Chin. Ann. Math.
35B(2), 2014, 237–260
DOI: 10.1007/s11401-014-0824-1

Chinese Annals of Mathematics, Series B

Atomic Decompositions of Triebel-Lizorkin Spaces with Local Weights and Applications*

Liguang LIU[1] Dachun YANG[2]

Abstract In this paper, the authors characterize the inhomogeneous Triebel-Lizorkin spaces $F^{s,w}_{p,q}(\mathbb{R}^n)$ with local weight w by using the Lusin-area functions for the full ranges of the indices, and then establish their atomic decompositions for $s \in \mathbb{R}$, $p \in (0,1]$ and $q \in [p,\infty)$. The novelty is that the weight w here satisfies the classical Muckenhoupt condition only on balls with their radii in $(0,1]$. Finite atomic decompositions for smooth functions in $F^{s,w}_{p,q}(\mathbb{R}^n)$ are also obtained, which further implies that a (sub)linear operator that maps smooth atoms of $F^{s,w}_{p,q}(\mathbb{R}^n)$ uniformly into a bounded set of a (quasi-)Banach space is extended to a bounded operator on the whole $F^{s,w}_{p,q}(\mathbb{R}^n)$. As an application, the boundedness of the local Riesz operator on the space $F^{s,w}_{p,q}(\mathbb{R}^n)$ is obtained.

Keywords Local weight, Triebel-Lizorkin space, Atom, Lusin-Area function, Riesz transform
2010 MR Subject Classification 46E35, 47B06, 42B20, 42B35

1 Introduction

The local weight class was introduced by Rychkov [18]. Recall that, for $p \in (1,\infty)$, the local weight class $\mathcal{A}^{\rm loc}_p(\mathbb{R}^n)$ consists of all non-negative locally integrable functions w such that

$$[w]_{\mathcal{A}^{\rm loc}_p(\mathbb{R}^n)} := \sup_{|Q|\le 1}\Big\{\frac{1}{|Q|}\int_Q w(x){\rm d}x\Big\}\Big\{\frac{1}{|Q|}\int_Q [w(x)]^{-\frac{1}{p-1}}{\rm d}x\Big\}^{p-1} < \infty,$$

where the supremum is taken over all cubes of $\mathbb{R}^n$ with the n-dimensional Lebesgue measure no more than 1 and with sides parallelling to the coordinate axis. If $p=1$, then the class $\mathcal{A}^{\rm loc}_1(\mathbb{R}^n)$ consists of all non-negative locally integrable functions w such that

$$[w]_{\mathcal{A}^{\rm loc}_1(\mathbb{R}^n)} := \sup_{|Q|\le 1}\Big\{\frac{1}{|Q|}\int_Q w(x){\rm d}x\Big\}\sup_{y\in Q}[w(y)]^{-1} < \infty.$$

Define $\mathcal{A}^{\rm loc}_\infty(\mathbb{R}^n) := \bigcup\limits_{1\le p<\infty}\mathcal{A}^{\rm loc}_p(\mathbb{R}^n)$. For any $q \in [1,\infty]$, let $q_w := \inf\{q : w \in \mathcal{A}^{\rm loc}_q(\mathbb{R}^n)\}$, which is called the critical index of the local weight w. Observe that the class $\mathcal{A}^{\rm loc}_\infty(\mathbb{R}^n)$

Manuscript received August 13, 2012.
[1]Department of Mathematics, School of Information, Renmin University of China, Beijing 100872, China.
E-mail: liuliguang@ruc.edu.cn
[2]Corresponding author. School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, China.
E-mail: dcyang@bnu.edu.cn
*Project supported by the National Natural Science Foundation of China (Nos. 11101425, 11171027) and the Specialized Research Fund for the Doctoral Program of Higher Education of China (No. 20120003110003).

consists of non-doubling weights, which may grow or decrease exponentially at infinity. Apart from the well-known Muckenhoupt weight class, an important example of $w \in \mathcal{A}_{\infty}^{\rm loc}(\mathbb{R}^n)$ is from Triebel [24, Chapter 6], wherein the author investigated weighted inhomogeneous Besov and Triebel-Lizorkin spaces associated to a weight w satisfying that there exist $\beta \in (0,1]$ and $C \in (0,\infty)$ such that, for all $x, y \in \mathbb{R}^n$, $0 < w(x) \le Cw(y)\mathrm{e}^{|x-y|^{\beta}}$; see also Schott [20–21] and Schmeißer-Triebel [19].

Rychkov [18] introduced and studied the inhomogeneous Besov and Triebel-Lizorkin spaces associated to a weight $w \in \mathcal{A}_{\infty}^{\rm loc}(\mathbb{R}^n)$. Izuki and Sawano [14–16] then investigated the wavelet characterizations of these function spaces. Also, Tang [22] established the maximal function characterization of the weighted local Hardy spaces $h_w^p(\mathbb{R}^n)$ with $w \in \mathcal{A}_{\infty}^{\rm loc}(\mathbb{R}^n)$, which is an extension of the results of Bui [1] and Goldberg [7]. Boundedness of some strongly singular integrals, pseudo-differential operators and their commutators on the weighted local Hardy spaces $h_w^p(\mathbb{R}^n)$ were also studied in [22–23]. For generalizations of the results in [22] to some Orlicz-type local Hardy spaces associated to the weight $w \in \mathcal{A}_{\infty}^{\rm loc}(\mathbb{R}^n)$ (see [25]). It should be mentioned that there are many works concerning the (in)homogeneous Besov or Triebel-Lizorkin spaces associated to the classical Muckenhoupt weights; see, for instance, [9–13, 24] and their references.

The main aim of this paper is to characterize the inhomogeneous Triebel-Lizorkin spaces $F_{p,q}^{s,w}(\mathbb{R}^n)$ with local weight $w \in \mathcal{A}_{\infty}^{\rm loc}(\mathbb{R}^n)$ in the sense of Rychkov [18] (see also Definition 1.1 below) by using atoms completely analogous to the classical atoms of Hardy spaces. In other words, for $s \in \mathbb{R}$, $p \in (0,1]$, $q \in [p,\infty)$ and $w \in \mathcal{A}_{\max\{q,1\}}^{\rm loc}(\mathbb{R}^n)$, we prove, in Theorem 1.2 below, that an element $f \in F_{p,q}^{s,w}(\mathbb{R}^n)$ if and only if it can be written as a linear combination of these weighted atoms with the coefficients belonging to ℓ^p. To this end, we first establish the Lusin-area function characterization of $F_{p,q}^{s,w}(\mathbb{R}^n)$ in Theorem 1.1 below. Moreover, finite atomic decompositions for smooth functions in $F_{p,q}^{s,w}(\mathbb{R}^n)$ are presented in Theorem 1.3 below. This allows us to deduce the following boundedness criteria in Theorem 1.4 below: If a (sub)linear operator maps atoms, which are infinitely differentiable, of $F_{p,q}^{s,w}(\mathbb{R}^n)$ into a (quasi-)Banach space uniformly, then it extends to a bounded (sub)linear operator on the whole $F_{p,q}^{s,w}(\mathbb{R}^n)$. As an application, the boundedness of the local Riesz operator on the space $F_{p,q}^{s,w}(\mathbb{R}^n)$ is obtained. It is expectable that Theorem 1.4 may have further more applications, say, in the study of the boundedness of operators on $F_{p,q}^{s,w}(\mathbb{R}^n)$ (see, for example, [2–3, 17]).

It should be mentioned that the definition of atoms of $F_{p,q}^{s,w}(\mathbb{R}^n)$ (see Definition 1.2 below) used by us throughout this paper is inspired by Han, Paluszyński and Weiss [8], in which atomic characterizations for the classical non-weighted homogeneous Triebel-Lizorkin space $\dot{F}_{p,q}^{s}(\mathbb{R}^n)$ were established, where $s \in \mathbb{R}$, $p \in (0,1]$ and $q \in [p,\infty)$. We also remark that atomic decompositions of the Triebel-Lizorkin spaces $F_{p,q}^{s,w}(\mathbb{R}^n)$ with local weights were also considered in [15], by using the machinery of the ϕ-transform of Frazier-Jawerth in [4–5]. The advantage of the atoms used in this paper is that it is more convenient for applications in the study on the boundedness of operators.

To recall the inhomogeneous Triebel-Lizorkin spaces with local weights introduced in [18], we need the following notation. Let $C_c^{\infty}(\mathbb{R}^n)$ be the set of all infinitely differentiable functions on $\mathbb{R}^n$ with compact support. Endow $C_c^{\infty}(\mathbb{R}^n)$ with the strict inductive topology, which is denoted by $\mathcal{D}(\mathbb{R}^n)$ and whose dual space by $\mathcal{D}'(\mathbb{R}^n)$. As in [18], let $\mathcal{S}'_e(\mathbb{R}^n)$ be the space of all $f \in \mathcal{D}'(\mathbb{R}^n)$ such that there exist positive constants A_f and N_f such that for all $\phi \in$

$\mathcal{D}(\mathbb{R}^n)$, $|\langle f,\phi\rangle| \le A_f \sup\{|D^\alpha\phi(x)|\mathrm{e}^{N_f|x|} : x\in\mathbb{R}^n, |\alpha|\le N_f\}$. For $p\in(0,\infty]$, we denote by $L^p_w(\mathbb{R}^n)$ the weighted Lebesgue space which consists of all functions f such that $\|f\|_{L^p_w(\mathbb{R}^n)} := \{\int_{\mathbb{R}^n} |f(x)|^p w(x)\mathrm{d}x\}^{\frac{1}{p}} < \infty$, and by $L^{p,\infty}_w(\mathbb{R}^n)$ the weighted weak-type Lebesgue space which consists of all functions f such that $\|f\|_{L^{p,\infty}_w(\mathbb{R}^n)} := \sup_{t>0} t[w(\{x\in\mathbb{R}^n : |f(x)|>t\})]^{\frac{1}{p}} < \infty$. For any $s\in\mathbb{R}$, we denote by $\lfloor s\rfloor$ the maximal integer no more than s.

Definition 1.1 *Let $w\in\mathcal{A}^{\mathrm{loc}}_\infty(\mathbb{R}^n)$, $s\in\mathbb{R}$, $p\in(0,\infty)$ and $q\in(0,\infty]$. Suppose that $\phi_0\in\mathcal{D}(\mathbb{R}^n)$ and $\phi := \phi_0 - 2^{-n}\phi_0(2^{-1}\cdot)$ satisfies that*

$$\int_{\mathbb{R}^n} x^\alpha \phi(x)\mathrm{d}x = 0 \quad \text{for all } |\alpha| \le \max\{-1, \lfloor s\rfloor\}. \tag{1.1}$$

For $j\in\mathbb{N}$, set $\phi_j := 2^{jn}\phi(2^j\cdot)$. The inhomogeneous Triebel-Lizorkin space $F^{s,w}_{p,q}(\mathbb{R}^n)$ is defined to be the collection of all $f\in\mathcal{S}'_e(\mathbb{R}^n)$ such that

$$\|f\|_{F^{s,w}_{p,q}(\mathbb{R}^n)} := \Big\|\Big[\sum_{j=0}^\infty 2^{jsq}|\phi_j * f|^q\Big]^{\frac{1}{q}}\Big\|_{L^p_w(\mathbb{R}^n)} < \infty$$

with a usual modification made when $q=\infty$.

Notice that $F^{s,w}_{p,q}(\mathbb{R}^n)$, with $w\in\mathcal{A}^{\mathrm{loc}}_\infty(\mathbb{R}^n)$, $s\in\mathbb{R}$, $p\in(0,\infty)$ and $q\in(0,\infty]$, are complete (quasi-)Banach spaces (see [18, Lemma 2.15]). By [18, Theorem 2.5], there exist positive constants A_0 and B_0, depending only on s,p,q and w, such that, when $A\ge A_0$ and $B\ge \frac{B_0}{p}$,

$$\|f\|_{F^{s,w}_{p,q}(\mathbb{R}^n)} \sim \Big\|\Big[\sum_{j=0}^\infty 2^{jsq}|\phi^*_{j,A,B}f|^q\Big]^{\frac{1}{q}}\Big\|_{L^p_w(\mathbb{R}^n)}, \tag{1.2}$$

where $\phi^*_{j,A,B}f$ denotes the Peetre-type maximal function of f, defined by

$$\phi^*_{j,A,B}f(x) := \sup_{y\in\mathbb{R}^n} \frac{|\phi_j * f(y)|}{(1+2^j|x-y|)^A \mathrm{e}^{B|x-y|}}, \quad x\in\mathbb{R}^n. \tag{1.3}$$

From this Peetre-type maximal function characterization for $F^{s,w}_{p,q}(\mathbb{R}^n)$, it follows easily that the space $F^{s,w}_{p,q}(\mathbb{R}^n)$ is independent of the choice of $\phi_0\in\mathcal{D}(\mathbb{R}^n)$ satisfying (1.1).

For any $a\in(0,\infty)$, $s\in\mathbb{R}$, $q\in(0,\infty)$ and $f\in\mathcal{S}'_e(\mathbb{R}^n)$, the Lusin-area functions $S^{a,s}_q(f)$ and $\widetilde{S}^{a,s}_q(f)$ are defined, respectively, by setting, for all $x\in\mathbb{R}^n$,

$$S^s_{a,q}(f)(x) := \Big[\sum_{j=0}^\infty \frac{1}{|B(x,2^{-j})|}\int_{|x-y|<a2^{-j}} |2^{js}\phi_j * f(y)|^q \mathrm{d}y\Big]^{\frac{1}{q}}$$

and

$$\widetilde{S}^s_{a,q}(f)(x) := \Big[\sum_{j=0}^\infty \sup_{|x-y|<a2^{-j}} |2^{js}\phi_j * f(y)|^q \mathrm{d}y\Big]^{\frac{1}{q}},$$

where $\{\phi_j\}_{j=0}^\infty$ are as in Definition 1.1. Applying the Peetre-type maximal function characterization of $F^{s,w}_{p,q}(\mathbb{R}^n)$ in (1.2), we can conclude the following Lusin-area function characterization of $F^{s,w}_{p,q}(\mathbb{R}^n)$, whose proof is presented in Section 3.

Theorem 1.1 *Let $w \in \mathcal{A}_\infty^{\rm loc}(\mathbb{R}^n)$, $a \in (0,\infty)$, $s \in \mathbb{R}^n$, $p \in (0,\infty)$ and $q \in (0,\infty)$. Then, there exists a positive constant C such that, for all $f \in \mathcal{S}_e'(\mathbb{R}^n)$,*

$$\frac{1}{C}\|f\|_{F_{p,q}^{s,w}(\mathbb{R}^n)} \le \|S_{a,q}^s(f)\|_{L_w^p(\mathbb{R}^n)} \le \|\widetilde{S}_{a,q}^s(f)\|_{L_w^p(\mathbb{R}^n)} \le C\|f\|_{F_{p,q}^{s,w}(\mathbb{R}^n)}.$$

Motivated by [8], we introduce atoms of the space $F_{p,q}^{s,w}(\mathbb{R}^n)$ as follows.

Definition 1.2 *Let $s \in \mathbb{R}$, $p \in (0,1]$, $q \in [p,\infty)$ and $w \in \mathcal{A}_{\max\{q,1\}}^{\rm loc}(\mathbb{R}^n)$. A distribution $a \in \mathcal{S}_e'(\mathbb{R}^n)$ is called a $(p,q,s)_w$-atom of $F_{p,q}^{s,w}(\mathbb{R}^n)$ if the following hold:*

(i) *a is supported on a cube $Q \subset \mathbb{R}^n$ centered at c_Q and of side length $\ell(Q)$.*

(ii) *$\|a\|_{F_{q,q}^{s,w}(\mathbb{R}^n)} \le [w(Q)]^{\frac{1}{q}-\frac{1}{p}}$.*

(iii) *If $|Q|<1$, then for any $g \in \mathcal{S}(\mathbb{R}^n)$, a polynomial P of degree at most $\mathcal{N} := \max\big\{\lfloor n\big(\frac{q_w}{p}-1\big)-s\rfloor, 0\big\}$ and a smooth cutoff function $\eta_Q \in \mathcal{S}(\mathbb{R}^n)$ such that $\eta_Q = 1$ on Q and $\eta_Q = 0$ outside $2Q$, $\langle a, g\rangle = \langle a, (g-P)\eta_Q\rangle$, here and hereafter, $2Q$ denotes the cube centered at c_Q and of side length $2\ell(Q)$.*

Now we give the following atomic characterization of the Triebel-Lizorkin spaces, which follows from the Calderón reproducing formula (see Lemma 2.3 below) and the Lusin-area function characterization of $F_{p,q}^{s,w}(\mathbb{R}^n)$ in Theorem 1.1 (see Section 4 for its proof).

Theorem 1.2 *Let $s \in \mathbb{R}$, $p \in (0,1]$, $q \in [p,\infty)$ and $w \in A_{\max\{q,1\}}^{\rm loc}(\mathbb{R}^n)$. Then, $f \in F_{p,q}^{s,w}(\mathbb{R}^n)$ if and only if $f = \sum_{k\in\mathbb{N}} \lambda_k a_k$ in $\mathcal{S}_e'(\mathbb{R}^n)$, where $\{\lambda_k\}_{k\in\mathbb{Z}} \in \ell^p$ and $\{a_k\}_{k\in\mathbb{N}}$ are $(p,q,s)_w$-atoms. Moreover, there exists a positive constant C such that, for all $f \in F_{p,q}^{s,w}(\mathbb{R}^n)$,*

$$\frac{1}{C}\|f\|_{F_{p,q}^{s,w}(\mathbb{R}^n)} \le \inf\Big\{\Big[\sum_{k\in\mathbb{N}}|\lambda_k|^p\Big]^{\frac{1}{p}}\Big\} \le C\|f\|_{F_{p,q}^{s,w}(\mathbb{R}^n)},$$

where the infimum is taken over all the decompositions of f as above.

Next, we show that functions in $C_c^\infty(\mathbb{R}^n) \cap F_{p,q}^{s,w}(\mathbb{R}^n)$ can be decomposed into finite linear combinations of $(p,q,s)_w$-atoms with their coefficients belonging to ℓ^p. The proof is given in Section 5 by invoking some ideas from [17].

Theorem 1.3 *Let $s \in \mathbb{R}$, $p \in (0,1]$, $q \in [p,\infty)$ and $w \in \mathcal{A}_{\max\{1,q\}}^{\rm loc}(\mathbb{R}^n)$. Then, every $f \in C_c^\infty(\mathbb{R}^n) \cap F_{p,q}^{s,w}(\mathbb{R}^n)$ admits an atomic decomposition $f = \sum_{k=1}^N \lambda_k a_k$, where $N \in \mathbb{N}$, $\{a_k\}_{k=1}^N$ are $(p,q,s)_w$-atoms such that each $a_k \in C_c^\infty(\mathbb{R}^n)$ and $\big[\sum_{k=1}^N |\lambda_k|^p\big]^{\frac{1}{p}} \le C\|f\|_{F_{p,q}^{s,w}(\mathbb{R}^n)}$ for some positive constant C independent of f and N.*

Indeed, $C_c^\infty(\mathbb{R}^n) \cap F_{p,q}^{s,w}(\mathbb{R}^n)$ is dense in $F_{p,q}^{s,w}(\mathbb{R}^n)$ (see Lemma 6.1 below). Consequently, we can establish a boundedness criteria for (sub)linear operators from $F_{p,q}^{s,w}(\mathbb{R}^n)$ to some (quasi)-Banach spaces as in [17, 26–27]. Before going into details, we first recall the following notion (see, for example, [17, 26–27]).

Definition 1.3 (i) *A quasi-Banach space $\mathcal{B}$ is a vector space endowed with a quasi-norm $\|\cdot\|_{\mathcal{B}}$ which is non-negative, non-degenerate (namely, $\|f\|_{\mathcal{B}} = 0$ if and only if $f = 0$), homogeneous, and obeys the quasi-triangle inequality, namely, there exists a constant $K \ge 1$ such that, for all f, $g \in \mathcal{B}$, $\|f+g\|_{\mathcal{B}} \le K[\|f\|_{\mathcal{B}} + \|g\|_{\mathcal{B}}]$.*

(ii) *Let $r \in (0,1]$. A quasi-Banach space $\mathcal{B}_r$ with the quasi-norm $\|\cdot\|_{\mathcal{B}_r}$ is called an r-quasi-Banach space if $\|f+g\|_{\mathcal{B}_r}^r \le \|f\|_{\mathcal{B}_r}^r + \|g\|_{\mathcal{B}_r}^r$ for all $f, g \in \mathcal{B}_r$.*

(iii) *For any r-quasi-Banach space $\mathcal{B}_r$ with $r \in (0,1]$ and a linear space $\mathcal{Y}$, an operator T from $\mathcal{Y}$ to $\mathcal{B}_r$ is said to be $\mathcal{B}_r$-sublinear if, for all f, $g \in \mathcal{Y}$ and λ, $\nu \in \mathbb{C}$,*

$$\|T(\lambda f + \nu g)\|_{\mathcal{B}_r} \le [|\lambda|^r \|T(f)\|_{\mathcal{B}_r}^r + |\nu|^r \|T(g)\|_{\mathcal{B}_r}^r]^{\frac{1}{r}}$$

and $\|T(f) - T(g)\|_{\mathcal{B}_r} \le \|T(f-g)\|_{\mathcal{B}_r}$.

Applying Theorem 1.3 and the density property of $C_c^\infty(\mathbb{R}^n) \cap F_{p,q}^{s,w}(\mathbb{R}^n)$ in $F_{p,q}^{s,w}(\mathbb{R}^n)$, we can establish a criterion for the boundedness of operators on $F_{p,q}^{s,w}(\mathbb{R}^n)$ (see Section 6 for its proof).

Theorem 1.4 *Let $s \in \mathbb{R}$, $p \in (0,1]$, $q \in [p,\infty)$ and $w \in \mathcal{A}_{\max\{1,q\}}^{\rm loc}(\mathbb{R}^n)$. Suppose that $\mathcal{B}_r$ is an r-quasi-Banach space with $r \in [p,1]$ and that $T:\ C_c^\infty(\mathbb{R}^n) \cap \dot{F}_{p,q}^{s,w}(\mathbb{R}^n) \to \mathcal{B}_r$ is a $\mathcal{B}_r$-sublinear operator satisfying that*

$$\sup\{\|Ta\|_{\mathcal{B}_r} : a \in C_c^\infty(\mathbb{R}^n) \text{ is any } (p,q,s)_w\text{-atom}\} < \infty.$$

Then T uniquely extends to a bounded $\mathcal{B}_r$-sublinear operator from $F_{p,q}^{s,w}(\mathbb{R}^n)$ to $\mathcal{B}_r$.

From Theorem 1.4, it follows the boundedness of the local Riesz operator on the Triebel-Lizorkin spaces with local weights. Let $\Phi \in \mathcal{D}(\mathbb{R}^n)$ satisfying $\Phi(x) = 1$ for $x \in B(0,1)$ and $\operatorname{supp}\Phi \subset B(0,2)$. For $j \in \{1,\cdots,n\}$, consider the local Riesz operator

$$\mathcal{R}_j^{\rm loc} f(x) := \text{p. v.} \int_{\mathbb{R}^n} \frac{y_j}{|y|^{n+1}} \Phi(y) f(x-y) \mathrm{d}y, \quad x \in \mathbb{R}^n$$

(see [22, 25]). It was proved in [22, Lemma 8.2] that $\mathcal{R}_j^{\rm loc}$ is bounded on $L_w^p(\mathbb{R}^n)$ when $p \in (1,\infty)$ and $w \in \mathcal{A}_p^{\rm loc}(\mathbb{R}^n)$, and from $L_w^1(\mathbb{R}^n)$ to $L_w^{1,\infty}(\mathbb{R}^n)$ when $w \in \mathcal{A}_1^{\rm loc}(\mathbb{R}^n)$. For $p \in (0,1]$, let $h_w^p(\mathbb{R}^n)$ be the weighted local Hardy space, which consists of all $f \in \mathcal{S}_e'(\mathbb{R}^n)$ such that $f^+ := \sup_{0<t\le 1} |\phi_t * f| \in L_w^p(\mathbb{R}^n)$, where $\phi \in C_c^\infty(\mathbb{R}^n)$ has a non-zero integral, and we define $\|f\|_{h_w^p(\mathbb{R}^n)} := \|f^+\|_{L_w^p(\mathbb{R}^n)}$. The operators $\{\mathcal{R}_j^{\rm loc}\}_{j=1}^n$ were used to characterize $h_w^1(\mathbb{R}^n)$ in [22]. Moreover, it was proved in [25, Theorem 8.2] that $\mathcal{R}_j^{\rm loc}$ is bounded on the Orlicz-type local Hardy spaces with local weights, which particularly implies that each $\mathcal{R}_j^{\rm loc}$ is bounded on the local weighted Hardy space $h_w^p(\mathbb{R}^n)$ with $p \in (0,1]$ and $w \in \mathcal{A}_\infty^{\rm loc}(\mathbb{R}^n)$.

Applying Theorem 1.4, we obtain the following conclusion (see Section 7 for its proof).

Theorem 1.5 *Let $s \in \mathbb{R}$, $p \in (0,1]$, $q \in [1,\infty)$ and $w \in \mathcal{A}_q^{\rm loc}(\mathbb{R}^n)$. Then, for all $j \in \{1,\cdots,n\}$, the operator $\mathcal{R}_j^{\rm loc}$ is bounded on $F_{p,q}^{s,w}(\mathbb{R}^n)$.*

Remark 1.1 (i) For $p \in (0,1]$, it was proved in [18, Theorem 2.25] that $F_{p,2}^{0,w}(\mathbb{R}^n) = h_w^p(\mathbb{R}^n)$ with equivalent quasi-norms. Thus, for all $p \in (0,1]$, if we take $s = 0$ and $q = 2$ in Theorem 1.5, then every $\mathcal{R}_j^{\rm loc}$ is bounded on the space $h_w^p(\mathbb{R}^n)$ if $w \in \mathcal{A}_2^{\rm loc}(\mathbb{R}^n)$.

(ii) The result in (i) is slightly weaker than the aforementioned corollary of [25, Theorem 8.2], which says that $\mathcal{R}_j^{\rm loc}$ is bounded on $h_w^p(\mathbb{R}^n)$ for all $w \in \mathcal{A}_\infty^{\rm loc}(\mathbb{R}^n)$. The reasons for this are as follows. The size condition of an $h_w^p(\mathbb{R}^n)$-atom in [25, Definition 3.4] can be given by any $\|\cdot\|_{L_w^r(\mathbb{R}^n)}$ norm with $r \in [1,\infty] \cap (p,\infty)$; meanwhile, any weight $w \in \mathcal{A}_\infty^{\rm loc}(\mathbb{R}^n)$ implies that $w \in \mathcal{A}_r^{\rm loc}(\mathbb{R}^n)$ for some $r \in (1,\infty)$. However, the size condition of a $(p,2,0)_w$-atom in Definition 1.2 is given by some fixed quasi-norm $\|\cdot\|_{F_{2,2}^{0,w}(\mathbb{R}^n)}$, so we are forced to use weights $w \in \mathcal{A}_2^{\rm loc}(\mathbb{R}^n)$.

This article is organized as follows. In Section 2, we first recall some known basic lemmas, including the properties of the local weight, Fefferman-Stein vector-valued inequalities associated to the local weights, and the Calderón reproducing formulae; we then prove a duality result related to the space $F^{s,w}_{p,q}(\mathbb{R}^n)$. The proof of Theorem 1.1 is presented in Section 3. The whole Section 4 focuses on the proof of Theorem 1.2, by using Theorem 1.1 and a series of auxiliary lemmas developed in Section 4. In Section 5, we prove Theorem 1.3 by using the atomic decomposition result in Theorem 1.2. Sections 6 is devoted to the proof of Theorem 1.4 by showing the density property of $C^\infty_c(\mathbb{R}^n)\cap F^{s,w}_{p,q}(\mathbb{R}^n)$ in $F^{s,w}_{p,q}(\mathbb{R}^n)$ (see Lemma 6.1). Finally, the proof of Theorem 1.5 is presented in Section 7, by using Theorem 1.5.

Throughout this paper, we use the following notation. Let $\mathbb{N} := \{1, \cdots\}$, $\mathbb{Z}_+ := \mathbb{N}\cup\{0\}$ and $\mathbb{Z} := \{0, \pm1, \cdots\}$. Denote by C a positive constant independent of the main parameters involved, which may vary at different occurrences. We use $f \lesssim g$ or $g \gtrsim f$ to denote $f \le Cg$ or $g \ge Cf$, respectively. If $f \lesssim g \lesssim f$, then we write $f \sim g$.

2 Preliminaries

For $\kappa\in(0,\infty)$, the local Hardy-Littlewood maximal operator $M^{\rm loc}_\kappa$ is defined by setting, for all locally integrable functions f and $x\in\mathbb{R}^n$,

$$M^{\rm loc}_\kappa(f)(x) := \sup_{\substack{Q\ni x\\ |Q|\le\kappa}} \frac{1}{|Q|}\int_Q |f(y)|\mathrm{d}y.$$

If $\kappa = 1$, then we simply write $M^{\rm loc}_\kappa$ as $M^{\rm loc}$. Also, for $B \ge 0$, all suitable functions f and $x\in\mathbb{R}^n$, let $K_B(f)(x) := \int_{\mathbb{R}^n} |f(y)|\mathrm{e}^{-B|x-y|}\mathrm{d}y$.

The following versions of the vector-valued Fefferman-Stein maximal inequalities associated to local weights were proved in [18, Lemma 2.11].

Lemma 2.1 *If $\kappa\in(0,\infty)$, $p\in(1,\infty)$, $q\in(1,\infty]$ and $w\in\mathcal{A}^{\rm loc}_p(\mathbb{R}^n)$, then, for any sequence of locally integrable functions $\{f_j\}_{j\in\mathbb{Z}}$, it holds true that*

$$\Big\|\Big\{\sum_{j\in\mathbb{Z}}[M^{\rm loc}_\kappa(f_j)]^q\Big\}^{\frac1q}\Big\|_{L^p_w(\mathbb{R}^n)} \le C\Big\|\Big\{\sum_{j\in\mathbb{Z}}|f_j|^q\Big\}^{\frac1q}\Big\|_{L^p_w(\mathbb{R}^n)} \tag{2.1}$$

and there exists a positive constant B_0, depending only on n and w, such that, when $B\ge\frac{B_0}{p}$,

$$\Big\|\Big\{\sum_{j\in\mathbb{Z}}[K_B(f_j)]^q\Big\}^{\frac1q}\Big\|_{L^p_w(\mathbb{R}^n)} \le C\Big\|\Big\{\sum_{j\in\mathbb{Z}}|f_j|^q\Big\}^{\frac1q}\Big\|_{L^p_w(\mathbb{R}^n)}, \tag{2.2}$$

where C is a positive constant depending only on n, κ, p, q, B and $[w]_{\mathcal{A}^{\rm loc}_p(\mathbb{R}^n)}$.

Some properties of the local weights are presented in the following lemma; whose proofs were given in [18, Lemma 1.4] and [22, Lemma 2.1 and Corollary 2.1].

Lemma 2.2 *Let $p\in[1,\infty]$, $w\in\mathcal{A}^{\rm loc}_p(\mathbb{R}^n)$ and $\kappa\in(0,\infty)$.*

(i) *There exists a positive constant c_w, which depends only on $[w]_{\mathcal{A}^{\rm loc}_p(\mathbb{R}^n)}$ and n, such that, for all $t\in[1,\infty)$ and cubes Q with $|Q| = 1$, $w(tQ)\le \mathrm{e}^{c_w t}w(Q)$.*

(ii) *There is a positive constant C, which depends only on $[w]_{\mathcal{A}^{\rm loc}_p(\mathbb{R}^n)}$ and n, such that, if $|Q|\le 1$, then $w(2Q)\le Cw(Q)$ and, if $|rQ| > 1$, then $w((r + 1)Q)\le Cw(rQ)$.*

(iii) $M_\kappa^{\rm loc}$ *is bounded from* $L_w^1(\mathbb{R}^n)$ *to* $L_w^{1,\infty}(\mathbb{R}^n)$ *if* $p=1$*, and bounded on* $L_w^p(\mathbb{R}^n)$ *if* $p\in(1,\infty]$.

(iv) *For* $p\in(1,\infty)$, $w\in\mathcal{A}_p^{\rm loc}(\mathbb{R}^n)$ *if and only if* $w^{-\frac{1}{p-1}}\in\mathcal{A}_{p'}^{\rm loc}(\mathbb{R}^n)$*, here and hereafter,* p' *denotes the conjugate index of* p.

The Calderón-type reproducing formula in the local case was essentially given in [18, Theorem 1.6]. Indeed, Lemma 2.3 for $j=0$ was proved in [18, Theorem 1.6] and the proofs for the general case $j\in\mathbb{N}$ are essentially the same. We omit the details.

Lemma 2.3 *Assume that* $\phi_0\in\mathcal{D}(\mathbb{R}^n)$ *has a nonzero integral. Let* $\phi:=\phi_0-2^{-n}\phi_0(2^{-1}\cdot)$. *Then, for any given* $L\in\mathbb{Z}_+$*, there exist functions* $\psi_0,\psi\in\mathcal{D}(\mathbb{R}^n)$ *such that* ψ_0 *has a nonzero integral,* ψ *has vanishing moments up to order* L *(namely,* $\int_{\mathbb{R}^n}x^\alpha\psi(x)\mathrm{d}x=0$ *for all multi-indices* α *with* $|\alpha|\le L$*) and, for all* $j\in\mathbb{Z}_+$ *and* $f\in\mathcal{D}'(\mathbb{R}^n)$,

$$f=(\psi_0)_j*(\phi_0)_j*f+\sum_{i=j+1}^{\infty}\psi_i*\phi_i*f \quad \text{in } \mathcal{D}'(\mathbb{R}^n), \tag{2.3}$$

where $\phi_i:=2^{in}\phi(2^i\cdot)$ *and* $\psi_i:=2^{in}\psi(2^i\cdot)$ *for all* $i\in\mathbb{N}$.

Finally, we conclude this section with the following duality result.

Proposition 2.1 *Let* $s\in\mathbb{R}$*,* $p\in(1,\infty)$*,* $q\in[1,\infty)$ *and* $w\in\mathcal{A}_p^{\rm loc}(\mathbb{R}^n)$*. Then*

$$(F_{p,q}^{s,\,w}(\mathbb{R}^n))^*=F_{p',q'}^{-s,\,w^{1-p'}}(\mathbb{R}^n).$$

Proof For $p\in(1,\infty)$, $q\in[1,\infty)$ and $w\in\mathcal{A}_p^{\rm loc}(\mathbb{R}^n)$, we denote by $L_w^p(\ell^q)(\mathbb{R}^n)$ the space of all sequences of functions $\{h_j\}_{j=0}^\infty$ such that $\|\{h_j\}_{j=0}^\infty\|_{L_w^p(\ell^q)(\mathbb{R}^n)}:=\left\|\left\{\sum_{j=0}^\infty|h_j|^q\right\}^{\frac1q}\right\|_{L_w^p(\mathbb{R}^n)}$ is finite. If $w=1$, then we simply write $L_w^p(\ell^q)(\mathbb{R}^n)$ as $L^p(\ell^q)(\mathbb{R}^n)$. By an argument similar to that used in the proof of $(L^p(\ell^q)(\mathbb{R}^n))^*=L^{p'}(\ell^{q'})(\mathbb{R}^n)$ (see [24, p. 177]), we conclude that

$$(L_w^p(\ell^q)(\mathbb{R}^n))^*=L_{w^{1-p'}}^{p'}(\ell^{q'})(\mathbb{R}^n).$$

Let $f\in F_{p',q'}^{-s,\,w^{-\frac{1}{p-1}}}(\mathbb{R}^n)$. For all $g\in F_{p,q}^{s,\,w}(\mathbb{R}^n)$, by Lemma 2.3, we have $g=\sum_{i=0}^\infty\psi_i*\phi_i*g$, where ψ_i, ϕ_i are as in Lemma 2.3. Without loss of generality, we may assume that $\{\phi_i\}_{i=1}^\infty$ has vanishing moments up to order $M>|s|$. From this and Hölder's inequality, it follows that

$$|\langle f,g\rangle|\le\sum_{i=0}^\infty|\langle\psi_i*f,\ \phi_i*g\rangle|\le\|f\|_{F_{p',q'}^{-s,\,w^{1-p'}}(\mathbb{R}^n)}\|g\|_{F_{p,q}^{s,\,w}(\mathbb{R}^n)}$$

with a usual modification made when $q=1$. Thus, $L_f(g):=\langle f,g\rangle$ induces a linear continuous functional on $F_{p,q}^{s,\,w}(\mathbb{R}^n)$ with $\|L_f\|\le\|f\|_{F_{p',q'}^{-s,\,w^{1-p'}}(\mathbb{R}^n)}$. Hence, $(F_{p,q}^{s,\,w}(\mathbb{R}^n))^*\supset F_{p',q'}^{-s,\,w^{1-p'}}(\mathbb{R}^n)$.

To show the converse, we assume that $L\in(F_{p,q}^{s,\,w}(\mathbb{R}^n))^*$. Since

$$f\in F_{p,q}^{s,\,w}(\mathbb{R}^n)\mapsto\{2^{js}\phi_j*f\}_{j=0}^\infty\in L_w^p(\ell^q)(\mathbb{R}^n)$$

is a one-to-one map from $F_{p,q}^{s,\,w}(\mathbb{R}^n)$ to a subspace of $L_w^p(\ell^q)(\mathbb{R}^n)$, it follows that the functional L can be interpreted as a functional on that subspace of $L_w^p(\ell^q)(\mathbb{R}^n)$. By the Hahn-Banach theorem, L can be extended to a continuous linear functional on $L^p_w(\ell^q)(\mathbb{R}^n)$ with the norm preserved, which is denoted by $\widetilde{L}$. By this and $(L^p_w(\ell^q)(\mathbb{R}^n))^* = L^{p'}_{w^{1-p'}}(\ell^{q'})(\mathbb{R}^n)$, we see that there exists $\{g_j\}_{j=0}^\infty \in L^{p'}_{w^{1-p'}}(\ell^{q'})(\mathbb{R}^n)$ such that, for all $f \in F^{s,\,w}_{p,q}(\mathbb{R}^n)$,

$$L(f) = \widetilde{L}(\{2^{js}\phi_j * f\}_{j=0}^\infty) = \sum_{j=0}^\infty 2^{js} \int_{\mathbb{R}^n} g_j(x)\phi_j * f(x)\mathrm{d}x = \sum_{j=0}^\infty 2^{js} \int_{\mathbb{R}^n} f(x)\widetilde{\phi}_j * g_j(x)\mathrm{d}x,$$

where $\widetilde{\phi}_j := \phi_j(-\cdot)$ for all $j \in \mathbb{Z}_+$ and $\|\{g_j\}_{j=0}^\infty\|_{L^{p'}_{w^{1-p'}}(\ell^{q'})(\mathbb{R}^n)} = \|\widetilde{L}\| = \|L\|$. If we let $g := \sum_{j=0}^\infty 2^{js}\phi_j * g_j$, then $L(f) = \int_{\mathbb{R}^n} g(x)f(x)\mathrm{d}x$ for all $f \in F^{s,\,w}_{p,q}(\mathbb{R}^n)$. Since $\{\phi_i\}_{i=0}^\infty$ have compact support and vanishing moments up to order $M > |s|$, it follows that, for all $x \in \mathbb{R}^n$,

$$|\phi_i * \phi_j * g_j(x)| \lesssim 2^{-|j-i|M} M^{\mathrm{loc}}_\kappa(g_j)(x)$$

with κ being a sufficiently large number depending on the support of $\{\phi_i\}_{i=0}^\infty$. Then,

$$\|g\|_{F^{-s,\,w^{1-p'}}_{p',q'}(\mathbb{R}^n)} \lesssim \Big\|\Big\{\sum_{i=0}^\infty \Big|\sum_{j=0}^\infty 2^{-|j-i|(M-|s|)} M^{\mathrm{loc}}_\kappa(g_j)\Big|^{q'}\Big\}^{\frac{1}{q'}}\Big\|_{L^{p'}_{w^{1-p'}}(\mathbb{R}^n)}.$$

By Hölder's inequality and $\sum_{j=0}^\infty 2^{-|j-i|(M-|s|)} \lesssim 1$, we see that the last term displayed above is bounded by

$$\Big\|\Big\{\sum_{i=0}^\infty \sum_{j=0}^\infty 2^{-|j-i|(M-|s|)} |M^{\mathrm{loc}}_\kappa(g_j)|^{q'}\Big\}^{\frac{1}{q'}}\Big\|_{L^{p'}_{w^{1-p'}}(\mathbb{R}^n)} \lesssim \Big\|\Big\{\sum_{j=0}^\infty |M^{\mathrm{loc}}_\kappa(g_j)|^{q'}\Big\}^{\frac{1}{q'}}\Big\|_{L^{p'}_{w^{1-p'}}(\mathbb{R}^n)}.$$

Finally, we apply (2.1) and Lemma 2.2(iv) to obtain

$$\|g\|_{F^{-s,\,w^{1-p'}}_{p',q'}(\mathbb{R}^n)} \lesssim \Big\|\Big\{\sum_{j=0}^\infty |g_j|^{q'}\Big\}^{\frac{1}{q'}}\Big\|_{L^{p'}_{w^{1-p'}}(\mathbb{R}^n)} \sim \|\{g_j\}_{j=0}^\infty\|_{L^{p'}_{w^{1-p'}}(\ell^{q'})(\mathbb{R}^n)} \sim \|L\|.$$

Hence, $(F^{s,\,w}_{p,q}(\mathbb{R}^n))^* \subset F^{-s,\,w^{1-p'}}_{p',q'}(\mathbb{R}^n)$, which completes the proof of Proposition 2.1.

3 Proof of Theorem 1.1

In this section, we show Theorem 1.1 by using the following estimate in [18, Lemma 2.9].

Lemma 3.1 *Assume that $\phi_0 \in \mathcal{D}(\mathbb{R}^n)$ has a nonzero integral. Let $\phi := \phi_0 - 2^{-n}\phi_0(2^{-1}\cdot)$. Then, for any $r \in (0,\infty)$, $A \ge 0$ and $B \ge 0$, there exists a positive constant C, depending only on n, r, ϕ_0, A and B, such that, for all $f \in \mathcal{S}'_e(\mathbb{R}^n)$, $j \ge 0$ and $x \in \mathbb{R}^n$,*

$$|\phi_j * f(x)| \le C\Big[\sum_{k=j}^\infty 2^{(j-k)Ar} 2^{kn} \int_{\mathbb{R}^n} \frac{|\phi_k * f(y)|^r}{(1+2^j|x-y|)^{Ar}\,\mathrm{e}^{Br|x-y|}}\mathrm{d}y\Big]^{\frac{1}{r}}.$$

Proof of Theorem 1.1 Since $S^s_{a,q}(f)(x) \le \widetilde{S}^s_{a,q}(f)(x)$ for all $x \in \mathbb{R}^n$, it follows that

$$\|S^s_{a,q}(f)\|_{L^p_w(\mathbb{R}^n)} \le \|\widetilde{S}^s_{a,q}(f)\|_{L^p_w(\mathbb{R}^n)}.$$

For all $A, B \in (0,\infty)$, $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$, by the definition of $\mathcal{S}'_e(\mathbb{R}^n)$, we obtain

$$\sup_{|x-y|<a2^{-k}} |\phi_k * f(y)| = \sup_{|y|<a2^{-k}} |\phi_k * f(x-y)| \le (1+a)^A 2^{aB} \sup_{y\in\mathbb{R}^n} \frac{|\phi_k * f(x-y)|}{(1+2^k|y|)^A\, 2^{B|y|}}.$$

Consequently,

$$\widetilde{S}_q^{a,s}(f)(x) \lesssim \left\{ \sum_{k=0}^{\infty} 2^{ksq} \left[\sup_{y\in\mathbb{R}^n} \frac{|\phi_k * f(x-y)|}{(1+2^k|y|)^A\, 2^{B|y|}} \right]^q \right\}^{\frac{1}{q}} \lesssim \left\{ \sum_{k=0}^{\infty} 2^{ksq} \left[\phi^*_{k,A,B} f(x) \right]^q \right\}^{\frac{1}{q}},$$

which, combined with (1.2), implies that $\|\widetilde{S}^s_{a,q}(f)\|_{L^p_w(\mathbb{R}^n)} \lesssim \|f\|_{F^{s,w}_{p,q}(\mathbb{R}^n)}$.

It remains to show that $\|f\|_{F^{s,w}_{p,q}(\mathbb{R}^n)} \lesssim \|S^s_{a,q}(f)\|_{L^p_w(\mathbb{R}^n)}$. To this end, we choose $r \in (0, \min\{p,q\})$, $A > \max\left\{\frac{n}{r} - s, \frac{n}{r}\right\}$ and $B > \frac{B_0}{p}$ (with B_0 as in Lemma 2.1). Then, from Lemma 3.1, we deduce that, for all $j \in \mathbb{Z}_+$ and $x \in \mathbb{R}^n$,

$$2^{js}|\phi_j * f(x)| \lesssim 2^{js} \left\{ \sum_{k=j}^{\infty} 2^{(j-k)Ar} 2^{2kn} \int_{|z|<a2^{-k}} \int_{\mathbb{R}^n} \frac{|\phi_k * f(y+z)|^r}{(1+2^j|x-y-z|)^{Ar} \mathrm{e}^{Br|x-y-z|}} \mathrm{d}y\mathrm{d}z \right\}^{\frac{1}{r}}.$$

Combining this with Fubini's theorem and the fact that, when $k \ge j \ge 0$ and $|z| \le a2^{-k}$,

$$\frac{1}{(1+2^j|x-y-z|)^{Ar}\mathrm{e}^{Br|x-y-z|}} \lesssim \frac{1}{(1+2^j|x-y|)^{Ar}\mathrm{e}^{Br|x-y|}},$$

we further conclude that, for all $x \in \mathbb{R}^n$,

$$2^{js}|\phi_j * f(x)| \lesssim \left\{ \sum_{k=j}^{\infty} 2^{(j-k)(A+s-\frac{n}{r})r} 2^{jn} \int_{\mathbb{R}^n} \frac{2^{kn}\int_{|z|<a2^{-k}} 2^{ksr}|\phi_k * f(y+z)|^r \mathrm{d}z}{(1+2^j|x-y|)^{Ar}\mathrm{e}^{Br|x-y|}} \mathrm{d}y \right\}^{\frac{1}{r}}.$$

Let

$$\Phi_{k,s,q}(f)(y) := \left[2^{kn} \int_{|z|<a2^{-k}} 2^{ksq} |\phi_k * f(y+z)|^q \mathrm{d}z \right]^{\frac{1}{q}}, \quad y \in \mathbb{R}^n.$$

Further, since $0 < r < \min\{p,q\}$ and $A + s - \frac{n}{r} > 0$, it follows, from Hölder's inequality, that

$$2^{js}|\phi_j * f(x)| \lesssim \left\{ \sum_{k=j}^{\infty} 2^{(j-k)(A+s-\frac{n}{r})r} \left[2^{jn} \int_{\mathbb{R}^n} \frac{[\Phi_{k,s,q}(f)(y)]^r}{(1+2^j|x-y|)^{Ar}\mathrm{e}^{Br|x-y|}} \mathrm{d}y \right]^{\frac{q}{r}} \right\}^{\frac{1}{q}}.$$

By $Ar > n$, we see that, for all $j \in \mathbb{Z}_+$ and $x \in \mathbb{R}^n$,

$$2^{jn} \int_{\mathbb{R}^n} \frac{[\Phi_{k,s,q}(f)(y)]^r}{(1+2^j|x-y|)^{Ar}\mathrm{e}^{Br|x-y|}} \mathrm{d}y \lesssim M^{\mathrm{loc}}(|\Phi_{k,s,q}(f)|^r)(x) + K_{Br}(|\Phi_{k,s,q}(f)|^r)(x),$$

which implies that

$$2^{js}|\phi_j * f(x)| \lesssim \left\{ \sum_{k=j}^{\infty} 2^{(j-k)(A+s-\frac{n}{r})r} [M^{\mathrm{loc}}(|\Phi_{k,s,q}(f)|^r)(x) + K_{Br}(|\Phi_{k,s,q}(f)|^r)(x)]^{\frac{q}{r}} \right\}^{\frac{1}{q}}.$$

Consequently, we have

$$\|f\|_{F^{s,w}_{p,q}(\mathbb{R}^n)} \lesssim \left\| \left\{ \sum_{k=0}^{\infty} [M^{\mathrm{loc}}(|\Phi_{k,s,q}(f)|^r)]^{\frac{q}{r}} \right\}^{\frac{1}{q}} + \left\{ \sum_{k=0}^{\infty} [K_{Br}(|\Phi_{k,s,q}(f)|^r)]^{\frac{q}{r}} \right\}^{\frac{1}{q}} \right\|_{L^p_w(\mathbb{R}^n)}.$$

Further, by $r < \min\{p,q\}$ and $B \ge \frac{B_0}{p}$, we apply (2.1) and (2.2) to obtain $\|f\|_{F^{s,w}_{p,q}(\mathbb{R}^n)} \lesssim \|S^s_{a,q}(f)\|_{L^p_w(\mathbb{R}^n)}$. This concludes the proof of Theorem 1.1.

4 Proof of Theorem 1.2

To prove Theorem 1.2, we need to establish a series of auxiliary lemmas.

Lemma 4.1 *Let* $s \in (-\infty, 0)$, $p \in (0, 1]$, $q \in [p, \infty)$ *and* $w \in A^{\rm loc}_{\max\{q,1\}}(\mathbb{R}^n)$. *Assume that*
(i) $\phi_0 \in \mathcal{D}(\mathbb{R}^n)$ *such that* $\operatorname{supp}(\phi_0) \subset B(0, 1)$;
(ii) *for any* $z \in \mathbb{R}^n$, $\phi(z) := \phi_0(z) - 2^{-n}\phi_0(2^{-1}z)$;
(iii) $a \in \mathcal{S}'_e(\mathbb{R}^n)$ *is a* $(p, q, s)_w$*-atom supported on a cube* Q, *with center* c_Q *and side length* $\ell(Q) \le \frac{1}{\sqrt{n}}$.
Then, for all $k \in \mathbb{Z}_+$, $\operatorname{supp}(\phi_k * a) \subset B(c_Q, 3)$. *Moreover, for all* $k \in \mathbb{Z}_+$ *and* $x \in B(c_Q, 3)$,

$$|\phi_k * a(x)| \le C 2^{kn+k(\mathcal{N}+1)}[w(Q)]^{-\frac{1}{p}}|Q|^{\min\{1,\frac{1}{q}\}+\frac{s+\mathcal{N}+1}{n}},$$

where C *is a positive constant independent of* k, a *and* x.

Proof It is easy to see that every $\operatorname{supp}(\phi_k * a) \subset B(c_Q, 3)$. Since $\ell(Q) \le \frac{1}{\sqrt{n}}$, it follows that there exists $i_0 \in \mathbb{Z}_+$ such that $2^{-i_0-1} < \sqrt{n}\ell(Q) \le 2^{-i_0}$. By Definition 1.2, the $(p, q, s)_w$-atom a has vanishing moments up to order $\mathcal{N} := \lfloor n\bigl(\frac{q_w}{p} - 1\bigr) - s\rfloor$. From the Calderón reproducing formula (2.3), it follows that there exist functions ψ_0,$\psi \in \mathcal{D}(\mathbb{R}^n)$ such that ψ_0 has a nonzero integral and ψ has vanishing moments up to order $\mathcal{N}$, and

$$a = (\psi_0)_{i_0} * (\phi_0)_{i_0} * a + \sum_{i=i_0+1}^{\infty} \psi_i * \phi_i * a \quad \text{in } \mathcal{D}'(\mathbb{R}^n). \tag{4.1}$$

Let $P_{\mathcal{N}}(\phi_k)$ be the Taylor polynomial given by that, for all $y, z \in \mathbb{R}^n$,

$$P_{\mathcal{N}}(\phi_k)(z; y) := \sum_{\substack{\gamma\in\mathbb{Z}^n_+\\ |\gamma|\le\mathcal{N}}} c_\gamma (c_Q - y)^\gamma (D^\gamma \phi_k)(z - c_Q),$$

where $\{c_\gamma\}_\gamma$ are coefficients. For any $y, z \in \mathbb{R}^n$, set $\Phi_{k,z}(y) := \phi_k(z - y) - P_{\mathcal{N}}(\phi_k)(z; y)$. Let η_Q be the smooth cutoff function associated to the cube Q as defined in Definition 1.2. For all $k \in \mathbb{Z}_+$ and $z \in \mathbb{R}^n$, by the vanishing moment condition of a, we have $\phi_k * a(z) = \langle a, \phi_k(z - \cdot)\eta_Q\rangle = \langle a, \Phi_{k,z}\eta_Q\rangle$. From this and (4.1), it follows that, for all $k \in \mathbb{Z}_+$ and $z \in \mathbb{R}^n$,

$$\phi_k * a(z) = \langle(\phi_0)_{i_0} * a,\ \widetilde{(\psi_0)_{i_0}} * (\Phi_{k,z}\eta_Q)\rangle + \sum_{i=i_0+1}^{\infty} \langle \phi_i * a,\ \widetilde{\psi}_i * (\Phi_{k,z}\eta_Q)\rangle,$$

where we used the notation $\widetilde{\varphi}(u) := \varphi(-u)$ for any function φ and $u \in \mathbb{R}^n$. By the choice of i_0 and the support conditions of ϕ and a, we conclude that $\operatorname{supp}((\phi_0)_{i_0} * a) \subset B(c_Q, 3\sqrt{n}\ell(Q))$ and $\operatorname{supp}(\phi_i * a) \subset B(c_Q, 3\sqrt{n}\ell(Q))$ for $i > i_0$. Thus, for all $k \in \mathbb{Z}_+$ and $z \in \mathbb{R}^n$,

$$\begin{aligned}\phi_k * a(z) &= \int_{B(c_Q,3\sqrt{n}\ell(Q))} (\phi_0)_{i_0} * a(y)\, \widetilde{(\psi_0)_{i_0}} * (\Phi_{k,z}\eta_Q)(y)\mathrm{d}y\\ &\quad + \sum_{i=i_0+1}^{\infty} \int_{B(c_Q,3\sqrt{n}\ell(Q))} \phi_i * a(y)\, \widetilde{\psi}_i * (\Phi_{k,z}\eta_Q)(y)\mathrm{d}y.\end{aligned} \tag{4.2}$$

For all multi-indices $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| \le \mathcal{N}$, the mean value theorem further implies that, for all $y \in B(c_Q, 3\sqrt{n}\ell(Q))$ and $z \in \mathbb{R}^n$,

$$|D^\alpha_y \Phi_{k,z}(y)| \lesssim 2^{kn+k(\mathcal{N}+1)}\ell(Q)^{\mathcal{N}-|\alpha|+1}. \tag{4.3}$$

From (4.3), it further follows that

$$|D_y^\alpha(\Phi_{k,z}\eta_Q)(y)| \lesssim \sum_{\substack{\beta\in\mathbb{Z}_+^n\\ \beta\le\alpha}} |D_y^\beta \Phi_{k,z}(y)| \lesssim 2^{kn+k(\mathcal{N}+1)}. \tag{4.4}$$

By (4.3) and $\ell(Q)\sim 2^{-i_0}$, we know that, for all $y\in B(c_Q, 3\sqrt{n}\ell(Q))$,

$$|\widetilde{(\psi_0)_{i_0}} * (\Phi_{k,z}\eta_Q)(y)| \lesssim 2^{kn+(k-i_0)(\mathcal{N}+1)}. \tag{4.5}$$

Since $\int_{\mathbb{R}^n}\psi(x)x^\alpha \mathrm{d}x=0$ for all $|\alpha|\le\mathcal{N}$, we see that, for all $i\ge i_0+1$ and $y\in B(c_Q,3\sqrt{n}\ell(Q))$,

$$\begin{aligned}&|\widetilde{\psi}_i * (\Phi_{k,z}\eta_Q)(y)|\\ &= \Big|\int_{\mathbb{R}^n}\psi_i(u-y)\Big[\Phi_{k,z}(u)\eta_Q(u)-\sum_{|\alpha|\le\mathcal{N}} c_\alpha(u-y)^\alpha D_y^\alpha(\Phi_{k,z}\eta_Q)(y)\Big]\mathrm{d}u\Big|.\end{aligned} \tag{4.6}$$

By the mean value theorem and (4.4), we see that the quality inside the bracket of the second line of (4.6) is dominated by

$$\sup_{\theta\in[0,1]}\sum_{|\alpha|=\mathcal{N}+1}|(u-y)^\alpha D^\alpha(\Phi_{k,z}\eta_Q)(\theta u+(1-\theta)y)| \lesssim |u-y|^{\mathcal{N}+1}2^{k(n+\mathcal{N}+1)}.$$

Inserting this into (4.6) gives that, when $i>i_0$,

$$|\widetilde{\psi}_i * (\Phi_{k,z}\eta_Q)(y)| \lesssim 2^{k(n+\mathcal{N}+1)}\int_{\mathbb{R}^n}|\psi_i(u-y)||u-y|^{\mathcal{N}+1}\mathrm{d}y \lesssim 2^{kn+(k-i)(\mathcal{N}+1)}. \tag{4.7}$$

Applying (4.2), (4.5) and (4.7), we conclude that, for all $k\in\mathbb{Z}_+$ and $z\in\mathbb{R}^n$,

$$\begin{aligned}|\phi_k * a(z)| &\lesssim 2^{kn+(k-i_0)(\mathcal{N}+1)}\int_{B(c_Q,3\sqrt{n}\ell(Q))}|(\phi_0)_{i_0} * a(y)|\mathrm{d}y\\ &\quad+\sum_{i=i_0+1}^\infty 2^{kn+(k-i)(\mathcal{N}+1)}\int_{B(c_Q,3\sqrt{n}\ell(Q))}|\phi_i * a(y)|\mathrm{d}y.\end{aligned} \tag{4.8}$$

Now we consider the following two cases.

Case 1 $q\in[1,\infty)$. Taking $z:=x\in B(c_Q,3)$ in (4.8) and applying the fact $(\phi_0)_{i_0}=\sum_{i=0}^{i_0}\phi_i$, we see that

$$\begin{aligned}|\phi_k * a(x)| &\lesssim 2^{kn+(k-i_0)(\mathcal{N}+1)}\int_{B(c_Q,3\sqrt{n}\ell(Q))}\sum_{i=0}^{i_0}|2^{is}\phi_i * a(y)|2^{-is}\,\mathrm{d}y\\ &\quad+2^{kn+k(\mathcal{N}+1)}\int_{B(c_Q,3\sqrt{n}\ell(Q))}\sum_{i=i_0+1}^\infty|2^{is}\phi_i * a(y)|2^{-i(s+\mathcal{N}+1)}\mathrm{d}y.\end{aligned}$$

Further, if we apply Hölder's inequality to each term on the right-hand side of the above inequality, and use the facts $s<0$ and $s+\mathcal{N}+1>0$, then

$$|\phi_k * a(x)| \lesssim 2^{kn+k(\mathcal{N}+1)}2^{-i_0(s+\mathcal{N}+1)}\int_{B(c_Q,3\sqrt{n}\ell(Q))}\Big\{\sum_{i=0}^\infty|2^{is}\phi_i * a(y)|^q\Big\}^{\frac1q}\mathrm{d}y. \tag{4.9}$$

Using Hölder's inequality and $w \in \mathcal{A}_q^{\mathrm{loc}}(\mathbb{R}^n)$, we further have

$$\int_{B(c_Q, 3\sqrt{n}\ell(Q))} \Big\{ \sum_{i=0}^{\infty} |2^{is}\phi_i * a(y)|^q \Big\}^{\frac{1}{q}} \mathrm{d}y \lesssim [w(Q)]^{-\frac{1}{p}} |Q|.$$

Inserting this estimate into (4.9), we conclude that, for all $x \in B(c_Q, 3)$,

$$|\phi_k * a(x)| \lesssim 2^{kn+k(\mathcal{N}+1)} [w(Q)]^{-\frac{1}{p}} |Q|^{1+\frac{s+\mathcal{N}+1}{n}},$$

which is the desired conclusion.

Case 2 $p \in (0,1)$ and $q \in [p,1)$. In this case, $w \in \mathcal{A}_1^{\mathrm{loc}}(\mathbb{R}^n)$ and $\mathcal{N} = \lfloor n(\frac{1}{p} - 1) - s \rfloor$. Since $w \in \mathcal{A}_1^{\mathrm{loc}}(\mathbb{R}^n)$, it follows that the ball $B(c_Q, 3\sqrt{n}\ell(Q))$ is covered by a finite number of (depending only on n) smaller cubes $\{\widetilde{Q}\}$ such that each $\widetilde{Q}$ has the same side length as that of Q and hence $w(\widetilde{Q}) \sim w(Q)$ by Lemma 2.2(ii), which further gives that

$$\sup_{y \in B(c_Q, 3\sqrt{n}\ell(Q))} |w^{-1}(y)| \le \sum_{\widetilde{Q}} \sup_{y \in \widetilde{Q}} |w^{-1}(y)| \lesssim \sum_{\widetilde{Q}} \frac{|\widetilde{Q}|}{w(\widetilde{Q})} \lesssim \frac{|Q|}{w(Q)}. \tag{4.10}$$

From (4.8), (4.10) and the fact $(\phi_0)_{i_0} = \sum_{i=0}^{i_0} \phi_i$, together with $s < 0$, we deduce that, for all $k \in \mathbb{Z}_+$ and $z \in \mathbb{R}^n$,

$$\begin{aligned} |2^{ks}\phi_k * a(z)| \lesssim{}& \frac{|Q|}{w(Q)} 2^{kn} 2^{(k-i_0)(s+\mathcal{N}+1)} \int_{B(c_Q, 3\sqrt{n}\ell(Q))} \sum_{i=0}^{i_0} |2^{is}\phi_i * a(y)| w(y) \mathrm{d}y \\ &+ \frac{|Q|}{w(Q)} \sum_{i=i_0+1}^{\infty} 2^{kn} 2^{(k-i)(s+\mathcal{N}+1)} \int_{B(c_Q, 3\sqrt{n}\ell(Q))} |2^{is}\phi_i * a(y)| w(y) \mathrm{d}y, \end{aligned}$$

which, together with $n + s + \mathcal{N} + 1 > 0$ and $q < 1$, gives that

$$\begin{aligned} &2^{-k(n+s+\mathcal{N}+1)} |2^{ks}\phi_k * a(z)| \\ &\lesssim \frac{|Q|}{w(Q)} 2^{i_0 n} 2^{-i_0(n+s+\mathcal{N}+1)q} \int_{B(c_Q, 3\sqrt{n}\ell(Q))} 2^{-i_0(n+s+\mathcal{N}+1)(1-q)} \sum_{i=0}^{i_0} |2^{is}\phi_i * a(y)| w(y)\, \mathrm{d}y \\ &\quad + \frac{|Q|}{w(Q)} \sum_{i=i_0+1}^{\infty} 2^{in} 2^{-i(n+s+\mathcal{N}+1)q} \int_{B(c_Q, 3\sqrt{n}\ell(Q))} 2^{-i(n+s+\mathcal{N}+1)(1-q)} |2^{is}\phi_i * a(y)| w(y) \mathrm{d}y. \end{aligned}$$

Since $q \ge p$ and $\mathcal{N} + 1 > n(\frac{1}{p} - 1) - s$, we have $(n+s+\mathcal{N}+1)q - n \ge (n+s+\mathcal{N}+1)p - n > 0$ and hence, for all $i \ge i_0$, $2^{in} 2^{-i(n+s+\mathcal{N}+1)q} \le |Q|^{[1+\frac{s+\mathcal{N}+1}{n}]q-1}$ by $2^{-i_0} \sim \ell(Q)$. Combining this with $B(c_Q, 3\sqrt{n}\ell(Q)) \subset B(c_Q, 3)$, we further know that, for all $k \in \mathbb{Z}_+$ and $z \in \mathbb{R}^n$,

$$\begin{aligned} &2^{-k(n+s+\mathcal{N}+1)} |2^{ks}\phi_k * a(z)| \\ &\lesssim \frac{|Q|^{[1+\frac{s+\mathcal{N}+1}{n}]q}}{w(Q)} \sum_{i=0}^{\infty} \int_{B(c_Q, 3)} 2^{-i(n+s+\mathcal{N}+1)(1-q)} |2^{is}\phi_i * a(y)| w(y) \mathrm{d}y. \end{aligned} \tag{4.11}$$

Let $\mathbb{A} := \sup_{k \in \mathbb{Z}_+} \sup_{z \in B(c_Q, 3)} 2^{-k(n+\mathcal{N}+1)} |\phi_k * a(z)|$ and $\phi^*_{k,A,B}(a)$ be the Peetre-type maximal function as defined in (1.3), where $A > A_0$ and $B > \frac{B_0}{q}$. By the facts that $w \in \mathcal{A}_1^{\mathrm{loc}}(\mathbb{R}^n)$ and

$n+\mathcal{N}+1+s>0$, together with $a\in F^{s,w}_{q,q}(\mathbb{R}^n)$ and (1.2), we see that

$$\mathbb{A}\le \frac{1}{[w(B(c_Q,3))]^{\frac{1}{q}}}\Big\{\sum_{k=0}^{\infty}\int_{\mathbb{R}^n}|2^{ks}\phi^*_{k,A,B}(x)|^q w(x)\mathrm{d}x\Big\}^{\frac{1}{q}}<\infty.$$

Since $\mathbb{A}$ is finite, by (4.11), we obtain

$$\mathbb{A}\lesssim \mathbb{A}^{1-q}\frac{|Q|^{q[1+\frac{s+\mathcal{N}+1}{n}]}}{w(Q)}\|a\|^q_{F^{s,w}_{q,q}}\lesssim \mathbb{A}^{1-q}|Q|^{q[1+\frac{s+\mathcal{N}+1}{n}]}w(Q)^{-\frac{q}{p}},$$

that is, $\mathbb{A}\lesssim [w(Q)]^{-\frac{1}{p}}|Q|^{1+\frac{s+\mathcal{N}+1}{n}}$. From this and the definition of $\mathbb{A}$, we deduce that, for all $k\in\mathbb{Z}_+$ and $x\in B(c_Q,3)$,

$$|2^{ks}\phi_k * a(x)|\lesssim 2^{k(n+\mathcal{N}+1)}[w(Q)]^{-\frac{1}{p}}|Q|^{1+\frac{s+\mathcal{N}+1}{n}},$$

which completes the proof of Lemma 4.1.

Lemma 4.2 *If $s\in\mathbb{R}$, $p,q\in(0,\infty)$ and $w\in\mathcal{A}^{\mathrm{loc}}_{\infty}(\mathbb{R}^n)$, then there exists a positive constant C such that, for all $f\in\mathcal{S}'_e(\mathbb{R}^n)$,*

$$\frac{1}{C}\|f\|_{F^{s,w}_{p,q}(\mathbb{R}^n)}\le\|f\|_{F^{s-1,w}_{p,q}(\mathbb{R}^n)}+\sum_{j=1}^{n}\|D_jf\|_{F^{s-1,w}_{p,q}(\mathbb{R}^n)}\le C\|f\|_{F^{s,w}_{p,q}(\mathbb{R}^n)}, \tag{4.12}$$

where $D_j:=\frac{\partial}{\partial x_j}$ for $j\in\{1,\cdots,n\}$.

Proof Fix $p,q\in(0,\infty)$ and $w\in\mathcal{A}^{\mathrm{loc}}_{\infty}(\mathbb{R}^n)$. For $m\in\mathbb{Z}_+$, it was proved by Rychkov [18, Theorem 2.20] that, for all $f\in\mathcal{S}'_e(\mathbb{R}^n)$,

$$\|f\|_{F^{m,w}_{p,q}(\mathbb{R}^n)}\sim\sum_{|\alpha|\le m}\|D^{\alpha}f\|_{F^{0,w}_{p,q}(\mathbb{R}^n)}, \tag{4.13}$$

where, for $\alpha:=(\alpha_1,\cdots,\alpha_n)\in\mathbb{Z}^n_+$, $D^{\alpha}:=\big(\frac{\partial}{\partial x_1}\big)^{\alpha_1}\cdots\big(\frac{\partial}{\partial x_n}\big)^{\alpha_n}$.

The proof for (4.12) is an easy adaption of (4.13) and the following lifting property in [18, Theorem 2.18]: There exists $t\in(0,\infty)$ sufficiently small, depending on p,n and w, such that, for all $a\in\mathbb{R}$ and $f\in\mathcal{S}'_e(\mathbb{R}^n)$,

$$\|\mathcal{J}^a_t f\|_{F^{s+a,w}_{p,q}(\mathbb{R}^n)}\sim\|f\|_{F^{s,w}_{p,q}(\mathbb{R}^n)}, \tag{4.14}$$

where $\mathcal{J}^t_a$ denotes the Bessel potential operator $\mathcal{J}^t_a:=(\mathrm{id}-t^2\Delta)^{-\frac{a}{2}}$, where id is the identity operator. Precisely, for any $s\in\mathbb{R}$, we choose $m\in\mathbb{Z}$ such that $m\le|s|<m+1$, and then applying (4.13)–(4.14) yields

$$\|f\|_{F^{s,w}_{p,q}(\mathbb{R}^n)}\sim\|\mathcal{J}^{1-s}_t f\|_{F^{1,w}_{p,q}(\mathbb{R}^n)}\sim\sum_{|\alpha|\le 1}\|D^{\alpha}(\mathcal{J}^{1-s}_t f)\|_{F^{0,w}_{p,q}(\mathbb{R}^n)}\sim\sum_{|\alpha|\le 1}\|D^{\alpha}f\|_{F^{s-1,w}_{p,q}(\mathbb{R}^n)},$$

which implies (4.12). This finishes the proof of the lemma.

Lemma 4.3 *Let $s\in\mathbb{R}$, $p\in(0,1]$, $q\in[p,\infty)$ and $w\in\mathcal{A}^{\mathrm{loc}}_{\max\{q,1\}}(\mathbb{R}^n)$. Then, there exists a positive constant C, depending only on s,p,q,n and w, such that, for all $(p,q,s)_w$-atoms a,*

$$\|a\|_{F^{s,w}_{p,q}(\mathbb{R}^n)}\le C. \tag{4.15}$$

Proof Let a be a $(p,q,s)_w$-atom supported on a cube Q with center c_Q and side length $\ell(Q)$. If $p=q$, then (4.15) follows trivially from Definition 1.2(ii), so it suffices to show (4.15) for $p<q$. To this end, we choose $\phi_0\in\mathcal{D}(\mathbb{R}^n)$ such that $\operatorname{supp}(\phi_0)\subset B(0,1)$, $\phi_0(x)=1$ when $|x|<\frac{1}{2}$, and $0\le\phi_0\le 1$, and let $\phi:=\phi_0-2^{-n}\phi_0(2^{-1}\cdot)$. First, we show that (4.15) holds when $s<0$ by considering the following two cases for $\ell(Q)$.

Case 1 $\frac{1}{\sqrt{n}}\le\ell(Q)$ and $s<0$. For any $k\in\mathbb{Z}_+$, we observe that $\operatorname{supp}(\phi_k)\subset B(0,2^{-k+1})$, which gives that $\operatorname{supp}(\phi_k*a)\subset Q(c_Q,\ell(Q)+4)$ by using $\operatorname{supp}a\subset Q$. Here and in what follows, we use $Q(x,r)$ to denote the cube in $\mathbb{R}^n$ with center $x\in\mathbb{R}^n$ and side length $r\in(0,\infty)$. From this and Hölder's inequality, together with Lemma 2.2(ii), it follows that

$$\|a\|_{F^{s,w}_{p,q}(\mathbb{R}^n)}\lesssim[w(Q)]^{\frac{1}{p}-\frac{1}{q}}\|a\|_{F^{s,w}_{q,q}(\mathbb{R}^n)}\lesssim 1.$$

This proves (4.15) for the case $\frac{1}{\sqrt{n}}\le\ell(Q)$.

Case 2 $\ell(Q)<\frac{1}{\sqrt{n}}$ and $s<0$. In this case, for any $k\in\mathbb{Z}_+$, by $\operatorname{supp}(\phi_k)\subset B(0,2^{-k+1})$ and $\operatorname{supp}a\subset Q$, we have $\operatorname{supp}(\phi_k*a)\subset B(c_Q,2+\sqrt{n}\ell(Q))\subset B(c_Q,3)$, which implies that

$$\begin{aligned}\|a\|_{F^{s,w}_{p,q}(\mathbb{R}^n)}&\lesssim\Big\{\int_{4\sqrt{n}Q}\Big[\sum_{k=0}^{\infty}2^{ksq}|\phi_k*a(x)|^q\Big]^{\frac{p}{q}}w(x)\mathrm{d}x\Big\}^{\frac{1}{p}}+\Big\{\int_{B(c_Q,3)\setminus 4\sqrt{n}Q}\cdots\Big\}^{\frac{1}{p}}\\&=:\mathrm{Z}_1+\mathrm{Z}_2.\end{aligned}$$

Following the argument used in Case 1, we apply Hölder's inequality and $w(4\sqrt{n}Q)\sim w(Q)$ to deduce that $\mathrm{Z}_1\lesssim 1$.

Now we estimate Z_2. If $x\in B(c_Q,3)\setminus 4\sqrt{n}Q$ such that $\phi_k*a(x)\ne 0$, then by $\phi_k*a(x)=\langle a,\phi_k(x-\cdot)\eta_Q\rangle$, we have $2\sqrt{n}\ell(Q)\le|x-c_Q|<\sqrt{n}\ell(Q)+2^{-k+1}$, which implies that $|x-c_Q|<2^{-k+2}$, that is, $k\le\lfloor\log_2\frac{4}{|x-c_Q|}\rfloor$. By this and Lemma 4.1, we see that

$$\mathrm{Z}_2\lesssim[w(Q)]^{-\frac{1}{p}}\Big\{\int_{\mathbb{R}^n}[M^{\mathrm{loc}}_\kappa(\chi_Q)(x)]^{\frac{p(s+n+\mathcal{N}+1)}{n}}w(x)\mathrm{d}x\Big\}^{\frac{1}{p}},$$

where κ is a positive constant independent of a and x. Since $\frac{p(s+n+\mathcal{N}+1)}{n}>q_w$, by the weighted boundedness properties of M_κ we further know that

$$\mathrm{Z}_2\lesssim[w(Q)]^{-\frac{1}{p}}\Big\{\int_{\mathbb{R}^n}[\chi_Q(x)]^{\frac{p(s+n+\mathcal{N}+1)}{n}}w(x)\mathrm{d}x\Big\}^{\frac{1}{p}}\lesssim 1.$$

Combining the estimates of Z_1 and Z_2, we see that (4.15) holds when $\ell(Q)\le\frac{1}{\sqrt{n}}$ and $s<0$.

Based on the conclusions in Cases 1–2, we know that (4.15) holds when $s<0$. The general case of (4.15) can be reduced to the case $s<0$ by using Lemma 4.2. Indeed, if $0\le s<1$ and a is a $(p,q,s)_w$-atom, then by Lemma 4.2, we see that

$$\|a\|_{F^{s,w}_{p,q}(\mathbb{R}^n)}\sim\|a\|_{F^{s-1,w}_{p,q}(\mathbb{R}^n)}+\sum_{j=1}^{n}\|D_ja\|_{F^{s-1,w}_{p,q}(\mathbb{R}^n)}. \tag{4.16}$$

If we can show that a and $\{D_ja\}_{j=1}^n$ are constant multiples of $(p,q,s-1)_w$-atoms, then applying the already proved conclusion that (4.15) holds for $s<0$ yields that $\|a\|_{F^{s-1,w}_{p,q}(\mathbb{R}^n)}\lesssim 1$ and $\|D_ja\|_{F^{s-1,w}_{p,q}(\mathbb{R}^n)}\lesssim 1$, which, combined with (4.16), gives that $\|a\|_{F^{s,w}_{p,q}(\mathbb{R}^n)}\lesssim 1$, that is, (4.15) holds for all $0\le s<1$. Repeating the above process yields (4.15) for all $s\in\mathbb{R}$.

To finish the proof of Lemma 4.3, we still need to show that if a is a $(p,q,s)_w$-atom, then a and $\{D_j a\}_{j=1}^n$ are constant multiples of $(p,q,s-1)_w$-atoms. It is obvious, from Definition 1.1, that $\|a\|_{F^{s-1,w}_{q,q}(\mathbb{R}^n)} \le \|a\|_{F^{s,w}_{q,q}(\mathbb{R}^n)}$, and hence a is a $(p,q,s-1)_w$-atom. Likewise, every $D_j a$ is a constant multiple of a $(p,q,s-1)_w$-atom provided that we can prove that

$$\|D_j a\|_{F^{s-1,w}_{q,q}(\mathbb{R}^n)} \lesssim \|a\|_{F^{s,w}_{q,q}(\mathbb{R}^n)}. \tag{4.17}$$

To show (4.17), we first shift the differential from a to ϕ, namely, $|D_j a * \phi_0| = |a * D_j\phi_0|$ and $|D_j a * \phi_i| = 2^i |a * (D_j\phi)_i|$ for $i \in \mathbb{N}$, and then use the Calderón reproducing formula (see Lemma 2.3) and the fact that, for all $k, i \in \mathbb{Z}_+$ and $x \in \mathbb{R}^n$,

$$|(D_j\phi)_i * \psi_k(x)| \lesssim 2^{(\min\{k,i\})n} 2^{-|k-i|L} \chi_{B(0,C2^{-\min\{k,i\}})}(x),$$

where $L \in (0,\infty)$ can be sufficiently large and C is a positive constant depending on the supports of ϕ and ψ. Based on these facts, (4.17) follows from a standard calculation (see the proof of [18, Theorem 2.5]) and we thus omit the details.

Summarizing all the above arguments, we complete the proof of Lemma 4.3.

Proof of Theorem 1.2 Fix $s \in \mathbb{R}$, $p \in (0,1]$, $q \in [p,\infty)$ and $w \in A^{\rm loc}_{\max\{q,1\}}(\mathbb{R}^n)$. If there exist $(p,q,s)_w$-atoms $\{a_k\}_{k\in\mathbb{N}}$ and $\{\lambda_k\}_{k\in\mathbb{Z}} \in \ell^p$ such that $f = \sum_{k\in\mathbb{N}} \lambda_k a_k$ in $\mathcal{S}'_e(\mathbb{R}^n)$, then we apply Lemma 4.3 to obtain $\|f\|^p_{F^{s,w}_{p,q}(\mathbb{R}^n)} \le \sum_{k\in\mathbb{N}} |\lambda_k|^p \|a_k\|^p_{F^{s,w}_{p,q}(\mathbb{R}^n)} \lesssim \sum_{k\in\mathbb{N}} |\lambda_k|^p$. This inequality implies that $\|f\|_{F^{s,w}_{p,q}(\mathbb{R}^n)} \lesssim \inf\{[\sum_{k\in\mathbb{N}} |\lambda_k|^p]^{\frac{1}{p}}\}$, where the infimum is taken over all the atomic decompositions of f as above.

To show the converse part, fix $f \in F^{s,w}_{p,q}(\mathbb{R}^n)$. Assume that $\phi_0 \in \mathcal{D}(\mathbb{R}^n)$ has a nonzero integral, $0 \le \phi_0 \le 1$, $\operatorname{supp}(\phi_0) \subset B(0,\frac{1}{2})$, $\phi_0(x) = 1$ whenever $|x| \le \frac{1}{4}$, and that $\phi := \phi_0 - 2^{-n}\phi_0(2^{-1}\cdot)$ satisfies (1.1). For all $x \in \mathbb{R}^n$, define

$$\widetilde{S}^s_q(f)(x) := \left\{\sum_{k=0}^\infty \sup_{|x-y|<\sqrt{n}2^{-k}} |2^{ks}\phi_k * f(y)|^q\right\}^{\frac{1}{q}}.$$

Then, by Theorem 1.1, we have $\|\widetilde{S}^s_q(f)\|_{L^p_w(\mathbb{R}^n)} \sim \|f\|_{F^{s,w}_{p,q}(\mathbb{R}^n)} < \infty$. Denote by $\mathcal{Q}$ the collection of all dyadic cubes of $\mathbb{R}^n$. For any $k \in \mathbb{Z}$, let $\Omega_k := \{x \in \mathbb{R}^n : \widetilde{S}^s_q(f)(x) > 2^k\}$ and

$$\Lambda_k := \left\{Q \in \mathcal{Q} : |Q| \le 1,\ w(Q\cap\Omega_k) \ge \frac{1}{2}w(Q) \text{ and } w(Q\cap\Omega_{k+1}) < \frac{1}{2}w(Q)\right\}. \tag{4.18}$$

Notice that Λ_k might be empty. It is easy to see that, for each $Q \in \mathcal{Q}$ with $|Q| \le 1$, there exists a unique $k \in \mathbb{Z}$ such that $Q \in \Lambda_k$. A dyadic cube $Q \in \Lambda_k$ is called a maximal dyadic cube in Λ_k if, for any dyadic cube $\widetilde{Q} \in \Lambda_k$, either $\widetilde{Q} \subset Q$ or $\widetilde{Q} \cap Q = \varnothing$. For each $k \in \mathbb{N}$, denote by $\{Q^i_k\}_{i\in I_k}$ the collection of all such maximal dyadic cubes in Λ_k, where I_k is the index set (which might be empty). Observe that $\{Q^i_k\}_{i\in I_k}$ are mutually disjoint. Moreover,

$$\{Q \in \mathcal{Q} : |Q| \le 1\} = \bigcup_{k\in\mathbb{Z}} \Lambda_k = \bigcup_{k\in\mathbb{Z}} \bigcup_{i\in I_k} \{Q \in \Lambda_k : Q \subset Q^i_k\}.$$

Given any $L \in \mathbb{Z}_+$ sufficiently large (to be determined later), by Lemma 2.3, we see that

$$f(x) = \sum_{\ell=0}^\infty \psi_\ell * \phi_\ell * f(x), \quad x \in \mathbb{R}^n, \tag{4.19}$$

where $\psi_0 \in \mathcal{D}(\mathbb{R}^n)$ has a nonzero integral and $\psi \in \mathcal{D}(\mathbb{R}^n)$ satisfies that $\int_{\mathbb{R}^n} x^\alpha \psi(x)\mathrm{d}x = 0$ for all multi-indices α with $|\alpha| \le L$. Without loss of generality, we may assume that both ψ_ℓ and ϕ_ℓ are supported on $B(0, 2^{-\ell})$ for all $\ell \in \mathbb{Z}_+$. For each $Q \in \mathcal{Q}$ with $|Q| \le 1$, there exists some $\ell \in \mathbb{Z}_+$ such that $\ell(Q) = 2^{-\ell}$, and then we define

$$\psi_Q(\cdot) := \psi_\ell(\cdot) = 2^{\ell n}\psi(2^\ell \cdot), \quad \phi_Q(\cdot) := \phi_\ell(\cdot) = 2^{\ell n}\phi(2^\ell \cdot). \tag{4.20}$$

With these notations, we rewrite (4.19) as

$$f(x) = \sum_{k\in\mathbb{Z}} \sum_{i \in I_k} \sum_{\substack{Q\subset Q_k^i\\ Q\in\Lambda_k}} \int_Q \psi_Q(x-y)(\phi_Q * f)(y)\mathrm{d}y, \quad x \in \mathbb{R}^n. \tag{4.21}$$

For each $k \in \mathbb{Z}$ and $i \in I_k$, let

$$\lambda_{k,\,i} := w(Q_k^i)^{\frac{1}{p}-\frac{1}{q}} \Big\{ \sum_{\substack{Q\subset Q_k^i\\ Q\in\Lambda_k}} w(Q)[\ell(Q)]^{-sq} \sup_{y\in Q} |(\phi_Q * f)(y)|^q \Big\}^{\frac{1}{q}}$$

and, for all $x \in \mathbb{R}^n$.

$$a_{k,\,i}(x) := \frac{1}{\lambda_{k,\,i}} \sum_{\substack{Q\subset Q_k^i\\ Q\in\Lambda_k}} \int_Q \psi_Q(x-y)(\phi_Q * f)(y)\mathrm{d}y.$$

Based on (4.21), we see that $f = \sum_{k\in\mathbb{Z}} \sum_{i\in I_k} \lambda_{k,\,i} a_{k,\,i}$ is the desired atomic decomposition of f, provided that we can show that every $a_{k,i}$ is a $(p, q, s)_w$-atom and

$$\Big\{ \sum_{k\in\mathbb{Z}} \sum_{i\in I_k} |\lambda_{k,\,i}|^p \Big\}^{\frac{1}{p}} \lesssim \|f\|_{F^{s,w}_{p,q}(\mathbb{R}^n)}. \tag{4.22}$$

To prove (4.22), we first show that, for all $k \in \mathbb{Z}$,

$$\sum_{Q\in\Lambda_k} w(Q)[\ell(Q)]^{-sq} \sup_{y\in Q} |(\phi_Q * f)(y)|^q \lesssim 2^{kq} w(\Omega_k). \tag{4.23}$$

To prove (4.23), let $\Omega_k^* := \big\{x \in \mathbb{R}^n : M^{\mathrm{loc}}(\chi_{\Omega_k})(x) > \frac{1}{2}\big\}$ for all $k \in \mathbb{Z}$. From (2.1) and Lemma 2.2(iii), it follows that $w(\Omega_k^*) \lesssim w(\Omega_k)$ and hence

$$\int_{\Omega_k^*\setminus\Omega_{k+1}} [\widetilde{S}_q^s(f)(x)]^q\, w(x)\mathrm{d}x \le 2^{(k+1)q} w(\Omega_k^*) \lesssim 2^{(k+1)q} w(\Omega_k). \tag{4.24}$$

On the other hand, for all $y \in Q \in \Lambda_k$, we have $Q \subset \Omega_k^*$ and

$$w((\Omega_k^* \setminus \Omega_{k+1}) \cap Q) \ge w(Q \setminus \Omega_{k+1}) = w(Q) - w(Q \cap \Omega_{k+1}) \ge \frac{w(Q)}{2},$$

which further implies that

$$\int_{\Omega_k^*\setminus\Omega_{k+1}} [\widetilde{S}_q^s(f)(x)]^q\, w(x)\mathrm{d}x \gtrsim \sum_{Q\in\Lambda_k} w(Q)\ell(Q)^{-sq} \sup_{y\in Q} |(\phi_Q * f)(y)|^q. \tag{4.25}$$

Combining (4.24)–(4.25), we obtain (4.23).

From Hölder's inequality, $w(Q_k^i \cap \Omega_k) \geq \frac{1}{2}w(Q_k^i)$, the fact that the family $\{Q_k^i\}_{i\in I_k}$ are mutually disjoint, and (4.23), we deduce that

$$\sum_{k\in\mathbb{Z}}\sum_{i\in I_k}|\lambda_{k,\,i}|^p \lesssim \sum_{k\in\mathbb{Z}} w(\Omega_k)2^{kp} \lesssim \|f\|^p_{F^{s,\,w}_{p,q}(\mathbb{R}^n)},$$

which shows (4.22).

Now we show that each $a_{k,i}$ is a constant multiple of some $(p,q,s)_w$-atom. Indeed, by $\operatorname{supp}(\phi_0) \subset B(0,1)$, it is easy to see that $\operatorname{supp}(a_{k,i}) \subset 4Q_k^i$. Also, notice that $a_{k,i}$ satisfies the vanishing moment condition provided that $|Q_k^i| < 1$. Thus, it remains to prove that $\|a_{k,i}\|_{F^{s,w}_{q,q}(\mathbb{R}^n)} \lesssim [w(4Q_k^i)]^{\frac{1}{q}-\frac{1}{p}}$. This can be done by considering the following two cases: $q \in (1,\infty)$ and $q \in [p,1]$.

Case 1 $q \in (1,\infty)$. By $w \in \mathcal{A}_q^{\mathrm{loc}}(\mathbb{R}^n)$, we see that, for all $Q \in \mathcal{Q}$ with $|Q| \leq 1$,

$$[w(Q)]^{\frac{1}{q}}[w^{1-q'}(Q)]^{\frac{1}{q'}} \sim |Q|. \tag{4.26}$$

From Theorem 1.1 and the fact that $Q \subset \{y \in \mathbb{R}^n : |x-y| < \sqrt{n}2^{-k}\}$ for all $x \in Q$ with $\ell(Q) = 2^{-k}$, we deduce that, for any $g \in F^{-s,w^{1-q'}}_{q',q'}(\mathbb{R}^n)$,

$$\|g\|_{F^{-s,w^{1-q'}}_{q',q'}(\mathbb{R}^n)} \gtrsim \Big\{\sum_{\substack{Q\subset Q_k^i\\ Q\in\Lambda_k}} w^{1-q'}(Q)\ell(Q)^{sq'}\sup_{y\in Q}|\widetilde{\psi}_Q * g(y)|^{q'}\Big\}^{\frac{1}{q'}}. \tag{4.27}$$

Applying (4.26)–(4.27) and Hölder's inequality, we conclude that, for all $g \in F^{-s,w^{1-q'}}_{q',q'}(\mathbb{R}^n)$ with a norm at most 1, $|\langle a_{k,i},\, g\rangle| \lesssim [w(Q_k^i)]^{\frac{1}{q}-\frac{1}{p}}$. From this and Proposition 2.1, it follows that $\|a_{k,i}\|_{F^{s,w}_{q,q}(\mathbb{R}^n)} \lesssim [w(Q_k^i)]^{\frac{1}{q}-\frac{1}{p}}$.

Case 2 $q \in [p,1]$. In this case, notice that $w \in \mathcal{A}_1^{\mathrm{loc}}(\mathbb{R}^n)$. By Theorem 1.1, we have

$$\|a_{k,i}\|_{F^{s,w}_{p,q}(\mathbb{R}^n)} \sim \Big\{\sum_{\substack{P\in\mathcal{Q}\\ |P|\leq 1}} \ell(P)^{-sq}\int_P \Big[\sup_{|x-z|<\ell(P)}|\phi_P * a_{k,i}(z)|^q\Big]w(x)\mathrm{d}x\Big\}^{\frac{1}{q}}. \tag{4.28}$$

For all $x \in P$ and $|x-z| < \ell(P)$, by $q \in [p,1]$ and the inequality that $\{\sum_{j\in\mathbb{N}}|b_j|\}^q \leq \sum_{j\in\mathbb{N}}|b_j|^q$ holds for all sequences $\{b_j\}_{j\in\mathbb{N}}$, we conclude that

$$|\phi_P * a_{k,i}(z)|^q \leq \frac{1}{|\lambda_{k,\,i}|^q}\sum_{\substack{Q\subset Q_k^i\\ Q\in\Lambda_k}}\sup_{y\in Q}|\phi_Q * f(y)|^q\Big[\int_Q|\phi_P * \psi_Q(z-y)|\mathrm{d}y\Big]^q. \tag{4.29}$$

Because ϕ and ψ satisfy the vanishing moment condition up to order L, we apply [6, Lemma 2 in p. 121 and Lemma 4 in p. 122] or [17, Corollary 3.1] to deduce that, for all $u \in \mathbb{R}^n$,

$$|\phi_P * \psi_Q(u)| \lesssim \Big[\min\Big\{\frac{\ell(P)}{\ell(Q)},\, \frac{\ell(Q)}{\ell(P)}\Big\}\Big]^L \frac{1}{\max\{|Q|,\,|P|\}}\Big[\frac{\max\{\ell(P),\,\ell(Q)\}}{\max\{\ell(P),\,\ell(Q)\}+|u|}\Big]^L.$$

Combining this with $\operatorname{supp}(\phi_P * \psi_Q) \subset \{u : |u| \leq 2\max\{\ell(P),\,\ell(Q)\}\}$ further implies that, for all $y \in Q$, $x \in P$ and $|x-z| < \ell(P)$, if $\phi_P * \psi_Q(z-y) \neq 0$, then

$$|c_P - c_Q| \leq 2(\sqrt{n}+1)\max\{\ell(P),\,\ell(Q)\} \leq 4\sqrt{n}\max\{\ell(P),\,\ell(Q)\} \tag{4.30}$$

and

$$|\phi_P * \psi_Q(z-y)| \lesssim \Big[\min\Big\{\frac{\ell(P)}{\ell(Q)}, \frac{\ell(Q)}{\ell(P)}\Big\}\Big]^L \frac{1}{\max\{|Q|, |P|\}} =: \mathbb{A}_{P,Q}. \tag{4.31}$$

Invoking (4.30)–(4.31), we continue to estimate (4.29) with

$$|\phi_P * a_{k,i}(z)|^q \lesssim \frac{1}{|\lambda_{k,\,i}|^q} \sum_{\substack{Q\subset Q_k^i \\ Q\in\Lambda_k \\ |c_P-c_Q|\le 4\sqrt{n}\max\{\ell(P),\,\ell(Q)\}}} \sup_{y\in Q} |\phi_Q * f(y)|^q (\mathbb{A}_{P,Q})^q\, |Q|^q.$$

Inserting this into (4.28) and interchanging the summations in P and Q, we obtain

$$\begin{aligned}\|a_{k,i}\|_{F^{s,w}_{p,q}(\mathbb{R}^n)} \lesssim \frac{1}{|\lambda_{k,\,i}|}\Big\{ &\sum_{\substack{Q\subset Q_k^i \\ Q\in\Lambda_k}} w(Q)\ell(Q)^{-sq} \sup_{y\in Q} |\phi_Q * f(y)|^q \\ &\times \Big(\sum_{\substack{P\in\mathcal{Q} \\ |P|\le 1 \\ |c_P-c_Q|\le 4\sqrt{n}\max\{\ell(P),\,\ell(Q)\}}} \Big[\frac{\ell(Q)}{\ell(P)}\Big]^{sq} \frac{w(P)}{w(Q)} (\mathbb{A}_{P,Q})^q\, |Q|^q \Big)\Big\}^{\frac{1}{q}},\end{aligned}$$

which gives $\|a_{k,i}\|_{F^{s,w}_{p,q}(\mathbb{R}^n)} \lesssim w(Q_k^i)^{\frac{1}{q}-\frac{1}{p}}$, provided that when $Q\subset Q_k^i$ and $Q\in\Lambda_k$,

$$\sum_{\substack{P\in\mathcal{Q} \\ |P|\le 1 \\ |c_P-c_Q|\le 4\sqrt{n}\max\{\ell(P),\,\ell(Q)\}}} \Big[\frac{\ell(Q)}{\ell(P)}\Big]^{sq} \frac{w(P)}{w(Q)} (\mathbb{A}_{P,Q})^q\, |Q|^q \lesssim 1. \tag{4.32}$$

To show (4.32), we define

$$\mathrm{Z}_i := \sum_{P\in W_i} \Big[\frac{\ell(Q)}{\ell(P)}\Big]^{sq} \frac{w(P)}{w(Q)} (\mathbb{A}_{P,Q})^q\, |Q|^q, \quad i\in\{1,2\},$$

where $W_1 := \{P\in\mathcal{Q} : \ell(P)\le\ell(Q),\, |P|\le 1,\, |c_P-c_Q|\le 4\sqrt{n}\max\{\ell(P),\,\ell(Q)\}\}$ and

$$W_2 := \{P\in\mathcal{Q} : \ell(P)>\ell(Q),\, |P|\le 1,\, |c_P-c_Q|\le 4\sqrt{n}\max\{\ell(P),\,\ell(Q)\}\}.$$

Notice that, for all $P\in W_1$, we have $\mathbb{A}_{P,Q}\le 1$ and $P\subset 5\sqrt{n}Q$, so that $\frac{w(P)}{w(Q)} \le \frac{w(5\sqrt{n}Q)}{w(Q)} \lesssim 1$ by Lemma 2.2(ii). Therefore, if we choose $L>|s|$, then

$$\mathrm{Z}_1 \lesssim \sum_{P\in W_1} \Big[\frac{\ell(P)}{\ell(Q)}\Big]^{Lq-sq} \frac{w(P)}{w(Q)} \lesssim \sum_{j=-\log_2\ell(Q)}^{\infty} \sum_{\substack{P\in\mathcal{Q} \\ \ell(P)=2^{-j} \\ |c_P-c_Q|\le 4\sqrt{n}\ell(Q)}} \Big[\frac{\ell(P)}{\ell(Q)}\Big]^{Lq-sq} \lesssim 1. \tag{4.33}$$

Now we estimate Z_2. Observe that $Q\subset 5\sqrt{n}P$ when $P\in W_2$. Since $\ell(P)\le 1$ and $\ell(Q)\le 1$, the fact $w\in\mathcal{A}_1^{\mathrm{loc}}(\mathbb{R}^n)$ implies that $\frac{w(5\sqrt{n}P)}{|5\sqrt{n}P|} \sim \inf_{x\in 5\sqrt{n}P} w(x)$ and $\frac{w(Q)}{|Q|} \sim \inf_{x\in Q} w(x)$. Thus,

$$\frac{w(P)}{w(Q)} \le \frac{w(5\sqrt{n}P)}{w(Q)} \sim \frac{|5\sqrt{n}P|}{|Q|} \frac{\inf_{x\in 5\sqrt{n}P} w(x)}{\inf_{x\in Q} w(x)} \lesssim \frac{|P|}{|Q|}.$$

By this and the expression of $\mathbb{A}_{P,Q}$, we see that, when L satisfies $(L+s)q+n(q-1)>0$,

$$\begin{aligned}\mathrm{Z}_2 &\lesssim \sum_{P\in W_2}\left[\frac{\ell(Q)}{\ell(P)}\right]^{(L+s)q+n(q-1)}\\ &\lesssim \sum_{j=0}^{-\log_2\ell(Q)}\sum_{\substack{P\in\mathcal{Q}\\ \ell(P)=2^{-j}\\ |c_P-c_Q|\le 4\sqrt{n}\ell(P)}}\left\{\frac{\ell(Q)}{\ell(P)}\right\}^{(L+s)q+n(q-1)}\lesssim 1.\end{aligned}\tag{4.34}$$

Combining (4.33)–(4.34) gives (4.32). Thus, we complete the proof of Theorem 1.2.

5 Proof of Theorem 1.3

Proof of Theorem 1.3 Let $f\in C_c^\infty(\mathbb{R}^n)\cap F_{p,q}^{s,w}(\mathbb{R}^n)$. With all the notation as in the proof of Theorem 1.2, we decompose f into $f=\sum_{k\in\mathbb{Z}}\sum_{i\in I_k}\lambda_{k,i}a_{k,i}$ in $\mathcal{S}_e'(\mathbb{R}^n)$, where each $a_{k,i}$ is a $(p,q,s)_w$-atom supported on $4Q_k^i$, defined by

$$a_{k,\,i}(x)=\frac{1}{\lambda_{k,\,i}}\sum_{\substack{Q\subset Q_k^i\\ Q\in\Lambda_k}}\int_Q\psi_Q(x-y)(\phi_Q*f)(y)\mathrm{d}y,\quad x\in\mathbb{R}^n,$$

and $\Big\{\sum_{k\in\mathbb{Z}}\sum_{i\in I_k}|\lambda_{k,\,i}|^p\Big\}^{\frac1p}\lesssim\|f\|_{F_{p,q}^{s,w}(\mathbb{R}^n)}$. Here, for any $k\in\mathbb{Z}$, Λ_k is as in (4.18); Q_k^i is the largest dyadic cube contained in the set Λ_k; the functions ψ_Q and ϕ_Q are as in (4.20). Without loss of generality, we may assume that ϕ_j,ψ_j with $j\in\mathbb{Z}_+$ are supported on $B(0,2^{-j})$, and ψ_j,ϕ_j with $j\in\mathbb{N}$ have the vanishing moments up to order L, where $L>|s|$.

For any $N\in\mathbb{N}$, let

$$\begin{aligned}Q(0,2^N)&:=\{(x_1,\cdots,x_n)\in\mathbb{R}^n:\,-1\le 2^{-N}x_i<1,\,i\in\{1,\cdots,n\}\},\\ W_1^N&:=\{Q\in\mathcal{Q}:\,Q\subset Q(0,2^N),2^{-N}\le\ell(Q)\le 1\},\\ W_2^N&:=\{Q\in\mathcal{Q}:\,\ell(Q)\le 1\}\setminus W_1^N.\end{aligned}$$

For each $Q\in W_1^N$, there exists a unique (k,i) such that $Q\subset Q_k^i$. Denote by J_N the collection of all such (k,i). Since W_1^N has finitely many elements, so does J_N. For each $(k,i)\in J_N$, let

$$\widetilde{a}_{k,\,i}(x)=\frac{1}{\lambda_{k,\,i}}\sum_{\substack{Q\subset Q_k^i\\ Q\in\Lambda_k\cap W_1^N}}\int_Q\psi_Q(x-y)(\phi_Q*f)(y)\mathrm{d}y,\quad x\in\mathbb{R}^n.\tag{5.1}$$

In a similar way to the arguments used in Cases 1–2 of the proof of Theorem 1.2, we conclude that each $\widetilde{a}_{k,i}$ is also a constant multiple of some $(p,q,s)_w$-atom. Since $f\in C_c^\infty(\mathbb{R}^n)$ and the summation in (5.1) has only finite terms, it follows that every $\widetilde{a}_{k,i}\in C_c^\infty(\mathbb{R}^n)$. For any $N\in\mathbb{N}$, set $f_N:=\sum_{(k,i)\in J_N}\lambda_{k,i}\widetilde{a}_{k,i}$ and $b_N:=f-f_N$. Then, both f_N and b_N are in $C_c^\infty(\mathbb{R}^n)$. Moreover, f_N is a linear combination of finite smooth $(p,q,s)_w$-atoms, and the proof of Theorem 1.2 shows that the ℓ^p-(quasi)norm of its coefficients is bounded by a constant multiple of $\|f\|_{F_{p,q}^{s,w}(\mathbb{R}^n)}$. Thus, to finish the proof of Theorem 1.3, we only need to prove that, when N is large enough, b_N is a small constant multiple of some $(p,q,s)_w$-atom.

For all $x \in \mathbb{R}^n$, noticing that $W_1^N = \bigcup_{(k,i)\in J_N} \{Q \subset Q_k^i :\ Q \in \Lambda_k \cap W_1^N\}$, we write

$$f_N(x) = \sum_{Q\in W_1^N} \int_Q \psi_Q(x-y)(\phi_Q * f)(y)\mathrm{d}y.$$

Then, applying (4.21), we see that, for all $x \in \mathbb{R}^n$,

$$b_N(x) = f(x) - f_N(x) = \sum_{Q\in W_2^N} \int_Q \psi_Q(x-y)(\phi_Q * f)(y)\mathrm{d}y. \tag{5.2}$$

Since f has compact support, there exists $N_0 \in \mathbb{N}$ such that $\operatorname{supp} f \subset Q(0, 2^{N_0})$. Notice that $\operatorname{supp}(\phi_Q) \subset Q$. For any $Q \in W_2^N$, to ensure that $\phi_Q * f$ is a non-zero function, we need $Q \cap Q(0, 2^{N_0+5}) \neq \varnothing$. But any cube Q in W_2^N is dyadic and satisfies $\ell(Q) \le 1$, and we therefore conclude that $Q\cap Q(0, 2^{N_0+5}) \neq \varnothing$ is the same as $Q \subset Q(0, 2^{N_0+5})$. Notice that, if $N > N_0 + 5$, then $\{Q \in W_2^N : Q \subset Q(0, 2^{N_0+5})\} = \{Q \in \mathcal{Q} : \ell(Q) < 2^{-N}\}$. This allows us to replace the summation in (5.2) with $\sum_{\substack{Q\in\mathcal{Q}\\ \ell(Q)<2^{-N}}}$ when $N > N_0 + 5$. Thus, when $N > N_0 + 5$, we have

$$b_N(x) = \sum_{\substack{Q\in\mathcal{Q}\\ \ell(Q)<2^{-N}}} \int_Q \psi_Q(x-y)(\phi_Q * f)(y)\mathrm{d}y = \sum_{\ell>N} \psi_\ell * \phi_\ell * f(x), \tag{5.3}$$

where ψ_ℓ and ϕ_ℓ are associated to ψ_Q and ϕ_Q as in (4.20). Clearly, $\operatorname{supp}(b_N) \subset Q(0, 2^{N_0+5})$. Now we estimate the quasi-norm $\|b_N\|_{F^{s,w}_{q,q}(\mathbb{R}^n)}$. From (5.3), it follows that

$$\|b_N\|_{F^{s,w}_{q,q}(\mathbb{R}^n)} = \Big\{\sum_{j=0}^{\infty} 2^{jsq} \int_{\mathbb{R}^n} \Big|\sum_{\ell>N} \phi_j * \psi_\ell * \phi_\ell * f(x)\Big|^q w(x)\mathrm{d}x\Big\}^{\frac{1}{q}}.$$

If $\phi_j * \psi_\ell * \phi_\ell * f \neq 0$, then by the support conditions of ϕ_j and ψ_ℓ we see that $|j-\ell| \le 2$. Thus,

$$\|b_N\|_{F^{s,w}_{q,q}(\mathbb{R}^n)} \lesssim \Big\{\sum_{j=N-2}^{\infty} \sum_{\substack{\ell>N\\ |\ell-j|\le 2}} 2^{jsq} \int_{\mathbb{R}^n} \Big|\phi_j * \psi_\ell * \phi_\ell * f(x)\Big|^q w(x)\mathrm{d}x\Big\}^{\frac{1}{q}}.$$

For the sake of simplicity, we only estimate the term $j = \ell$. By $\operatorname{supp}(\phi_j * \psi_j * \phi_j * f) \subset Q(0, 2^{N_0+5})$ and the estimate that, for all $\varphi, g \in \mathcal{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \varphi(x)x^\alpha \mathrm{d}z = 0$ for all multi-indices $|\alpha| \le L$,

$$|\varphi_j * g(x)| \lesssim 2^{-jL}(1+|x|)^{-(n+1)}, \qquad x \in \mathbb{R}^n$$

(see [17, Lemma 3.3(i)] or [6, p. 121, Lemma 2]), we have

$$\Big\{\sum_{j=N-2}^{\infty} 2^{jsq} \int_{\mathbb{R}^n} |\phi_j * \psi_j * \phi_j * f(x)|^q w(x)\mathrm{d}x\Big\}^{\frac{1}{q}} \lesssim 2^{-N(L-|s|)}[w(Q(0, 2^{N_0+5}))]^{\frac{1}{q}},$$

which is bounded by a constant multiple of $2^{-\frac{N(L-|s|)}{2}}[w(Q(0, 2^{N_0+5}))]^{\frac{1}{q}-\frac{1}{p}}$ provided that N is large enough (depending on w, N_0, p, q, s and L). Therefore,

$$\|b_N\|_{F^{s,w}_{q,q}(\mathbb{R}^n)} \le c2^{-\frac{N(L-|s|)}{2}}[w(Q(0, 2^{N_0+5}))]^{\frac{1}{q}-\frac{1}{p}}$$

for some positive constant c depending only on w, N_0, n, p, q, L and s, and

$$a_N := c^{-1}2^{\frac{N(L-|s|)}{2}}b_N$$

is a $(p,q,s)_w$-atom supported on $Q(0,2^{N_0+5})$ by observing that a_N does not need to satisfy any vanishing moment since $|Q(0,2^{N_0+5})| \geq 1$, which implies that

$$f = f_N + b_N = \sum_{(k,i)\in J_N} \lambda_{k,i}a_{k,i} + c2^{-\frac{N(L-|s|)}{2}}a_N$$

is a finite atomic decomposition with $(p,q,s)_w$-atoms $a_{k,i}, a_N$ in $C_c^\infty(\mathbb{R}^n)$ and the coefficients satisfying

$$\sum_{(k,i)\in J_N} |\lambda_{k,i}|^p + |c2^{-N\sigma}|^p \lesssim \|f\|^p_{F^{s,w}_{p,q}(\mathbb{R}^n)} + 2^{-\frac{N(L-|s|)}{2p}} \lesssim \|f\|^p_{F^{s,w}_{p,q}(\mathbb{R}^n)}.$$

This finishes the proof of Theorem 1.3.

6 Proof of Theorem 1.4

The goal of this section is to show Theorem 1.4. We need the following density lemma.

Lemma 6.1 *Let $s \in \mathbb{R}$, $p, q \in (0,\infty)$ and $w \in \mathcal{A}^{\rm loc}_\infty(\mathbb{R}^n)$. Then, $C_c^\infty(\mathbb{R}^n) \cap F^{s,w}_{p,q}(\mathbb{R}^n)$ is dense in $F^{s,w}_{p,q}(\mathbb{R}^n)$.*

Proof By the localization principle in [18, Theorem 2.21], for all $f \in F^{s,w}_{p,q}(\mathbb{R}^n)$, we have

$$\|f\|_{F^{s,w}_{p,q}(\mathbb{R}^n)} \sim \Big[\sum_{k\in\mathbb{Z}^n} \|\gamma^k f\|^p_{F^{s,w}_{p,q}(\mathbb{R}^n)}\Big]^{\frac{1}{p}}, \tag{6.1}$$

where $\gamma \in \mathcal{D}(\mathbb{R}^n)$ such that its integer translates $\gamma^k(x) := \gamma(x-k)$ for all $x \in \mathbb{R}^n$ with $k \in \mathbb{Z}^n$ form a partition of unity, that is, $\sum_{k\in\mathbb{Z}^n} \gamma^k(x) = 1$ for all $x \in \mathbb{R}^n$. Notice that $\operatorname{supp}(\gamma^k)$ has finite overlapping property (bounded by a positive constant depending only on $\operatorname{supp}\gamma$). For all $N \in \mathbb{N}$, define $f_N := \sum_{\substack{j\in\mathbb{Z}^n\\ |j|\leq N}} \gamma^j f$. Observe that every f_N has compact support. Moreover, the sequence $\{f_N\}_{N\in\mathbb{N}}$ converges to f as $N \to \infty$. Indeed, by (6.1), we see that, when $N \to \infty$,

$$\|f - f_N\|_{F^{s,w}_{p,q}(\mathbb{R}^n)} \lesssim \Big\{\sum_{\substack{k\in\mathbb{Z}^n\\ |k|\geq N-c}} \|\gamma^k f\|^p_{F^{s,w}_{p,q}(\mathbb{R}^n)}\Big\}^{\frac{1}{p}} \to 0,$$

where c is a positive constant depending only on $\operatorname{supp}\gamma$. Thus, to finish the proof of this lemma, we may as well assume that $f \in F^{s,w}_{p,q}(\mathbb{R}^n)$ has compact support.

Now suppose that $f \in F^{s,w}_{p,q}(\mathbb{R}^n)$ has compact support. By (2.3), write $f = \sum_{i=0}^\infty \psi_i * \phi_i * f$ with ψ_i and ϕ_i as in Lemma 2.3. Without loss of generality, we may assume that both ψ_i and ϕ_i are supported on the ball $B(0,2^{-i})$ for $i \in \mathbb{Z}_+$. For all $N \in \mathbb{N}$, let $g_N := \sum_{i=1}^N \psi_i * \phi_i * f$. Since f is assumed to have compact support, it follows that every $g_N \in C_c^\infty(\mathbb{R}^n)$. It remains to show that $g_N \to f$ in $F^{s,w}_{p,q}(\mathbb{R}^n)$ as $N \to \infty$. Notice that

$$\|g_N - f\|_{F^{s,w}_{p,q}(\mathbb{R}^n)} = \Big\|\Big\{\sum_{k=0}^\infty 2^{ksq}\Big|\sum_{i=N+1}^\infty \phi_k * \psi_i * \phi_i * f\Big|^q\Big\}^{\frac{1}{q}}\Big\|_{L^p_w(\mathbb{R}^n)}, \quad N \in \mathbb{N}.$$

For all $x \in \mathbb{R}^n$, from the previous assumptions that both ψ_i and ϕ_i are supported on the ball $B(0, 2^{-i})$, the definition of the Peetre-type maximal function, and [17, Corollary 3.1] (see also [6, Lemma 2 in p. 121, and Lemma 4 in p. 122]), we deduce that, for all $x \in \mathbb{R}^n$,

$$|\phi_k * \psi_i * \phi_i * f(x)| \lesssim 2^{-|k-i|M}(1 + 2^{k-i})^A \phi^*_{i,A,B} f(x),$$

where A and B are positive constants satisfying $A \ge A_0$ and $B \ge \frac{B_0}{p}$, and $M > A + |s|$ is a sufficiently large constant. By this and Hölder's inequality, we further conclude that

$$\|g_N - f\|_{F^{s,w}_{p,q}(\mathbb{R}^n)} \lesssim \left\| \left\{ \sum_{i=N+1}^{\infty} 2^{isq} |\phi^*_{i,A,B} f|^q \right\}^{\frac{1}{q}} \right\|_{L^p_w(\mathbb{R}^n)},$$

which tends to 0 as $N \to \infty$ in terms of (1.2). Thus, $\{g_N\}_{N\in\mathbb{N}} \subset C^\infty_c(\mathbb{R}^n) \cap F^{s,w}_{p,q}(\mathbb{R}^n)$ and it converges to f in $F^{s,w}_{p,q}(\mathbb{R}^n)$ as $N \to \infty$, which completes the proof of Lemma 6.1.

Proof of Theorem 1.4 For any $f \in C^\infty_c(\mathbb{R}^n) \cap F^{s,w}_{p,q}(\mathbb{R}^n)$, by Theorem 1.3, f admits a finite atomic decomposition $f = \sum_{k=1}^{N} \lambda_k a_k$, where $N \in \mathbb{N}$, $\{a_k\}_{k=1}^N$ are $(p, q, s)_w$-atoms in $C^\infty_c(\mathbb{R}^n)$ and $\{\sum_{k=1}^{N} |\lambda_k|^p\}^{\frac{1}{p}} \lesssim \|f\|_{F^{s,w}_{p,q}(\mathbb{R}^n)}$. By the assumption of Theorem 1.4, we see that $\|Ta_k\|_{\mathcal{B}_r} \lesssim 1$ for all $k \in \{1, \cdots, N\}$. Moreover, by $r \in [p, 1]$ and the fact that T is $\mathcal{B}_r$-sublinear, we have

$$\|Tf\|^r_{\mathcal{B}_r} = \sum_{k=1}^{N} |\lambda_k|^r \|Ta_k\|^r_{\mathcal{B}_r} \lesssim \sum_{k=1}^{N} |\lambda_k|^r \lesssim \Big[\sum_{k=1}^{N} |\lambda_k|^p\Big]^{\frac{r}{p}} \lesssim \|f\|^r_{F^{s,w}_{p,q}(\mathbb{R}^n)}. \tag{6.2}$$

In general, for any $f \in F^{s,w}_{p,q}(\mathbb{R}^n)$, by Lemma 6.1, there exists $\{g_m\}_{m\in\mathbb{N}} \subset C^\infty_c(\mathbb{R}^n) \cap F^{s,w}_{p,q}(\mathbb{R}^n)$ that converges to f in $F^{s,w}_{p,q}(\mathbb{R}^n)$ as $m \to \infty$. By (6.2) and Definition 1.3(iii), $\{Tg_m\}_{m\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{B}_r$ and hence it converges to some element in $\mathcal{B}_r$, which we denote by $\widetilde{T}f$, namely, $\widetilde{T}f := \lim_{m\to\infty} Tg_m$ in $\mathcal{B}_r$. Notice that $\widetilde{T}$ is well defined based on (6.2) and Definition 1.3(iii). Consequently, $\|\widetilde{T}f\|_{\mathcal{B}_r} = \lim_{m\to\infty} \|Tg_m\|_{\mathcal{B}_r} \lesssim \lim_{m\to\infty} \|g_m\|_{F^{s,w}_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{F^{s,w}_{p,q}(\mathbb{R}^n)}$, which completes the proof of Theorem 1.4.

7 Proof of Theorem 1.5

In this section, we apply Theorem 1.4 to show Theorem 1.5.

Proof of Theorem 1.5 Notice that $F^{s,w}_{p,q}(\mathbb{R}^n)$, for $p \in (0, 1]$ and $q \in [p, \infty)$, is a p-quasi-Banach space. By Theorem 1.4, it suffices to show that, for any $(p, q, s)_w$-atom a supported on a cube Q,

$$\|\mathcal{R}^{\mathrm{loc}}_j a\|_{F^{s,w}_{p,q}(\mathbb{R}^n)} \lesssim 1. \tag{7.1}$$

To this end, by Theorem 1.1, we only need to show that $\|\widetilde{S}^s_{1,q}(\mathcal{R}^{\mathrm{loc}}_j a)\|_{L^p_w(\mathbb{R}^n)} \lesssim 1$, where

$$\widetilde{S}^s_{1,q}(\mathcal{R}^{\mathrm{loc}}_j a)(x) = \Big[\sum_{k=0}^{\infty} \sup_{|x-y|<2^{-k}} |2^{ks} \phi_k * (\mathcal{R}^{\mathrm{loc}}_j a)(y)|^q\Big]^{\frac{1}{q}}, \quad x \in \mathbb{R}^n.$$

Here $\phi_0 \in \mathcal{D}(\mathbb{R}^n)$, $\phi := \phi_0 - 2^{-n}\phi_0(2^{-1}\cdot)$ satisfies the vanishing moments up to order $L \ge \max\{-1, \lfloor s \rfloor\}$, and $\phi_k := 2^{nk}\phi(2^k\cdot)$ for $k \in \mathbb{N}$. Without loss of generality, we may assume that every $\operatorname{supp}(\phi_k) \subset B(0, 2^{-k})$ and $L > |s|$.

By the support conditions of a and ϕ_k, together with the definition of $\mathcal{R}_j^{\rm loc}$, we see that $\mathrm{supp}\,(\widetilde{S}_{1,q}^s(\mathcal{R}_j^{\rm loc}\,a)) \subset \{x \in \mathbb{R}^n : |x - c_Q| < \ell(Q) + 3\}$. Applying Hölder's inequality with $\frac{1}{\frac{q}{p}} + \frac{1}{(\frac{q}{p})'} = 1$ and Lemma 2.2(ii), we know that

$$\begin{aligned}
&\|\widetilde{S}_{1,q}^s(\mathcal{R}_j^{\rm loc}\,a)\|_{L_w^p(\mathbb{R}^n)}\\
&\lesssim \left\{\int_{|x-c_Q|<\ell(Q)+3}\left[\sum_{k=0}^{\infty}\sup_{|x-y|<2^{-k}}|2^{ks}\phi_k * (\mathcal{R}_j^{\rm loc}\,a)(y)|^q\right]w(x)\mathrm{d}x\right\}^{\frac{1}{q}}[w(Q)]^{\frac{1}{p}-\frac{1}{q}}.
\end{aligned}$$

By the Calderón reproducing formulae and the fact that $\mathcal{R}_j^{\rm loc}$ commutates with the convolution operator, we conclude that, for all $y \in \mathbb{R}^n$,

$$\phi_k * (\mathcal{R}_j^{\rm loc}\,a)(y) = \sum_{i=0}^{\infty}\phi_k * \psi_i * (\mathcal{R}_j^{\rm loc}\,(\phi_i * a))(y),$$

where ψ_i and ϕ_i are as in Lemma 2.3 (here we also assume that ψ_i and ϕ_i are supported on $B(0, 2^{-i})$, and $\{\psi_i\}_{i=1}^{\infty}$ have vanishing moments up to order $L > |s|$). Furthermore,

$$|\phi_k * \mathcal{R}_j^{\rm loc}\,a(y)| \lesssim \sum_{i=0}^{\infty}2^{-|k-i|L}\frac{1}{(2^{-k}+2^{-i})^n}\int_{|y-z|<2^{-k}+2^{-i}}|\mathcal{R}_j^{\rm loc}\,(\phi_i * a)(z)|\mathrm{d}z,$$

where we used again [6, Lemma 2 in p. 121 and Lemma 4 in p. 122] (see also [17, Corollary 3.1]). As $q > 1$, using Hölder's inequality and the definition of $\mathcal{A}_q^{\rm loc}\,(\mathbb{R}^n)$ gives that, when $|x-y| < 2^{-k}$,

$$\begin{aligned}
&\frac{1}{(2^{-k}+2^{-i})^n}\int_{|y-z|<2^{-k}+2^{-i}}|\mathcal{R}_j^{\rm loc}\,(\phi_i * a)(z)|\mathrm{d}z\\
&\lesssim \left[\frac{1}{w(B(x, 2^{-k}+2^{-i}))}\int_{|x-z|<2^{-k+1}+2^{-i}}|\mathcal{R}_j^{\rm loc}\,(\phi_i * a)(z)|^q w(z)\mathrm{d}z\right]^{\frac{1}{q}}.
\end{aligned}$$

Notice that L is taken to be larger than $|s|$. Then, applying Hölder's inequality again, we obtain

$$\begin{aligned}
&\sup_{|x-y|<2^{-k}}|2^{ks}\phi_k * (\mathcal{R}_j^{\rm loc}\,a)(y)|^q\\
&\lesssim \sum_{i=0}^{\infty}\frac{2^{isq}2^{-|k-i|L}}{w(B(x, 2^{-k}+2^{-i}))}\int_{|x-z|<2^{-k+1}+2^{-i}}|\mathcal{R}_j^{\rm loc}\,(\phi_i * a)(z)|^q w(z)\mathrm{d}z.
\end{aligned}$$

By this and Fubini's theorem, together with the fact that

$$\int_{|x-z|<2^{-k+1}+2^{-i}}\frac{1}{w(B(x, 2^{-k}+2^{-i}))}w(x)\mathrm{d}x \lesssim 1,$$

we see that

$$\begin{aligned}
&\left\{\int_{|x-c_Q|<\ell(Q)+3}\left[\sum_{k=0}^{\infty}\sup_{|x-y|<2^{-k}}|2^{ks}\phi_k * (\mathcal{R}_j^{\rm loc}\,a)(y)|^q\right]w(x)\mathrm{d}x\right\}^{\frac{1}{q}}\\
&\lesssim \left\{\sum_{i=0}^{\infty}2^{isq}\int_{\mathbb{R}^n}|\mathcal{R}_j^{\rm loc}\,(\phi_i * a)(z)|^q w(z)\mathrm{d}z\right\}^{\frac{1}{q}}.
\end{aligned}$$

Then, using the fact that $\mathcal{R}_j^{\rm loc}$ is bounded on $L_w^q(\mathbb{R}^n)$ and the size condition of a $(p, q, s)_w$-atom, we see that the last quality in the above estimate is bounded by

$$\left\{\sum_{i=0}^{\infty}2^{isq}\int_{\mathbb{R}^n}|\phi_i * a(z)|^q w(z)\mathrm{d}z\right\}^{\frac{1}{q}} \lesssim \|a\|_{F_{q,q}^{s,w}(\mathbb{R}^n)} \lesssim [w(Q)]^{\frac{1}{q}-\frac{1}{p}}.$$

Thus, $\|\widetilde{S}_{1,q}^s(\mathcal{R}_j^{\rm loc}\,a)\|_{L_w^p(\mathbb{R}^n)} \lesssim 1$. Hence, (7.1) holds and we complete the proof of the theorem.

References

[1] Bui, H., Weighted Hardy spaces, *Math. Nachr.*, **103**, 1981, 45–62.

[2] Cao, W., Chen, J. and Fan, D., Boundedness of an oscillating multiplier on Triebel-Lizorkin spaces, *Acta Math. Sin.* (*Engl. Ser.*), **26**, 2010, 2071–2084.

[3] Chen, J., Yu, X., Zhang, Y. and Wang, H., Hypersingular parameterized Marcinkiewicz integrals with variable kernels on Sobolev and Hardy-Sobolev spaces, *Appl. Math. J. Chinese Univ. Ser. B*, **23**, 2008, 420–430.

[4] Frazier, M. and Jawerth, B., Decomposition of Besov spaces, *Indiana Univ. Math. J.*, **34**, 1985, 777–799.

[5] Frazier, M. and Jawerth, B., A discrete transform and decompositions of distribution spaces, *J. Funct. Anal.*, **93**, 1990, 34–170.

[6] Frazier, M., Jawerth, B. and Weiss, G., Littlewood-Paley Theory and the Study of Function Spaces, CBMS Regional Conference Series in Mathematics, **79**, Amer. Math. Soc., Providence, R. I., 1991.

[7] Goldberg, D., A local version of real Hardy spaces, *Duke Math.*, **46**, 1979, 27–42.

[8] Han, Y., Paluszyński, M. and Weiss, G., A new atomic decomposition for the Triebel-Lizorkin spaces, *Contemp. Math.*, **189**, 1995, 235–249,

[9] Haroske, D. D. and Piotrowska, I., Atomic decompositions of function spaces with muckenhoupt weights, and some relation to fractal analysis, *Math. Nachr.*, **281**, 2008, 1476–1494.

[10] Haroske, D. D. and Skrzypczak, L., Entropy and approximation numbers of embeddings of function spaces with muckenhoupt weights, I, *Rev. Mat. Complut.*, **21**, 2008, 135–177.

[11] Haroske, D. D. and Skrzypczak, L., Spectral theory of some degenerate elliptic operators with local singularities, *J. Math. Anal. Appl.*, **371**, 2010, 282–299.

[12] Haroske, D. D. and Skrzypczak, L., Entropy and approximation numbers of embeddings of function spaces with muckenhoupt weights, II. general weights, *Ann. Acad. Sci. Fenn. Math.*, **36**, 2011, 111–138.

[13] Haroske, D. D. and Skrzypczak, L., Entropy numbers of embeddings of function spaces with muckenhoupt weights, III. Some limiting cases, *J. Funct. Spaces Appl.*, **9**, 2011, 129–178.

[14] Izuki, M. and Sawano, Y., Wavelet bases in the weighted Besov and Triebel-Lizorkin spaces with A_p^{loc}-weights, *J. Approx. Theory*, **161**, 2009, 656–673.

[15] Izuki, M. and Sawano, Y., Atomic decomposition for weighted Besov and Triebel-Lizorkin spaces, *Math. Nachr.*, **285**, 2012, 103–126.

[16] Izuki, M., Sawano, Y. and Tanaka, H., Weighted Besov-Morrey spaces and Triebel-Lizorkin spaces, in: Harmonic analysis and nonlinear partial differential equations, 21–60, RIMS Kôkyûroku Bessatsu, Ser. B, **22**, Res. Inst. Math. Sci. (RIMS), Kyoto, 2010.

[17] Liu, L. and Yang, D., Boundedness of sublinear operators in Triebel-Lizorkin spaces via atoms, *Studia Math.*, **190**, 2009, 163–183.

[18] Rychkov, V. S., Littlewood-Paley theory and function spaces with A_p^{loc} weights, *Math. Nachr.*, **224**, 2001, 145–180.

[19] Schmeißer, H.-J. and Triebel, H., Topics in Fourier Analysis and Function Spaces, John Wiley and Sons, Ltd., Chichester, 1987.

[20] Schott, T., Function spaces with exponential weights, I, *Math. Nachr.*, **189**, 1998, 221–242.

[21] Schott, T., Function spaces with exponential weights, II, *Math. Nachr.*, **196**, 1998, 231–250.

[22] Tang, L., Weighted local Hardy spaces and their applications, *Illinois J. Math.* (to appear).

[23] Tang, L., Weighted norm inequalities for pseudo-differential operators with smooth symbols and their commutators, *J. Funct. Anal.*, **262**, 2012, 1603–1629.

[24] Triebel, H., Theory of Function Spaces, Birkhäuser Verlag, Basel, 1983.

[25] Yang, D. and Yang, S., Weighted local Orlicz Hardy spaces with applications to pseudo-differential operators, *Dissertationes Math.*, **478**, 2011, 1–78.

[26] Yang, D. and Zhou, Y., Boundedness of sublinear operators in Hardy spaces on RD-spaces via atoms, *J. Math. Anal. Appl.*, **339**, 2008, 622–635.

[27] Yang, D. and Zhou, Y., A boundedness criterion via atoms for linear operators in Hardy spaces, *Constr. Approx.*, **29**, 2009, 207–218.