Chin. Ann. Math.
35B(2), 2014, 261–270
DOI: 10.1007/s11401-014-0823-2

Chinese Annals of Mathematics, Series B

3-Lie Algebras Realized by Cubic Matrices*

Ruipu BAI[1] Hui LIU[1] Meng ZHANG[1]

Abstract Associative multiplications of cubic matrices are provided. The N^3-dimensional 3-Lie algebras are realized by cubic matrices, and structures of the 3-Lie algebras are studied.

Keywords 3-Lie algebra, Cubic matrix, Multiplication
2000 MR Subject Classification 15B99, 17D99

1 Introduction

Kerner in [1] presented the survey which scattered in a series of papers whose common denominator is the use of cubic relations in various algebraic structures, and constructed the non-associative ternary algebra of cubic matrices (the object with elements having three indices). For cubic matrices $A = (a_{ijk})$, $B = (b_{ijk})$, $C = (c_{ijk})$, $1 \leq i, j, k \leq 2$, Kerner defined the ternary multiplication of three cubic matrices $(A \oslash B \oslash C)_{ijk} = \sum\limits_{pqr} a_{ipq}b_{pjr}c_{rqk}$, and $(A * B * C)_{ijk} = \sum\limits_{pqr} a_{piq}b_{qjr}c_{rkj}$ in [1]. The symmetry properties of the ternary algebras were studied, and the Z_3-graded generalization of Grassmann algebras, a ternary generalization of Clifford algebras, also the description of quark fields were discussed. Awata, Li, Minicc and Yoneyad in [2] presented several nontrivial examples of the 3-dimensional quantum Nambu bracket with cubic matrices. For cubic matrices $A = (a_{ijk})$, $B = (b_{ijk})$, $C = (c_{ijk})$, $1 \leq i, j, k \leq N$, they defined the traces functions: $\langle A\rangle = \sum\limits_{p,m=1}^{N} a_{pmp}$, $\langle AB\rangle = \sum\limits_{p,m,q=1}^{N} a_{pmq}b_{qmp}$ and $\langle ABC\rangle = \sum\limits_{p,m,q,r=1}^{N} a_{pmq}b_{qmr}c_{rmp}$, which satisfy

$$\langle AB\rangle = \langle BA\rangle, \quad \langle ABC\rangle = \langle BCA\rangle = \langle CAB\rangle,$$

defined the multiplication of three cubic matrices (ABC) by $(ABC)_{ijk} = \sum\limits_{pqm} A_{ijp}B_{qmq}C_{pjk}$, then obtain a skew-symmetric quantum Nambu bracket

$$[A, B, C] = (ABC) + (BCA) + (CAB) - (ACB) - (BAC) - (CBA),$$

which satisfies the generalized Jacobi identity (3.1).

Manuscript received May 7, 2012. Revised April 2, 2013.
[1]College of Mathematics and Computer Science, Hebei University, Baoding 071002, Heibei, China.
E-mail: bairuipu@hbu.cn liuhuiabcxyz@163.com zhangmeng447@126.com
*Project supported by the National Natural Science Foundation of China (No. 11371245) and the Natural Science Foundation of Hebei Province (No. A2010000194).

The multi-index constants are often used to describe structures of algebras. For example, if L is a 3-Lie algebra (see [3]) with a basis $x_1, \cdots, x_N$, the entire multiplication table can be described by the structure constants a_{ijl}^k which occur in the expression $[x_i, x_j, x_l] = \sum\limits_{k=1}^{N} a_{ijl}^k x_k$. Then we obtain a four indices matrix $A = (a_{ijl}^k)$ which satisfies $a_{i_1 i_2 i_3}^k = \mathrm{sgn}(\sigma) a_{i_{\sigma(1)} i_{\sigma(2)} i_{\sigma(3)}}^k$ and $\sum\limits_{k=1}^{N} (a_{ijl}^k a_{kmn}^t + a_{imn}^k a_{klj}^t + a_{jmn}^k a_{ilk}^t) = 0$, where σ is arbitrary 3-ary permutation. In [4], the authors studied structures of 2-step nilpotent metric 3-Lie algebras by means of four indices matrices. By the properties of four indices matrices, it is proved that there do not exist 2, 4, 6 and 10-dimensional 2-step nilpotent metric 3-Lie algebras with corank zero, there exist 8, or greater than 10-dimensional 2-step nilpotent metric 3-Lie algebras with corank zero. And up to isomorphisms only one 8-dimensional 2-step nilpotent metric 3-Lie algebras with corank zero.

In this paper, we discuss the multiplications of two cubic matrices which satisfy the associative law, and define the sth-determinant, and "traces" of cubic matrix. By means of cubic matrices, we construct 3-Lie algebras.

Throughout this paper, we assume that the cubic matrices are over a field F of characteristic zero, and the symbol $\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j \end{cases}$ for positive integers i and j.

2 Multiplications of Cubic Matrix

An N-order cubic matrix $A = (a_{ijk})$ over a field F is an ordered object which the elements with 3 indices i, j, k and $1 \leq i, j, k \leq N$. The element in the position (i, j, k) is denoted by $(A)_{ijk} = a_{ijk}$, $1 \leq i, j, k \leq N$.

Denote the set of all cubic matrices over a field F by Ω. Then Ω is an N^3-dimensional vector space over F with

$$A + B = (a_{ijk} + b_{ijk}) \in \Omega, \quad \lambda A = (\lambda a_{ijk}) \in \Omega,$$

$\forall A = (a_{ijk}), B = (b_{ijk}) \in \Omega,\ \lambda \in F$, that is, $(A + B)_{ijk} = a_{ijk} + b_{ijk}$, $(\lambda A)_{ijk} = \lambda a_{ijk}$.

Denote E_{ijk} a cubic matrix which the element in the position (i, j, k) is 1 and elsewhere are zero, that is, $E_{ijk} = (a_{lmn})$ with $a_{lmn} = \delta_{il}\delta_{jm}\delta_{kn},\ 1 \leq l, m, n \leq N;\ E_i = \sum\limits_{j=1}^{N} E_{ijj}$, $1 \leq j \leq N$. Then $\{E_{ijk},\ 1 \leq i, j, k \leq N\}$ is a basis of Ω, and for every $A = (a_{ijk}) \in \Omega$, $A = \sum\limits_{1 \leq i,j,k \leq N} a_{ijk} E_{ijk},\ a_{ijk} \in F$.

Every cubic matrix A can be written as the following three types of blocking forms:

$$A = (A_1^1, \cdots, A_N^1), \quad A = (A_1^2, \cdots, A_N^2), \quad A = (A_1^3, \cdots, A_N^3),$$

where $A_i^1 = (a_{ijk})$, $A_i^2 = (a_{jik})$, $A_i^3 = (a_{jki})$ are usual $(N \times N)$-order matrices with the elements a_{ijk}, a_{jik}, a_{jki} at the position of the jth-row and the kth-column, respectively, $1 \leq i \leq N$.

Define the multiplications $*_{rs}$ of cubic matrices are as follows: $\forall A = (a_{ijk}), B = (b_{ijk}) \in \Omega$,

$A *_{rs} B = ((A *_{rs} B)_{ijk})$, $1 \le i,j,k \le N$, where

$$\begin{cases}(A *_{11} B)_{ijk} = \sum_{p=1}^{N} a_{ijp}b_{ipk}, \quad (A *_{12} B)_{ijk} = \sum_{p=1}^{N} a_{pjk}b_{ijp},\\(A *_{13} B)_{ijk} = \sum_{p=1}^{N} a_{ipk}b_{pjk}, \quad (A *_{21} B)_{ijk} = \sum_{p,q=1}^{N} a_{qjp}b_{ipk},\\(A *_{22} B)_{ijk} = \sum_{p,q=1}^{N} a_{pjk}b_{iqp}, \quad (A *_{23} B)_{ijk} = \sum_{p,q=1}^{N} a_{ipq}b_{pjk},\\(A *_{24} B)_{ijk} = \sum_{p,q=1}^{N} a_{ijp}b_{qpk}, \quad (A *_{25} B)_{ijk} = \sum_{p,q=1}^{N} a_{pqk}b_{ijp},\\(A *_{26} B)_{ijk} = \sum_{p,q=1}^{N} a_{ipk}b_{pjq},\\(A *_{27} B)_{ijk} = \Big(\sum_{q=1}^{N} A_q^1\Big)_{jk}\Big(\sum_{p=1}^{N} B_p^2\Big)_{ik} = \sum_{p,q=1}^{N} a_{qjk}b_{ipk},\\(A *_{28} B)_{ijk} = \Big(\sum_{p=1}^{N} A_p^2\Big)_{ik}\Big(\sum_{q=1}^{N} B_q^3\Big)_{ij} = \sum_{p,q=1}^{N} a_{iqk}b_{ijp},\\(A *_{29} B)_{ijk} = \Big(\sum_{p=1}^{N} A_p^3\Big)_{ij}\Big(\sum_{q=1}^{N} B_q^1\Big)_{jk} = \sum_{p,q=1}^{N} a_{ijq}b_{pjk},\\(A *_{31} B)_{ijk} = \sum_{p,q=1}^{N} a_{ipq}\Big(\sum_{r=1}^{N} B_r^1\Big)_{jk} = \sum_{p,q,r=1}^{N} a_{ipq}b_{rjk},\\(A *_{32} B)_{ijk} = \sum_{p,q=1}^{N} a_{pjq}\Big(\sum_{r=1}^{N} B_r^2\Big)_{ik} = \sum_{p,q,r=1}^{N} a_{qjp}b_{irk},\\(A *_{33} B)_{ijk} = \sum_{p,q=1}^{N} a_{pqk}\Big(\sum_{r=1}^{N} B_r^3\Big)_{ij} = \sum_{p,q,r=1}^{N} a_{pqk}b_{ijr}.\end{cases} \tag{2.1}$$

Then the multiplications of two cubic matrices defined as above in the blocking form are as follows:

$$A *_{11} B = ((A *_{11} B)_1^1, \cdots, (A *_{11} B)_N^1) = (A_1^1B_1^1, \cdots, A_N^1B_N^1),$$
$$A *_{12} B = ((A *_{12} B)_1^2, \cdots, (A *_{12} B)_N^2) = (B_1^2A_1^2, \cdots, B_N^2A_N^2),$$
$$A *_{13} B = ((A *_{13} B)_1^3, \cdots, (A *_{13} B)_N^3) = (A_1^3B_1^3, \cdots, A_N^3B_N^3),$$
$$A *_{21} B = ((A *_{21} B)_1^1, \cdots, (A *_{21} B)_N^1) = \Big(\sum_{p=1}^{N} A_p^1B_1^1, \cdots, \sum_{p=1}^{N} A_p^1B_N^1\Big),$$
$$A *_{22} B = ((A *_{22} B)_1^2, \cdots, (A *_{22} B)_N^2) = \Big(\sum_{p=1}^{N} B_p^2A_1^2, \cdots, \sum_{p=1}^{N} B_p^2A_N^2\Big),$$
$$A *_{23} B = ((A *_{23} B)_1^3, \cdots, (A *_{23} B)_N^3) = \Big(\sum_{p=1}^{N} A_p^3B_1^3, \cdots, \sum_{p=1}^{N} A_p^3B_N^3\Big),$$
$$A *_{24} B = ((A *_{24} B)_1^1, \cdots, (A *_{24} B)_N^1) = \Big(A_1^1\sum_{p=1}^{N} B_p^1, \cdots, A_N^1\sum_{p=1}^{N} B_p^1\Big),$$

$$A *_{25} B = ((A *_{25} B)_1^2, \cdots, (A *_{25} B)_N^2) = \Big(B_1^2 \sum_{p=1}^N A_p^2, \cdots, B_N^2 \sum_{p=1}^N A_p^2\Big),$$

$$A *_{26} B = ((A *_{26} B)_1^3, \cdots, (A *_{26} B)_N^3) = \Big(A_1^3 \sum_{p=1}^N B_p^3, \cdots, A_N^3 \sum_{p=1}^N B_p^3\Big),$$

where $A_i^s B_j^t$ is the product of two $(N \times N)$-order matrices A_i^s and B_j^t.

Define the linear isomorphism $\tau : \Omega \to \Omega$, that is, $\forall A = (a_{ijk}) \in \Omega$,

$$\tau(A) = A^\tau = (a_{ijk}^\tau), \quad (A^\tau)_{ijk} = a_{ijk}^\tau = a_{kij} = (A)_{kij}, \quad 1 \le i, j, k \le N.$$

Then τ satisfies $\tau^3 = \mathrm{Id}_\Omega$, and

$$\begin{array}{lll}
\tau(A *_{11} B) = \tau(A) *_{12} \tau(B), & \tau(A *_{12} B) = \tau(A) *_{13} \tau(B), & \tau(A *_{13} B) = \tau(A) *_{11} \tau(B), \\
\tau(A *_{21} B) = \tau(A) *_{25} \tau(B), & \tau(A *_{25} B) = \tau(A) *_{23} \tau(B), & \tau(A *_{23} B) = \tau(A) *_{21} \tau(B), \\
\tau(A *_{22} B) = \tau(A) *_{26} \tau(B), & \tau(A *_{26} B) = \tau(A) *_{24} \tau(B), & \tau(A *_{24} B) = \tau(A) *_{22} \tau(B), \\
\tau(A *_{27} B) = \tau(A) *_{28} \tau(B), & \tau(A *_{28} B) = \tau(A) *_{29} \tau(B), & \tau(A *_{29} B) = \tau(A) *_{27} \tau(B), \\
\tau(A *_{31} B) = \tau(A) *_{32} \tau(B), & \tau(A *_{32} B) = \tau(A) *_{33} \tau(B), & \tau(A *_{33} B) = \tau(A) *_{31} \tau(B).
\end{array}$$

Theorem 2.1 *$(\Omega, *_{11})$, $(\Omega, *_{21})$, $(\Omega, *_{22})$, $(\Omega, *_{27})$ and $(\Omega, *_{31})$ are non-isomorphic associative algebras. And there exists the unit element $U(1) = (u_{ijk})$ in $(\Omega, *_{11})$, where $u_{ijk} = \delta_{jk}$, $1 \le i, j, k \le N$, that is, for every $A \in (\Omega, *_{11})$, $A *_{11} U(1) = U(1) *_{11} A = A$. And the multiplication tables of the associative algebras in the basis $\{E_{ijk},\ 1 \le i, j, k \le N\}$ are as follows, respectively,*

$$\begin{cases} E_{ijk} *_{11} E_{lmn} = \delta_{il}\delta_{km} E_{ijn}, \\ E_{ijk} *_{21} E_{lmn} = \delta_{km} E_{ljn}, \\ E_{ijk} *_{22} E_{lmn} = \delta_{in} E_{ljk}, \\ E_{ijk} *_{27} E_{lmn} = \delta_{kn} E_{ljk}, \\ E_{ijk} *_{31} E_{lmn} = E_{imn}, \quad 1 \le i,\ j, k, l, m, n \le N. \end{cases} \tag{2.2}$$

Proof The result follows from the direct computations.

The determinant $|A|$ of a cubic matrix A is defined as

$$|A| = \det(A_1^1) \cdots \det(A_N^1) = \prod_{t=1}^N \det(A_t^1). \tag{2.3}$$

Then we have $|A *_{11} B| = |B *_{11} A| = |A||B|$, $|U(1)| = 1$. If $|A| \ne 0$, then A is called a non-degenerate cubic matrix, the inverse cubic matrix of A is denoted by A^{-1}, that is,

$$A *_{11} A^{-1} = A^{-1} *_{11} A = U(1).$$

And for arbitrary non-degenerate cubic matrices A and B, we have $(A *_{11} B)^{-1} = B^{-1} *_{11} A^{-1}$, $|A^{-1}| = \frac{1}{|A|}$.

For constructing 3-Lie algebras by cubic matrices according to the multiplications $*_{11}$, $*_{21}$, $*_{22}$, $*_{27}$, $*_{31}$, we define the "sth-trace" linear functions $\langle\ \rangle_s : \Omega \to F$, $s = 1, 2, 3, 4$ as follows: $\forall A = (a_{ijk}) \in \Omega$,

$$\langle A \rangle_1 = \sum_{p,q=1}^N a_{pqq}, \quad \langle A \rangle_2 = \sum_{p,q=1}^N a_{pqp}, \quad \langle A \rangle_3 = \sum_{p,q=1}^N a_{ppq}, \quad \langle A \rangle_4 = \sum_{p,q,r=1}^N a_{pqr}. \tag{2.4}$$

Theorem 2.2 *For arbitrary cubic matrices $A, B \in \Omega$, we have*

$$\langle E_{ijk}\rangle_1 = \delta_{jk}, \quad \langle E_{ijk}\rangle_2 = \delta_{ik}, \quad \langle E_{ijk}\rangle_3 = \delta_{ij}, \quad \langle E_{ijk}\rangle_4 = 1,$$
$$\langle A *_{11} B\rangle_1 = \langle B *_{11} A\rangle_1, \quad \langle A *_{21} B\rangle_1 = \langle B *_{21} A\rangle_1, \quad \langle A *_{22} B\rangle_2 = \langle B *_{22} A\rangle_2,$$
$$\langle A *_{27} B\rangle_4 = \langle B *_{27} A\rangle_4, \quad \langle A *_{31} B\rangle_4 = \langle B *_{31} A\rangle_4.$$

Proof The result follows from the direct computations.

3 Construction of 3-Lie Algebras by Cubic Matrix

In this section we construct 3-Lie algebras by cubic matrices. First we introduce some notions on n-Lie algebras (see [3]).

An n-Lie algebra J over a field F is a vector space endowed with an n-ary multilinear skew-symmetric multiplication which satisfies the n-Jacobi identity: $\forall x_1, \cdots, x_n, y_2, \cdots, y_n \in J$,

$$[[x_1, \cdots, x_n], y_2, \cdots, y_n] = \sum_{i=1}^{n} [x_1, \cdots, [x_i, y_2, \cdots, y_n], \cdots, x_n]. \tag{3.1}$$

The n-ary skew-symmetry of the operation $[x_1, \cdots, x_n]$ means that

$$[x_1, \cdots, x_n] = \mathrm{sgn}(\sigma)[x_{\sigma(1)}, \cdots, x_{\sigma(n)}], \quad \forall x_1, \cdots, x_n \in J$$

for any permutation $\sigma \in S_n$. A subspace W of J is called a subalgebra if $[W, \ldots, W] \subseteq W$. In particular, the subalgebra generated by the vectors $[x_1, \cdots, x_n]$ for any $x_1, \cdots, x_n \in J$ is called the derived algebra of J, which is denoted by J^1. If $J^1 = 0$, then J is called an abelian n-Lie algebra.

An ideal of an n-Lie algebra J is a subspace I such that $[I, J, \cdots, J] \subseteq I$.

An ideal I of an n-Lie algebra J is called nilpotent, if $I^s = 0$ for some $s \geq 0$, where $I^0 = I$ and I^s is defined as $I^s = [I^{s-1}, I, J, \cdots, J]$ for $s \geq 1$. If $I = J$, then J is nilpotent n-Lie algebra. If

$$J^2 = [J^1, J, \cdots, J] = 0,$$

then J is called a 2-step nilpotent n-Lie algebra.

The subset $Z(J) = \{x \in L \mid [x, y_1, \cdots, y_{n-1}] = 0, \ \forall y_1, \cdots, y_{n-1} \in L\}$ is called the center of J.

Now we define the 3-ary linear multiplications on Ω as follows according to the multiplications $*_{11}$, $*_{21}$, $*_{22}$, $*_{27}$ and $*_{31}$: $\forall A, B, C \in \Omega$,

$$\begin{cases} [A,B,C]_{11} = \langle A\rangle_1(B *_{11} C - C *_{11} B) + \langle B\rangle_1(C *_{11} A - A *_{11} C) \\ \qquad\qquad\quad + \langle C\rangle_1(A *_{11} B - B *_{11} A), \\ [A,B,C]_{21} = \langle A\rangle_1(B *_{21} C - C *_{21} B) + \langle B\rangle_1(C *_{21} A - A *_{21} C) \\ \qquad\qquad\quad + \langle C\rangle_1(A *_{21} B - B *_{21} A), \\ [A,B,C]_{22} = \langle A\rangle_2(B *_{22} C - C *_{22} B) + \langle B\rangle_2(C *_{22} A - A *_{22} C) \\ \qquad\qquad\quad + \langle C\rangle_2(A *_{22} B - B *_{22} A), \\ [A,B,C]_{27} = \langle A\rangle_4(B *_{27} C - C *_{27} B) + \langle B\rangle_4(C *_{27} A - A *_{27} C) \\ \qquad\qquad\quad + \langle C\rangle_4(A *_{27} B - B *_{27} A), \\ [A,B,C]_{31} = \langle A\rangle_4(B *_{31} C - C *_{31} B) + \langle B\rangle_4(C *_{31} A - A *_{31} C) \\ \qquad\qquad\quad + \langle C\rangle_4(A *_{31} B - B *_{31} A). \end{cases} \tag{3.2}$$

Theorem 3.1 *The 3-ary algebras* $(\Omega,[,,]_{11})$*,* $(\Omega,[,,]_{21})$*,* $(\Omega,[,,]_{22})$*,* $(\Omega,[,,]_{27})$ *and* $(\Omega,[,,]_{31})$ *are 3-Lie algebras, which are denoted by* J_{11}*,* J_{21}*,* J_{22}*,* J_{27} *and* J_{31}*, respectively.*

Proof Thanks to Proposition 2.2, $\forall A,B\in\Omega$,

$$\langle A *_{22} B - B *_{22} A\rangle_2 = 0,\quad \langle A *_{11} B - B *_{11} A\rangle_1 = \langle A *_{21} B - B *_{21} A\rangle_1 = 0,$$
$$\langle A *_{27} B - B *_{27} A\rangle_4 = \langle A *_{31} B - B *_{31} A\rangle_4 = 0.$$

Following from Theorems 2.1 and 3.1 in [5], $(\Omega,[,,]_{11})$, $(\Omega,[,,]_{21})$, $(\Omega,[,,]_{22})$, $(\Omega,[,,]_{27})$ and $(\Omega,[,,]_{31})$ are 3-Lie algebras.

Define the linear isomorphism ω: $\Omega\to\Omega$, that is, $\forall A=(a_{ijk})\in\Omega$, $\omega(A)=(a'_{ijk})$, where $(\omega(A))_{ijk}=a'_{ijk}=a_{jik}=(A)_{jik}$, $1\le i,j,k\le N$. Then by Theorems 2.1–2.2, for arbitrary $A,B\in\Omega$, $\omega(A *_{22} B)=\omega(B) *_{21}\omega(A)$, $\langle A\rangle_2=\langle\omega(A)\rangle_1$. Then

$$\begin{aligned}\omega([A,B,C]_{22}) &= \omega(\langle A\rangle_2(B *_{22} C - C *_{22} B) + \langle B\rangle_2(C *_{22} A - A *_{22} C)\\ &\quad + \langle C\rangle_2(A *_{22} B - B *_{22} A))\\ &= -\langle A\rangle_2(\omega(B) *_{21}\omega(C) - \omega(C) *_{21}\omega(B)) - \langle B\rangle_2(\omega(C) *_{21}\omega(A)\\ &\quad - \omega(A) *_{21}\omega(C)) - \langle C\rangle_2(\omega(A) *_{21}\omega(B) - \omega(B) *_{21}\omega(A))\\ &= -[\omega(A),\omega(B),\omega(C)]_{21}.\end{aligned}$$

Therefore, 3-Lie algebra J_{22} is anti-isomorphic to 3-Lie algebra J_{21} in the isomorphism ω : $J_{22}\to J_{21}$, that is, $\forall A,B,C\in\Omega$, $\omega([A,B,C]_{22})=-[\omega(A),\omega(B),\omega(C)]_{21}$.

Following from the multiplications $*_{11}$, $*_{21}$, $*_{27}$ and $*_{31}$, the multiplication tables of the 3-Lie algebras in the basis $\{E_{ijk}, 1\le i,j,k\le N\}$ are as follows:

$$\begin{aligned}[E_{ijk},E_{lmn},E_{pqr}]_{11} &= \delta_{jk}\delta_{lp}(\delta_{nq}E_{lmr}-\delta_{rm}E_{lqn}) + \delta_{mn}\delta_{pi}(\delta_{rj}E_{iqk}-\delta_{kq}E_{ijr})\\ &\quad + \delta_{qr}\delta_{li}(\delta_{km}E_{ijn}-\delta_{nj}E_{imk}), \qquad (3.3)\\ [E_{ijk},E_{lmn},E_{pqr}]_{21} &= \delta_{jk}\delta_{nq}E_{pmr}-\delta_{mn}\delta_{kq}E_{pjr}+\delta_{qr}\delta_{km}E_{ljn}\\ &\quad - \delta_{jk}\delta_{rm}E_{lqn}+\delta_{mn}\delta_{rj}E_{iqk}-\delta_{qr}\delta_{nj}E_{imk}, \qquad (3.4)\\ [E_{ijk},E_{lmn},E_{pqr}]_{27} &= \delta_{rn}(E_{pmn}-E_{lqn})+\delta_{kr}(E_{iqr}-E_{pjr})+\delta_{kn}(E_{ljk}-E_{imk}), \qquad (3.5)\\ [E_{ijk},E_{lmn},E_{rpq}]_{31} &= E_{lpq}-E_{ipq}+E_{imn}-E_{rmn}+E_{rjk}-E_{ljk}, \qquad (3.6)\end{aligned}$$

where $1\le i,j,k,l,m,n,p,q,r\le N$.

4 Structures of 3-Lie Algebras J_{rs}

Now we study the structures of the 3-Lie algebras J_{11}, J_{21}, J_{27} and J_{31}. First, we study the structure of 3-Lie algebra J_{11}. Denote

$$J_{11_l}=\{A\in\Omega\mid A=(A^1_1,\cdots,A^1_N)=(0,\cdots,A^1_l,\cdots,0)\},\quad l=1,\cdots,N.$$

For every $A\in J_{11}$, $A=(A^1_1,0,\cdots,0)+(0,A^1_2,0,\cdots,0)+\cdots+(0,\cdots,0,A^1_N)$, then

$$J_{11}=J_{11_1}\dot{+}\cdots\dot{+}J_{11_N}$$

as the direct sum of subspaces. From the multiplication (3.3), we have

$$\begin{cases} [E_{lnm}, E_{ljj}, E_{lmn}]_{11} = -E_{lnn} + E_{lmm}, & 1 \le m \ne n \le N, \\ [E_{lpm}, E_{ljj}, E_{lmn}]_{11} = -E_{lpn}, & 1 \le p \ne m, p \ne n, m \ne n \le N, \\ [E_{lnq}, E_{ljj}, E_{lmn}]_{11} = E_{lmq}, & 1 \le q \ne m, q \ne n, m \ne n \le N, \\ [E_{lpm}, E_{ljj}, E_{lmm}]_{11} = -E_{lpm}, & 1 \le p \ne m, p \ne j, m \ne j \le N, \\ [E_{ljm}, E_{ljj}, E_{lmm}]_{11} = -2E_{ljm}, & 1 \le m \ne j \le N, \\ [E_{lmq}, E_{ljj}, E_{lmm}]_{11} = E_{lmq}, & 1 \le q \ne m, m \ne j, q \ne j \le N, \\ [E_{ljq}, E_{ljj}, E_{lmm}]_{11} = -E_{ljq}, & 1 \le q \ne j, q \ne m, m \ne j \le N, \\ [E_{lpj}, E_{ljj}, E_{lmm}]_{11} = E_{lpj}, & 1 \le p \ne j, p \ne m, m \ne j \le N, \\ [E_{ijj}, E_{lqm}, E_{lmn}]_{11} = E_{lqn}, & 1 \le q \ne n,\ i \ne l \le N. \end{cases} \tag{4.1}$$

Then $J_{11_l}, 1 \le l \le N$ are subalgebras and satisfy

$$J^1_{11_l} = \sum_{\substack{i,j=1 \\ i \ne j}}^{N} FE_{lij} \dotplus \sum_{i=1}^{N-1} F(E_{lii} - E_{li+1i+1}), \quad 1 \le l \le N, \tag{4.2}$$

$$[J_{11_i}, J_{11_j}, J_{11_l}]_{11} = 0, \quad [J_{11_l}, J_{11_l}, J_{11_j}]_{11} = J^1_{11_l}, \quad 1 \le i \ne j,\ j \ne l,\ i \ne l \le N. \tag{4.3}$$

And the center of J_{11} is

$$Z(J_{11}) = \sum_{i=2}^{N} F(E_1 - E_i), \quad E_i = \sum_{j=1}^{N} E_{ijj}, \quad 1 \le i \le N. \tag{4.4}$$

Summarizing above discussions, we obtain the following result.

Theorem 4.1 (1) *The 3-Lie algebra J_{11} can be decomposed into the direct sum of subalgebras*

$$J_{11} = J_{11_1} \dotplus J_{11_2} \dotplus \cdots \dotplus J_{11_N}, \quad J^1_{11} = J^1_{11_1} \dotplus J^1_{11_2} \dotplus \cdots \dotplus J^1_{11_N}, \quad [J^1_{11}, J^1_{11}, J^1_{11}]_{11} = 0,$$

and the derived algebra $J^1_{11_l}$ of J_{11_l} are minimal ideals of J_{11}, $1 \le l \le N$.

(2) *For arbitrary $1 \le p_1 < p_2 < \cdots < p_k \le N$, the subspace $J_{11_{p_1}} \dotplus \cdots \dotplus J_{11_{p_k}}$ is a subalgebra of J_{11}.*

(3) *J_{11} is the semidirect product $J_{11} = I \dotplus FE_1$, where $I = J^1_{11} \dotplus Z(J_{11})$ is the maximal ideal of J_{11} with codimension one.*

Proof The result follows from identities (4.1)–(4.4).

Next we study the structure of the 3-Lie algebra J_{21}. For every $A \in \Omega$,

$$A = (A^1_1, \cdots, A^1_N) = \Big(A^1_1 + \sum_{i=2}^{N} A^1_i, 0, \cdots, 0\Big) + \Big(-\sum_{i=2}^{N} A^1_i, A^1_2, \cdots, A^1_N\Big).$$

Denote

$$\Phi = \{A \in \Omega \mid A = (A^1_1 \cdots, A^1_N) = (A^1_1, 0, \cdots, 0)\} = \sum_{m,n=1}^{N} FE_{1mn},$$

$$\Psi = \Big\{A \in \Omega \mid A = (A^1_1 \cdots, A^1_N), \quad \sum_{i=1}^{N} A^1_i = 0\Big\}.$$

Then $\Omega=\Phi\dot{+}\Psi$ as the direct sum of subspaces and for every $A\in\Psi$, $\langle A\rangle_1=\sum\limits_{p,q=1}^{N}a_{pqq}=0$.

Since identity (3.4) and $\langle A\rangle_1=0$ for $A\in\Psi$, Ψ is an (N^3-N^2)-dimensional abelian ideal of J_{21}, that is, $[\Psi,\Psi,J_{21}]_{21}=0$, and Φ is a subalgebra with

$$\Phi^1=[\Phi,\Phi,\Phi]_{21}=\sum_{\substack{m,n=1\\ m\neq n}}^{N}FE_{1mn}+\sum_{m=2}^{N}F(E_{111}-E_{1mm}).$$

Summarizing the above discussions, we obtain the following result.

Theorem 4.2 (1) *The 3-Lie algebra $J_{21}=J_{21}^1+FE_{111}$, where*

$$J_{21}^1=\sum_{h,i,j=1,\ i\neq j}^{N}FE_{hij}+\sum_{h=2}^{N}\sum_{i=1}^{N}F(E_{111}-E_{hii})+\sum_{i=2}^{N}F(E_{111}-E_{1ii}),\tag{4.5}$$
$$[J_{21}^1,J_{21}^1,J_{21}]_{21}=J_{21}^1,\quad [J_{21}^1,J_{21}^1,J_{21}^1]_{21}=0.$$

(2) *The 3-Lie algebra J_{21} is the semidirect product $J_{21}=\Psi\dot{+}\Phi$, and $\Psi=\sum\limits_{i=2}^{N}\sum\limits_{n=1}^{N}Q_i^n$ is an (N^3-N^2)-dimensional abelian ideal of J_{21}, and Φ is a subalgebra with*

$$\Phi^1=[\Phi,\Phi,\Phi]_{21}=\sum_{\substack{m,n=1\\ m\neq n}}^{N}FE_{1mn}+\sum_{m=2}^{N}F(E_{111}-E_{1mm}).$$

(3) *The abelian ideal Ψ has a decomposition $\Psi=\sum\limits_{i=2}^{N}\sum\limits_{n=1}^{N}Q_i^n$, where*

$$Q_i^n=\sum_{m=1}^{N}F(E_{1mn}-E_{imn}),\quad i=2,\cdots,N,\ n=1,\cdots,N$$

are $(N-1)$-dimensional minimal ideals of J_{21}. Therefore, $Q_i^n,\ i=2,\cdots,N,\ n=1,\cdots,N$, are irreducible modules of J_{21}.

Proof The results (1) and (2) follow from the identity (3.4). Denote $Q_i^n=\sum\limits_{m=1}^{N}F(E_{1mn}-E_{imn})$, $i=2,\cdots,N$, $n=1,\cdots,N$, then $\Psi=\sum\limits_{i=2}^{N}\sum\limits_{n=1}^{N}Q_i^n$ is the direct sum of subspaces. Since the identity (2.2) and Proposition 2.2,

$$\begin{aligned}&[E_{1mn}-E_{imn},E_{ljk},E_{rpq}]_{21}\\ &=\langle E_{ljk}\rangle_1(E_{rpq}*_{21}(E_{1mn}-E_{imn})-(E_{1mn}-E_{imn})*_{21}E_{rpq})\\ &\quad+\langle E_{rpq}\rangle_1((E_{1mn}-E_{imn})*_{21}E_{ljk}-E_{ljk}*_{21}(E_{1mn}-E_{imn}))\\ &=\delta_{jk}\delta_{qm}(E_{1pn}-E_{ipn})+\delta_{pq}\delta_{km}(E_{1jn}-E_{ijn}).\end{aligned}$$

Therefore, $Q_i^n,\ i=2,\cdots,N,\ n=1,\cdots,N$, are minimal ideals of J_{21}.

Now we study the structure of the 3-Lie algebra J_{27}. Denote

$$J_{27_p}=\{A\in\Omega\mid A=(\delta_{pk}a_{ijk})=(0,\cdots,A_p^3,0,\cdots,0)\},\quad Q_i^p=\sum_{j=1}^{N}FE_{ijp},\ 1\le i,p\le N.$$

Since identity (3.5),

$$\begin{aligned}&[E_{ijk}, E_{lmn}, E_{pqr}]_{27}\\ &=\begin{cases}0, & k\neq r,\ k\neq n,\ n\neq r,\\ E_{pmn}-E_{lqn}, & k\neq n, n=r,\\ E_{pmn}-E_{lqn}+E_{iqn}-E_{pjn}+E_{ljn}-E_{imn}, & k=n=r.\end{cases}\end{aligned} \tag{4.6}$$

Then J_{27_p}, $1\leq p\leq N$, are subalgebras and $J^1_{27_p}=\sum\limits_{i+j>2}F(E_{11p}-E_{ijp})$. Since

$$[E_{ijk}, E_{imk}, E_{ink}]_{27}=-E_{imk}+E_{ijk}-E_{ink}+E_{imk}-E_{ijk}+E_{ink}=0,$$

$[Q^i_p, Q^i_p, Q^i_p]_{27}=0$. Therefore, Q^p_i are abelian subalgebras, $1\leq i,p\leq N$. And

$$J^1_{27}=\sum_{k=1}^{N}\sum_{i+j>2}^{N}F(E_{11k}-E_{ijk}),\quad \dim J^1_{27}=N^3-1.$$

Theorem 4.3 *The 3-Lie algebra J_{27} is an indecomposable 3-Lie algebra, and J_{27} can be decomposed into the direct sum of subalgebras*

$$J_{27}=J_{27_1}\dot{+}\cdots\dot{+}J_{27_p}\dot{+}\cdots\dot{+}J_{27_N}.$$

For arbitrary $1\leq p_1,\cdots,p_m\leq N$, $J_{27_{p_1}}\dot{+}\cdots\dot{+}J_{27_{p_m}}$ and $Q^{p_1}_i\dot{+}Q^{p_2}_i\dot{+}\cdots\dot{+}Q^{p_m}_i$ are subalgebras of J_{27}.

Proof The result follows from identities (3.5) and (4.6).

Lastly, we study the structure of the 3-Lie algebra J_{31}. Denote

$$J_{31_p}=\{A\in J_{31}\mid A=(A^1_1,\cdots,A^1_p,\cdots,A^1_N)=(0,\cdots,0,A^1_p\cdots,0)\},\quad 1\leq p\leq N.$$

Theorem 4.4 (1) *The 3-Lie algebra J_{31} is a 2-step-nilpotent 3-Lie algebra,*

$$J^1_{31}=\sum_{p,j=2}^{N}\sum_{k=1}^{N}F(E_{1jk}-E_{pjk}+E_{p11}-E_{111})+\sum_{p,k=2}^{N}F(E_{11k}-E_{p1k}+E_{p11}-E_{111}). \tag{4.7}$$

(2) *The 3-Lie algebra J_{31} can be decomposed into the direct sum of abelian subalgebras*

$$J_{31}=J_{31_1}\dot{+}\cdots\dot{+}J_{31_p}\dot{+}\cdots\dot{+}J_{31_N}.$$

Proof The identity (4.7) follows from the direct computation according to the multiplication (3.6). By the identities (2.2) and (2.4), we have $E_{ijk}*_{31}E_{lmn}=E_{imn}$ and $\langle E_{ijk}\rangle_4=1$, $1\leq i,j,k,l,m,n\leq N$. Then for arbitrary $E_{ijk}, E_{lmn}, E_{rpq}$, E_{xyz}, E_{uvw},

$$\begin{aligned}&[[E_{ijk}, E_{lmn}, E_{rpq}]_{31}, E_{xyz}, E_{uvw}]_{31}\\ &=[E_{lpq}-E_{rmn}-E_{ljk}+E_{imn}-E_{ipq}+E_{rjk}, E_{xyz}, E_{uvw}]_{31}\\ &=(E_{lpq}-E_{rmn}-E_{ljk}+E_{imn}-E_{ipq}+E_{rjk})*_{31}E_{xyz}\\ &\quad -E_{xyz}*_{31}(E_{lpq}-E_{rmn}-E_{ljk}+E_{imn}-E_{ipq}+E_{rjk})\\ &\quad +E_{uvw}*_{31}(E_{lpq}-E_{rmn}-E_{ljk}+E_{imn}-E_{ipq}+E_{rjk})\\ &\quad -(E_{lpq}-E_{rmn}-E_{ljk}+E_{imn}-E_{ipq}+E_{rjk})*_{31}E_{uvw}=0.\end{aligned}$$

Therefore, J_{31} is 2-step-nilpotent. The result (1) holds.

It is clear that

$$J_{31} = J_{31_1} \dot{+} \cdots \dot{+} J_{31_p} \dot{+} \cdots \dot{+} J_{31_N}$$

as the direct sum of subspaces. For $1 \leq i, j, k, l, m, n, p, q, r \leq N$, from identity (3.6),

$$[E_{ijk}, E_{imn}, E_{ipq}]_{31} = E_{ipq} - E_{ipq} + E_{imn} - E_{imn} + E_{ijk} - E_{ljk} = 0,$$

then J_{31_p}, $1 \leq p \leq N$ are abelian subalgebras but non-ideals. It follows the result (2).

References

[1] Kerner, R., The cubic chessboard, arXiv: 0004031v1.

[2] Awata, R., Li, M., Minicc, D. and Yoneyad, T., On the quantization of Nambu brackets, *J. High Energy Phys.*, 2001, DOI: 10.1066/1126-6708/2001/02/013.

[3] Filippov, V., n-Lie algebras, *Sib. Mat. Zh.*, **26**(6), 1985, 126–140.

[4] Bai, R., Han, W. and Liu, H., Structures of 2-step nilpotent 3-Lie algebras, *Antarctica J. Math.*, **9**(1), 2012, 23–40.

[5] Bai, R., Bai, C. and Wang, J., Realizations of 3-Lie algebras, *J. Math. Phys.*, **51**, 2010, 063505.