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f-Harmonic Morphisms Between Riemannian Manifolds*

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Abstract f-Harmonic maps were first introduced and studied by Lichnerowicz in 1970. In this paper, the author studies a subclass of f-harmonic maps called f-harmonic morphisms which pull back local harmonic functions to local f-harmonic functions. The author proves that a map between Riemannian manifolds is an f-harmonic morphism if and only if it is a horizontally weakly conformal f-harmonic map. This generalizes the well-known characterization for harmonic morphisms. Some properties and many examples as well as some non-existence of f-harmonic morphisms are given. The author also studies the f-harmonicity of conformal immersions.

Keywords f-Harmonic maps, f-Harmonic morphisms, F-Harmonic maps, Harmonic morphisms, p-Harmonic morphisms
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1 f-Harmonic Maps vs. F-Harmonic Maps

1.1 f-Harmonic maps

Let $f : (M, g) \to (0, \infty)$ be a smooth function. An f-harmonic map is a map $\phi : (M^m, g) \to (N^n, h)$ between Riemannian manifolds such that $\phi|_\Omega$ is a critical point of the f-energy (see [10, 18]),

$$E_f(\phi) = \frac{1}{2}\int_\Omega f\,|\mathrm{d}\phi|^2 \mathrm{d}v_g$$

for every compact domain $\Omega \subseteq M$. The Euler-Lagrange equation gives the f-harmonic map equation (see [7, 23])

$$\tau_f(\phi) \equiv f\tau(\phi) + \mathrm{d}\phi(\mathrm{grad}\, f) = 0, \tag{1.1}$$

where $\tau(\phi) = \mathrm{Tr}_g \nabla \mathrm{d}\phi$ is the tension field of ϕ vanishing of which means ϕ is a harmonic map.

Example 1.1 Let $\varphi, \psi, \phi : \mathbb{R}^3 \to \mathbb{R}^2$ be defined as

$$\begin{aligned}\varphi(x, y, z) &= (x, y),\\ \psi(x, y, z) &= (3x, xy),\\ \phi(x, y, z) &= (x, y + z).\end{aligned}$$

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Then, one can easily check that both φ and ψ are f-harmonic map with $f = e^z$, φ is a horizontally conformal submersion whilst ψ is not. Also, ϕ is an f-harmonic map with $f = e^{y-z}$, which is a submersion but not horizontally weakly conformal.

1.2 *F*-Harmonic map

Let $F : [0, +\infty) \to [0, +\infty)$ be a C^2-function, strictly increasing on $(0, +\infty)$, and let $\varphi : (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds. Then φ is said to be an F-harmonic map if $\varphi|_\Omega$ is a critical point of the F-energy functional

$$E_F(\varphi) = \int_\Omega F\Big(\frac{|\mathrm{d}\varphi|^2}{2}\Big)\, v_g$$

for every compact domain $\Omega \subseteq M$. The equation of F-harmonic maps is given by (see [2])

$$\tau_F(\varphi) \equiv F'\Big(\frac{|\mathrm{d}\varphi|^2}{2}\Big)\tau(\varphi) + \varphi_*\Big(\mathrm{grad}\, F'\Big(\frac{|\mathrm{d}\varphi|^2}{2}\Big)\Big) = 0, \tag{1.2}$$

where $\tau(\varphi)$ denotes the tension field of φ.

Harmonic maps, p-harmonic maps and exponential harmonic maps are examples of F-harmonic maps with $F(t) = t$, $F(t) = \frac{1}{p}(2t)^{\frac{p}{2}}$ $(p > 4)$, and $F(t) = e^t$, respectively (see [2]).

In particular, p-harmonic map equation can be written as

$$\tau_p(\varphi) = |\mathrm{d}\varphi|^{p-2}\tau(\varphi) + \mathrm{d}\varphi(\mathrm{grad}|\mathrm{d}\varphi|^{p-2}) = 0. \tag{1.3}$$

1.3 Relationship between *f*-harmonic and *F*-harmonic maps

We can see from (1.1) that an f-harmonic map with $f = \mathrm{const} > 0$ is nothing but a harmonic map, so both f-harmonic maps and F-harmonic maps are generalizations of harmonic maps. Though we were warned in [7] that f-harmonic maps should not be confused with F-harmonic maps and p-harmonic maps, we observe that, apart from critical points, any F-harmonic map is a special f-harmonic map. More precisely we have the following corollary.

Corollary 1.1 *Any F-harmonic map $\varphi : (M, g) \to (N, h)$ without critical points, i.e., $|\mathrm{d}\varphi_x| \neq 0$ for all $x \in M$, is an f-harmonic map with $f = F'\big(\frac{|\mathrm{d}\varphi|^2}{2}\big)$. In particular, a p-harmonic map without critical points is an f-harmonic map with $f = |\mathrm{d}\varphi|^{p-2}$.*

Proof Since F is a C^2-function and strictly increasing on $(0, +\infty)$ we have $F'(t) > 0$ on $(0, +\infty)$. If the F-harmonic map $\varphi : (M, g) \to (N, h)$ has no critical points, i.e., $|\mathrm{d}\varphi_x| \neq 0$ for all $x \in M$, then the function $f : (M, g) \to (0, +\infty)$ with $f = F'\big(\frac{|\mathrm{d}\varphi|^2}{2}\big)$ is smooth and we see from (1.1)–(1.2) that the F-harmonic map φ is an f-harmonic map with $f = F'\big(\frac{|\mathrm{d}\varphi|^2}{2}\big)$. The second statement follows from the fact that for a p-harmonic map, $F(t) = \frac{1}{p}(2t)^{\frac{p}{2}}$ and hence

$$f = F'\Big(\frac{|\mathrm{d}\varphi|^2}{2}\Big) = |\mathrm{d}\varphi|^{p-2}.$$

Another relationship between f-harmonic maps and harmonic maps can be characterized as follows.

Corollary 1.2 (see [18]) *A map $\phi : (M^m, g) \to (N^n, h)$ with $m \neq 2$ is f-harmonic if and only if $\phi : (M^m, f^{\frac{2}{m-2}} g) \to (N^n, h)$ is a harmonic map.*

1.4 A physical motivation for the study of f-harmonic maps

In physics, the equation of motion of a continuous system of spins with inhomogeneous neighbor Heisenberg interaction (such a model is called the inhomogeneous Heisenberg ferromagnet) is given by

$$\frac{\partial u}{\partial t} = f(x)(u \times \Delta u) + \nabla f \cdot (u \times \nabla u), \tag{1.4}$$

where $\Omega \subseteq \mathbb{R}^m$ is a smooth domain in the Euclidean space, f is a real-valued function defined on Ω, $u(x,t) \in S^2$, $\times$ denotes the cross products in $\mathbb{R}^3$ and Δ is the Laplace operator on $\mathbb{R}^m$. Physically, the function f is called the coupling function, and is the continuum limit of the coupling constants between the neighboring spins. Since u is a map into S^2 it is well-known that the tension field of u can be written as $\tau(u) = \Delta u + |\nabla u|^2 u$, and one can easily check that the right-hand side of the inhomogeneous Heisenberg spin system (1.4) can be written as $u \times (f\tau(u) + \nabla f \cdot \nabla u)$. It follows that u is a smooth stationary solution of (1.4) if and only if $f\tau(u) + \nabla f \cdot \nabla u = 0$, i.e., u is an f-harmonic map. So there exists a 1-1 correspondence between the set of the stationary solutions of the inhomogeneous Heisenberg spin system (1.4) on the domain Ω and the set of f-harmonic maps from Ω into 2-sphere. The above inhomogeneous Heisenberg spin system (1.4) is also called inhomogeneous Landau-Lifshitz system (see [5–6, 9, 14, 16–17]).

Using Corollary 1.2 we have the following example which provides many stationary solutions of the inhomogeneous Heisenberg spin system defined on $\mathbb{R}^3$.

Example 1.2 $u : (\mathbb{R}^3, \mathrm{d}s_0) \to (N^n, h)$ is an f-harmonic map if and only if

$$u : (\mathbb{R}^3, f^2 \mathrm{d}s_0) \to (N^n, h)$$

is a harmonic map. In particular, there is a 1-1 correspondence between harmonic maps from 3-sphere

$$S^3 \setminus \{N\} \equiv \left(\mathbb{R}^3, \frac{4\mathrm{d}s_0}{(1+|x|^2)^2}\right) \to (N^n, h)$$

and f-harmonic maps with $f = \frac{2}{1+|x|^2}$ from Euclidean 3-space $\mathbb{R}^3 \to (N^n, h)$. When $(N^n, h) = S^2$, we have a 1-1 correspondence between the set of harmonic maps $S^3 \to S^2$ and the set of stationary solutions of the inhomogeneous Heisenberg spin system on $\mathbb{R}^3$. Similarly, there exists a 1-1 correspondence between harmonic maps from hyperbolic 3-space

$$H^3 \equiv \left(D^3, \frac{4\mathrm{d}s_0}{(1-|x|^2)^2}\right) \to (N^n, h)$$

and f-harmonic maps $(D^3, \mathrm{d}s_0) \to (N^n, h)$ with $f = \frac{2}{1-|x|^2}$ from the unit disk in Euclidean 3-space.

1.5 A little more about f-harmonic maps

Corollary 1.3 *If $\phi : (M^m, g) \to (N^n, h)$ is an f_1-harmonic map and also an f_2-harmonic map, then* $\operatorname{grad}(f_1/f_2) \in \ker \mathrm{d}\phi$.

Proof This follows from

$$\tau_{f_1}(\phi) \equiv f_1\tau(\phi) + \mathrm{d}\phi(\operatorname{grad} f_1) = 0,$$
$$\tau_{f_2}(\phi) \equiv f_2\tau(\phi) + \mathrm{d}\phi(\operatorname{grad} f_2) = 0,$$

and hence

$$\mathrm{d}\phi(\operatorname{grad} \ln (f_1/f_2)) = 0.$$

Proposition 1.1 *A conformal immersion $\phi : (M^m, g) \to (N^n, h)$ with $\phi^* h = \lambda^2 g$ is f-harmonic if and only if it is m-harmonic and $f = C\lambda^{m-2}$. In particular, an isometric immersion is f-harmonic if and only if $f = const$ and hence it is harmonic.*

Proof It is not difficult to check (see [24]) that for a conformal immersion $\phi : (M^m, g) \to (N^n, h)$ with $\phi^* h = \lambda^2 g$, the tension field is given by

$$\tau(\phi) = m\lambda^2\eta + (2 - m)\mathrm{d}\phi(\operatorname{grad} \ln \lambda),$$

so we can compute the f-tension field to have

$$\tau_f(\phi) = f[m\lambda^2\eta + \mathrm{d}\phi(\operatorname{grad} \ln(\lambda^{2-m} f))],$$

where η is the mean curvature vector of the submanifold $\phi(M) \subset N$. Noting that η is normal part whilst $\mathrm{d}\phi(\operatorname{grad} \ln \lambda^{2-m} f)$ is the tangential part of $\tau_f(\phi)$, we conclude that $\tau_f(\phi) = 0$ if and only if

$$\begin{cases} m\lambda^2\eta = 0, \\ \mathrm{d}\phi(\operatorname{grad} \ln(\lambda^{2-m} f)) = 0. \end{cases}$$

It follows that $\eta = 0$ and $\operatorname{grad}(\ln(\lambda^{2-m} f)) = 0$ since ϕ is an immersion. From these we see that ϕ is a minimal conformal immersion which means it is an m-harmonic map (see [24]) and that $f = C\lambda^{m-2}$. Thus, we obtain the first statement. The second statement follows from the first one with $\lambda = 1$.

2 f-Harmonic Morphisms

A horizontally weakly conformal map is a map $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds such that for each $x \in M$ at which $\mathrm{d}\varphi_x \neq 0$, the restriction $\mathrm{d}\varphi_x|_{H_x} : H_x \to T_{\varphi(x)}N$ is conformal and surjective, where the horizontal subspace H_x is the orthogonal complement of $V_x = \ker \mathrm{d}\varphi_x$ in T_xM. It is not difficult to see that there exists a number $\lambda(x) \in (0, \infty)$ such that $h(\mathrm{d}\varphi(X), \mathrm{d}\varphi(Y)) = \lambda^2(x)g(X, Y)$ for any $X, Y \in H_x$. At the point $x \in M$ where $\mathrm{d}\varphi_x = 0$ one can let $\lambda(x) = 0$ and obtain a continuous function $\lambda : M \to \mathbb{R}$ which is called the dilation of a horizontally weakly conformal map φ. A non-constant horizontally weakly conformal map φ is called horizontally homothetic if the gradient of $\lambda^2(x)$ is vertical meaning that $X(\lambda^2) \equiv 0$ for any horizontal vector field X on M. Recall that a C^2 map $\varphi : (M, g) \to (N, h)$ is a p-harmonic morphism ($p > 1$) if it preserves the solutions of p-Laplace equation in the sense that for any p-harmonic function $f : U \to \mathbb{R}$, defined on an open subset U of N with $\varphi^{-1}(U)$ non-empty, $f \circ \varphi : \varphi^{-1}(U) \to \mathbb{R}$ is a p-harmonic function. A p-harmonic morphism can be characterized as a horizontally weakly conformal p-harmonic map (see [3, 11, 15, 19, 21]).

Definition 2.1 *Let $f : (M, g) \to (0, \infty)$ be a smooth function. A C^2-function $u : U \to \mathbb{R}$ defined on an open subset U of M is called f-harmonic if*

$$\Delta_f^M u \equiv f\Delta^M u + g(\mathrm{grad} f, \mathrm{grad}\, u) = 0. \tag{2.1}$$

A continuous map $\phi : (M^m, g) \to (N^n, h)$ is called an f-harmonic morphism if for every harmonic function u defined on an open subset V of N such that $\phi^{-1}(V)$ is non-empty, the composition $u \circ \phi$ is f-harmonic on $\phi^{-1}(V)$.

Theorem 2.1 *Let $\phi : (M^m, g) \to (N^n, h)$ be a smooth map. Then, the following are equivalent*:

(1) *ϕ is an f-harmonic morphism*;

(2) *ϕ is a horizontally weakly conformal f-harmonic map*;

(3) *There exists a smooth function λ^2 on M such that*

$$\Delta_f^M (u \circ \phi) = f\lambda^2 (\Delta^N u) \circ \phi$$

for any C^2-function u defined on (an open subset of) N.

Proof We will need the following lemma to prove the theorem.

Lemma 2.1 (see [15]) *For any point $q \in (N^n, h)$ and any constants C_σ, $C_{\alpha\beta}$ with $C_{\alpha\beta} = C_{\beta\alpha}$ and $\sum_{\alpha=1}^{n} C_{\alpha\alpha} = 0$, there exists a harmonic function u on a neighborhood of q such that $u_\sigma(q) = C_\sigma, u_{\alpha\beta}(q) = C_{\alpha\beta}$.*

Let $\phi : (M^m, g) \to (N^n, h)$ be a map and let $p \in M$. Suppose that

$$\phi(x) = (\phi^1(x), \phi^2(x), \cdots, \phi^n(x))$$

is the local expression of ϕ with respect to the local coordinates $\{x^i\}$ in the neighborhood $\phi^{-1}(V)$ of p and $\{y^\alpha\}$ in a neighborhood V of $q = \phi(p) \in N$. Let $u : V \to \mathbb{R}$ be defined on an open subset V of N. Then, a straightforward computation gives

$$\begin{aligned} \Delta_f^M (u \circ \phi) &= f\Delta^M (u \circ \phi) + \mathrm{d}(u \circ \phi)(\mathrm{grad} f) \\ &= f u_{\alpha\beta} g(\mathrm{grad}\phi^\alpha, \mathrm{grad}\phi^\beta) + f u_\alpha \Delta^M \phi^\alpha + \mathrm{d}(u \circ \phi)(\mathrm{grad} f) \\ &= f g(\mathrm{grad}\phi^\alpha, \mathrm{grad}\phi^\beta) u_{\alpha\beta} + [f\, \Delta^M \phi^\sigma + (\mathrm{grad} f)\phi^\sigma] u_\sigma. \end{aligned} \tag{2.2}$$

By Lemma 2.1, we can choose a local harmonic function u on $V \subset N$ such that $u_\sigma(q) = C_\sigma = 0,\ \forall \sigma = 1, 2, \cdots, n,\ u_{\alpha\beta}(q) = 1\ (\alpha \neq \beta),$ and all other $u_{\rho\sigma}(q) = C_{\rho\sigma} = 0$, and substitute it into

(2.2) to have

$$g(\mathrm{grad}\phi^\alpha, \mathrm{grad}\phi^\beta) = 0, \quad \forall \alpha \neq \beta = 1, 2, \cdots, n. \tag{2.3}$$

Note that the choice of such functions implies

$$h^{\alpha\beta}(\phi(p)) = 0, \quad \forall \alpha \neq \beta = 1, 2, \cdots, n. \tag{2.4}$$

Another choice of harmonic function u with $C_{11} = 1$, $C_{\alpha\alpha} = -1$ $(\alpha \neq 1)$ and all other $C_\sigma, C_{\alpha\beta} = 0$ for (2.2) gives

$$g(\mathrm{grad}\phi^1, \mathrm{grad}\phi^1) - g(\mathrm{grad}\phi^\alpha, \mathrm{grad}\phi^\alpha) = 0, \quad \forall \alpha \neq \beta = 2, 3, \cdots, n. \tag{2.5}$$

Note also that for these choices of harmonic functions u we have

$$h^{11}(\phi(p)) - h^{\alpha\alpha}(\phi(p)) = 0, \quad \forall \alpha \neq \beta = 2, 3, \cdots, n. \tag{2.6}$$

It follows from (2.3)–(2.6) that the f-harmonic morphism ϕ is a horizontally weakly conformal map

$$g(\mathrm{grad}\phi^\alpha, \mathrm{grad}\phi^\beta) = \lambda^2 h^{\alpha\beta} \circ \phi. \tag{2.7}$$

Substituting horizontal conformality equation (2.7) into (2.2), we have

$$\begin{aligned} \Delta_f^M(u \circ \phi) &= f\lambda^2 (h^{\alpha\beta} \circ \phi) u_{\alpha\beta} + [f\,\Delta^M \phi^\sigma + (\mathrm{grad} f)\phi^\sigma] u_\sigma \\ &= f\lambda^2 (\Delta^N u) \circ \phi + [f\,\Delta^M \phi^\sigma + f\lambda^2 (h^{\alpha\beta} \overline{\Gamma}^\sigma_{\alpha\beta}) \circ \phi + (\mathrm{grad} f)\phi^\sigma] u_\sigma \\ &= f\lambda^2 (\Delta^N u) \circ \phi + \mathrm{d}u\,(\tau_f(\phi)) \end{aligned} \tag{2.8}$$

for any function u defined (locally) on N. By special choice of harmonic function u we conclude that the f-harmonic morphism is an f-harmonic map. Thus, we obtain the implication "(1) $\Rightarrow$ (2)". Note that the only assumption we used to obtain (2.8) is the horizontal conformality (2.7). Therefore, it follows from (2.8) that "(2) $\Rightarrow$ (3)". Finally, "(3) $\Rightarrow$ (1)" is clearly true. Thus, we complete the proof of Theorem 2.1.

Similar to harmonic morphisms we have the following regularity result.

Corollary 2.1 *For $m \geq 3$, an f-harmonic morphism $\phi : (M^m, g) \to (N^n, h)$ is smooth.*

Proof In fact, by Corollary 1.1, if $m \neq 2$ and $\phi : (M^m, g) \to (N^n, h)$ is an f-harmonic morphism, then $\phi : (M^m, f^{\frac{2}{m-2}} g) \to (N^n, h)$ is a harmonic map and hence a harmonic morphism, which is known to be smooth (see [4]).

It is well-known that the composition of harmonic morphisms is again a harmonic morphism. The composition law for f-harmonic morphisms, however, will need to be modified accordingly. In fact, by the definitions of harmonic morphisms and f-harmonic morphisms we have the following result.

Corollary 2.2 *Let $\phi : (M^m, g) \to (N^n, h)$ be an f-harmonic morphism with dilation λ_1 and $\psi : (N^n, h) \to (Q^l, k)$ a harmonic morphism with dilation λ_2. Then the composition $\psi \circ \phi : (M^m, g) \to (Q^l, k)$ is an f-harmonic morphism with dilation $\lambda_1(\lambda_2 \circ \phi)$.*

More generally, we can prove that f-harmonic morphisms pull back harmonic maps to f-harmonic maps.

Proposition 2.1 *Let $\phi : (M^m, g) \to (N^n, h)$ be an f-harmonic morphism with dilation λ and $\psi : (N^n, h) \to (Q^l, k)$ a harmonic map. Then the composition $\psi \circ \phi : (M^m, g) \to (Q^l, k)$ is an f-harmonic map.*

Proof It is well-known (see [4, Proposition 3.3.12]) that the tension field of the composition map is given by

$$\tau(\psi \circ \phi) = \mathrm{d}\psi(\tau(\phi)) + \mathrm{Tr}_g \nabla \mathrm{d}\psi(\mathrm{d}\phi, \mathrm{d}\phi),$$

from which we have the f-tension of the composition $\psi \circ \phi$ given by

$$\tau_f(\psi \circ \phi) = \mathrm{d}\psi(\tau_f(\phi)) + f \mathrm{Tr}_g \nabla \mathrm{d}\psi(\mathrm{d}\phi, \mathrm{d}\phi). \tag{2.9}$$

Since ϕ is an f-harmonic morphism and hence a horizontally weakly conformal f-harmonic map with dilation λ, we can choose local orthonormal frames $\{e_1, \cdots, e_n, e_{n+1}, \cdots, e_m\}$ around $p \in M$ and $\{\epsilon_1, \cdots, \epsilon_n\}$ around $\phi(p) \in N$ so that

$$\begin{cases} \mathrm{d}\phi(e_i) = \lambda \epsilon_i, & i = 1, \cdots, n, \\ \mathrm{d}\phi(e_\alpha) = 0, & \alpha = n+1, \cdots, m. \end{cases}$$

Using these local frames we compute

$$\begin{aligned} \mathrm{Tr}_g \nabla \mathrm{d}\psi(\mathrm{d}\phi, \mathrm{d}\phi) &= \sum_{i=1}^{m} \nabla \mathrm{d}\psi(\mathrm{d}\phi e_i, \mathrm{d}\phi e_i) = \lambda^2 \Big(\sum_{i=1}^{n} \nabla \mathrm{d}\psi(\epsilon_i, \epsilon_i) \Big) \circ \phi \\ &= \lambda^2 \tau(\psi) \circ \phi. \end{aligned}$$

Substituting this into (2.9) we have

$$\begin{aligned} \tau_f(\psi \circ \phi) &= f \mathrm{d}\psi(\tau(\phi)) + f \lambda^2 \tau(\psi) \circ \phi + \mathrm{d}(\psi \circ \phi)(\mathrm{grad}\, f) \\ &= \mathrm{d}\psi(\tau_f(\phi)) + f \lambda^2 \tau(\psi) \circ \phi, \end{aligned}$$

from which the proposition follows.

Theorem 2.2 *Let $\phi : (M^m, g) \to (N^n, h)$ be a horizontally weakly conformal map with $\varphi^* h = \lambda^2 g|_{\mathcal{H}}$. Then, any two of the following conditions imply the other one*:

(1) *ϕ is an f-harmonic map and hence an f-harmonic morphism.*

(2) $\mathrm{grad}(f\lambda^{2-n})$ *is vertical.*

(3) *ϕ has minimal fibers.*

Proof It can be check (see [4]) that the tension field of a horizontally weakly conformal map $\phi : (M^m, g) \to (N^n, h)$ is given by

$$\tau(\phi) = -(m-n)\mathrm{d}\phi(\mu) + (2-n)\mathrm{d}\phi(\mathrm{grad} \ln \lambda),$$

where λ is the dilation of the horizontally weakly conformal map ϕ and μ is the mean curvature vector field of the fibers. It follows that the f-tension field of ϕ can be written as

$$\tau_f(\phi) = -(m-n)f\mathrm{d}\phi(\mu) + f\mathrm{d}\phi(\mathrm{grad}\ln\lambda^{2-n}) + \mathrm{d}\phi(\mathrm{grad}\, f),$$

or, equivalently,

$$\tau_f(\phi) = f[-(m-n)\mathrm{d}\phi(\mu) + \mathrm{d}\phi(\mathrm{grad}\ln(f\lambda^{2-n}))] = 0.$$

From this we obtain the theorem.

An immediate consequence is the following result.

Corollary 2.3 (a) *A horizontally homothetic map* (*in particular, a Riemannian submersion*) $\phi : (M^m, g) \to (N^n, h)$ *is an f-harmonic morphism if and only if* $-(m-n)\mu + \mathrm{grad}\ln f$ *is vertical.*

(b) *A weakly conformal map* $\phi : (M^m, g) \to (N^m, h)$ *with conformal factor* λ *of same dimension spaces is f-harmonic and hence an f-harmonic morphism if and only if* $f = C\lambda^{m-2}$ *for some constant* $C > 0$.

(c) *A horizontally weakly conformal map* $\phi : (M^m, g) \to (N^2, h)$ *is an f-harmonic map and hence an f-harmonic morphism if and only if* $-(m-2)\mu + \mathrm{grad}\ln f$ *is vertical.*

Using the characterizations of f-harmonic morphisms and p-harmonic morphisms and Corollary 1.1 we have the following corollary which provides many examples of f-harmonic morphisms.

Corollary 2.4 *A map* $\phi : (M^m, g) \to (N^n, h)$ *between Riemannian manifolds is a p-harmonic morphism without critical points if and only if it is an f-harmonic morphism with* $f = |\mathrm{d}\phi|^{\mathrm{p}-2}$.

Example 2.1 The Möbius transformation $\phi : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^m \setminus \{0\}$ defined by

$$\phi(x) = a + \frac{r^2}{|x-a|^2}(x-a)$$

is an f-harmonic morphism with $f(x) = C\big(\frac{r}{|x-a|}\big)^{2(m-2)}$. In fact, it is well-known that the Möbius transformation is a conformal map between the same dimensional spaces with the dilation $\lambda = \frac{r^2}{|x-a|^2}$. It follows from [20] that ϕ is an m-harmonic morphism, and hence by Corollary 2.4, the inversion is an f-harmonic morphism with

$$f = |\mathrm{d}\phi|^{m-2} = (\sqrt{m}\lambda)^{m-2} = C\Big(\frac{r}{|x-a|}\Big)^{2(m-2)}.$$

The next example is an f-harmonic morphism that does not come from a p-harmonic morphism.

Example 2.2 The map from Euclidean 3-space into the hyperbolic plane $\phi : (\mathbb{R}\times\mathbb{R}\times\mathbb{R}^+, \mathrm{d}s_0^2) \to H^2 \equiv \big(\mathbb{R}\times\{0\}\times\mathbb{R}^+, \frac{1}{z^2}\mathrm{d}s_0^2\big)$ with $\phi(x,y,z) = (x, 0, \sqrt{y^2+z^2})$ is an f-harmonic morphism with $f = \frac{1}{z}$. Similarly, we know from [12] that the map $\phi : H^3 \equiv \big(\mathbb{R}\times\mathbb{R}\times$

$\mathbb{R}^+, \frac{1}{z^2}\mathrm{d}s_0^2) \to H^2 \equiv \big(\mathbb{R}\times\{0\}\times\mathbb{R}^+, \frac{1}{z^2}\mathrm{d}s_0^2\big)$ with $\phi(x,y,z) = (x, 0, \sqrt{y^2+z^2}\,)$ is a harmonic morphism. It follows from Example 1.2 that the map from Euclidean space into the hyperbolic plane $\phi : (\mathbb{R}\times\mathbb{R}\times\mathbb{R}^+, \mathrm{d}s_0^2) \to H^2 \equiv \big(\mathbb{R}\times\{0\}\times\mathbb{R}^+, \frac{1}{z^2}\mathrm{d}s_0^2\big)$ with $\phi(x,y,z) = (x,0,\sqrt{y^2+z^2}\,)$ is an f-harmonic map with $f = \frac{1}{z}$. Since this map is also horizontally weakly conformal it is an f-harmonic morphism by Theorem 2.1.

Example 2.3 Any harmonic morphism $\phi : (M^m, g) \to (N^n, h)$ is an f-harmonic morphism for a positive function f on M with vertical gradient, i.e., $\mathrm{d}\phi(\mathrm{grad} f) = 0$. In particular, the radial projection $\phi : \mathbb{R}^{m+1}\setminus\{0\} \to S^m,\ \ \phi(x) = \frac{x}{|x|}$ is an f-harmonic morphism for $f = \alpha(|x|)$, where $\alpha : (0,\infty) \to (0,\infty)$ is any smooth function. In fact, we know from [4] that the radial projection is a harmonic morphisms and on the other hand, one can check that the function $f = \alpha(|x|)$ is positive and has vertical gradient.

Using the property of f-harmonic morphisms and Sacks-Uhlenbeck's well-known result on the existence of harmonic 2-spheres we have the following proposition which gives many examples of f-harmonic maps from Euclidean 3-space into a manifold whose universal covering space is not contractible.

Proposition 2.2 *For any Riemannian manifold whose universal covering space is not contractible, there exists an f-harmonic map $\phi : \big(\mathbb{R}^3, \mathrm{d}s_0^2\big) \to (N^n, h)$ from Euclidean 3-space with $f(x) = \frac{2}{1+|x|^2}$.*

Proof Let $\mathrm{d}s_0^2$ denote the Euclidean metric on $\mathbb{R}^3$. It is well-known that we can use the inverse of the stereographic projection to identify $\big(\mathbb{R}^3, \frac{4\mathrm{d}s_0^2}{(1+|x|^2)^2}\big)$ with

$$S^3\setminus\{N\} = \Big\{(u_1,u_2,u_3,u_4)\Big|\sum_{i=1}^{4} u_i^2 = 1,\ u_4 \neq 1\Big\},$$

the Euclidean 3-sphere minus the north pole. In fact, the identification is given by the isometry

$$\sigma : \Big(\mathbb{R}^3, \frac{4\mathrm{d}s_0^2}{(1+|x|^2)^2}\Big) \to S^3\setminus\{N\} \subseteq \mathbb{R}^4$$

with

$$\sigma(x_1,x_2,x_3) = \Big(\frac{2x_1}{1+|x|^2},\ \frac{2x_2}{1+|x|^2},\ \frac{2x_3}{1+|x|^2},\ \frac{|x|^2-1}{1+|x|^2}\Big).$$

One can check that under this identification, the Hopf fiberation

$$\phi : \Big(\mathbb{R}^3, \frac{4\mathrm{d}s_0^2}{(1+|x|^2)^2}\Big) \cong S^3\setminus\{N\} \to S^2$$

can be written as

$$\phi(x_1,x_2,x_3) = (|z|^2 - |w|^2, 2zw),$$

where

$$z = \frac{2x_1}{1+|x|^2} + \mathrm{i}\frac{2x_2}{1+|x|^2}, \quad w = \frac{2x_3}{1+|x|^2} + \mathrm{i}\frac{|x|^2-1}{1+|x|^2}.$$

It is well-known (see [4]) that the Hopf fiberation ϕ is a harmonic morphism with dilation $\lambda = 2$. So, by Corollary 1.1, $\phi : (\mathbb{R}^3, \mathrm{d}s_0^2) \to S^2$ is an f-harmonic map with $f = \frac{2}{1+|x|^2}$. It is easy to see that this map is also horizontally conformal submersion and hence, by Theorem 2.1, it is an f-harmonic morphism. On the other hand, by a well-known result of Sacks-Uhlenbeck's, we know that there exists a harmonic map $\rho : S^2 \to (N^n, h)$ from 2-sphere into a manifold whose covering space is not contractible. It follows from Proposition 2.1 that the composition $\rho \circ \phi : (\mathbb{R}^3, \mathrm{d}s_0^2) \to (N^n, h)$ is an f-harmonic map with $f = \frac{2}{1+|x|^2}$.

Remark 2.1 We notice that the authors in [8] and [14] used the heat flow method to study the existence of f-harmonic maps from closed unit disk $D^2 \to S^2$ sending boundary to a single point. The f-harmonic morphism $\phi : (\mathbb{R}^3, \mathrm{d}s_0^2) \to S^2$ in Proposition 2.2 clearly restrict to an f-harmonic map $\phi : (D^3, \mathrm{d}s_0^2) \to S^2$ from 3-dimensional open disk into S^2. It would be interesting to know if there exists any f-harmonic map from higher dimensional closed disk into two-sphere. Though we know that $\phi : (M^m, g) \to (N^n, h)$ being f-harmonic implies

$$\phi : (M^m, f^{\frac{2}{m-2}} g) \to (N^n, h)$$

being harmonic we need to be careful trying to use results from harmonic maps theory since a conformal change of metric may change the curvature and the completeness of the original manifold (M^m, g).

As we remark in Example 2.3 that any harmonic morphism is an f-harmonic morphism provided f is positive with vertical gradient, however, such a function need not always exist as the following proposition shows.

Proposition 2.3 *A Riemannian submersion $\phi : (M^m, g) \to (N^n, h)$ from non-negatively curved compact manifold with minimal fibers is an f-harmonic morphism if and only if $f = C > 0$. In particular, there exists no nonconstant positive function on S^{2n+1} so that the Hopf fiberation $\phi : S^{2n+1} \to (N^n, h)$ is an f-harmonic morphism.*

Proof By Corollary 2.3, a Riemannian submersion $\phi : (M^m, g) \to (N^n, h)$ with minimal fibers is an f-harmonic morphism if and only if $\mathrm{grad} \ln f$ is vertical, i.e., $\mathrm{d}\phi(\mathrm{grad} \ln f) = 0$. This, together with the following lemma will complete the proof of the proposition.

Lemma 2.1 *Let $\phi : (M^m, g) \to (N^n, h)$ be any Riemannian submersion of a compact positively curved manifold M. Then, there exists no (nonconstant) function $f : M \to \mathbb{R}$ such that $\mathrm{d}\phi(\mathrm{grad} \ln f) = 0$.*

Proof Suppose that $f : (M^m, g) \to \mathbb{R}$ has vertical gradient. Consider

$$(M, \mathrm{e}^{\varepsilon f} g),$$

where $\varepsilon > 0$ is a sufficiently small constant.

If ε is small enough, then $\mathrm{e}^{2\varepsilon f} g$ is positively curved. One can check that

$$\phi : (M, \mathrm{e}^{2\varepsilon f} g) \to (N, h) \tag{2.10}$$

is a horizontally homothetic submersion with dilation $\lambda^2 = e^{-2\varepsilon f}$ since f has vertical gradient. By the main theorem in [22] we conclude that the map ϕ defined in (2.10) is a Riemannian submersion, which implies that the dilation and hence the function f has to be a constant.

Remark 2.2 It would be very interesting to know if there exists any f-harmonic morphism (or f-harmonic map) $\phi : S^{2n+1} \to (N^n, h)$ with non-constant f. Note that for the case of $n = 2$, the problem of classifying all f-harmonic morphisms $\phi : (S^3, g_0) \to (N^2, h)$ (where g_0 denotes the standard Euclidean metric on the 3-sphere) amounts to classifying all harmonic morphisms $\phi : (S^3, f^2 g_0) \to (N^2, h)$ from conformally flat 3-spheres. A partial result on the latter problem was given in [13] in which the author proved that a submersive harmonic morphism $\phi : (S^3, f^2 g_0) \to (N^2, h)$ with non-vanishing horizontal curvature is the Hopf fiberation up to an isometry of (S^3, g_0). This implies that there exists no submersive f-harmonic morphism $\phi : (S^3, g_0) \to (N^2, h)$ with non-constant f and the horizontal curvature $K_{\mathcal{H}}(f^2 g_0) \neq 0$.

Proposition 2.4 *For $m > n \geq 2$, a polynomial map* (*i.e., a map whose component functions are polynomials*) *$\phi : \mathbb{R}^m \to \mathbb{R}^n$ is an f-harmonic morphism if and only if ϕ is a harmonic morphism and f has vertical gradient.*

Proof Let $\phi : \mathbb{R}^m \to \mathbb{R}^n$ be a polynomial map (i.e., a map whose component functions are polynomials). If ϕ is an f-harmonic morphism, then, by Theorem 2.1, it is a horizontally weakly conformal f-harmonic map. It was proved in [1] that any horizontally weakly conformal polynomial map between Euclidean spaces has to be harmonic. This implies that ϕ is also a harmonic morphism, and in this case we have $d\phi(\text{grad}\, f) = 0$ from (1.1).

Example 2.4 $\phi : \mathbb{R}^3 \cong \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ with $\phi(t, z) = p(z)$, where $p(z)$ is any polynomial function in z, is an f-harmonic morphism with $f(t, z) = \alpha(t)$ for any positive smooth function α.

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