indeed, as a consequence of (2.21), the coassociativity of δ_H and the naturality of c, we have

$$\rho_A\circ\widetilde{h}=(\rho_A\circ f)\wedge(\rho_A\circ g^{-1})$$

$$
\begin{aligned}
&= ((f \otimes H) \circ \delta_H) \wedge ((g^{-1} \otimes \lambda_H) \circ c_{H,H} \circ \delta_H) \\
&= ((\mu_A \circ (f \otimes g^{-1})) \otimes \Pi_H^L) \circ (H \otimes (c_{H,H} \circ \delta_H)) \circ \delta_H \\
&= ((\mu_A \circ (f \otimes g^{-1})) \otimes (\Pi_H^L \circ \Pi_H^L)) \circ (H \otimes (c_{H,H} \circ \delta_H)) \circ \delta_H \\
&= (A \otimes \Pi_H^L) \circ \rho_A \circ \widetilde{h},
\end{aligned}
$$

and then there exists a morphism $h : H \to A_H$, such that $\widetilde{h} = i_A \circ h$. Note that, in the conditions of this theorem, $f \wedge f^{-1} = g \wedge g^{-1}$ and $f^{-1} \wedge f = g^{-1} \wedge g$. Then,

$$
\varphi_{A_H} \circ (H \otimes \eta_{A_H}) = \phi_{A_H} \circ (H \otimes \eta_{A_H}) \tag{3.21}
$$

because $i_A \circ \varphi_{A_H} \circ (H \otimes \eta_{A_H}) = q_A \circ f = f \wedge f^{-1}$ and $i_A \circ \phi_{A_H} \circ (H \otimes \eta_{A_H}) = q'_A \circ g = g \wedge g^{-1}$, where q_A is the morphism defined in Remark 2.2 and q'_A is the analogous for g.

On the other hand, using that f and g are convolution invertible total integrals, we have

$$
\widetilde{h} \circ \eta_H = \mu_A \circ (A \otimes g^{-1}) \circ \rho_A \circ f \circ \eta_H = \mu_A \circ (A \otimes g^{-1}) \circ \rho_A \circ g \circ \eta_H = (g \wedge g^{-1}) \circ \eta_H = \eta_A.
$$

Therefore, taking into account that $\eta_A = i_A \circ \eta_{A_H}$, we obtain that $h \circ \eta_H = \eta_{A_H}$.

The morphism $\widetilde{h}^{-1} = g \wedge f^{-1}$ admits a factorization through the equalizer i_A (the proof is similar to the one developed for $\widetilde{h}$) and the factorization h^{-1} is the convolution inverse of h. As a consequence h is in $\mathrm{Reg}_{\varphi_{A_H}}(H, A_H) \cap \mathrm{Reg}_{\phi_{A_H}}(H, A_H)$. Indeed, first note that

$$
\begin{aligned}
i_A \circ (h \wedge h^{-1}) &= \widetilde{h} \wedge \widetilde{h}^{-1} = f \wedge g^{-1} \wedge g \wedge f^{-1} = f \wedge f^{-1} \wedge f \wedge f^{-1} \\
&= \varphi_A \circ (H \otimes \eta_A) = i_A \circ \varphi_{A_H} \circ (H \otimes \eta_{A_H})
\end{aligned}
$$

and using (3.21), $h \wedge h^{-1} = \varphi_{A_H} \circ (H \otimes \eta_{A_H}) = \phi_{A_H} \circ (H \otimes \eta_{A_H})$. Similarly, $h^{-1} \wedge h = \varphi_{A_H} \circ (H \otimes \eta_{A_H}) = \phi_{A_H} \circ (H \otimes \eta_{A_H})$. Moreover, $i_A \circ (h \wedge h^{-1} \wedge h) = \widetilde{h} \wedge \widetilde{h}^{-1} \wedge \widetilde{h} = \widetilde{h} = i_A \circ h$ and $i_A \circ (h^{-1} \wedge h \wedge h^{-1}) = \widetilde{h}^{-1} \wedge \widetilde{h} \wedge \widetilde{h}^{-1} = \widetilde{h}^{-1} = i_A \circ h^{-1}$. Then $h \wedge h^{-1} \wedge h = h$ and $h^{-1} \wedge h \wedge h^{-1} = h^{-1}$.

The proof of (3.12) follows by the definition of h and ϕ_{A_H}. In order to get (3.13), we begin by showing the equality $\mu_A \circ \mu_{A \otimes A} \circ (f \otimes (g^{-1} \wedge g) \otimes (f \wedge g^{-1}) \otimes g) \circ (\delta_H \otimes \delta_H) = \mu_A \circ (f \otimes f)$, which follows because f and g are convolution invertible integrals, A is a right H-comodule algebra, (2.8) and the equality $f^{-1} \wedge f = g^{-1} \wedge g$. Using this equality and arguments similar to the ones developed above, we will finish the proof by showing (3.13) as

$$
\begin{aligned}
& i_A \circ \mu_{A_H} \circ (\mu_{A_H} \otimes h^{-1}) \circ (\mu_{A_H} \otimes \tau_{A_H} \otimes \mu_H) \circ (h \otimes \phi_{A_H} \otimes \delta_{H \otimes H}) \\
& \quad \circ (\delta_H \otimes h \otimes H \otimes H) \circ \delta_{H \otimes H} \\
&= \mu_A \circ ((\mu_A \circ (\mu_A \otimes g^{-1}) \circ ((f \wedge f^{-1} \wedge f) \otimes (f \wedge g^{-1}) \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)) \\
& \quad \otimes (\mu_A \circ (A \otimes (f^{-1} \wedge f \wedge f^{-1})) \circ \rho_A \circ \mu_A \circ (g \otimes g))) \circ \delta_{H \otimes H} \\
&= (\mu_A \circ \mu_{A \otimes A} \circ (f \otimes (g^{-1} \wedge g) \otimes (f \wedge g^{-1}) \otimes g) \circ (\delta_H \otimes \delta_H)) \wedge (f^{-1} \circ \mu_H) \\
&= (\mu_A \circ (f \otimes f)) \wedge (f^{-1} \circ \mu_H) \\
&= i_A \circ \sigma_{A_H}.
\end{aligned}
$$

Corollary 3.2 *Let $A_H \hookrightarrow A$, $A_H \hookrightarrow B$ be two equivalent H-cleft extensions with cleaving morphisms f and g, respectively. Then the corresponding crossed systems $(\varphi_{A_H}, \sigma_{A_H})$ and (ϕ_{A_H}, τ_{A_H}) are equivalent.*

Proof If $A_H \hookrightarrow A$ and $A_H \hookrightarrow B$ are equivalent, there exists an isomorphism of right H-comodule algebras $T : A \to B$, such that $i_B = T \circ i_A$, and, as a consequence, $l = T \circ f$ is a convolution invertible total integral for $A_H \hookrightarrow B$ with inverse $l^{-1} = T \circ f^{-1}$. Therefore, as a consequence of Proposition 3.8, the crossed system $(\psi_{A_H}, \omega_{A_H})$ associated to $A_H \hookrightarrow B$ for l is equivalent to (ϕ_{A_H}, τ_{A_H}). Moreover, if p_A is the factorization through i_A of the morphism $q_A = \mu_A \circ (A \otimes f^{-1}) \circ \rho_A$ and p_B is the factorization through i_B of the morphism $q_B = \mu_A \circ (B \otimes l^{-1}) \circ \rho_B$. By using (2.27), $\psi_{A_H} = p_B \circ \mu_B \circ (l \otimes i_B)$. Then

$$\psi_{A_H} = p_B \circ \mu_B \circ (T \otimes T) \circ (f \otimes i_A) = p_B \circ T \circ \mu_A \circ (f \otimes i_A) = p_A \circ \mu_A \circ (f \otimes i_A) = \varphi_{A_H}.$$

On the other hand, by (3.18), $\omega_{A_H} = p_B \circ \mu_B \circ (l \otimes l)$. As a consequence,

$$\omega_{A_H} = p_B \circ \mu_B \circ (T \otimes T) \circ (f \otimes f) = p_B \circ T \circ \mu_A \circ (f \otimes f) = p_A \circ \mu_A \circ (f \otimes f) = \sigma_{A_H}.$$

Therefore $(\varphi_{A_H}, \sigma_{A_H})$ and (ϕ_{A_H}, τ_{A_H}) are equivalent.

Proposition 3.9 *If (φ_A, α) and (ϕ_A, β) are two equivalent crossed systems, so are the associated H-cleft extensions $A \hookrightarrow A \times_\alpha H$ and $A \hookrightarrow A \times_\beta H$.*

Proof We will begin to show that this correspondence is well defined. Let h be the morphism in $\mathrm{Reg}_{\varphi_A}(H, A) \cap \mathrm{Reg}_{\phi_A}(H, A)$ satisfying conditions (3.12)–(3.13). We denote by $A \hookrightarrow A \times_\alpha H$ and $A \hookrightarrow A \times_\beta H$ the H-cleft extensions defined by (φ_A, α) and (ϕ_A, β), respectively. We will show that $T = p_{A\otimes H} \circ (\mu_A \otimes H) \circ (A \otimes h \otimes H) \circ (A \otimes \delta_H) \circ i_{A\otimes H}$ is a morphism of H-comodule algebras, such that $T \circ i_{A\times_\alpha H} = i_{A\times_\beta H}$. First of all, note that the idempotent morphisms defined by the two crossed systems coincide. We denote it by $\nabla_{A\otimes H}$. Moreover, using (2.14) and (3.4), it is easy to prove that $T \circ \eta_{A\times_\alpha H} = \eta_{A\times_\beta H}$. In order to see the multiplicative condition for T, we need to fix a new notation and get two auxiliary identities. First note that, as a consequence of Remark 3.2, the crossed systems (φ_A, α) and (ϕ_A, β) define two quadruples $(A, H, \psi^A_{H,\alpha} = (\varphi_A \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A), \sigma^A_{H,\alpha} = (\alpha \otimes \mu_H) \circ \delta_{H\otimes H})$ and $(A, H, \psi^A_{H,\beta} = (\phi_A \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A),\ \sigma^A_{H,\beta} = (\beta \otimes \mu_H) \circ \delta_{H\otimes H})$, respectively, which induce the corresponding weak crossed products. On the other hand, by using the equivalence between (φ_A, α) and (ϕ_A, β) and that $h \in \mathrm{Reg}_{\varphi_A}(H, A)$ and $\beta \in \mathrm{Reg}_{\varphi_A}(H \otimes H, A)$, the equality $\mu_A \circ (A \otimes h) \circ \sigma^A_{H,\alpha} = \mu_A \circ ((\mu_A \circ (h \otimes \phi_A) \circ (\delta_H \otimes h)) \otimes \beta) \circ \delta_{H\otimes H}$ holds. Taking into account that $\nabla_{A\otimes H} \circ \psi^A_H = \psi^A_H$, it is easy to prove the equality

$$\begin{aligned}
&(\nabla_{A\otimes H} \otimes H) \circ ((\mu_A \circ (h \otimes \phi_A) \circ (\delta_H \otimes A)) \otimes \delta_H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A) \\
&= (\mu_A \otimes \delta_H) \circ (h \otimes \psi^A_{H,\beta}) \circ (\delta_H \otimes A).
\end{aligned} \tag{3.22}$$

In a routine way, we can check that T is a multiplicative morphism.

Using that $i_{A\otimes H} \circ T = (\mu_A \otimes H) \circ (A \otimes h \otimes H) \circ (A \otimes \delta_H) \circ i_{A\otimes H}$, it is easy to see that T is a morphism of right H-comodules. Moreover, by (2.14) and (3.4), we have that $T \circ i_{A\times_\alpha H} = i_{A\times_\beta H}$ and the associated H-cleft extensions $A \hookrightarrow A \times_\alpha H$ and $A \hookrightarrow A \times_\beta H$ are equivalent.

Remark 3.4 In the conditions of the previous result, T is an isomorphism with inverse

$$T^{-1} = p_{A\otimes H} \circ (\mu_A \otimes H) \circ (A \otimes h^{-1} \otimes H) \circ (A \otimes \delta_H) \circ i_{A\otimes H}.$$

Theorem 3.3 *Two H-cleft extensions $A_H \hookrightarrow A$, $A_H \hookrightarrow B$ are equivalent if and only if so are their respective associated crossed systems.*

Proof The "if" part is a consequence of Corollary 3.2. Moreover, if $A_H \hookrightarrow A$, $A_H \hookrightarrow B$ are two weak H-cleft extensions with equivalent crossed systems $(\varphi_{A_H}, \sigma_{A_H})$ and (ϕ_{A_H}, τ_{A_H}), as a consequence of Proposition 3.9, we know that the associated H-cleft extensions $A_H \hookrightarrow A_H \times_{\sigma_{A_H}} H$ and $A_H \hookrightarrow A_H \times_{\tau_{A_H}} H$ are equivalent. Therefore, using Theorem 3.2, we obtain $A_H \hookrightarrow A \approx A_H \hookrightarrow A_H \times_{\sigma_{A_H}} H \approx A_H \hookrightarrow A_H \times_{\tau_{A_H}} H \approx A_H \hookrightarrow B$ which proves the theorem.

Now we can give the main result of this section which is a generalization of [14, Theorem 2.7].

Corollary 3.3 *Let (A, ρ_A) be a right H-comodule algebra. There exists a bijective correspondence between the equivalence classes of H-cleft extensions $A_H \hookrightarrow B$ and the equivalence classes of crossed systems for H over A_H.*

Proof If $\mathrm{CS}(H, A_H)$ denotes the set of equivalence classes of crossed systems of H over A_H and $\mathrm{Cleft}(A_H)$ denotes the set of equivalence classes of H-cleft extensions $A_H \hookrightarrow B$, using Proposition 3.9 and Corollary 3.2 we have two maps $F : \mathrm{CS}(H, A_H) \to \mathrm{Cleft}(A_H)$ and $G : \mathrm{Cleft}(A_H) \to \mathrm{CS}(H, A_H)$ defined by $F([(\varphi_{A_H}, \sigma_{A_H})]) = [A_H \hookrightarrow A_H \times_{\sigma_{A_H}} H]$ and by $G([A_H \hookrightarrow B]) = [(\phi_{A_H}, \tau_{A_H})]$, respectively. The map G is the inverse of F, because, by using Proposition 3.7, $(G \circ F)\ ([(\varphi_{A_H}, \sigma_{A_H})]) = G([A_H \hookrightarrow A_H \times_{\sigma_{A_H}} H]) = [(\varphi_{A_H}, \sigma_{A_H})]$. And as a consequence of Theorem 3.2, $(F \circ G)([A_H \hookrightarrow B]) = F([(\phi_{A_H}, \tau_{A_H})]) = [A_H \hookrightarrow A_H \times_{\tau_{A_H}} H] = [A_H \hookrightarrow B]$.

4 Crossed Systems and Cohomology

In [5], we developed a cohomology theory of algebras over cocommutative weak Hopf algebras which generalizes the one given in [23] for Hopf algebras. The main result contained in [5] (see Theorem 3.11) asserts that if (A, φ_A) is a commutative left H-module algebra, there exists a bijection between the second cohomology group, denoted by $H^2_{\varphi_A}(H, A)$, and the equivalence classes of weak crossed products $A \otimes_\alpha H$, where $\alpha : H \otimes H \to A$ satisfies the 2-cocycle and the normal conditions. In this section, for an H-cleft extension $A_H \hookrightarrow A$, we will establish a bijection between the set of equivalence classes of crossed systems with a fixed weak H-module algebra structure and the second cohomology group $H^2_{\varphi_{\mathcal{Z}(A_H)}}(H, Z(A_H))$, where $Z(A_H)$ is the center of the subalgebra of coinvariants A_H. Our results generalize to the weak Hopf algebra setting, the ones proved by Doi for Hopf algebras in [14].

Proposition 4.1 *Let $A_H \hookrightarrow A$ be an H-cleft extension. We denote by $(\varphi_{A_H}, \sigma_{A_H})$ the corresponding crossed system defined by the convolution invertible total integral $f : H \to A$. Then $(Z(A_H), \varphi_{Z(A_H)})$ is a left H-module algebra, where $\varphi_{Z(A_H)}$ is the factorization through the equalizer $i_{Z(A_H)}$ of the morphism $\varphi_{A_H} \circ (H \otimes i_{Z(A_H)})$.*

Proof We define $\psi_A : H \otimes A_H \to A$ as $\psi_A = \mu_A \circ (A \otimes (\mu_A \circ c_{A,A})) \circ (((f^{-1} \otimes f) \circ \delta_H) \otimes i_A)$. In a similar way to Proposition 2.4, it is easy to see that ψ_A factorizes through the equalizer i_A, and then there exists a morphism $\psi_{A_H} : H \otimes A_H \to A_H$, such that $i_A \circ \psi_{A_H} = \psi_A$. On the other hand, the following equalities hold:

$$\mu_A \circ (f^{-1} \otimes i_A) = \mu_A \circ (\psi_A \otimes f^{-1}) \circ (H \otimes c_{H,A_H}) \circ (\delta_H \otimes A_H), \tag{4.1}$$

$$\mu_A \circ c_{A,A} \circ (f \otimes i_A) = \mu_A \circ (\psi_A \otimes f) \circ (\delta_H \otimes A_H). \tag{4.2}$$

Indeed, using Definition 3.1(1), (3) and that u_1 factorizes through the center of A (which follows because if H is cocommutative, (4) and (5) of Definition 2.6 coincide),

$$\begin{aligned}\mu_A\circ(f^{-1}\otimes i_A) &= \mu_A\circ((f^{-1}\wedge u_1)\otimes i_A)\\ &= \mu_A\circ(\mu_A\otimes A)\circ(f^{-1}\otimes i_A\otimes u_1))\circ(H\otimes c_{H,A_H})\circ(\delta_H\otimes A_H)\\ &= \mu_A\circ(\mu_A\otimes A)\circ(f^{-1}\otimes i_A\otimes(f\wedge f^{-1}))\circ(H\otimes c_{H,A_H})\circ(\delta_H\otimes A_H)\\ &= \mu_A\circ(\psi_A\otimes f^{-1})\circ(H\otimes c_{H,A_H})\circ(\delta_H\otimes A_H).\end{aligned}$$

The proof of (4.2) follows a similar pattern.

Now we can prove that $\varphi_{A_H}\circ(H\otimes i_{Z(A_H)})$ factorizes through the center of A_H. Indeed, using (4.1)–(4.2) and the properties of the center of A_H, we have

$$\begin{aligned}& i_A\circ\mu_{A_H}\circ((\varphi_{A_H}\circ(H\otimes i_{Z(A_H)}))\otimes A_H)\\ =\;& \mu_A\circ((\mu_A\circ(f\otimes i_A))\otimes(\mu_A\circ(\psi_A\otimes f^{-1})\circ(H\otimes c_{H,A_H})\circ(\delta_H\otimes A_H))\circ(H\otimes c_{H,A_H}\otimes A_H)\\ &\circ(\delta_H\otimes i_{Z(A_H)}\otimes A_H)\\ =\;& \mu_A\circ(\mu_A\otimes A)\circ(f\otimes(\mu_A\circ c_{A,A}\circ((i_A\circ i_{Z(A_H)})\otimes(i_A\circ\psi_{A_H})))\otimes f^{-1})\\ &\circ(H\otimes Z(A_H)\otimes H\otimes c_{H,A_H})\circ(H\otimes Z(A_H)\otimes\delta_H\otimes A_H)\\ &\circ(H\otimes c_{H,Z(A_H)}\otimes A_H)\circ(\delta_H\otimes Z(A_H)\otimes A_H)\\ =\;& \mu_A\circ((\mu_A\circ c_{A,A}\circ(f\otimes i_A))\otimes(\mu_A\circ((i_A\circ i_{Z(A_H)})\otimes f^{-1}))\circ(H\otimes c_{Z(A_H),A_H}\otimes H)\\ &\circ(H\otimes Z(A_H)\otimes c_{H,A_H})\circ(H\otimes c_{H,Z(A_H)}\otimes A_H)\circ(\delta_H\otimes Z(A_H)\otimes A_H)\\ =\;& i_A\circ\mu_{A_H}\circ c_{A_H,A_H}\circ((\varphi_{A_H}\circ(H\otimes i_{Z(A_H)}))\otimes A_H).\end{aligned}$$

Then there exists a morphism $\varphi_{Z(A_H)}: H\otimes Z(A_H)\to Z(A_H)$, such that $i_{Z(A_H)}\circ\varphi_{Z(A_H)}=\varphi_{A_H}\circ(H\otimes i_{Z(A_H)})$. Trivially, $\varphi_{Z(A_H)}$ satisfies the conditions of Definition 2.6. Moreover,

$$\varphi_{Z(A_H)}\circ(H\otimes\varphi_{Z(A_H)})=\varphi_{Z(A_H)}\circ(\mu_H\otimes Z(A_H)), \tag{4.3}$$

because $i_{Z(A_H)}\circ\varphi_{Z(A_H)}\circ(H\otimes\varphi_{Z(A_H)})=\varphi_{A_H}\circ(\mu_H\otimes i_{Z(A_H)})$, and then $(Z(A_H),\varphi_{Z(A_H)})$ is a left H-module algebra.

The following technical lemma will be useful for the last theorem of this paper.

Lemma 4.1 *Let $A_H\hookrightarrow A$ be an H-cleft extension. We denote by φ_{A_H} the weak H-module algebra structure defined for A_H by a convolution invertible total integral $f: H\to A$, and by ψ_{A_H} the morphism defined in Proposition* 4.1*. Then the equality $\varphi_{A_H}\circ(H\otimes\psi_{A_H})\circ(\delta_H\otimes A_H)=\mu_{A_H}\circ((\varphi_{A_H}\circ(H\otimes\eta_{A_H}))\otimes A_H)$ holds.*

Proof By composing the left part with the monomorphism i_A, we obtain that

$$i_A\circ\varphi_{A_H}\circ(H\otimes\psi_{A_H})\circ(\delta_H\otimes A_H)=i_A\circ\mu_{A_H}\circ((\varphi_{A_H}\circ(H\otimes\eta_{A_H}))\otimes A_H).$$

Taking into account that i_A is a monomorphism, we get the equality.

Now we show the main result of this section.

Theorem 4.1 *Let $A_H\hookrightarrow A$ be an H-cleft extension. We denote by $(\varphi_{A_H},\sigma_{A_H})$ the corresponding crossed system defined by the convolution invertible integral $f: H\to A$. Then there is a bijective correspondence between the second cohomology group $H^2_{\varphi_{Z(A_H)}}(H,Z(A_H))$ and the equivalence classes of crossed systems for H over A_H having φ_{A_H} as weak H-module algebra structure.*

Proof Let $[\tau]$ be in $H^2_{\varphi_{A_H}}(H, Z(A_H))$. Using the properties of the center of A_H, it is easy to prove that the morphism $\sigma_{A_H} \wedge (i_{Z(A_H)} \circ \tau)$ satisfies Definition 3.2(1)–(2). As far as Definition 3.1(3), note that

$$\begin{aligned}
&(\sigma_{A_H} \wedge (i_{Z(A_H)} \circ \tau)) \circ (\Pi^L_H \otimes H) \circ \delta_H \\
&= \mu_{A_H} \circ ((\sigma_{A_H} \circ (\Pi^L_H \otimes H) \circ \delta_H) \otimes ((i_{Z(A_H)} \circ \tau) \circ (\Pi^L_H \otimes H) \circ \delta_H)) \circ \delta_H \\
&= u_1 \wedge u_1 = u_1.
\end{aligned}$$

Using the equivalence between (3.5) and (3.6), we have that $u_1 = (\sigma_{A_H} \wedge (i_{Z(A_H)} \circ \tau)) \circ (\eta_H \otimes H)$. In a similar way, $u_1 = (\sigma_{A_H} \wedge (i_{Z(A_H)} \circ \tau)) \circ (H \otimes \eta_H)$, and then $(\varphi_{A_H}, \sigma_{A_H} \wedge (i_{Z(A_H)} \circ \tau))$ is a crossed system for A over H.

Conversely, let (φ_{A_H}, γ) be a crossed system for H over A_H. Then the morphism $\sigma^{-1}_{A_H} \wedge \gamma$ factorizes through the equalizer $i_{Z(A_H)}$. Indeed,

$$\begin{aligned}
&\mu_{A_H} \circ (A_H \otimes \sigma^{-1}_{A_H} \wedge \gamma) \\
&= \mu_{A_H} \circ (A_H \otimes (u_2 \wedge u_2 \wedge \sigma^{-1}_{A_H} \wedge \gamma)) \\
&= \mu_{A_H} \circ (\mu_{A_H} \otimes A_H) \circ (u_2 \wedge u_2 \otimes A_H \otimes \sigma^{-1}_{A_H} \wedge \gamma) \circ (H \otimes c_{A_H,H} \otimes H \otimes H) \\
&\quad \circ (c_{A_H,H} \otimes H \otimes H \otimes H) \circ (A_H \otimes \delta_{H\otimes H}) \\
&= \mu_{A_H} \circ (\mu_{A_H} \otimes H) \circ (\sigma^{-1}_{A_H} \wedge \sigma_{A_H} \wedge u_2 \otimes A_H \otimes \sigma^{-1}_{A_H} \wedge \gamma) \circ (H \otimes c_{A_H,H} \otimes H \otimes H) \\
&\quad \circ (c_{A_H,H} \otimes H \otimes H \otimes H) \circ (A_H \otimes \delta_{H\otimes H}) \\
&= \mu_{A_H} \circ (\mu_{A_H} \otimes H) \circ (\sigma^{-1}_{A_H} \wedge \sigma_{A_H} \otimes (\varphi_{A_H} \circ (H \otimes \psi_{A_H}) \circ ((\delta_H \circ \mu_H) \otimes A_H)) \otimes \sigma^{-1}_{A_H} \wedge \gamma) \\
&\quad \circ (\delta_{H\otimes H} \otimes A_H \otimes H \otimes H) \circ (H \otimes c_{A_H,H} \otimes H \otimes H) \circ (c_{A_H,H} \otimes H \otimes H \otimes H) \circ (A_H \otimes \delta_{H\otimes H}) \\
&= \mu_{A_H} \circ (\mu_{A_H} \otimes A_H) \circ ((A_H \otimes (\mu_{A_H} \circ ((\varphi_{A_H} \circ (H \otimes \varphi_{A_H})) \otimes A_H) \circ (H \otimes H \otimes c_{A_H,A_H}) \\
&\quad \circ (H \otimes H \otimes \sigma_{A_H} \otimes A_H) \circ (\delta_{H\otimes H} \otimes A_H)) \otimes A_H) \\
&\quad \circ (\sigma^{-1}_{A_H} \otimes H \otimes H \otimes (\varphi_{A_H} \circ (\mu_H \otimes A_H)) \otimes \sigma^{-1}_{A_H} \wedge \gamma) \circ (H \otimes H \otimes \delta_{H\otimes H} \otimes A_H \otimes H \otimes H) \\
&\quad \circ (\delta_{H\otimes H} \otimes A_H \otimes H \otimes H) \circ (H \otimes c_{A_H,H} \otimes H \otimes H) \\
&\quad \circ (c_{A_H,H} \otimes H \otimes H \otimes H) \circ (A_H \otimes \delta_{H\otimes H}) \\
&= \mu_{A_H} \circ (\mu_{A_H} \otimes A_H) \circ (A_H \otimes (\varphi_{A_H} \circ (H \otimes \varphi_{A_H})) \otimes A_H) \\
&\quad \circ (\sigma^{-1}_{A_H} \otimes H \otimes H \otimes (\varphi_{A_H} \circ (\mu_H \otimes A_H)) \otimes \sigma_{A_H} \wedge \sigma^{-1}_{A_H} \wedge \gamma) \\
&\quad \circ (H \otimes H \otimes \delta_{H\otimes H} \otimes A_H \otimes H \otimes H) \circ (\delta_{H\otimes H} \otimes A_H \otimes H \otimes H) \\
&\quad \circ (H \otimes c_{A_H,H} \otimes H \otimes H) \circ (c_{A_H,H} \otimes H \otimes H \otimes H) \circ (A_H \otimes \delta_{H\otimes H}) \\
&= \mu_{A_H} \circ (A_H \otimes (\mu_{A_H} \circ ((\varphi_{A_H} \circ (H \otimes \varphi_{A_H})) \otimes A_H) \circ (H \otimes H \otimes c_{A_H,A_H}) \circ (H \otimes H \otimes \gamma \otimes A_H) \\
&\quad \circ (\delta_{H\otimes H} \otimes A_H))) \circ (A_H \otimes H \otimes H \otimes (\varphi_{A_H} \circ (\mu_H \otimes A_H))) \circ (\sigma^{-1}_{A_H} \otimes \delta_{H\otimes H} \otimes A_H) \\
&\quad \circ (H \otimes H \otimes H \otimes c_{A_H,H}) \circ (H \otimes H \otimes c_{A_H,H} \otimes H) \circ (H \otimes c_{A_H,H} \otimes H \otimes H) \\
&\quad \circ (c_{A_H,H} \otimes H \otimes H \otimes H) \circ (A_H \otimes \delta_{H\otimes H}) \\
&= \mu_{A_H} \circ (A_H \otimes (\mu_{A_H} \circ (\gamma \otimes (\varphi_{A_H} \circ (\mu_H \otimes A_H))))) \circ (\delta_{H\otimes H} \otimes A_H)) \\
&\quad \circ (A_H \otimes H \otimes H \otimes (\varphi_{A_H} \circ (\mu_H \otimes A_H))) \circ (\sigma^{-1}_{A_H} \otimes \delta_{H\otimes H} \otimes A_H) \circ (H \otimes H \otimes H \otimes c_{A_H,H}) \\
&\quad \circ (H \otimes H \otimes c_{A_H,H} \otimes H) \circ (H \otimes c_{A_H,H} \otimes H \otimes H) \circ (c_{A_H,H} \otimes H \otimes H \otimes H) \circ (A_H \otimes \delta_{H\otimes H}) \\
&= \mu_{A_H} \circ (\sigma^{-1}_{A_H} \wedge \gamma \wedge u_2 \otimes A_H) \circ (H \otimes c_{A_H,H}) \circ (c_{A_H,H} \otimes H) \\
&= \mu_{A_H} \circ (\sigma^{-1}_{A_H} \wedge \gamma \otimes A_H) \circ (H \otimes c_{A_H,H}) \circ (c_{A_H,H} \otimes H).
\end{aligned}$$

In the above computations, the first and the third equalities follow because $\sigma_{A_H}^{-1}$ is in $\mathrm{Reg}_{\varphi_{A_H}}(H, A_H)$; the second one holds because u_2 factorizes through the center of A_H; in the fourth one and the nineth ones, we use (4.3); the fifth and eighth one rely on Definition 3.2(1) for σ_{A_H} and γ, respectively; the sixth one follows by cocommutativity; the seventh one uses that H is cocommutative and σ_{A_H} is a morphism in $\mathrm{Reg}_{\varphi_{A_H}}(H, A_H)$; finally, the last equality follows by Definition 3.1(4)–(5) for γ.

As a consequence, by using that H is cocommutative, $\sigma_{A_H}^{-1} \wedge \gamma \wedge \sigma_{A_H} = \sigma_{A_H} \wedge \sigma_{A_H}^{-1} \wedge \gamma = \gamma$, and therefore $\sigma_{A_H}^{-1} \wedge \gamma = \gamma \wedge \sigma_{A_H}^{-1}$.

The proof of the condition Definition 3.2(2) follows a similar pattern to the one developed in [14] and will be omitted. As far as Definition 3.2(3), the proof follows in a similar way to the one giving for $\sigma_{A_H} \wedge (i_{Z(A_H)} \circ \tau)$ using Proposition 3.3. Finally, we have to show that the correspondence is well defined. Let $[\tau]$ and $[\tau']$ be in $H^2(H, Z(A_H))$, such that the crossed systems $(\varphi_{A_H}, \sigma_{A_H} \wedge (i_{Z(A_H)} \circ \tau))$ and $(\varphi_{A_H}, \sigma \wedge (i_{Z(A_H)} \circ \tau'))$ are equivalent. Let h be the morphism in $\mathrm{Reg}_{\varphi_{A_H}}(H, A_H)$ satisfying conditions (3.12)–(3.13). Then h factorizes through the center of A_H. Indeed,

$$\begin{aligned}
&\mu_{A_H} \circ (h \otimes A_H) \\
&= \mu_{A_H} \circ ((h \wedge u_1 \wedge u_1) \otimes A_H) \\
&= \mu_{A_H} \circ ((h \wedge u_1) \otimes (\mu_{A_H} \circ c_{A_H,A_H})) \circ (H \otimes u_1 \otimes A_H) \circ (\delta_H \otimes A_H) \\
&= \mu_{A_H} \circ (\mu_{A_H} \otimes u_1) \circ (h \otimes (\mu_{A_H} \circ ((\varphi_{A_H} \circ (H \otimes \eta_{A_H})) \otimes A_H))) \otimes H) \\
&\quad \circ (\delta_H \otimes c_{H,A_H}) \circ (\delta_H \otimes A_H) \\
&= \mu_{A_H} \circ (\mu_{A_H} \otimes u_1) \circ (h \otimes (\varphi_{A_H} \circ (H \otimes \varphi_{A_H}) \circ (\delta_H \otimes A_H)) \otimes H) \\
&\quad \circ (\delta_H \otimes c_{H,A_H}) \circ (\delta_H \otimes A_H) \\
&= \mu_{A_H} \circ ((\mu_{A_H} \circ (\mu_{A_H} \otimes A_H) \circ (h \otimes \varphi_{A_H} \otimes h^{-1}) \circ (\delta_H \otimes c_{H,A_H}) \circ (\delta_H \otimes A_H)) \otimes h) \\
&\quad \circ (H \otimes \varphi_{A_H} \otimes H) \circ (\delta_H \otimes c_{H,A_H}) \circ (\delta_H \otimes A_H) \\
&= \mu_{A_H} \circ (\varphi_{A_H} \otimes h) \circ (H \otimes \varphi_{A_H} \otimes H) \circ (\delta_H \otimes c_{H,A_H}) \circ (\delta_H \otimes A_H) \\
&= \mu_{A_H} \circ ((\mu_{A_H} \circ (u_1 \otimes A_H)) \otimes h) \circ (H \otimes c_{H,A_H}) \circ (\delta_H \otimes A_H) \\
&= \mu_{A_H} \circ (A_H \otimes (u_1 \wedge h)) \circ c_{H,A_H} \\
&= \mu_{A_H} \circ (A_H \otimes h) \circ c_{H,A_H}.
\end{aligned}$$

Using that h factorizes through the center of A_H and (3.13), it is easy to see that τ and τ' are cohomologous.

Conversely, if τ and τ' are cohomologous, using the properties of the center of A_H, we get that the corresponding crossed systems $(\varphi_{A_H}, \sigma_{A_H} \wedge (i_{Z(A_H)} \circ \tau))$ and $(\varphi_{A_H}, \sigma_{A_H} \wedge (i_{Z(A_H)} \circ \tau'))$ are equivalent, and we conclude the proof.

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