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Inhomogeneous Quantum Codes (III): The Asymmetric Case*

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Abstract The stabilizer (additive) method and non-additive method for constructing asymmetric quantum codes have been established. In this paper, these methods are generalized to inhomogeneous quantum codes.

Keywords Inhomogeneous quantum codes, Mixed classical codes, Asymmetric quantum codes
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1 Introduction

After the works of Shor [11] and Steane [12–13] in 1995–1996, the theory of quantum error-correcting codes developed rapidly. In 1998, Calderbank et al. [2] presented systematic methods to construct binary quantum codes, called stabilizer codes or additive codes, from classical error-correcting codes. At the same time, the stabilizer method was generalized to non-binary quantum codes and new methods were found to construct non-additive quantum codes. Recently, a number of new types of quantum codes, such as convolutional quantum codes, subsystem quantum codes and asymmetric quantum codes, were studied and the stabilizer method was extended to these variations of quantum codes. In particular, there were intensive activities in the area of asymmetric quantum codes (see [1, 3, 8–9, 16]).

The current paper concentrates on the asymmetric quantum codes which deal with the case, in which dephasing errors (Z-errors) happen more frequently than qubit-flipping errors (X-errors) (see [12–13]). Such codes are used in fault tolerant operations of a quantum computer carrying controlled and measured quantum information over asymmetric channels (see [14]). The characterization of non-additive symmetric quantum codes was given in [5–6] to the asymmetric case, and several examples of good asymmetric quantum codes were shown in [15].

In this paper, we deal with the asymmetric case of the more general quantum codes, called inhomogeneous quantum codes. An inhomogeneous quantum code is a subspace of $\mathbb{C}^{q_1} \otimes \mathbb{C}^{q_2} \otimes \cdots \otimes \mathbb{C}^{q_n}$, where $q_1, \cdots, q_n$ may take different positive integers. Inhomogeneous quantum codes were researched as early as in 1999 (see [7, 9]), and it seems that such kind of quantum codes has not been well-developed since then. The general formulation of symmetric inhomogeneous quantum codes was defined, and the stabilizer (additive) construction and non-additive construction were extended to the symmetric inhomogeneous case in [4, 15]. The main

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aim of this paper is to establish the stabilizer and non-additive constructions for asymmetric inhomogeneous quantum codes and present several series of good asymmetric inhomogeneous quantum codes by using these constructions.

This paper is organized as follows. In Section 2, we recall the basic facts on mixed classical codes and inhomogeneous (symmetric) quantum codes introduced in [4]. Then we define asymmetric inhomogeneous quantum codes as a slight variation of the symmetric ones. In Section 3, we establish the stabilizer construction of asymmetric inhomogeneous quantum codes and show some examples of good (additive) asymmetric inhomogeneous quantum codes by using the stabilizer method. In Section 4, we present a new characterization of asymmetric inhomogeneous quantum codes and show some non-additive asymmetric inhomogeneous quantum codes as an application of the new characterization.

2 Preliminaries

2.1 Mixed classical codes

We recall some basic facts on mixed classical codes introduced in [4]. These facts will be used to construct asymmetric inhomogeneous quantum codes in the next two sections. Let $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ be a finite abelian group and $|A_i| = q_i$ $(1 \le i \le n)$. We assume that

$$2 \le q_1 \le q_2 \le \cdots \le q_n. \tag{2.1}$$

For $a = (a_1, \cdots, a_n)$ and $b = (b_1, \cdots, b_n)$ in A $(a_i, b_i \in A_i), 1 \le i \le n$, we define the Hamming weight of a by

$$w_H(a) = \sharp\{i:\ 1 \le i \le n,\ a_i \ne 0\},$$

and the Hamming distance between a and b by

$$d_H(a, b) = \sharp\{i:\ 1 \le i \le n,\ a_i \ne b_i\} = w_H(a - b).$$

Definition 2.1 *A (classical) mixed code C on A is a subset of A with the size $K = |C| \ge 2$. The minimal distance of C is defined by*

$$d = d(C) = \min\{w_H(c - c'):\ c, c' \in,\ c \ne c'\}.$$

We call (A, K, d) the parameters of C. A mixed code C on A is called additive if C is a subgroup of A. For the additive code C, we have $d(C) = \min\{w_H(C):\ 0 \ne c \in C\}$. A mixed code C with the minimal distance d can detect $\le d-1$ digits of errors or correct $\le [\frac{\mathrm{d}-1}{2}]$ digits of errors. One of the basic problems in the classical coding theory is to construct mixed codes having larger efficiency $\frac{|C|}{|A|} = \frac{K}{N}$ $(N = q_1 \cdots q_n)$ and larger $d = d(C)$. We have the following Singleton bound (see [4, Lemma 3.2]):

$$K \le q_1 q_2 \cdots q_{n-d+1}. \tag{2.2}$$

One of the best classes of mixed codes is called MDS code which meets the Singleton bound: $K = q_1 q_2 \cdots q_{n-d+1}$. Some examples of the mixed MDS (algebraic geometry) codes were presented in [4, Theorem 4.3].

Let $\widehat{A} = \mathrm{Hom}(A, \mathbb{C}^*)$ be the group of characters of A. It is known that we have an isomorphism of groups $A \to \widehat{A},\ a \mapsto \chi_a$ and such isomorphism can be chosen to satisfy

$$\chi_a(b) = \chi_b(a), \quad a, b \in A. \tag{2.3}$$

For an additive code C on A, the dual code $C^{\perp}$ of C is defined by

$$C^{\perp} = \{a \in A : \ \chi_a(c) = 1 \text{ for all } c \in C\},$$

where $C^{\perp}$ is additive and $|C| \cdot |C^{\perp}| = |A|$. In the next section, we use the following "symplectic" mapping $(\ , \ )_s$ to analyse and construct asymmetric inhomogeneous quantum codes

$$(\ , \ )_s : \ A^2 \times A^2 \to \mathbb{C}^*(= \mathbb{C}\backslash\{0\}),$$

where for $v = (a|b), v' = (a'|b')$ $(a, a', b, b' \in A)$,

$$(v, v')_s = \chi_a(b')^{-1}\chi_b(a').$$

It can be seen that this mapping is a non-degenerate pairing which means that

(1) $(v_1 + v_2, v')_s = (v_1, v')_s(v_2, v')_s, \ (v, v')_s = (v', v)_s^{-1}$,

(2) $(v, v')_s = 1$ for all $v' \in A^2$ if and only if $v = 0$.

For a subgroup G of A^2, the set

$$G^{\perp_s} = \{v \in A^2 : \ (v, v')_s = 1 \text{ for all } v' \in G\}$$

is also a subgroup of A^2, called the symplectic dual of G. It can be seen that $|G| \cdot |G^{\perp_s}| = |A^2|$ and $(G^{\perp_s})^{\perp_s} = G$.

2.2 Asymmetric inhomogeneous quantum codes

Now we recall the notations on (symmetric) inhomogeneous quantum codes given in [4]. Let

$$V = V_1 \otimes V_2 \otimes \cdots \otimes V_n, \quad V_i = \mathbb{C}^{q_i}, \quad 1 \le i \le n.$$

For each i, let $\{|c\rangle : \ c \in A_i\}$ be a fixed orthonormal basis of V_i. Namely, for $c, c' \in A_i$,

$$\langle c \mid c' \rangle = \begin{cases} 1, & \text{if } c = c', \\ 0, & \text{otherwise,} \end{cases}$$

where $\langle \ | \ \rangle$ denotes the Hermitian inner product on the complex vector space V_i. Then V has the following orthonormal basis:

$$\{|c\rangle = |c_1 c_2 \cdots c_n\rangle = |c_1\rangle \otimes |c_2\rangle \otimes \cdots \otimes |c_n\rangle : c = (c_1, c_2, \cdots, c_n) \in A\} \tag{2.4}$$

and $|c_i\rangle$ is called the i-th quantum digit qubit of $|c\rangle$. An A-ary (inhomogeneous) quantum state is a non-zero vector in V which is uniquely expressed by

$$|v\rangle = \sum_{c \in A} \varphi(c)|c\rangle, \quad \varphi(c) \in \mathbb{C}.$$

In quantum physics, two quantum states $|v\rangle = \sum_{c} \varphi(c)|c\rangle$ and $|u\rangle = \sum_{c} \psi(c)|c\rangle$ are called indistinguishable if $|v\rangle = \alpha|u\rangle$ for some nonzero complex number α, namely, $\varphi(c) = \alpha\psi(c)$ for all $c \in A$. Such two quantum states are assumed the same in the quantum world. On the other hand, $|v\rangle$ and $|u\rangle$ are called totally distinguishable if $\langle v|u\rangle = \sum_{c \in A} \overline{\varphi}(c)\psi(c) = 0$, where $\overline{\varphi}(c)$ is the complex conjugate of $\varphi(c)$.

Now we introduce the quantum error group acting on V. Each quantum error is a unitary (linear) operator on the complex vector space V. At each qudit, there are two types of errors

$X(a_i)$ and $Z(b_i)$ $(a_i, b_i \in V_i)$ acting on $V_i = \mathbb{C}^{q_i}$ defined by their action on the basis $\{|c\rangle :\ c \in A_i\}$:

$$X(a_i)|c\rangle = |a_i + c\rangle, \quad Z(b_i)|c\rangle = \chi_{b_i}(c)|c\rangle. \tag{2.5}$$

On the quantum state space $V = V_1 \otimes V_2 \otimes \cdots \otimes V_n$, we have quantum error operators $X(a)$ and $Z(b)$, $a = (a_1, \cdots, a_n)$, $b = (b_1, \cdots, b_n) \in A$ $(a_i, b_i \in A_i)$ defined by their action on the basis (2.4) as, for $|c\rangle = |c_1 c_2 \cdots c_n\rangle$ $(c_i \in A_i)$,

$$\begin{aligned} X(a)|c\rangle &= X(a_1)|c_1\rangle \otimes X(a_2)|c_2\rangle \otimes \cdots \otimes X(a_n)|c_n\rangle \\ &= |a_1 + c_1\rangle \otimes |a_2 + c_2\rangle \otimes \cdots \otimes |a_n + c_n\rangle = |a + c\rangle, \\ Z(b)|c\rangle &= Z(b_1)|c_1\rangle \otimes Z(b_2)|c_2\rangle \otimes \cdots \otimes Z(b_n)|c_n\rangle \\ &= \chi_{b_1}(c_1)|c_1\rangle \otimes \chi_{b_2}(c_2)|c_2\rangle \otimes \cdots \otimes \chi_{b_n}(c_n)|c_n\rangle \\ &= \chi_b(c)|c\rangle. \end{aligned} \tag{2.6}$$

Let m be the exponent of the group A. Namely, m is the smallest positive integer, such that $ma = 0$ for all $a \in A$. Then the values of all characters $\chi \in \widehat{A}$ are the power of $\omega = \mathrm{e}^{\frac{2\pi\sqrt{-1}}{m}}$. The (non-abelian) quantum error group of V is

$$E_n = \{\omega^\lambda X(a)Z(b) :\ \lambda \in \{0, 1, \cdots, m-1\},\ a, b \in A\}.$$

The mapping

$$E_n \to \overline{E}_n = A^2, \quad \varepsilon = \omega^\lambda X(a)Z(b) \mapsto \overline{\varepsilon} = (a \mid b)$$

is an epimorphism from the group E_n to the additive group $A^2 = A \oplus A$. The following result is a starting point on the stabilizer construction of inhomogeneous quantum codes.

Lemma 2.1 *A subgroup G of E_n is abelian if and only if $\overline{G} = \{\overline{e} :\ e \in G\}$ is a symplectic self-orthogonal subgroup of A^2 (namely, $\overline{G} \subseteq (\overline{G}^{\perp_s})$).*

Definition 2.2 *For a quantum error $e = \omega^\lambda X(a)Z(b) \in E_n$ and $\overline{e} = (a \mid b) \in \overline{E}_n = A^2$ $(a = (a_1, \cdots, a_n),\ b = (b_1, \cdots b_n),\ a_i, b_i \in A)$, we define their quantum weight w_Q, X-weight w_X and Z-weight w_Z by*

$$\begin{aligned} w_Q(e) &= w_Q(\overline{e}) = \sharp\{i :\ 1 \le i \le n,\ (a_i, b_i) \ne (0,0) \in A_i^2\}, \\ w_X(e) &= w_X(\overline{e}) = \sharp\{i :\ 1 \le i \le n,\ a_i \ne 0\}, \\ w_Z(e) &= w_Z(\overline{e}) = \sharp\{i :\ 1 \le i \le n,\ b_i \ne 0\}, \end{aligned}$$

respectively. Namely, w_Q is the number of qubits, where the quantum error $X(a_i)Z(b_i) \ne I$ occurs. w_X and w_Z are the numbers of qubits, where the X-error and the Z-error occur respectively.

Now we define the inhomogeneous quantum codes over A.

Definition 2.3 *An inhomogeneous quantum code Q over A is a complex vector subspace of V with dimension $K = \dim_{\mathbb{C}} Q \ge 1$. Each non-zero vector in Q is called a codeword.*

For $K \ge 2$, we call that a quantum error $e \in E_n$ can be detected by Q, if for any $|v\rangle$ and $|v'\rangle$ in Q such that $\langle v|v'\rangle = 0$ (i.e., $|v\rangle$ and $|v'\rangle$ are totally distinguishable), we have $\langle v|e|v'\rangle = 0$ (i.e., $|v\rangle$ and $e|v'\rangle$ are distinguishable). More generally, for a subset S of E_n, we call that S can be detected by Q, if each $e \in S$ can be detected by Q.

A (symmetric) inhomogeneous quantum code Q over A is called to have parameters (A, K, d) $(d \geq 1)$ if $K = \dim Q$ and the set of quantum errors

$$E_n(d-1) = \{e \in E_n : \ w_Q(e) \leq d-1\}$$

can be detected by Q.

The Asymmetric quantum code is a slight variation of the symmetric one. Let $d_X, d_Z \geq 1$.

Definition 2.4 *An inhomogeneous quantum code Q over A is called asymmetric with parameters $\left(A, K, \frac{d_z}{d_x}\right)$, if the set of quantum errors*

$$E_n\Big(d_z - \frac{1}{d_x} - 1\Big) = \{e \in E_n : \ w_Z(e) \leq d_z - 1, \ w_X(e) \leq d_x - 1\}$$

can be detected by Q. Such a code Q is called pure for $\frac{d_z}{d_x}$, if for any $|v\rangle$ and $|v'\rangle$ in Q (not necessarily $\langle v|v'\rangle = 0$) and any $e \in E_n\big(d_z - \frac{1}{d_x} - 1\big)$, $w_Q(e) \geq 1$, we have $\langle v|e|v'\rangle = 0$. When $K = 1$, an asymmetric inhomogeneous quantum code with parameters $\left(A, 1, \frac{d_z}{d_x}\right)$ is always pure for $\frac{d_z}{d_x}$.

One of the basic problems in the quantum code theory is to construct asymmetric inhomogeneous quantum codes with large efficiency $\frac{\dim Q}{\dim V} = \frac{K}{N}$ $(N = |A| = q_1 \cdots q_n)$ and larger d_X, d_Z (good ability to detect and correct quantum errors). As the cases of usual classical and quantum codes, we have bounds of the parameters to judge the goodness of asymmetric inhomogeneous quantum codes. Let Q be a pure quantum code with parameters $\left(A, K, \frac{d_z}{d_x}\right)$. By the definition of pure quantum codes and the inequalities $w_X(ee') \leq w_X(e) + w_X(e')$ and $w_Z(ee') \leq w_Z(e) + w_Z(e')$ for $e, e' \in E_n$, we have that the $N(d_z, d_x)$ subspaces of V

$$e(Q) = \{e|v\rangle : \ |v\rangle \in Q\}, \quad e \in \overline{E}_n\Bigg(\frac{\left[\frac{\mathrm{d}_z - 1}{2}\right]}{\left[\frac{\mathrm{d}_x - 1}{2}\right]}\Bigg)$$

are orthogonal to each other, where

$$N(d_z, d_x) = \left|\overline{E}_n\Bigg(\frac{\left[\frac{\mathrm{d}_z - 1}{2}\right]}{\left[\frac{\mathrm{d}_x - 1}{2}\right]}\Bigg)\right| = N(d_z)N(d_x)$$

with

$$N(d) = \sum_{\lambda=0}^{\left[\frac{\mathrm{d}-1}{2}\right]} \sum_{1 \leq i_1 < \cdots < i_\lambda \leq n} (q_{i_1} - 1) \cdots (q_{i_\lambda} - 1) \binom{n}{\lambda}.$$

Therefore, we get the following Hamming bound for pure quantum codes:

$$|A| (= q_1 \cdots q_n) \geq K \cdot N(d_x) N(d_z).$$

The quantum code with parameters $\left(A, K, \frac{d_z}{d_x}\right)$ is called perfect if $|A| = K \cdot N(d_x)N(d_z)$. On the other hand, we will show that for some asymmetric quantum codes constructed in Sections 3 and 4, we have the following Singleton bound:

$$K \leq \frac{q_1 q_2 \cdots q_{n-d_X+1} q_1 q_2 \cdots q_{n-d_Z+1}}{q_1 q_2 \cdots q_n}. \tag{2.7}$$

It seems that this Singleton bound may be true for all asymmetric inhomogeneous quantum codes. As in the classical case, an asymmetric inhomogeneous quantum code is called an MDS code, if the equality in (2.7) holds.

In the next section, we establish the stabilizer construction of asymmetric inhomogeneous quantum codes and present examples of perfect or MDS codes by using this construction.

At the end of this section, we remark that the asymmetric quantum code with parameters $\left(A, K, \frac{d_z}{d_x}\right)$ has more powerful ability than the symmetric quantum code with parameters (A, K, d), since the detected set $E_n\left(d - \frac{1}{d} - 1\right)$ is usually larger than the set $E_n(d-1)$.

3 Stabilizer (Additive) Asymmetric Inhomogeneous Quantum Codes

In this section, we give the stabilizer construction of asymmetric inhomogeneous quantum codes as a generalization of such construction in the symmetric case (see [2, 4]) and the asymmetric usual case (see [16] for $A_i = \mathbb{F}_q$ $(1 \le i \le n)$). As in Section 2, let $A = A_1 \oplus \cdots \oplus A_n$, A_i be an abelian group and $|A_i| = q_i$. We fix an isomorphism $A \to \widehat{A}$, $a \mapsto \chi_a$, such that formula (2.3) holds.

Theorem 3.1 *If there exists a classical additive mixed code C in $A \oplus A$ and $C \subseteq C^{\perp_s}$, then there exists an asymmetric inhomogeneous quantum code Q in V with* $\dim Q = \frac{|A|}{|C|}$ *and $S = \{e \in E_n :\ w_X(e) \le w_X(C^{\perp_s} \backslash C) - 1$ or $w_Z(e) \le w_Z(C^{\perp_s} \backslash C) - 1\}$ can be detected by Q. Particularly, Q has parameters $\left(A, K, \frac{d_z}{d_x}\right)$, where*

$$K = \frac{|A|}{|C|}, \quad d_x = w_X(C^{\perp_s} \backslash C), \quad d_z = w_Z(C^{\perp_s} \backslash C),$$

where for a subset S of $A \oplus A$,

$$w_X(S) = \min\{w_X(v) :\ v \in S\}, \quad w_Z(S) = \min\{w_Z(v) :\ v \in S\}.$$

Moreover, Q is pure for $\frac{d'_z}{d'_x}$, where $d'_x = w_X(C^{\perp_s} \backslash \{0\})$, $d'_z = w_Z(C^{\perp_s} \backslash \{0\})$.

Proof The following proof is a minor modification of the proof in [2; 4, Theorem 4.1]. We omit some details. Firstly by Lemma 2.2 and the assumption $C \subseteq C^{\perp_s}$, we can lift C to an abelian subgroup G of E_n, such that $\overline{G} = C$ and $|G| = |C|$. Then the complex vector space $V = V_1 \otimes V_2 \otimes \cdots \otimes V_n$ $(V_i = \mathbb{C}^{q_i})$ has the following orthogonal decomposition:

$$V = \bigoplus_{\chi \in \widehat{G}} V(\chi),$$

where for each $\chi \in \widehat{G}$,

$$V(\chi) = \{|v\rangle \in V :\ g|v\rangle = \chi(g)|v\rangle \text{ for each } g \in G\}.$$

We show that each $Q = V(\chi)$ is a quantum code with the required parameters $\left(A, K, \frac{d_z}{d_x}\right)$.

In order to determine the dimension $\dim Q = \dim V(\chi)$, it can be proved that each quantum error operator $e \in E_n$ is a permutation of the set $\Sigma = \{V(\chi) : \chi \in \widehat{G}\}$ and the group E_n acts on Σ transitively. Therefore, all $V(\chi)$ $(\chi \in \widehat{G})$ have the same dimension, so that $\dim Q = \dim V(\chi) = \frac{\dim V}{|\widehat{G}|} = \frac{|A|}{|C|} = K$.

Next we determine the parameters d_x and d_z. We need to show that if $|v\rangle, |v'\rangle \in Q = V(\chi)$ and $\langle v|v'\rangle = 0$, then for any $\overline{e} \in \overline{E_n} = A \oplus A$ with $w_X(\overline{e}) \le w_X(C^{\perp_s} \backslash C) - 1$ and $w_Z(\overline{e}) \le w_Z(C^{\perp_s} \backslash C) - 1$, we have $\langle v|\overline{e}|v'\rangle = 0$. If $\overline{e} \in C = \overline{G}$, then $\overline{e}|v'\rangle = \chi(\overline{e})|v'\rangle$, so that $\langle v|\overline{e}|v'\rangle = \chi(\overline{e})\langle v|v'\rangle = 0$. If $\overline{e} \notin C$, then $\overline{e} \notin C^{\perp_s}$ which means that there exists an $\overline{e}' \in C$, such that $\langle \overline{e}, \overline{e}' \rangle_s \ne 1$. For $|v\rangle \in Q = V(\chi)$,

$$\overline{e}'(\overline{e}|v\rangle) = (\overline{e}'\overline{e})|v\rangle = \langle \overline{e}, \overline{e}' \rangle_s \overline{e}\,\overline{e}'|v\rangle = \chi(\overline{e}')\langle \overline{e}, \overline{e'} \rangle_s \overline{e}|v\rangle = \chi'(\overline{e}')\overline{e}|v\rangle,$$

where $\chi' \in \widehat{G}$ and $\chi' \neq \chi$. Therefore, $\overline{e}$ is a mapping from $Q = V(\chi)$ to $V(\chi')$. Since $|v\rangle \in V(\chi)$, $\overline{e}|v'\rangle \in V(\chi')$ and $V(\chi) \perp V(\chi')$, we have $\langle v|\overline{e}|v'\rangle = 0$.

At last we show that $Q = V(\chi)$ is pure for $d_x = w_X(C^{\perp_s}\backslash\{0\})$ and $d_z = w_Z(C^{\perp_s}\backslash\{0\})$. By the definition, for any $|v\rangle, |v'\rangle \in Q$, any $0 \neq \overline{e} \in \overline{E_n}$ with $w_X(\overline{e}) \leq w_X(C^{\perp_s}\backslash\{0\}) - 1$ and $w_Z(\overline{e}) \leq w_Z(C^{\perp_s}\backslash\{0\}) - 1$, we need to show that $\langle v|\overline{e}|v'\rangle = 0$. From the assumption, we know that $\overline{e} \notin C^{\perp_s}$. Then, by the above argument, $\overline{e}|v'\rangle \in V(\chi')$, $\chi' \neq \chi$, so that $\langle v|\overline{e}|v'\rangle = 0$. This completes the proof of Theorem 3.1.

As a usual case, the asymmetric inhomogeneous quantum codes constructed by Theorem 3.1 are called additive codes since they come from the classical additive codes over $A \oplus A$. We need C to be symplectic self-orthogonal and $\frac{d_z}{d_x}$ to be determined by the minimum quantum weights w_X and w_Z of $C^{\perp_s}\backslash C$. The next result shows that we can get asymmetric inhomogeneous quantum codes from a pair of classical additive codes C_1 and C_2 over A with $C_2^{\perp} \subseteq C_1$, where $C_2^{\perp}$ is the usual dual of C_2 and $\frac{d_z}{d_x}$ can be determined by the Hamming weights of C_1 and C_2.

Theorem 3.2 *If there exist* (*mixed*) *additive codes C_1 and C_2 with parameters (A, K_1, d_1) and (A, K_2, d_2) respectively and $C_2^{\perp} \subseteq C_1$, then there exists an asymmetric inhomogeneous quantum code Q with parameters $\left(A, \frac{K_1K_2}{|A|}, \frac{d_z}{d_x}\right)$, where $d_x = w_H(C_2\backslash C_1^{\perp})$ and $d_z = w_H(C_1\backslash C_2^{\perp})$. Moreover, Q is pure for $d_x' = d_2$ and $d_z' = d_1$.*

Proof Consider the additive code $C = C_1^{\perp} \oplus C_2^{\perp}$ over $A \oplus A$. The assumption $C_2^{\perp} \subseteq C_1$ implies $C_1^{\perp} \subseteq C_2$ and $C^{\perp_s} = C_2 \oplus C_1 \supseteq C$. By Theorem 3.1, we have the asymmetric quantum code Q with parameters $\left(A, K, \frac{d_z}{d_x}\right)$, where

$$K = \frac{|A|}{|C|} = \frac{|A| \cdot |C_1| \cdot |C_2|}{|A|^2} = \frac{K_1K_2}{|A|},$$

and Q can detect the set

$$\begin{aligned} &\{\overline{e} \in \overline{E}_n = A \oplus A: \ w_X(\overline{e}) \leq w_X(C_2 \oplus C_1\backslash C_1^{\perp} \oplus C_2^{\perp}) - 1 \text{ and} \\ &w_Z(\overline{e}) \leq w_Z(C_2 \oplus C_1\backslash C_1^{\perp} \oplus C_2^{\perp}) - 1\} \\ = &\{\overline{e} \in A \oplus A: \ w_X(\overline{e}) \leq w_X(C_2\backslash C_1^{\perp}) - 1 \text{ and } w_Z(\overline{e}) \leq w_Z(C_1\backslash C_2^{\perp}) - 1\}. \end{aligned}$$

Thus we can take $d_x = w_X(C_2\backslash C_1^{\perp})$ and $d_z = w_Z(C_1\backslash C_2^{\perp})$. Moreover, for any $|v\rangle$ and $|v'\rangle \in Q$ we have $\langle v|\overline{e}|v'\rangle = 0$ for any $\overline{e}$ in the set

$$\begin{aligned} &\{0 \neq \overline{e} \in \overline{E}_n = A \oplus A: \ w_X(\overline{e}) \leq w_X(C_2 \oplus C_1\backslash\{0\}) - 1 \text{ and} \\ &w_Z(\overline{e}) \leq w_Z(C_2 \oplus C_1\backslash\{0\}) - 1\} \\ \supseteq &\{0 \neq \overline{e} \in \overline{E}_n: \ w_X \leq w_H(C_2\backslash\{0\}) - 1 = d_2 - 1 \text{ and } w_Z(\overline{e}) \leq w_H(C_1\backslash\{0\}) - 1 = d_1 - 1\}. \end{aligned}$$

Therefore, Q is pure for $d_x' = d_2$ and $d_z' = d_1$.

Remark 3.1 In Theorem 3.2, if $d_1^{\perp} = w_H(C_1^{\perp}\backslash\{0\}) > w_H(C_2\backslash C_1^{\perp})$ and $d_2^{\perp} = w_H(C_2^{\perp}\backslash\{0\}) > w_H(C_1\backslash C_2^{\perp})$, then Q is pure quantum code with parameters $\left(A, K, \frac{d_z}{d_x}\right)$, $d_x = d_2$ and $d_z = d_1$. For the classical mixed codes C_1 and C_2, we have the Singleton bound

$$K_1 \leq q_1q_2 \cdots q_{n-d_z+1}, \quad K_2 \leq q_1q_2 \cdots q_{n-d_x+1}.$$

Thus the asymmetric quantum code Q satisfies the Singleton bound (2.7),

$$K = \frac{K_1K_2}{|A|} \leq \frac{q_1q_2 \cdots q_{n-d_x+1}q_1q_2 \cdots q_{n-d_z+1}}{q_1q_2 \cdots q_n}.$$

In fact, Q is an MDS code if and only if both C_1 and C_2 are MDS codes. On the other hand, we can see similarly that the quantum code Q is perfect ($|A| = K \cdot N(d_1)N(d_2)$) if and only if both the classical codes C_1 and C_2 are perfect ($|A| = K_1 N(d_1)$ and $|A| = K_2 N(d_2)$).

(2) Taking $C = C_2^{\perp} \oplus C_1^{\perp}$ in the proof of Theorem 3.2, we get the asymmetric quantum code Q with parameters $\left(A, K, \frac{d_z}{d_x}\right)$, where $d_x = w_H(C_1 \backslash C_2^{\perp})$, $d_z = w_H(C_2 \backslash C_1^{\perp})$, and the code Q' is pure for $d_x = d_1$ and $d_z = d_2$.

Example 3.1 (Perfect Quantum Codes) Suppose that there exists a perfect (MDS) additive code C in A with parameters (A, K, d), $K = q_1 \cdots q_{n-d+1}$. Take $C_1 = C$ and $C_2 = A$ in Theorem 3.2. Then $C_2^{\perp} = \{0\} \subseteq C_1$ and C_2 is a trivial perfect (MDS) code with parameters $(A, |A|, 1)$. By Remark 3.1, we get a perfect (MDS) quantum code with parameters $\left(A, K, \frac{d_z}{d_x}\right)$ where $\{d_x, d_z\} = \{d, 1\}$. Such a quantum code can detect only the X-error (Z-error) for $d_x = d$ and $d_z = 1$ (for $d_x = 1$ and $d_z = d$).

It is known that for the usual q-ary case ($A = A_1 \oplus \cdots \oplus A_n = \mathbb{F}_q^n, A_i = \mathbb{F}_q, 1 \le i \le n$), all the nontrivial parameters of perfect additive classical codes are

$$\begin{aligned} &(\mathbb{F}_q^n, q^{n-m}, 3), \quad n = \frac{q^m - 1}{q - 1}, \quad \text{Hamming codes,} \\ &(\mathbb{F}_2^{23}, 2^{11}, 7), \quad (\mathbb{F}_3^{11}, 3^5, 5), \quad \text{Golay codes.} \end{aligned}$$

For the more general mixed case, Herzog and Schonheim [17] presented a group-partition method to construct classical mixed codes with $d = 3$. We introduce this construction briefly now.

Let G be a finite (additive) abelian group and $G_1, \cdots, G_n$ be subgroups of G. $\{G_1, \cdots, G_n\}$ is called a partition of G, if $G_i \backslash \{0\}$ $(1 \le i \le n)$ is a partition of $G \backslash \{0\}$. Namely,

$$G \backslash \{0\} = \bigcup_{i=1}^{n} (G_i \backslash \{0\}) \quad \text{(disjoint)},$$

which implies that

$$(*) \quad |G| - 1 = \sum_{i=1}^{n} (|G_i| - 1).$$

For a partition $\{G_1, \cdots, G_n\}$ of G, consider the mapping

$$\begin{aligned} \varphi: \ & A = G_1 \oplus G_2 \oplus \cdots \oplus G_n \to G, \\ & (g_1, g_2, \cdots, g_n) \mapsto g_1 + g_2 + \cdots + g_n. \end{aligned}$$

Then φ is an epimorphism of groups, so that $C = \ker \varphi$ is an additive code in A and $K = |C| = \frac{|A|}{|G|} = \frac{|G_1| \cdots |G_n|}{|G|}$. By a simple computation and $(*)$, we know that C is a perfect code with parameters $(A, K, 3)$.

It is proved that if G has a partition ($n \ge 2$), then G should be an elementary p-group. Namely, G is an additive group $\mathbb{F}_p^m$ for some prime number p and $m \ge 2$. Several partitions of $(\mathbb{F}_q^m, +)$ were constructed in [17–18, 20–21]. From these constructions, we get several perfect quantum codes with parameters $\left(A, K, \frac{d_z}{d_x}\right)$ for some group $A = A_1 \oplus \cdots \oplus A_n, A_i = (\mathbb{F}_p^{m_i}, +)$ $(1 \le i \le n)$ and $\{d_x, d_z\} = \{1, 3\}$.

Example 3.2 (MDS Quantum Codes) By using the Riemann-Roch theorem for a function field M with a constant field $\mathbb{F}_q$, the following classical mixed (algebraic-geometric) codes were constructed in [4].

Lemma 3.1 (see [4, Theorem 3.2] or [19]) *Let* $A = \mathbb{F}_{q_1} \oplus \mathbb{F}_{q_2} \oplus \cdots \oplus \mathbb{F}_{q_n}$, $q_i = q^{m_i}$ $(1 \le i \le n)$ *and* $m_1 \le m_2 \le \cdots \le m_n$*,* M *be a function field with a constant field* $\mathbb{F}_q$*,* $g = g(M)$ *be the genus of* M*,* P_i $(1 \le i \le n)$ *be distinct prime divisors of* M*,* $\deg P_i = m_i$ $(1 \le i \le n)$*,* $D = P_1 + P_2 + \cdots + P_n$*, and* $m = \deg D = m_1 + m_2 + \cdots + m_n$*. Let* G *be a divisor of* M *and* $v_{P_i}(G) = 0$ $(1 \le i \le n)$*. Then*

(1) *If* $\deg G \le m - 1$*, then*

$$C(D, G) = \{c_f = (f(P_1), f(P_1), \cdots, f(P_n)) \in A : \ f \in L(G)\}$$

is an $\mathbb{F}_q$*-linear code with parameters* (A, K, d)*, where* $K = q^k$*,* $k = l(G) \ge \deg G + 1 - g$ *and* $d \ge t$*, where* t *is determined by*

$$m_1 + m_2 + \cdots + m_{n-t} - 1 < \deg G \le m_1 + m_2 + \cdots + m_{n-t+1} - 1.$$

Moreover, $l(G) = \deg G + 1 - g$ *if* $\deg G \ge 2g - 1$*; and* $d = t$ *if* $\deg G > m_1 + m_2 + \cdots + m_{n-t} - 1 + g$*.*

(2) *If* $\deg G \ge 2g - 1$*, then*

$$C'(D, G) = \{c_\omega = (\mathrm{res}_{P_1}\omega, \mathrm{res}_{P_2}\omega, \cdots, \mathrm{res}_{P_n}\omega) \in A : \ \omega \in \Omega(G - D)\}$$

is an $\mathbb{F}_q$*-linear code with parameters* (A, K', d')*, where* $K' = q^{k'}$*,* $k' = l(W + D - G) = \deg D - \deg G + g - 1 + l(G - D) \ge \deg D - \deg G + g - 1$ *and* $d' \ge t'$*, where* t' *is determined by*

$$m_n + m_{n-1} + \cdots + m_{n-t'+2} + 2g - 1 \le \deg G < m_n + \cdots + m_{n-t'+2} + m_{n-t'+1} + 2g - 1.$$

Moreover, $k' = \deg D - \deg G + g - 1$ *if* $\deg G \le m - 1$*; and* $d' = t'$ *if* $\deg G < m_n + m_{n-1} + \cdots + m_{n-t'+1} + g - 1$*.*

(3) *If* $2g - 1 \le \deg G \le m - 1$*, then* $C(D, G)^\perp = C'(D, G)$*.*

Wang and Feng [4] constructed a class of (symmetric) inhomogeneous quantum codes by the classical mixed algebraic-geometric codes.

Theorem 3.3 *Let* $q = p^s$*, where* p *is a prime number and* $s \ge 1$*. Let* t *be a positive integer. Let* $d_1, d_2, \cdots, d_l$ *be all the positive divisors of* t*, such that*

$$1 = d_1 < d_2 < \cdots < d_l = t,$$

and $m_1, m_2, \cdots, m_n$ *be the following integers:*

$$\begin{aligned} &m_1 = m_2 = \cdots = m_{N_q(d_1)} = d_1 (= 1), \\ &m_{N_q(d_1)+1} = m_{N_q(d_1)+2} = \cdots = m_{N_q(d_1)+N_q(d_2)} = d_2, \\ &\qquad\qquad\vdots \\ &m_{N_q(d_1)+\cdots+N_q(d_{l-1})+1} = \cdots = m_n = d_l, \end{aligned}$$

where

$$n = \sum_{\lambda=1}^{l} N_q(d_\lambda) = \sum_{e|d} N_q(e).$$

$N_q(e)$ *is the number of monic irreducible polynomials of degree* e *in* $\mathbb{F}_q[x]$ *and the number of finite prime divisors of degree* e *in the rational function field* $\mathbb{F}_q(x)$*. Let* $A = A_1 \oplus \cdots \oplus A_n$*,* *where* $A_i = \mathbb{F}_{q^{m_i}} (1 \le i \le n)$*. Then for each integer* k, $\frac{q^d}{2} \le k \le q^d$*, there exists a mixed additive quantum code* Q *with parameters* (A, K, d)*, where* $K = q^{2k-q^d}$*,* d *is determined by*

$$m_1 + m_2 \cdots + m_{n-d} < k \le m_1 + m_2 + \cdots + m_{n-d} + m_{n-d+1},$$

and Q *is pure for* d*. Moreover, if* $k = m_1 + m_2 + \cdots + m_{n-d+1}$*, then* Q *is an MDS code.*

Actually, the inhomogeneous quantum code constructed in Theorem 3.3 is also an asymmetric inhomogeneous quantum code with parameters $\left(A, K, \frac{d_z}{d_x}\right)$. In the proof of this theorem in [4], an additive mixed classical code C_k was constructed. C_k has parameters (A, q^k, d), $C_k^{\perp} \subseteq C_k$ and $w_H(C_k \backslash C_k^{\perp}) = w_H(C_k \backslash \{0\}) = d$. Letting $C_1 = C_2 = C_k$, we can get an asymmetric inhomogeneous quantum code with parameters $\left(A, K, \frac{d_z}{d_x}\right)$.

4 Non-additive Asymmetric Inhomogeneous Quantum Codes

In this section, we present a new characterization of asymmetric inhomogeneous quantum codes and show some methods to construct such non-additive codes. The new characterization is a generalization of symmetric cases given in [16].

Each A-ary quantum state $|v\rangle = \sum\limits_{c \in A} \alpha_c |c\rangle$ can be identified with a nonzero mapping $\varphi : A \to \mathbb{C}$ defined by $\varphi(c) = \alpha_c$ for all $c \in A$. For a subset S of $\{1, 2, \cdots, n\}$ and $c = (c_1, c_2, \cdots, c_n) \in A\,(c_i \in A_i)$, c_S is the sub vector of c whose coordinate positions belong to S. Namely, $c_S = (c_i)_{i \in S}$. And $A_S = \bigoplus\limits_{i \in S} A_i$ can be viewed as a subgroup of A. For $\varphi, \psi : A \to \mathbb{C}$, we define their Hermitian inner product by

$$(\varphi, \psi) = \sum_{c \in A} \overline{\varphi(c)} \psi(c) \in \mathbb{C},$$

where $\overline{\varphi(c)}$ stands for the conjugate of the complex number $\varphi(c)$.

Let $\widehat{A} = \{\chi_a : \ a \in A\}$ be the character group of A. For a function $f : A \to \mathbb{C}$, the Fourier transform of f is $F : A \to \mathbb{C}$, where

$$F(b) = \sum_{a \in A} f(a) \chi_b(a),$$

and we have the following inverse transform:

$$f(a) = \frac{1}{|A|} \sum_{b \in A} F(b) \overline{\chi_a(b)}.$$

In the proof of the following Theorem 4.2, we need the following two simple facts on Fourier transform.

Lemma 4.1 *Let* $F : A \to \mathbb{C}$ *be the Fourier transform of* $f : A \to \mathbb{C}$*. Then*
(1) $F \equiv 0$ *if and only if* $f \equiv 0$.
(2) $F(a) = 0$ *for all* $0 \ne a \notin A$ *if and only if* f *is a constant.*

Theorem 4.1 (i) *There exists an asymmetric inhomogeneous quantum code with parameters* $\left(A, K, \frac{d_z}{d_x}\right) (K \ge 2,\ d_x, d_z \ge 1)$ *if and only if there exist* K *nonzero mappings*

$$\varphi_i : A \to \mathbb{C}, \quad 1 \le i \le K, \tag{4.1}$$

satisfying the following conditions: for each d*,* $1 \le d \le \min\{d_x, d_z\}$ *and each partition of* $\{1, 2, \cdots, n\}$,

$$\begin{cases} \{1, 2, \cdots, n\} = S \cup X \cup Z \cup T, \\ |S| = d - 1, \quad |X| = d_x - d, \\ |Z| = d_z - d, \quad |T| = n + d - d_x - d_z + 1, \end{cases} \tag{4.2}$$

and $c_S, c'_S \in A_S$*,* $c_Z \in A_Z$*,* $a_X \in A_X$*, we have the equality*

$$\sum_{c_X \in A_X, c_T \in A_T} \overline{\varphi}_i(c_S, c_X, c_Z, c_T)\varphi_j(c'_S, c_X - a_X, c_Z, c_T) = \begin{cases} 0 & \text{for } i \ne j, \\ f(c_S, c'_S, c_Z, a_X) & \text{for } i = j, \end{cases} \tag{4.3}$$

where the complex number $f(c_S, c'_S, c_Z, a_X)$ *is independent of* i*.*

(ii) *There exists a pure asymmetric inhomogeneous quantum code with parameters* $\left(A, K, \frac{d_z}{d_x}\right)$ $(K, d_x, d_z \ge 1)$ *if and only if there exist* K *non-zero mappings* φ_i $(1 \le i \le K)$ *as shown in* (4.1) *such that*

(a) φ_i $(1 \le i \le K)$ *are linear independent, namely, the rank of the* $K \times |A|$ *matrix* $(\varphi_i(a))_{1 \le i \le K, a \in A}$ *is* K*,*

(b) *for each* d*,* $1 \le d \le \min\{d_x, d_z\}$*, a partition* (4.2) *and* $c_S, a_S \in A_S$*,* $c_Z \in A_Z$*,* $a_X \in A_X$*,*

$$\sum_{c_X \in A_X, c_T \in A_T} \overline{\varphi}_i(c_S, c_X, c_Z, c_T)\varphi_j(c_S + a_S, c_Z + a_X, c_Z, c_T) = \begin{cases} 0 & \text{for } (a_S, a_X) \ne (0, 0), \\ \dfrac{(\varphi_i, \varphi_j)}{|A_{Z \cup S}|} & \text{for } (a_S, a_X) = (0, 0). \end{cases} \tag{4.4}$$

Proof We follow the argument in the proof of [5, Theorem 2.2] or [16, Theorem 2.1]. We omit some computational details.

(i) Let Q be a K-dimensional subspace of $V = \mathbb{C}^{q_1} \otimes \cdots \otimes \mathbb{C}^{q_n}$ with the orthogonal basis

$$|v_i\rangle = \sum_{a \in A} \varphi_i(a)|a\rangle, \quad 1 \le i \le K.$$

Then

$$(\varphi_i, \varphi_j) = \sum_{a \in A} \overline{\varphi}_i(a)\varphi_j(a) = \langle v_i | v_j \rangle = \begin{cases} 0, & \text{if } i \ne j, \\ 1, & \text{if } i = j. \end{cases}$$

For two vectors in Q,

$$|u\rangle = \sum_{i=1}^{K} \alpha_i |v_i\rangle, \quad |u'\rangle = \sum_{i=1}^{K} \alpha'_i |v_i\rangle, \quad \alpha_i, \alpha'_i \in \mathbb{C},$$

we have

$$\langle u | u' \rangle = \sum_{i,j=1}^{K} \overline{\alpha}_i \alpha'_i.$$

For each $e = X(a)Z(b)$ $(a, b \in A)$ with $w_X(e) \le d_x - 1$ and $w_Z(e) \le d_z - 1$, we can find a partition (4.2), such that e can be expressed by

$$e = X(a_S, a_X, 0_Z, 0_T)Z(b_S, 0_X, b_Z, 0_T). \tag{4.5}$$

The action of e on $|u'\rangle$ can be computed by (2.5)

$$e|u'\rangle = \sum_{j=1}^{K} \alpha'_j \sum_{c_S,c_X,c_Z,c_T} \varphi_j(c_S - a_S, c_X - a_X, c_Z, c_T) \cdot \chi_{b_S}(c_S)\chi_{b_Z}(c_Z)|c_S, c_X, c_Z, c_T\rangle.$$

By Definition 2.4, Q has parameters $\left(A, K, \frac{d_z}{d_x}\right)$ if and only if

$$\begin{aligned}0 &= \langle u|e|u'\rangle \\ &= \chi_{b_S}(-a_S) \sum_{i,j=1}^{K} \overline{\alpha}_i \alpha'_j \\ &\quad \times \sum_{c_S,c_X,c_Z,c_T} \overline{\varphi}_i(c_S, c_X, c_Z, c_T)\varphi_j(c_S - a_S, c_X - a_X, c_Z, c_B)\chi_{b_S}(c_S)\chi_{b_Z}(c_Z).\end{aligned}$$

Since b_S and b_Z are arbitrary elements in A_S and A_Z, respectively, by Lemma 4.1(1), we know that the above equality is equivalent to

$$\sum_{i,j=1}^{K} \overline{\alpha}_i \alpha'_j \sum_{c_X,c_T} \overline{\varphi}_i(c_S, c_X, c_Z, c_T)\varphi_j(c'_S, c_X - a_X, c_Z, c_T) = 0$$

for any c_S, c'_S, a_X and a_Z. Consider the matrix

$$M = (m_{ij})_{1 \le i,j \le K}, \quad m_{ij} = \sum_{c_X,c_T} \overline{\varphi}_i(c_S, c_X, c_Z, c_T)\varphi_j(c'_S, c_X - a_X, c_Z, c_T).$$

Our statement now becomes that for any $\alpha, \alpha' \in \mathbb{C}^K$, $\overline{\alpha} \cdot \alpha'^{\mathrm{T}} = 0$ implies $\overline{\alpha} M \alpha'^{\mathrm{T}} = 0$. It is easy to see that under the assumption $K \ge 2$, $M = fI$, where I is the identity matrix and $f = f(c_S, c'_S, c_Z, a_X) \in \mathbb{C}$. This is the condition (4.3).

(b) can be proved by the same argument as in the proof of [16, Theorem 2.1(ii)].

Now we give an interesting application of Theorem 4.1, where the parameters d_x and d_z are symmetric.

Theorem 4.2 *Let $d_1, d_2 \ge 1$. Then there exists a (pure) quantum code Q with parameters (A, K, d_z, d_x), $d_x = d_1$ and $d_z = d_2$ if and only if there exists a (pure) quantum code $\widehat{Q}$ with parameters $\left(A, K, \frac{d_z}{d_x}\right)$, $d_x = d_2$ and $d_z = d_1$.*

Proof (i) If Q is a quantum code with parameters $\left(A, K, \frac{d_z}{d_x}\right)$, where $K \ge 2$, $d_x = d_1$ and $d_z = d_2$. By Theorem 4.1, we have K nonzero mappings $\varphi_i : A \to \mathbb{C}$ $(1 \le i \le K)$ satisfying the condition (4.3) in Theorem 4.1. Let $\Phi_i : A \to \mathbb{C}$ be the Fourier transform of φ_i,

$$\Phi_i(b) = \sum_{a \in A} \chi_b(a)\varphi_i(a), \quad b \in A, \ 1 \le i \le K.$$

We show that Φ_i $(1 \le i \le K)$ give the required quantum codes. Namely, for each partition (4.2), we need to show that

$$\begin{aligned}&\sum_{c_Z,c_T} \overline{\Phi}_i(c_S, c_X, c_Z, c_T)\Phi_j(c_S + a_S, c_X, c_Z + a_Z, c_T) \\ &= \begin{cases} 0 & \text{for } i \ne j, \\ I(c_S, c_X, a_S, a_Z) & \text{for } i = j. \end{cases}\end{aligned} \tag{4.6}$$

The left-hand side of (4.6) is

$$\sum_{c_Z,c_T}\Big[\sum_{\alpha_S,\alpha_X,\alpha_Z,\alpha_T}\overline{\chi_{c_S}}(\alpha_S)\overline{\chi_{c_X}}(\alpha_X)\overline{\chi_{c_Z}}(\alpha_Z)\overline{\chi_{c_T}}(\alpha_T)\overline{\varphi}_i(\alpha_S,\alpha_X,\alpha_Z,\alpha_T)$$
$$\times\sum_{\beta_S,\beta_X,\beta_Z,\beta_T}\chi_{c_S+a_S}(\beta_S)\chi_{c_X}(\beta_X)\chi_{c_Z+a_Z}(\beta_Z)\chi_{c_T}(\beta_T)\varphi_j(\beta_S,\beta_X,\beta_Z,\beta_T)\Big]$$
$$=\sum_{\alpha_S,\alpha_X,\alpha_Z,\alpha_T,\beta_S,\beta_X,\beta_Z,\beta_T}\chi_{c_S}(\beta_S-\alpha_S)\chi_{a_S}(\beta_S)\chi_{c_X}(\beta_X-\alpha_X)\chi_{a_Z}(\beta_Z)$$
$$\times\overline{\varphi}_i(\alpha_S,\alpha_X,\alpha_Z,\alpha_T)\varphi_j(\beta_S,\beta_X,\beta_Z,\beta_T)\sum_{c_Z,c_T}\chi_{c_Z}(\beta_Z-\alpha_Z)\chi_{c_T}(\beta_T-\alpha_T)$$
$$=|A_{Z\cup T}|\sum_{\alpha_S,\alpha_X,\alpha_Z,\alpha_T,\beta_S,\beta_X}\chi_{c_S}(\beta_S-\alpha_S)\chi_{a_S}(\beta_S)\chi_{c_X}(\beta_X-\alpha_X)\chi_{a_Z}(\alpha_Z)$$
$$\times\overline{\varphi}_i(\alpha_S,\alpha_X,\alpha_Z,\alpha_T)\varphi_j(\beta_S,\beta_X,\alpha_Z,\alpha_T)$$
$$=\sum_{\alpha_S,\alpha_Z,b_S,b_X}\chi_{c_S}(b_S)\chi_{a_S}(\alpha_S+b_S)\chi_{c_X}(b_X)\chi_{a_Z}(\alpha_Z)$$
$$\times\sum_{\alpha_X,\alpha_T}\overline{\varphi}_i(\alpha_S,\alpha_X,\alpha_Z,\alpha_T)\varphi_j(\alpha_S+b_S,\alpha_X+b_X,\alpha_Z,\alpha_T). \tag{4.7}$$

By (4.3), the right-hand side of (4.7) is zero for $i\neq j$, and for $i=j$ it is independent of i. Therefore, the equality (4.6) is true. If Q is pure, then φ_i $(1\le i\le K)$ satisfies condition (4.4). φ_i $(1\le i\le K)$ are linear independent, so are Φ_i $(1\le i\le K)$. Then we need to show that, for each partition (4.2),

$$\sum_{c_Z,c_T}\overline{\Phi}_i(c_S,c_X,c_Z,c_T)\Phi_j(c_S+a_S,c_X,c_Z+a_Z,c_T)=\begin{cases}0 & \text{for } (a_S,a_Z)\neq(0,0),\\ I_{ij} & \text{for } (a_S,a_Z)=(0,0),\end{cases} \tag{4.8}$$

where I_{ij} is independent of c_S and c_X.

We also have (4.7). Since φ_i $(1\le i\le K)$ satisfies (4.4), the right-hand side of (4.7) is

$$\sum_{\alpha_S,\alpha_Z}N_{ij}\chi_{a_S}(\alpha_S)\chi_{a_Z}(\alpha_Z)=\begin{cases}0 & \text{for } (a_S,a_Z)\neq(0,0),\\ I_{ij} & \text{for } (a_S,a_Z)=(0,0),\end{cases}$$

where $I_{ij}=N_{ij}|A_{S\cup Z}|$ and $N_{ij}=\frac{(\varphi_i,\varphi_j)}{|A_{Z\cup S}|}$. This completes the proof of Theorem 4.2.

By this result, from now on we denote the parameter $\frac{d_z}{d_x}$ by $\{d_x,d_z\}$. Now we give another application of Theorem 4.1.

Theorem 4.3 *Let C be a mixed classical additive code in A, $d^{\perp}$ be the minimal distance of the dual code $C^{\perp}$ of C, and $V=\{v_i:\ 1\le i\le K\}$ be a set of K distinct vectors in A, such that*

$$d_V=\min\{w_H(v_i-v_j+c):\ 1\le i\neq j\le K, c\in C\}\ge 1.$$

Then there exists a pure asymmetric inhomogeneous code with parameters $(A,K,\{d^{\perp},d_V\})$.

Proof The proof is similar to that of [16, Theorem 3.2] (for the asymmetric q-ary case) or [15, Theorem 3.4] (for the inhomogeneous symmetric case). We omit the details.

Example 4.1 Let $d,n\ge 2$ and $A=A_1\oplus\cdots\oplus A_n,\ A_i=Z_d(=\mathbb{Z}/d\mathbb{Z})$ $(1\le i\le n)$. Let $A_d(n,2,l)$ be the maximum size of the d-ary constant weight codes with length n, distance 2 and weight l.

Let $C(l)$ be a d-ary constant weight code with length n, distance 2, weight l and size $A_d(n, 2, l)$. Taking

$$V = \bigcup_{i=0}^{[\frac{n-2}{4}]} C\Big(\Big[\frac{n}{2}\Big] - 2i - 1\Big)$$

in Theorem 4.4, we have $d_V \geq 2$. Then we get a pure quantum code with parameters $(A, K, \{2, d_V\})$, where $K = \sum_{i=0}^{[\frac{n-2}{4}]} A_d(n, 2, [\frac{n}{2}] - 2i - 1)$, $d_V \geq 2$.

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