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The Extension of the H^k Mean Curvature Flow in Riemannian Manifolds*

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Abstract In this paper, the authors consider a family of smooth immersions $F_t : M^n \to N^{n+1}$ of closed hypersurfaces in Riemannian manifold N^{n+1} with bounded geometry, moving by the H^k mean curvature flow. The authors show that if the second fundamental form stays bounded from below, then the H^k mean curvature flow solution with finite total mean curvature on a finite time interval $[0, T_{\max})$ can be extended over $T_{\max}$. This result generalizes the extension theorems in the paper of Li (see "On an extension of the H^k mean curvature flow, *Sci. China Math.*, **55**, 2012, 99–118").

Keywords H^k mean curvature flow, Riemannian manifold, Sobolev type inequality, Moser iteration
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1 Introduction

Let M^n be a compact n-dimensional hypersurface without boundary, and let $F_0 : M^n \to N^{n+1}$ be a smooth immersion of M^n into a Riemannian manifold N^{n+1}. Consider the generalized mean curvature flow (abbreviated for GMCF), namely, a smooth one-parameter family of immersions

$$F(\cdot, t) : M^n \to N^{n+1}$$

satisfying the evolution equation

$$\begin{cases} \dfrac{\partial}{\partial t} F(\cdot, t) = -f(H(\cdot, t))\nu(\cdot, t), \\ F(\cdot, 0) = F_0(\cdot), \end{cases} \tag{1.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a smooth function, depending only on the mean curvature of the immersed surface, and $\nu(\cdot, t)$ is the outer unit normal on $M_t := F(M, t)$ at $F(\cdot, t)$. If $f' > 0$ along the GMCF, then the short time existence has been established in [10]. It is easy to prove that (1.1) admits a smooth solution on a maximal time interval $[0, T_{\max})$ with $T_{\max} < \infty$.

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If f is the identity function, then (1.1) is the classical mean curvature flow. If we choose f to be the power function x^k, then (1.1) is the H^k mean curvature flow. In this paper, we mainly pay our attention to the H^k mean curvature flow, also we get some results on the GMCF.

The long time existence, convergence, blow up and extension properties are of great interest subjects in curvature flow. Recently, many efforts have been made on the extension theorem for the mean curvature flow under some curvature conditions (see [1, 6, 11–12]). Le and Sesum [6] showed that if the second fundamental form stays bounded from below all the way to T, then some integral condition of mean curvature is enough to extend the mean curvature flow past time T. This extension theorem had also been generalized to the setting when the outer space is Riemannian manifold (see [11–12]). In arbitrary codimension, Han and Sun [1] gave an integral condition under which the mean curvature flow can be extended and then they investigated some properties of type I singularity. In [7], Li proved an extension theorem for the H^k mean curvature flow in $\mathbb{R}^n$. Motivated by his idea, we prove the following main theorems in our Riemannian setting.

Theorem 1.1 *Let M be a compact n-dimensional hypersurface without boundary, smoothly immersed into N^{n+1} with bounded geometry by F_0. Let $[0, T_{\max})$ be the maximal time interval of the H^k mean curvature flow with $T_{\max} < \infty$, and $H(\cdot, 0) \geq \delta_0 > 0$. Then the quantity $\max\limits_{M_t} |B|^2$ becomes unbounded as $t \to T_{\max}$.*

Along mean curvature flow, Huisken [3–4] proved that if $T < \infty$ is the first singularity time for a compact MCF, then $\sup\limits_{M_t} |A|(\cdot, t) \to \infty$ as $t \to T$. The above theorem is natural for GMCF.

Theorem 1.2 *Assume $k, n \in \mathbb{N}, k, n \geq 2$ and $n+1 \geq k$. Let M be a compact n-dimensional hypersurface without boundary, smoothly immersed into N^{n+1} with bounded geometry by F_0. Consider the H^k mean curvature flow on M,*

$$\frac{\partial}{\partial t} F(\cdot, t) = -H^k(\cdot, t) \cdot \nu(\cdot, t), \quad F(\cdot, 0) = F_0(\cdot).$$

If

(1) *$h_{ij} \geq C g_{ij}(t)$ along the H^k mean curvature flow for a uniform constant $C > 0$,*
(2) *for some $\alpha \geq n + k + 1$,*

$$\|H\|_{L^\alpha(M \times [0, T_{\max}))} := \Big(\int_0^{T_{\max}} \int_{M_t} |H(t)|^\alpha \mathrm{d}\mu \mathrm{d}t \Big)^{\frac{1}{\alpha}} < \infty,$$

then the flow can be extended over the time $T_{\max}$.

2 Preliminaries

In the following, the induced metric and the second fundamental form on M will be denoted by $g = \{g_{ij}\}$ and $B = \{h_{ij}\}$. The mean curvature of M is the trace of the second fundamental form, i.e.,

$$H = g^{ij} h_{ij}.$$

The square of the second fundamental form is

$$|B|^2 = g^{ij} g^{kl} h_{ik} h_{jl}.$$

The Riemann curvature tensor of N and its covariant derivative will be denoted by $\overline{R}m = \{\overline{R}_{\alpha\beta\gamma\delta}\}$ and $\overline{\nabla}\overline{R}m = \{\overline{\nabla}_\sigma \overline{R}_{\alpha\beta\gamma\delta}\}$, respectively. We write $Rm = \{R_{ijkl}\}$ for the curvature tensor of M. Let ν be the unit outer normal to M_t, then for a fixed time t, we can choose a local field of frame $e_0, e_1, \cdots, e_n$ in N, such that restricted to M_t, we have

$$e_0 = \nu, \quad e_i = \frac{\partial F}{\partial x_i}.$$

The relation between B, Rm and $\overline{R}m$ is then given by the equations of Gauss and Codazzi:

$$\begin{aligned} &R_{ijkl} = \overline{R}_{ijkl} + h_{ik}h_{jl} - h_{il}h_{jk}, \\ &\nabla_k h_{ij} - \nabla_j h_{ik} = \overline{R}_{0ijk}. \end{aligned}$$

We have the following proposition.

Proposition 2.1 (see [4])

$$\begin{aligned} \Delta h_{ij} = {} & \nabla_i \nabla_j H + H h_{il} h^l_j - |B|^2 h_{ij} + H \overline{R}_{0i0j} \\ & - h_{ij} \overline{R}_{0l0}{}^l + h_{jl} \overline{R}^l{}_{mi}{}^m + h_{il} \overline{R}^l{}_{mj}{}^m \\ & - 2h_{lm} \overline{R}^l{}_i{}^m{}_j + \nabla_j \overline{R}_{0li}{}^l + \nabla_l \overline{R}_{0ij}{}^l, \end{aligned} \tag{2.1}$$

$$\begin{aligned} \frac{1}{2}\Delta |B|^2 = {} & \langle h_{ij}, \nabla_i \nabla_j H \rangle + |\nabla B|^2 + H \cdot h_{ik} h^k{}_l h^{li} - |B|^2 \\ & + H h^{ij} \overline{R}_{0i0j} - |B|^2 \overline{R}_{0l0}{}^l + 2h^{ij} h_{jl} \overline{R}^l{}_{mi}{}^m - 2h^{ij} h^{lm} \overline{R}_{limj} \\ & + h^{ij} (\nabla_j \overline{R}_{0li}{}^l + \nabla_l \overline{R}_{0ij}{}^l). \end{aligned} \tag{2.2}$$

3 The Evolution Equations

Theorem 3.1 *For the GMCF in Riemannian manifold, we have the following evolution equations:*

$$\frac{\partial}{\partial t} F(t) = -f(H(t))\nu(t), \tag{3.1}$$

$$\frac{\partial}{\partial t} g_{ij}(t) = -2f(H(t))h_{ij}, \tag{3.2}$$

$$\frac{\partial}{\partial t} \nu(t) = f'(H(t))\nabla H(t), \tag{3.3}$$

$$\begin{aligned} \frac{\partial}{\partial t} h_{ij}(t) = {} & f'(H(t))\Delta h_{ij} + f''(H(t)) \nabla_{e_i} H \nabla_{e_j} H \\ & - [f(H(t)) + H f'(H(t))] g^{kl} h_{ik} h_{jl} + f'(H(t))|B|^2 h_{ij} \\ & + [f(H(t)) - H f'(H(t))] \overline{R}_{0i0j} + f'(H(t)) h_{ij} \overline{R}_{0l0}{}^l \\ & + f'(H(t))(2h_{lm} \overline{R}^l{}_i{}^m{}_j - h_{jl} \overline{R}^l{}_{mi}{}^m - h_{il} \overline{R}^l{}_{mj}{}^m - \nabla_j \overline{R}_{0li}{}^l - \nabla_l \overline{R}_{0ij}{}^l), \end{aligned} \tag{3.4}$$

$$\frac{\partial}{\partial t} H(t) = f'(H(t))\Delta H + f''(H(t))|\nabla H|^2 + f(H(t))(|B|^2 + \overline{\mathrm{Ric}}(\nu, \nu)), \tag{3.5}$$

$$\frac{\partial}{\partial t} \mathrm{d}\mu(t) = -2f(H(t))H(t)\mathrm{d}\mu(t). \tag{3.6}$$

Proof Let us first prove (3.2).

$$\begin{aligned}\frac{\partial}{\partial t}g_{ij}&=\frac{\partial}{\partial t}\langle F_*e_i,F_*e_j\rangle=\Big\langle\overline{\nabla}_{e_i}\frac{\partial F}{\partial t},F_*e_j\Big\rangle+\Big\langle\overline{\nabla}_{e_j}\frac{\partial F}{\partial t},F_*e_i\Big\rangle\\&=-\langle\overline{\nabla}_{e_i}(f(H(t))\nu),F_*e_j\rangle-\langle\overline{\nabla}_{e_j}(f(H(t))\nu),F_*e_i\rangle\\&=-f(H(t))(\langle\nabla_{e_i}\nu,F_*e_j\rangle+\langle\nabla_{e_j}\nu,F_*e_i\rangle)\\&=-f(H(t))h_{ij}.\end{aligned}$$

Next we prove (3.3).

$$\begin{aligned}\frac{\partial}{\partial t}\nu&=g^{ij}\Big\langle\frac{\partial}{\partial t}\nu,F_*e_i\Big\rangle F_*e_j=g^{ij}\Big(-\Big\langle\nu,\overline{\nabla}_{e_i}\frac{\partial F}{\partial t}\Big\rangle\Big)F_*e_j\\&=-g^{ij}\langle\nu,\overline{\nabla}_{e_i}(-f(H(t))\nu)\rangle F_*e_j\\&=g^{ij}\langle\nu,(\overline{\nabla}_{e_i}f(H(t)))\nu+f(H(t))\overline{\nabla}_{e_i}\nu\rangle F_*(e_j)\\&=g^{ij}\langle\nu,(\overline{\nabla}_{e_i}f(H(t)))\nu\rangle F_*e_j\\&=g^{ij}\overline{\nabla}_{e_i}f(H(t))F_*e_j\\&=\nabla f(H(t))=f'(H(t))\nabla H(t).\end{aligned}$$

Using (2.1) of Proposition 2.1, we have

$$\begin{aligned}\frac{\partial}{\partial t}h_{ij}(t)&=\frac{\partial}{\partial t}\langle\overline{\nabla}_{e_i}\nu,F_*e_j\rangle=\langle\overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\nabla}_{e_i}\nu,F_*e_j\rangle+\langle\overline{\nabla}_{e_i}\nu,\overline{\nabla}_{\frac{\partial}{\partial t}}F_*e_j\rangle\\&=\langle\overline{\nabla}_{e_i}\overline{\nabla}_{\frac{\partial}{\partial t}}\nu-f(H(t))\overline{R}(e_i,\nu)\nu,F_*e_j\rangle+\langle\overline{\nabla}_{e_i}\nu,-\overline{\nabla}_{e_j}(f(H(t))\nu)\rangle\\&=\langle\overline{\nabla}_{e_i}(f'(H(t))\nabla H),e_j\rangle-f(H(t))\langle\overline{R}(e_i,\nu)\nu,e_j\rangle\\&\quad-\langle\overline{\nabla}_{e_i}\nu,\overline{\nabla}_{e_j}(f(H(t)))\nu\rangle-\langle\overline{\nabla}_{e_i}\nu,f(H(t))\overline{\nabla}_{e_j}\nu\rangle\\&=\langle(f''(H(t))\overline{\nabla}_{e_i}H(t))\nabla H+f'(H(t))\overline{\nabla}_{e_i}\nabla H,e_j\rangle\\&\quad+f(H(t))\overline{R}_{0i0j}-f(H(t))g^{kl}h_{ik}h_{jl}\\&=f''(H(t))\nabla_{e_i}H\nabla_{e_j}H+f'(H(t))\nabla_{e_i}\nabla_{e_j}H-f(H(t))g^{kl}h_{ik}h_{jl}+f(H(t))\overline{R}_{0i0j}\\&=f''(H(t))\nabla_{e_i}H\nabla_{e_j}H+f'(H(t))(\Delta h_{ij}-Hh_{il}h^l_j+|B|^2h_{ij}-H\overline{R}_{0i0j}\\&\quad+h_{ij}\overline{R}_{0l0}{}^l-h_{jl}\overline{R}^l{}_{mi}{}^m-h_{il}\overline{R}^l{}_{mj}{}^m+2h_{lm}\overline{R}^l{}_i{}^m{}_j-\nabla_j\overline{R}_{0li}{}^l-\nabla_l\overline{R}_{0ij}{}^l)\\&\quad-f(H(t))g^{kl}h_{ik}h_{jl}+f(H(t))\overline{R}_{0i0j}\\&=f'(H(t))\Delta h_{ij}+f''(H(t))\nabla_{e_i}H\nabla_{e_j}H\\&\quad-[f(H(t))+Hf'(H(t))]g^{kl}h_{ik}h_{jl}+f'(H(t))|B|^2h_{ij}\\&\quad+[f(H(t))-Hf'(H(t))]\overline{R}_{0i0j}+f'(H(t))h_{ij}\overline{R}_{0l0}{}^l\\&\quad+f'(H(t))(2h_{lm}\overline{R}^l{}_i{}^m{}_j-h_{jl}\overline{R}^l{}_{mi}{}^m-h_{il}\overline{R}^l{}_{mj}{}^m-\nabla_j\overline{R}_{0li}{}^l-\nabla_l\overline{R}_{0ij}{}^l),\end{aligned}$$

this proves (3.4).

To prove (3.5), it is easy to get

$$\frac{\partial}{\partial t}g^{il}=2f(H(t))g^{ij}h_{jk}g^{kl}.$$

Hence

$$\frac{\partial}{\partial t}H=\frac{\partial g^{ij}}{\partial t}h_{ij}+g^{ij}\frac{\partial h_{ij}}{\partial t}$$

$$
\begin{aligned}
&= 2f(H(t))g^{ij}h_{jk}g^{kl}h_{ij} + g^{ij}(f'(H(t))\Delta h_{ij} + f''(H(t))\nabla_{e_{i2000}}H\nabla_{e_j}H \\
&\quad - [f(H(t)) + Hf'(H(t))]g^{kl}h_{ik}h_{jl} + f'(H(t))|B|^2 h_{ij} \\
&\quad + [f(H(t)) - Hf'(H(t))]\overline{R}_{0i0j} + f'(H(t))h_{ij}\overline{R}_{0l0}{}^{l} \\
&\quad + f'(H(t))(2h_{lm}\overline{R}^{l}{}_{i}{}^{m}{}_{j} - h_{jl}\overline{R}^{l}{}_{mi}{}^{m} - h_{il}\overline{R}^{l}{}_{mj}{}^{m} - \nabla_j\overline{R}_{0li}{}^{l} - \nabla_l\overline{R}_{0ij}{}^{l})) \\
&= 2f(H(t))|B|^2 + f'(H(t))\Delta H + f''(H(t))|\nabla H|^2 - f(H(t))|B|^2 - Hf'(H(t))|B|^2 \\
&\quad + f'(H(t))|B|^2 H + [f(H(t)) - Hf'(H(t))]\overline{R}_{0i0}{}^{i} + f'(H(t))H\overline{R}_{0l0}{}^{l} \\
&\quad + f'(H(t))(2h_{lm}\overline{R}^{l}{}_{i}{}^{m}{}_{j}g^{ij} - h_{jl}\overline{R}^{l}{}_{mi}{}^{m}g^{ij} - h_{il}\overline{R}^{l}{}_{mj}{}^{m}g^{ij} - \nabla_j\overline{R}_{0li}{}^{l}g^{ij} - \nabla_l\overline{R}_{0ij}{}^{l}g^{ij}) \\
&= f'(H(t))\Delta H + f''(H(t))|\nabla H|^2 + f(H(t))(|B|^2 + \overline{\mathrm{Ric}}(\nu,\nu)),
\end{aligned}
$$

this proves (3.5). It is easy to obtain (3.6), we omit the concrete computation.

4 Sobolev Inequalities for the GMCF

Li [7] obtained a Sobolev inequality for the power mean curvature flow by using Michael-Simon inequality (see [8]), which is crucial for the Moser iteration in his situation. In our setting, we also need an inequality which is similar to Michael-Simon inequality. Hence, in this section we first introduce the Hoffman-Spruck Sobolev inequality.

Lemma 4.1 (see [2]) *Let $M \to N$ be an isometric immersion of Riemannian manifolds of dimension n and $n+p$ ($p \geq 1$), respectively. Assume $K_N \leq b^2$ and let h be a nonnegative C^1 function on M vanishing on ∂M. Then*

$$
\Big(\int_M h^{\frac{n}{n-1}}\mathrm{d}\mu\Big)^{\frac{n-1}{n}} \leq c_n \int_M (|\nabla h| + h|H|)\mathrm{d}\mu, \tag{4.1}
$$

provided

$$
b^2(1-\alpha)^{-\frac{2}{n}}(\omega_n^{-1}\mathrm{Vol}(\mathrm{Supp}h))^{\frac{2}{n}} \leq 1 \tag{4.2}
$$

and

$$
2\rho_0 \leq \overline{R}(M), \tag{4.3}
$$

where $\overline{R}(M)$ is the injectivity radius of N restricted to M and

$$
\rho_0 = \begin{cases} b^{-1}\sin^{-1}[b(1-\alpha)^{-\frac{1}{n}}(\omega_n^{-1}\mathrm{Vol}(\mathrm{Supp}h))^{\frac{1}{n}}] & \text{for } b \text{ real}, \\ (1-\alpha)^{-\frac{1}{n}}(\omega_n^{-1}\mathrm{Vol}(\mathrm{Supp}h))^{\frac{1}{n}} & \text{for } b \text{ arbitrary}. \end{cases} \tag{4.4}
$$

Here α is a free parameter, $0 < \alpha < 1$, and

$$
c_n := \pi \cdot 2^{n-1}\alpha^{-1}(1-\alpha)^{-\frac{1}{n}}\frac{n}{n-1}\omega_n^{-\frac{1}{n}}. \tag{4.5}
$$

Following the proof of Theorem 3.4 in [7] and using Lemma 4.1, we obtain the following general result.

Theorem 4.1 *Suppose that $k, n \in \mathbb{N}$, $k, n \geq 2$, or $k = 1$ and $n = 2$. Set*

$$
Q_k = \frac{kn}{kn-(k+1)} = \frac{n}{n - \dfrac{k+1}{k}}.
$$

Let M be a compact n-dimensional hypersurface without boundary, which is smoothly embedded in N^{n+1}. Assume $K_N \le b^2$. Then for all nonnegative Lipschitz functions v on M, we have

$$\begin{aligned}\|v\|^{k+1}_{L^{\frac{k+1}{k}Q_k}(M)} &\le A_{n,k}(\|\nabla v\|^{k+1}_{L^{\frac{k+1}{k}}(M)} + \|H\|^{n+k+1}_{L^{n+k+1}(M)}\|v\|^{k+1}_{L^{\frac{k+1}{k}}(M)}) \\ &\le \widehat{A}_{n,k}(\|\nabla v\|^{k+1}_{L^2(M)} + \|H\|^{n+k+1}_{L^{n+k+1}(M)}\|v\|^{k+1}_{L^2(M)}),\end{aligned} \tag{4.6}$$

provided that the function $h := v^{\frac{(k+1)(n-1)}{kn-(k+1)}}$ satisfies (4.2)–(4.3)*, where H is the mean curvature of M and*

$$\begin{aligned}A_{n,k} &= 2^{\frac{(n-1)(k+1)(n+k+1)}{kn-(k+1)}}(2c_{n,k})^{n+k+1} \quad \Big(c_{n,k} = c_n \cdot \frac{(k+1)(n-1)}{kn-(k+1)}\Big), \\ \widehat{A}_{n,k} &= A_{n,k}\mathrm{Vol}(M)^{\frac{k-1}{2}}.\end{aligned}$$

Corollary 4.1 *Under the conditions of Theorem* 4.1*, for any nonnegative Lipschitz function v, we have*

$$\|v\|^2_{L^{2Q_k}(M)} \le \widetilde{A}_{n,k}(\|v\|^{\frac{k-1}{k}}_{L^2(M)} \cdot \|\nabla v\|^{\frac{k+1}{k}}_{L^2(M)} + (\|H\|^{n+k+1}_{L^{n+k+1}(M)})^{\frac{1}{k}}\|v\|^2_{L^2(M)}),$$

where

$$\widetilde{A}_{n.k} = A^{\frac{1}{k}}_{n,k} \cdot \Big(\frac{2k}{k+1}\Big)^{\frac{k+1}{k}}.$$

Similar to the proof of Theorem 3.6 in [7], using Corollary 4.1 and Hölder's inequality, we obtain the following Sobolev type inequality for the GMCF.

Theorem 4.2 *Suppose that $k, n \in \mathbb{N}$, $k, n \ge 2$. Let M be a compact n-dimensional hypersurface without boundary, which is smoothly embedded in N^{n+1}. Assume $K_N \le b^2$. Consider the GMCF*

$$\frac{\partial}{\partial t}F(\cdot,t) = -f(H(\cdot,t))\nu(\cdot,t), \quad 0 \le t \le T \le T_{\max} < \infty,$$

where $f \in C^\infty(\Omega)$, $\Omega \subset \mathbb{R}$. Suppose $f'(x) > 0$, and $f(x)\cdot x \ge 0$ along the GMCF. Then for all nonnegative Lipschitz functions v, we have

$$\begin{aligned}\|v\|^{\beta}_{L^{\beta}(M\times[0,T])} \le B_{n,k,T} \cdot \max_{0\le t\le T}\|v\|^{\frac{(k+1)^2}{k^2n}+\frac{k-1}{k}}_{L^2(M_t)}\Big(&\|\nabla v\|^{\frac{k+1}{k}}_{L^2(M\times[0,T])} \\ &+ \max_{0\le t\le T}\|v\|^{\frac{k+1}{k}}_{L^2(M_t)}(\|H\|^{n+k+1}_{L^{n+k+1}(M\times[0,T])})^{\frac{1}{k}}\Big),\end{aligned}$$

provided that the function $h := v^{\frac{(k+1)(n-1)}{kn-(k+1)}}$ satisfies (4.2)–(4.3)*, where*

$$B_{n,k,T} = \widetilde{A}_{n,k} \cdot \mathrm{Vol}(M)^{\frac{(k-1)(k+1)}{2k^2n}} \cdot \max\{T^{\frac{k-1}{k}}, T^{\frac{k-1}{2k}}\}$$

and $\beta = 2 + \frac{k+1}{k}\cdot\frac{k+1}{kn} > 2$.

Remark 4.1 If $k = 1$, then $\frac{k+1}{k} = 2$. Thus we do not need to use Hölder inequality to control the L^2-norm, and in this case, $B_{n,k,T} = B_{n,1,T} = A_{n,1}$ is a constant.

5 Reverse Hölder and Harnack Inequalities

In this section, we can follow the lines of [7] and [11], and easily derive a soft version of reverse Hölder inequality and a Harnack inequality for parabolic inequality along the GMCF in Riemannian manifolds. Suppose that $f \in C^\infty(\Omega)$ for an open set $\Omega \subset \mathbb{R}$, and that v is a smooth function on $M \times [0, T]$ such that its image is contained in Ω.

We start with the following differential inequality:

$$\Big(\frac{\partial}{\partial t} - \Delta_{f,t}\Big) v \leq (G + C) f(v) + f''(v) \|\nabla_t v\|^2, \quad v \geq 0, \tag{5.1}$$

where the function $G + C$ has bounded $L^q(M \times [0, T])$-norm with

$$q > \frac{\gamma}{\gamma - 2}, \quad \gamma = 2 + \frac{(k+1)^2}{k^2 n},$$

C is a fixed positive constant and $\Delta_{f,t}(\cdot) = f'(\cdot)\Delta_t(\cdot)$. Let $\eta(x, t)$ be a smooth function on $M \times [0, T]$ with the property that $\eta(x, 0) = 0$ for all $x \in M$.

Let $\mathcal{S}$ be the set of all functions $f \in C^\infty(\Omega)$ ($\Omega \subset \mathbb{R}$) satisfying the following conditions:

(a) f satisfies the differential inequality (5.1),

(b) $f'(x) > 0$ for all $x \in \Omega$,

(c) $f(x) \geq 0$ whenever $x \geq 0$,

(d) $f(H(t))H(t) \geq 0$ along the GMCF,

(e) $f'(v) \geq C_2 > 0$ on $M \times [0, T]$ for some uniform constant C_2.

Lemma 5.1 *Let M be a compact n-dimensional hypersurface without boundary, which is smoothly embedded in N^{n+1}. Consider the differential inequality* (5.1). *Let $\beta \geq 2$ be a fixed number. Then*

$$\begin{aligned}
&C_2 \int_0^s \int_{M_t} |\nabla_t(\eta f^{\frac{\beta}{2}}(v))|^2 \mathrm{d}\mu \mathrm{d}t + \int_{M_s} f^\beta(v)\eta^2 \mathrm{d}\mu \\
\leq\ &\frac{\beta}{\beta - 1} \int_0^s \int_{M_t} f^\beta(v) \Big\{ 2\eta \Big(\frac{\partial}{\partial t} - f'(v)\Delta_t\Big)\eta \\
&+ \Big[\frac{1}{\beta} \frac{f(v) f''(v)}{f'(v)} + \frac{8\beta^2 - 2\beta + 2}{\beta(\beta - 1)} f'(v)\Big] |\nabla_t \eta|^2 \Big\} \mathrm{d}\mu \mathrm{d}t \\
&+ \frac{\beta^2}{\beta - 1} \|(G + C) f'(v)\|_{L^q(M \times [0,T])} \cdot \|\eta^2 f^\beta(v)\|_{L^{\frac{q}{q-1}}(M \times [0,T])}.
\end{aligned}$$

Proof Multiplying (5.1) by $\eta^2 f'(v) f^{\beta - 1}(v)$, then for any $s \in [0, T]$, we have

$$\begin{aligned}
&\int_0^s \int_{M_t} (-\Delta_{f,t} v)\eta^2 f'(v) f^{\beta-1}(v) \mathrm{d}\mu \mathrm{d}t + \int_0^s \int_{M_t} \frac{\partial v}{\partial t} \cdot \eta^2 f'(v) f^{\beta - 1}(v) \mathrm{d}\mu \mathrm{d}t \\
\leq &\int_0^s \int_{M_t} |G + C| \eta^2 f'(v) f^\beta(v) \mathrm{d}\mu \mathrm{d}t + \int_0^s \int_{M_t} \eta^2 f'(v) f''(v) f^{\beta - 1}(v) |\nabla_t v|^2 \mathrm{d}\mu \mathrm{d}t.
\end{aligned}$$

Using the integration by parts, the properties of η and (3.6), we conclude that

$$\int_0^s \int_{M_t} [2\langle \nabla_t v, \nabla_t \eta\rangle \eta (f'(v))^2 f^{\beta - 1}(v) + \eta^2 (2 f'(v) f''(v) f^{\beta - 1}(v)$$

$$
\begin{aligned}
&+(\beta-1)(f'(v))^3 f^{\beta-2}(v))|\nabla_t v|^2]\mathrm{d}\mu\mathrm{d}t+\frac{1}{\beta}\int_{M_s} f^\beta(v)\eta^2\mathrm{d}\mu(s)\\
\leq&\ \frac{1}{\beta}\int_0^s\int_{M_t} f^\beta(v)\Big[2\eta\frac{\partial\eta}{\partial t}-\eta^2 f(H(t))H(t)\Big]\mathrm{d}\mu\mathrm{d}t\\
&+\int_0^s\int_{M_t}|G+C|\eta^2 f'(v)f^\beta(v)\mathrm{d}\mu\mathrm{d}t+\int_0^s\int_{M_t}\eta^2 f'(v)f''(v)f^{\beta-1}(v)|\nabla_t v|^2\mathrm{d}\mu\mathrm{d}t\\
\leq&\ \frac{1}{\beta}\int_0^s\int_{M_t} f^\beta(v)2\eta\frac{\partial\eta}{\partial t}\mathrm{d}\mu\mathrm{d}t+\int_0^s\int_{M_t}|G+C|\eta^2 f'(v)f^\beta(v)\mathrm{d}\mu\mathrm{d}t\\
&+\int_0^s\int_{M_t}\eta^2 f'(v)f''(v)f^{\beta-1}(v)|\nabla_t v|^2\mathrm{d}\mu\mathrm{d}t.
\end{aligned}
$$

Direct calculation gives

$$
\begin{aligned}
&\frac{1}{\beta}\int_0^s\int_{M_t} f^\beta(v)2\eta\frac{\partial\eta}{\partial t}\mathrm{d}\mu\mathrm{d}t\\
=&\ \frac{1}{\beta}\int_0^s\int_{M_t}\Big[f^\beta(v)2\eta\Big(\frac{\partial}{\partial t}-f'(v)\Delta_t\Big)\eta+f^\beta(v)f'(v)2\eta\Delta_t\eta\Big]\mathrm{d}\mu\mathrm{d}t\\
=&\ \frac{1}{\beta}\int_0^s\int_{M_t}\Big[f^\beta(v)2\eta\Big(\frac{\partial}{\partial t}-f'(v)\Delta_t\Big)\eta-2f^\beta(v)f'(v)|\nabla_t\eta|^2\Big]\mathrm{d}\mu\mathrm{d}t\\
&-\frac{2}{\beta}\int_0^s\int_{M_t}\eta[\beta f^{\beta-1}(f'(v))^2+f^\beta(v)f''(v)]\langle\nabla_t v,\nabla_t\eta\rangle\mathrm{d}\mu\mathrm{d}t.
\end{aligned}
$$

And the Cauchy-Schwartz inequality implies

$$
\begin{aligned}
&4\int_0^s\int_{M_t}\eta(f'(v))^2 f^{\beta-1}(v)\langle\nabla_t v,\nabla_t\eta\rangle\mathrm{d}\mu\mathrm{d}t\\
\geq&\ -2\epsilon^2\int_0^s\int_{M_t}\eta^2(f'(v))^3 f^{\beta-2}(v)|\nabla_t v|^2\mathrm{d}\mu\mathrm{d}t\\
&-\frac{2}{\epsilon^2}\int_0^s\int_{M_t} f'(v)f^\beta(v)|\nabla_t\eta|^2\mathrm{d}\mu\mathrm{d}t
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{2}{\beta}\int_0^s\int_{M_t}\eta f^\beta(v)f''(v)\langle\nabla_t v,\nabla_t\eta\rangle\mathrm{d}\mu\mathrm{d}t\\
\geq&\ -\int_0^s\int_{M_t}\eta^2 f(v)f'(v)f''(v)f^{\beta-1}(v)|\nabla_t v|^2\mathrm{d}\mu\mathrm{d}t\\
&-\frac{1}{\beta^2}\int_0^s\int_{M_t}\frac{f(v)f''(v)}{f'(v)}f^\beta(v)|\nabla_t\eta|^2\mathrm{d}\mu\mathrm{d}t.
\end{aligned}
$$

Note that

$$
|\nabla_t(f^{\frac{\beta}{2}}(v))|^2=\Big|\frac{\beta}{2}f^{\frac{\beta}{2}-1}(v)f'(v)\nabla_t(v)\Big|^2=\frac{\beta^2}{4}f^{\beta-2}(v)(f'(v))^2|\nabla_t v|^2.
$$

If we choose $\epsilon^2=\frac{\beta-1}{4}$, then we can obtain that

$$
\begin{aligned}
&\frac{2(\beta-1)}{\beta}\int_0^s\int_{M_t}|\nabla_t(f^{\frac{\beta}{2}}(v))|^2 f'(v)\eta^2\mathrm{d}\mu\mathrm{d}t+\int_{M_s} f^\beta(v)\eta^2\mathrm{d}\mu\\
\leq&\ \int_0^s\int_{M_t} f^\beta(v)\Big\{2\eta\Big(\frac{\partial}{\partial t}-f'(v)\Delta_t\Big)\eta
\end{aligned}
$$

$$
\begin{aligned}
&+\Big[\frac{1}{\beta}\frac{f(v)f''(v)}{f'(v)}-2f'(v)+\frac{8\beta}{\beta-1}f'(v)\Big]|\nabla_t\eta|^2\Big\}\mathrm{d}\mu\mathrm{d}t\\
&+\beta\int_0^s\int_{M_t}|G+C|\eta^2 f'(v)f^\beta(v)\mathrm{d}\mu\mathrm{d}t.
\end{aligned}
$$

Combining the above estimates with

$$|\nabla_t(\eta f^{\frac{\beta}{2}}(v))|^2\le 2f^\beta(v)|\nabla_t\eta|^2+2\eta^2|\nabla_t f^{\frac{\beta}{2}}(v)|^2$$

gives

$$
\begin{aligned}
&C_2\int_0^s\int_{M_t}|\nabla_t(\eta f^{\frac{\beta}{2}}(v))|^2\mathrm{d}\mu\mathrm{d}t+\int_{M_s}f^\beta(v)\eta^2\mathrm{d}\mu\\
\le{}&\frac{\beta}{\beta-1}\int_0^s\int_{M_t}f^\beta(v)\Big\{2\eta\Big(\frac{\partial}{\partial t}-f'(v)\Delta_t\Big)\eta\\
&+\Big[\frac{1}{\beta}\frac{f(v)f''(v)}{f'(v)}+\frac{8\beta^2-2\beta+2}{\beta(\beta-1)}f'(v)\Big]|\nabla_t\eta|^2\Big\}\mathrm{d}\mu\mathrm{d}t\\
&+\frac{\beta^2}{\beta-1}\|(G+C)f'(v)\|_{L^q(M\times[0,T])}\cdot\|\eta^2f^\beta(v)\|_{L^{\frac{q}{q-1}}(M\times[0,T])}.
\end{aligned}
$$

Theorem 5.1 *Let M be a compact n-dimensional hypersurface without boundary, which is smoothly embedded in N^{n+1}. Assume $K_N\le b^2$, and $k,n\in\mathbb{N}$, $k,n\ge 2$. Consider the differential inequality* (5.1). *Let*

$$
\begin{aligned}
C_{0,q}&=\|(G+C)f'(v)\|_{L^q(M\times[0,T])},\\
C_1&=(1+\|H\|^{\frac{n+k+1}{k}}_{L^{n+k+1}(M\times[0,T])})^{\frac{2}{\gamma}},
\end{aligned}
$$

and $\beta\ge 2$ be a fixed number. Then there exists a positive constant $C_{n,k,T}(C_{0.q},C_1,\beta,q)$ depending only on $n,k,T,\beta,q,C_{0,q},C_1$ and $\mathrm{Vol}(M)$, such that for any $f\in\mathcal{S}$,

$$
\begin{aligned}
&\|\eta^2f^\beta(v)\|_{L^{\frac{\gamma}{2}}(M\times[0,T])}\\
\le{}&C_{n,k,T}(C_{0.q},C_1,\beta,q)\Big\|f^\beta(v)\Big[\eta^2+2\eta\Big(\frac{\partial}{\partial t}-f'(v)\Delta_t\Big)\eta\\
&+\Big(\frac{1}{\beta}\frac{f(v)f''(v)}{f'(v)}+\frac{8\beta^2-2\beta+2}{\beta(\beta-1)}f'(v)\Big)|\nabla_t\eta|^2\Big]\Big\|_{L^1(M\times[0,T])},
\end{aligned}
\tag{5.2}
$$

provided that the function $(\eta f^{\frac{\beta}{2}}(v))^{\frac{(k+1)(n-1)}{kn-(k+1)}}$ satisfies the conditions (4.2)–(4.3) *for any $t\in[0,T]$, where*

$$
\begin{aligned}
&C_{n,k,T}(C_{0.q},C_1,\beta,q)\\
={}&\frac{\beta}{\beta-1}\max\Big\{2(\widetilde{B}_{n,k,T})^{\frac{2}{\gamma}}C_1,\Big[2(\widetilde{B}_{n,k,T})^{\frac{2}{\gamma}}C_1C_{0,q}\frac{\beta^2}{\beta-1}\Big]^{1+\nu}\Big\}
\end{aligned}
$$

and

$$\nu=\frac{\gamma}{(\gamma-2)q-\gamma}.$$

In particular, if $(G+C)f'(v) \in L^\infty(M\times[0,T])$*, then letting* $q\to\infty$*, we have*

$$\begin{aligned}
&C_{n,k,T}(C_{0,\infty}, C_1, \beta, q)\\
&= \frac{2\beta}{\beta-1}\max\Big\{1, \frac{C_{0,\infty}\beta^2}{\beta-1}\Big\}\cdot(\widetilde{B}_{n,k,T})^{\frac{2}{\gamma}}C_1\\
&\le \frac{2\beta}{\beta-1}\Big(\frac{C_{0,\infty}\beta^2}{\beta-1}+1\Big)(\widetilde{B}_{n,k,T})^{\frac{2}{\gamma}}C_1\\
&\le \frac{2\beta^2}{(\beta-1)^2}(C_{0,\infty}\beta+1)(\widetilde{B}_{n,k,T})^{\frac{2}{\gamma}}C_1\\
&\le 8(1+C_{0,\infty})\beta(\widetilde{B}_{n,k,T})^{\frac{2}{\gamma}}C_1,
\end{aligned}$$

where

$$\begin{aligned}
&\widetilde{B}_{n,k,T} = B_{n,k,T}\cdot\max\Big\{\Big(\frac{1}{C_2}\Big)^{\frac{k+1}{2k}}, 1\Big\},\\
&C_{0,\infty} = \|(G+C)f'(v)\|_{L^\infty(M\times[0,T])}.
\end{aligned}$$

In this case, we obtain

$$\begin{aligned}
&\|\eta^2 f^\beta(v)\|_{L^{\frac{\gamma}{2}}(M\times[0,T])}\\
&\le D_{n,k,T}\beta C_1\Big\|f^\beta(v)\Big[\eta^2+2\eta\Big(\frac{\partial}{\partial t}-f'(v)\Delta_t\Big)\eta\\
&\quad+\Big(\frac{1}{\beta}\frac{f(v)f''(v)}{f'(v)}+\frac{8\beta^2-2\beta+2}{\beta(\beta-1)}f'(v)\Big)|\nabla_t\eta|^2\Big]\Big\|_{L^1(M\times[0,T])},
\end{aligned}\tag{5.3}$$

provided that the function $(\eta f^{\frac{\beta}{2}}(v))^{\frac{(k+1)(n-1)}{kn-(k+1)}}$ *satisfies the conditions* (4.2)–(4.3) *for any* $t\in[0,T]$*, where*

$$D_{n,k,T} = 8(1+C_{0,\infty})(\widetilde{B}_{n,k,T})^{\frac{2}{\gamma}}.$$

Proof Denote

$$\begin{aligned}
A := &\frac{\beta}{\beta-1}\int_0^s\int_{M_t} f^\beta(v)\Big\{2\eta\Big(\frac{\partial}{\partial t}-f'(v)\Delta_t\Big)\eta\\
&+\Big[\frac{1}{\beta}\frac{f(v)f''(v)}{f'(v)}+\frac{8\beta^2-2\beta+2}{\beta(\beta-1)}f'(v)\Big]|\nabla_t\eta|^2\Big\}\mathrm{d}\mu\mathrm{d}t\\
&+\frac{\beta^2}{\beta-1}\|(G+C)f'(v)\|_{L^q(M\times[0,T])}\cdot\|\eta^2 f^\beta(v)\|_{L^{\frac{q}{q-1}}(M\times[0,T])}
\end{aligned}$$

and

$$\begin{aligned}
\Lambda := &\frac{\beta}{\beta-1}\int_0^s\int_{M_t} f^\beta(v)\Big\{2\eta\Big(\frac{\partial}{\partial t}-f'(v)\Delta_t\Big)\eta\\
&+\Big[\frac{1}{\beta}\frac{f(v)f''(v)}{f'(v)}+\frac{8\beta^2-2\beta+2}{\beta(\beta-1)}f'(v)\Big]|\nabla_t\eta|^2\Big\}\mathrm{d}\mu\mathrm{d}t.
\end{aligned}$$

By Lemma 5.1, we have

$$\begin{aligned}
&\|\eta f^{\frac{\beta}{2}}(v)\|_{L^2(M_s)} \le A^{\frac12},\\
&\|\nabla_t(\eta f^{\frac{\beta}{2}}(v))\|_{L^2(M\times[0,T])} \le \Big(\frac{A}{C_2}\Big)^{\frac12}.
\end{aligned}$$

Let $S := M \times [0,T]$ and let the norm $\|\cdot\|_{L^p(M\times[0,T])}$ be abbreviated by $\|\cdot\|_{L^p(S)}$. If the function $(\eta f^{\frac{\beta}{2}}(v))^{\frac{(k+1)(n-1)}{kn-(k+1)}}$ satisfies the conditions (4.2)–(4.3) for any $t \in [0,T]$, applying Theorem 4.2 to $\eta f^{\frac{\beta}{2}}(v)$, we have the following estimate:

$$
\begin{aligned}
&\|\eta^2 f^\beta(v)\|_{L^{\frac{\gamma}{2}}(S)} = (\|\eta f^{\frac{\beta}{2}}(v)\|^{\gamma}_{L^\gamma(S)})^{\frac{2}{\gamma}} \\
&\le [\widetilde{B}_{n,k,T} \cdot A^{\frac{\gamma}{2}}(1+\|H\|^{\frac{n+k+1}{k}}_{L^{n+k+1}(S)})]^{\frac{2}{\gamma}} = (\widetilde{B}_{n,k,T})^{\frac{2}{\gamma}} C_1 A \\
&= (\widetilde{B}_{n,k,T})^{\frac{2}{\gamma}} C_1 \Big(\Lambda + \frac{\beta^2}{\beta-1}\|(G+C)f'(v)\|_{L^q(S)} \cdot \|\eta^2 f^\beta(v)\|_{L^{\frac{q}{q-1}}(S)}\Big).
\end{aligned}
$$

Since $1 < \frac{q}{q-1} < \frac{\gamma}{2}$, by using the interpolation inequality

$$
\|\eta^2 f^\beta(v)\|_{L^{\frac{q}{q-1}}(S)} \le \varepsilon \|\eta^2 f^\beta(v)\|_{L^{\frac{\gamma}{2}}(S)} + \varepsilon^{-\nu}\|\eta^2 f^\beta(v)\|_{L^1(S)},
$$

where

$$
\nu = \frac{1-\dfrac{q-1}{q}}{\dfrac{q-1}{q}-\dfrac{2}{\gamma}} = \frac{\gamma}{(\gamma-2)q-\gamma}.
$$

Hence if we choose

$$
\varepsilon = \frac{\beta-1}{2(\widetilde{B}_{n,k,T})^{\frac{2}{\gamma}} C_1 \beta^2 C_{0,q}},
$$

then we have

$$
\begin{aligned}
&\|\eta^2 f^\beta(v)\|_{L^{\frac{\gamma}{2}}(S)} \\
&\le 2(\widetilde{B}_{n,k,T})^{\frac{2}{\gamma}} C_1 \Lambda + \Big[2\frac{\beta^2}{\beta-1} C_{0,q} (\widetilde{B}_{n,k,T})^{\frac{2}{\gamma}} C_1\Big]^{1+\nu} \cdot \|\eta^2 f^\beta(v)\|_{L^1(S)} \\
&\le \max\Big\{2(\widetilde{B}_{n,k,T})^{\frac{2}{\gamma}} C_1, \Big[2\frac{\beta^2}{\beta-1} C_{0,q}(\widetilde{B}_{n,k,T})^{\frac{2}{\gamma}} C_1\Big]^{1+\nu}\Big\} \cdot (\Lambda + \|\eta^2 f^\beta(v)\|_{L^1(S)}) \\
&:= \widetilde{C}_{n,k,T}(c_{0,q}, C_1, \beta, q) \cdot (\Lambda + \|\eta^2 f^\beta(v)\|_{L^1(S)}),
\end{aligned}
$$

where $\widetilde{C}_{n,k,T}(c_{0,q}, C_1, \beta, q)$ is the constant depending only on $n, k, T, \beta, q, C_{0,q}, C_1, \mathrm{Vol}\,(M)$. From the definition of Λ and noting that $1 < \frac{\beta}{\beta-1} \le 2$, we obtain

$$
\begin{aligned}
&\|\eta^2 f^\beta(v)\|_{L^{\frac{\gamma}{2}}(S)} \\
&\le C_{n,k,T}(C_{0,q}, C_1, \beta, q)\Big\| f^\beta(v)\Big[\eta^2 + 2\eta\Big(\frac{\partial}{\partial t} - f'(v)\Delta_t\Big)\eta \\
&\quad + \Big(\frac{1}{\beta}\frac{f(v)f''(v)}{f'(v)} + \frac{8\beta^2-2\beta+2}{\beta(\beta-1)} f'(v)\Big)|\nabla_t \eta|^2\Big]\Big\|_{L^1(S)},
\end{aligned}
$$

where

$$
C_{n,k,T} = \widetilde{C}_{n,k,T} \cdot \frac{\beta}{\beta-1}.
$$

Next, we shall show that an L^∞-norm of $f(v)$ over a smaller domain can be bounded by an L^β-norm of $f(v)$ over the whole manifold $M \times [0,T]$.

Corollary 5.1 *Let M be a compact n-dimensional hypersurface without boundary, which is smoothly embedded in N^{n+1}. Assume $-c^2 \le K_N \le b^2$, and $k, n \in \mathbb{N},\ k, n \ge 2$. Consider the differential inequality* (5.1). *Let*

$$C_{0,\infty} = \|(G+C)f'(v)\|_{L^\infty(M\times[0,T])},$$
$$C_1 = (1 + \|H\|^{\frac{n+k+1}{k}}_{L^{n+k+1}(M\times[0,T])})^{\frac{2}{\gamma}},$$

and $\beta \ge 2$ be a fixed number. Then there exists a uniform constant $C_n > 0$ depending only on n, such that for any $f \in \mathcal{S}$, we have

$$\|f(v)\|_{L^\infty(M\times[\frac{T}{2},T])} \le E_{n,k,T}(\beta)\cdot C_1^{\frac{1}{\beta}\cdot\frac{\gamma}{\gamma-2}}\|f(v)\|_{L^\beta(M\times[0,T])}, \tag{5.4}$$

where

$$E_{n,k,T}(\beta) = (D_{n,k,T}C_n\beta)^{\frac{1}{\beta}\cdot\frac{\gamma}{\gamma-2}}\cdot\Big(\frac{\gamma}{2}\Big)^{\frac{1}{\beta}\cdot\frac{2\gamma}{(\gamma-2)^2}}\cdot 4^{\frac{1}{\beta}\cdot\frac{\gamma^2}{(\gamma-2)^2}}.$$

Proof Set

$$t_i = \frac{T}{2}\Big(1 - \frac{1}{4^{p+i}}\Big),\quad r_i = \frac{1}{2^p} + \frac{1}{2^{p+i+1}},\quad i = 0, 1, 2, \cdots.$$

Let $\eta_i(x,t)$ be smooth functions satisfying the following properties:

$$\eta_i|_{[t_i,T]} = 1,\quad \eta_i|_{[0,t_{i-1}]} = 0,\quad 0 \le \eta_i \le 1,\quad \Big|\frac{\mathrm{d}}{\mathrm{d}t}\eta_i\Big| \le C_n 4^{p+i},$$
$$\eta_i|_{M\cap B(x_0,r_i)} = 1,\quad \eta_i|_{M-M\cap B(x_0,r_{i-1})} = 0,\quad 0 \le \eta_i|_M \le 1,\quad |\nabla_t\eta_i| \le C_n 4^{p+i+1}.$$

Set $I_i = [t_i, T]$. Now we claim that $(\eta_i f^{\frac{\beta}{2}}(v))^{\frac{(k+1)(n-1)}{kn-(k+1)}}$ satisfies the conditions (4.2)–(4.3) for any $t \in [0,T]$.

In fact, under the GMCF, we observe that

$$\mathrm{Vol}_{g(t)}(B(R)) \le \mathrm{Vol}_{g(0)}(B(R))$$

for any $t \in [0,T]$ by (3.6). For $g(0)$, there exists a non-positive constant $K = K\big(n, \max_{x\in M_0}|B|, N\big)$ such that the sectional curvature of M_0 is bounded from below by K. Then by the Bishop-Gromov Volume comparison theorem, we have

$$\mathrm{Vol}_{g(0)}(B(R)) \le \mathrm{Vol}_K(B(R)),$$

where $\mathrm{Vol}_K(B(R))$ denotes the volume of the ball with radius R in the n-dimensional complete simply connected space form with constant curvature K. Hence

$$\mathrm{Vol}_{g(t)}(B(R)) \le \mathrm{Vol}_K(B(R)).$$

Therefore, we can choose R sufficiently small such that

$$b^2(1-\alpha)^{-\frac{2}{n}}(\omega_n^{-1}\cdot\mathrm{Vol}_K(B(R)))^{\frac{2}{n}} \le 1,\quad 2\rho_0 \le \overline{R}(M),$$

where ρ_0 is defined by (4.4). Here the sufficient smallness of R can be achieved by choosing a sufficiently large p. So $(\eta_i f^{\frac{\beta}{2}}(v))^{\frac{(k+1)(n-1)}{kn-(k+1)}}$ satisfies the conditions (4.2)–(4.3) for any $t \in [0, T]$. Since $\|(G + C)f'(v)\|_{L^\infty(M\times[0,T])}$ exists, using Theorem 5.1, we have

$$\|f^\beta(v)\|_{L^{\frac{\gamma}{2}}(M\times I_i)} \le (D_{n,k,T} C_n 4^{p+i+1}) \cdot \beta \cdot C_1 \|f^\beta(v)\|_{L^1(M\times I_{i-1})}.$$

Then by the standard Moser iteration process, we have

$$\|f(v)\|_{L^\infty(M\times[\frac{T}{2},T])} \le E_{n,k,T}(\beta) \cdot C_1^{\frac{1}{\beta}\cdot\frac{\gamma}{\gamma-2}} \|f(v)\|_{L^\beta(M\times[0,T])}.$$

Corollary 5.2 *Let M be a compact n-dimensional hypersurface without boundary, which is smoothly embedded in N^{n+1} with bounded geometry. Suppose $n, k \in \mathbb{N}$, $k, n \ge 2$, and $n+1 \ge k$. Consider the H^k mean curvature flow*

$$\frac{\partial}{\partial t} F(\cdot, t) = -H^k(\cdot, t) \cdot \nu(\cdot, t), \quad 0 \le t \le T \le T_{\max} < \infty.$$

If

$$H(t) \ge \Big(\frac{C_2}{k}\Big)^{\frac{1}{k-1}} > 0, \quad \|kH^{k-1}(t)(|B|^2 + C)\|_{L^\infty(M\times[0,T])} < \infty$$

along the H^k mean curvature flow for some uniform constant $C_2 > 0$, then there exists a uniform constant C_n, depending only on n, such that

$$\begin{aligned}
&\|H(t)\|_{L^\infty(M\times[\frac{T}{2},T])} \\
&\le E^{\frac{1}{k}}_{n,k,T}\Big(\frac{n+k+1}{k}\Big)(1 + \|H\|^{\frac{n+k+1}{k}}_{L^{n+k+1}(M\times[0,T])})^{\frac{2}{\gamma-2}\frac{1}{n+k+1}} \cdot \|H(t)\|_{L^{n+k+1}(M\times[0,T])} \\
&\le F_{n,k,T_{\max}} \cdot \|H(t)\|_{L^{n+k+1}(M\times[0,T])},
\end{aligned}$$

where

$$F_{n,k,T_{\max}} = E^{\frac{1}{k}}_{n,k,T_{\max}}\Big(\frac{n+k+1}{k}\Big)(1 + \|H\|^{\frac{n+k+1}{k}}_{L^{n+k+1}(M\times[0,T])})^{\frac{2}{\gamma-2}\frac{1}{n+k+1}}.$$

Proof Let

$$f(x) = x^k : \mathbb{R}_+ \to \mathbb{R}.$$

From the evolution equation of $H(t)$, i.e., (3.5), we have

$$\begin{aligned}
\Big(\frac{\partial}{\partial t} - \Delta_{f,t}\Big)H(t) &= f''(H(t))|\nabla_t H(t)|^2 + f(H(t))(|B|^2 + \overline{\mathrm{Ric}}(\nu,\nu)) \\
&\le f''(H(t))|\nabla_t H(t)|^2 + f(H(t))(|B|^2 + C).
\end{aligned}$$

By Corollary 5.1, there exists a uniform constant $C_n > 0$, such that

$$\|H^k(t)\|_{L^\infty(M\times[\frac{T}{2},T])} \le E_{n,k,T}(\beta) C_1^{\frac{1}{\beta}\frac{\gamma}{\gamma-2}} \|H^k(t)\|_{L^\beta(M\times[0,T])},$$

i.e.,

$$\|H(t)\|_{L^\infty(M\times[\frac{T}{2},T])} \le E^{\frac{1}{k}}_{n,k,T}(\beta) C_1^{\frac{1}{k\beta}\frac{\gamma}{\gamma-2}} \|H(t)\|_{L^{k\beta}(M\times[0,T])}.$$

Choose $\beta = \frac{n+k+1}{k} \ge 2$, then it follows that

$$\|H(t)\|_{L^\infty(M\times[\frac{T}{2},T])} \le E^{\frac{1}{k}}_{n,k,T}\Big(\frac{n+k+1}{k}\Big) C_1^{\frac{\gamma}{\gamma-2}\frac{1}{n+k+1}} \cdot \|H(t)\|_{L^{n+k+1}(M\times[0,T])}.$$

Remark 5.1 When $k = 1$, $n + 1 \ge k$ is obvious, but for $k \ge 2$, this assumption is needed in our proof.

6 Proof of Main Theorem

Proof of Theorem 1.1 We shall follow the basic ideas of Schulze [9]. If Theorem 1.1 is false, then there exists some $C < \infty$ such that

$$\max_{M_t} |B|^2 \leq C$$

on $0 \leq t < T_{\max}$. Using the evolution equation and the upper bound for H, it follows that for $p \in U \subset M$, $0 < \sigma < \rho < T_{\max}$, we have that

$$\operatorname{dist}(F(p,\rho), F(p,\sigma)) \leq \int_\sigma^\rho H^k(p,\tau)\mathrm{d}\tau \leq C(\rho - \sigma)$$

and $F(\cdot, t)$ converges uniformly to some continuous limit function $F(\cdot, T_{\max})$. We want to show that $F(\cdot, T_{\max})$ actually represents a smooth limit surface $M_{T_{\max}}$. This is then a contradiction to the maximality of $T_{\max}$. In order to show that $F(\cdot, T_{\max})$ represents a smooth surface $M_{T_{\max}}$, we only have to establish uniform bounds for all derivatives of the second fundamental form on M_t, $0 \leq t < T_{\max}$.

In the following, we denote the metric of N and $\mathrm{Graph}(u)$ by $\overline{g}$ and g respectively. For $k \leq 1$, since H^k is concave in ${h^i}_j$ and it has uniform C^2-bound, then using the estimate of [5] (see [5, Theorem 2 in Chapter 5.5]), we can obtain the uniform $C^{2,\alpha}$-bounds. For $k > 1$, let S be a fixed reference hypersurface which is tangent to the hypersurface $F(\cdot, t_0)$ at some point $p \in N$, and assume that we have Gaussian coordinates $\{x_1, \cdots, x_n\}$ in a neighborhood of p on S. Then there exists a local coordinate in the neighborhood of p in N constructed from the above coordinate. Suppose that U is a neighborhood of p such that for every point $q \in U$ there exists a unique minimal geodesic

$$\gamma : [0, d(q,S)] \to N, \quad \gamma(0) = q, \quad \gamma(1) = q' \in S$$

to the hypersurface S satisfying $L(\gamma(t)) = d(q,S)$. The coordinate of q is set to be

$$(x_1(q), \cdots, x_n(q), x_{n+1}(q)) = (x_1(q'), \cdots, x_n(q'), d(q,S)).$$

By the construction, $\overline{g}(\dot\gamma(1), v) = 0$ for any $v \in T_{q'}S$. Given $\partial_{x_i} \in T_qN$, $1 \leq i \leq n$, there exists a curve γ_q such that $\gamma_q(0) = q$, $\dot\gamma_q(0) = \partial_{x_i}$ and $d(\gamma_q(s), S) = d(q,S)$. For any point $\gamma_q(s), -\delta < s < \delta$, there exists a unique minimal geodesic from q to S. Hence we have a family of minimal geodesics

$$F : [0, d(q,S)] \times (-\delta, \delta) \to N,$$

such that $F(\cdot, s) : [0, d(q,S)] \to N$ is the minimal geodesic from $\gamma_q(s)$ to S. Hence the vector field $v(t) = \mathrm{d}F(\partial_s)(t,0)$ is a Jacobi field with $v(0) = \partial_{x_i} \in T_qN$ and $\overline{g}\big(v(1), \frac{\mathrm{d}}{\mathrm{d}t}F(t,0)|_{t=1}\big) = 0$. Hence

$$\overline{g}\Big(v(0), \frac{\mathrm{d}}{\mathrm{d}t}F(t,0)\Big|_{t=0}\Big) = 0,$$

that is $\overline{g}(\partial_{x_i}, \partial_{x_{n+1}}) = 0,\ i = 1, \cdots, n$. Since $\partial_{x_{n+1}} = \frac{\mathrm{d}}{\mathrm{d}t}F(t,0)\big|_{t=0}$, we have

$$\overline{g}(\partial_{x_{n+1}}, \partial_{x_{n+1}}) = 1.$$

Under this coordinate $\{x_1, \cdots, x_{n+1}\}$, locally around p we can write $F(\cdot, t)$ for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ (for some $\varepsilon > 0$) as graphs of function $u(t)$ on S (see [4, 13]). Set

$$\mathrm{Graph}(u) := \{(x_1, \cdots, x_n, u(x_1, \cdots, x_n)) \mid (x_1, \cdots, x_n) \in S\}. \tag{6.1}$$

Then $\{E_i = \partial_{x_i} + u_{x_i}\partial_{x_{n+1}} \mid 1 \le i \le n\}$ gives a basis for the tangent space to Graph(u). It is easy to see that

$$\nu := \frac{1}{\sqrt{1 + |\nabla u|^2}}\Big(\sum_{i,j} -\overline{g}^{ij} u_{x_i}\partial_j + \partial_{n+1}\Big) \tag{6.2}$$

is the unit inner normal vector on $F(\cdot, t)$ and u satisfies the following evolution equation:

$$\frac{\partial}{\partial t}u = \sqrt{1 + |\nabla u|^2} H^k. \tag{6.3}$$

By direct calculation, we have that

$$\begin{aligned} h_{ij} &= \overline{g}(\overline{\nabla}_{E_i}E_j, \nu) = \overline{g}(\overline{\nabla}_{\partial_i + u_i\partial_{n+1}}(\partial_j + u_j\partial_{n+1}), \nu) \\ &= \frac{1}{\sqrt{1 + |\nabla u|^2}}\{u_{ij} + \Gamma^{n+1}_{ij} - u_l\Gamma^l_{ij} - u_j u_l \Gamma^1_{i,n+1} - u_i u_l \Gamma^l_{j,n+1}\}, \end{aligned}$$

since $\Gamma^p_{n+1,n+1} = 0,\ \Gamma^{n+1}_{i,n+1} = 0$ for $i = 1, \cdots, n$, and $p = 1, \cdots, n+1$, where Γ is the Christoffel symbol of N. Using the expression of E_i, we compute that

$$g^{ij} = \overline{g}^{ij} - \frac{1}{1 + |\nabla u|^2}\overline{g}^{ir}\overline{g}^{js}u_{x_r}u_{x_s}. \tag{6.4}$$

Hence

$$\begin{aligned} H = g^{ij}h_{ij} &= \frac{1}{\sqrt{1 + |\nabla u|^2}}\Big(\overline{g}^{ij} - \frac{1}{1 + |\nabla u|^2}\overline{g}^{ir}\overline{g}^{js}u_{x_r}u_{x_s}\Big) \\ &\quad \cdot (u_{ij} + \Gamma^{n+1}_{ij} - u_l\Gamma^l_{ij} - 2u_i u_l \Gamma^l_{j,n+1}). \end{aligned} \tag{6.5}$$

Therefore (6.3) and (6.5) imply that

$$\begin{aligned} \frac{\partial}{\partial t}u &= (\sqrt{1 + |\nabla u|^2})^{1-k} \\ &\quad \cdot \Big(\Big(\overline{g}^{ij} - \frac{1}{1 + |\nabla u|^2}\overline{g}^{ir}\overline{g}^{js}u_r u_s\Big)(u_{ij} + \Gamma^{n+1}_{ij} - u_l\Gamma^l_{ij} - 2u_i u_l\Gamma^l_{j,n+1})\Big)^k. \end{aligned} \tag{6.6}$$

According to Theorem 2, Chapter 5.3 in [5], with the assumption that $|B|$ is bounded, we can obtain the uniform Hölder-estimates in space and time for $\frac{\partial}{\partial t}u$. Similarly, by Theorem 4, Chapter 5.2 in [5], we can also have the Hölder-estimates for ∇u. On the other hand, the mean curvature H satisfies the evolution equation

$$\begin{aligned} \frac{\partial}{\partial t}H &= kH^{k-1}\Delta H + k(k-1)H^{k-2}|\nabla H|^2 + H^k(|B|^2 + \overline{\mathrm{Ric}}(\nu, \nu)) \\ &\ge kH^{k-1}\Delta H + k(k-1)H^{k-2}|\nabla H|^2 + \frac{1}{n}H^{k+2} - CH^k. \end{aligned}$$

Then let ϕ be the solution of the ODE

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -C\phi^k, \quad \phi(0) = H_{\min}(0) \ge \delta > 0.$$

Then we have

$$\phi^{k-1}(t) = \frac{\phi^{k-1}(0)}{1 + C(k-1)\phi^{k-1}(0)t}.$$

Since $k > 1$, $\phi(t) > 0$ for all $t > 0$.

If we consider ϕ as a function on $M \times [0, T_{\max})$, we have

$$\frac{\partial}{\partial t}(H - \phi) \geq kH^{k-1}\Delta(H - \phi) + k(k-1)H^{k-2}|\nabla(H - \phi)|^2 + \frac{1}{n}H^{k+2} - C(H^k(x,t) - \phi^k(t)).$$

Suppose t_0 be the first time that

$$H(p, t_0) - \phi(t_0) = \min_{x \in M}(H(x,t) - \phi(t))$$

attaining zero. Then at (p, t_0), we have

$$0 \geq \frac{\partial}{\partial t}(H - \phi),$$

$$\Delta(H - \phi) \geq 0.$$

By $H(p, t_0) = \phi(t_0) > 0$, we have

$$0 \geq \frac{\partial}{\partial t}(H - \phi) \geq \frac{1}{n}H^{k+2} > 0,$$

which is a contradiction. Hence $H(x,t) > \phi(t) > \delta(T_{\max}) > 0$, where

$$\delta(T_{\max}) = \frac{\phi^{k-1}(0)}{1 + C(k-1)\phi^{k-1}(0)T_{\max}}.$$

(6.6) implies that

$$v := (\sqrt{1 + |\nabla u|^2})^{1-k}$$

and

$$w := \Big(\Big(\overline{g}^{ij} - \frac{1}{1 + |\nabla u|^2}\overline{g}^{ir}\overline{g}^{js}u_r u_s\Big)(u_{ij} + \Gamma_{ij}^{n+1} - u_l\Gamma_{ij}^l - 2u_i u_l \Gamma_{j,n+1}^l)\Big)^{k-1} \tag{6.7}$$

are also uniformly Hölder-continuous in space and time. Therefore we can write (6.6) as a linear, strictly parabolic PDE

$$\frac{\partial}{\partial t}u - a^{ij}u_{ij} + b^i u_i + f = 0$$

with coefficients a^{ij}, b^i, f in C^β in space and time. The interior Schauder estimates then lead to $C^{2,\beta}$-bounds. In both cases, namely, $k \leq 1$ and $k > 1$, using again parabolic Schauder estimates, we get a bound on all the higher C^l-norms.

Proof of Theorem 1.2 It is sufficient to prove the theorem for $\alpha = n+k+1$ since by the Hölder inequality, $\|H(t)\|_{L^\alpha(M \times [0,T])} < \infty$ implies $\|H(t)\|_{L^{n+k+1}(M \times [0,T])} < \infty$ if $\alpha > n+k+1$. Note that $\|H(t)\|_{L^\alpha(M \times [0,T])}$ is invariant under the rescaling of the H^k mean curvature flow.

$$\widetilde{F}(p,t) = Q^{\frac{1}{k+1}} \cdot F\Big(p, \frac{t}{Q}\Big) \quad \text{for } Q > 0.$$

We argue by contradiction. Suppose that the solution to the H^k mean curvature flow can not be extended over $T_{\max}$. Then $B(t)$ is unbounded as $t \to T_{\max}$. Let λ_i $(i = 1, \cdots, n)$ be the principal curvatures. Then

$$|B|^2 = \sum_{i=1}^{n} \lambda_i^2 \leq \Big(\sum_{i=1}^{n} \lambda_i\Big)^2 = H^2(t).$$

Since $h_{ij} \geq c g_{ij}$ $(c > 0)$, thus $H^{k+1}(x, t)$ is also unbounded as $t \to T_{\max}$. Namely,

$$\sup_{(x,t)\in M\times[0,T_{\max})} H^{k+1}(x,t) = \infty.$$

Choose an increasing time sequence $\{t^{(i)}\}_{i=1}^{\infty}$, such that $\lim\limits_{i\to\infty} t^{(i)} = T_{\max}$. We take a sequence of points $x^{(i)} \in M$, satisfying

$$Q^{(i)} = H^{k+1}(x^{(i)}, t^{(i)}) = \max_{(x,t)\in M\times[0,t^{(i)}]} H^{k+1}(x,t),$$

then

$$\lim Q^{(i)} = \infty.$$

Therefore there exists a positive integer i_0 such that $(Q^{(i)})^{\frac{2}{k+1}} t^{(i)} \geq 1$ and $Q^{(i)} \geq 1$ for $i \geq i_0$.

For $i \geq i_0$ and $t \in [0, 1]$, we consider the rescaled flows

$$F^{(i)}(x,t) = (Q^{(i)})^{\frac{1}{k+1}} F\Big(x, \frac{t-1}{(Q^{(i)})^{\frac{2}{k+1}}} + t^{(i)}\Big) : (M, g^{(i)}(t)) \to (N, Q^{(i)} h).$$

Then a simple calculation shows that

$$\begin{aligned}
g^{(i)}(x,t) &= (Q^{(i)})^{\frac{2}{k+1}} g\Big(x, \frac{t-1}{(Q^{(i)})^{\frac{2}{k+1}}} + t^{(i)}\Big),\\
h^{(i)}_{pq}(x,t) &= (Q^{(i)})^{\frac{2}{k+1}} h_{pq}\Big(x, \frac{t-1}{(Q^{(i)})^{\frac{2}{k+1}}} + t^{(i)}\Big),\\
H^{(i)}(x,t) &= (Q^{(i)})^{-\frac{1}{k+1}} H\Big(x, \frac{t-1}{(Q^{(i)})^{\frac{2}{k+1}}} + t^{(i)}\Big),
\end{aligned}$$

where $g^{(i)}, h^{(i)}_{pq}$ and $H^{(i)}$ are the corresponding induced metric, second fundamental forms, and the mean curvature, respectively. From the definition of $Q^{(i)}$ we must have

$$(H^{(i)}(x,t))^{k+1} \leq 1, \quad 0 \leq h^{i}_{pq}(x,t) \leq 1, \quad (x,t) \in M \times [0,1].$$

As in [12], we can find a subsequence of $(M, g^{(i)}(t), F^{(i)}(t), x^{(i)})$, $t \in [0,1]$, converges to a Riemannian manifold $(\widetilde{M}, \widetilde{g}(t), \widetilde{F}(t), \widetilde{x})$, where $\widetilde{F}(t) : \widetilde{M} \to \mathbb{R}^{n+1}$ is an immersion.

Since

$$(H^{(i)}(x,t))^{k+1} \leq 1 \quad \text{on } M \times [0,1] \quad \text{for all } i \geq i_0,$$

it follows that $k(H^{(i)}(x,t))^{k-1}(B^{(i)}(x,t))^2$ is also bounded on $M\times[0,1]$ for any $i \geq i_0$. And since (N, h) has bounded geometry and $Q^{(i)} \geq 1$ for $i \geq i_0$, $(N, Q^{(i)}h)$ also has bounded geometry with the same bounding constants as (N, h) for each $i \geq i_0$. It follows from Corollary 5.2 that

$$\max_{(x,t)\in M^{(i)}\times[\frac{1}{2},1]} (H^{(i)}(x,t))^{k+1} \leq C\Big(\int_0^1 \int_{M_t} |H^{(i)}(x,t)|^{n+k+1} \mathrm{d}\mu_{g^{(i)}(t)} \mathrm{d}t\Big)^{\frac{k+1}{n+k+1}},$$

where C is a constant independent of i for $i \geq i_0$. Hence

$$\max_{(x,t)\in\widetilde{M}\times[\frac{1}{2},1]} \widetilde{H}^{k+1}(x,t) = \lim_{i\to\infty} \max_{(x,t)\in M^{(i)}\times[\frac{1}{2},1]} (H^{(i)}(x,t))^{k+1}$$
$$\leq \lim_{i\to\infty} C\Big(\int_0^1 \int_{M_t} |H^{(i)}(x,t)|^{n+k+1} \mathrm{d}\mu_{g^{(i)}(t)} \mathrm{d}t\Big)^{\frac{k+1}{n+k+1}}$$
$$\leq \lim_{i\to\infty} C\Big(\int_{t^{(i)}-\frac{1}{(Q^{(i)})^{\frac{2}{k+1}}}}^{t^{(i)}} \int_{M_t} |H(x,t)|^{n+k+1} \mathrm{d}\mu \mathrm{d}t\Big)^{\frac{k+1}{n+k+1}} = 0,$$

since $\big(\int_0^T \int_{M_t} |H|^{n+k+1}\mathrm{d}\mu\mathrm{d}t\big)^{\frac{1}{n+k+1}} < \infty$ and $\lim\limits_{i\to\infty} Q^{(i)} = \infty$.

On the other hand, by our construction, we have

$$\widetilde{H}^{k+1}(\widetilde{x},1) = \lim_{i\to\infty} (H^{(i)}(x^{(i)},1))^{k+1} = 1.$$

This is a contradiction. We complete the proof of Theorem 1.2.

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