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Banach Algebra Dynamical Systems*

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Abstract This paper generalizes the C^*-dynamical system to the Banach algebra dynamical system (A, G, α) and define the crossed product $A \rtimes_\alpha G$ of a given Banach algebra dynamical system (A, G, α). Then the representation of $A \rtimes_\alpha G$ is described when A admits a bounded left approximate identity. In a natural way, the authors define the reduced crossed product $A \rtimes_{\alpha,r} G$ and discuss the question when $A \rtimes_\alpha G$ coincides with $A \rtimes_{\alpha,r} G$.

Keywords Banach algebra dynamical systems, Representation, Crossed product, Left multiplier, Reduced crossed product
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1 Introduction

In quantum physics, time evolution or spacial translation of the observables is described by a (non-commutative) C^*-dynamical system. A C^*-dynamical system is a locally compact group G acting by automorphisms on a C^*-algebra A. It is a triple (A, G, α), where A is a C^*-algebra, G is a locally compact group, and α is a strongly continuous action of G on A as involutive automorphisms. A crossed product is a C^*-algebra built out of a C^*-dynamical system. The theory of crossed products of C^*-algebras started with the papers by Turumaru [1] and Zeller-Meier [2]. Given a C^*-dynamical system (A, G, α), the corresponding crossed product $A \rtimes_\alpha G$ is a C^*-algebra. The crossed product construction provides a means to construct new examples of C^*-algebras. One of the basic facts for a crossed product C^*-algebra $A \rtimes_\alpha G$ is that the non-degenerate representations of this algebra on Hilbert spaces are one-to-one correspondences with the non-degenerate involutive covariant representation of (A, G, α), i.e., with the pair (π, U), where π is a non-degenerate involutive representation of A on a Hilbert space, and U is a unitary strongly continuous representation of G on the same space, such that the covariance condition $\pi(\alpha_s(a)) = U_s\pi(a)U_s^{-1}$ is satisfied for $a \in A$ and $s \in G$.

For a given C^*-dynamical system (A, G, α), besides the "full" crossed product C^*-algebra $A \rtimes_\alpha G$, there is another important crossed product C^*-algebra, the reduced crossed product $A \rtimes_{\alpha,r} G$, which was defined by Zeller-Meyer for discrete groups in [2] and generalized by Takai in [3]. They correspond to different completions of $C_c(G, A)$. The former corresponds to the universal representation, and the latter corresponds to the so-called regular representation which can be regarded as a subrepresentation of the universal one. In general, the two crossed product C^*-algebras are different, but Landstad [4] proved that if (A, G, α) is a C^*-dynamical system with G amenable, then $A \rtimes_\alpha G$ is equal to $A \rtimes_{\alpha,r} G$. This is an important theorem in

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the C^*-dynamical systems theory, since the reduced crossed product is more concrete and many familiar groups are amenable, such as the abelian group. It serves as a major step towards the duality theorem for the crossed product, and it is also a key point in Connes' Thom isomorphism for $A \rtimes_\alpha \mathbb{R}$ (see [5]) and Pimsner and Voiculescu's results (see [6]).

This paper generalizes C^*-dynamical systems to the general Banach algebra setting. We define a Banach algebra dynamical system (A, G, α), where A is a Banach algebra, G is a locally compact group, and α is a strongly continuous action of G on A as isometric automorphisms. We construct the Banach algebra of the crossed product $A \rtimes_\alpha G$ from these data. We also study that the representations of $A \rtimes_\alpha G$ are all the representations of the integrated form. There are some differences, since the representations of Banach algebras may not be contractive, unlike the case of C^*-algebras. But in some sense, roughly speaking, we can still prove that the non-degenerate covariant representations of (A, G, α) are in bijection with the non-degenerate representation of this crossed product Banach algebra $A \rtimes_\alpha G$. To do it, we generalize the methods in [7]. We notice that, Sjoerd Dirksen, Marcel De Jeu and Marten Wortel also constructed a kind of crossed product Banach algebra theory (see [8]). They started from a semi-norm on the algebra $C_c(G, A)$, and it is difficult to supply a non-trivial example under their definition. But in our construction, there are many interesting examples and we can prove that the semi-norm which we define on $C_c(G, A)$ is in fact a norm, and this will bring us much convenience for further discussion.

We also construct the reduced crossed product Banach algebra $A \rtimes_{\alpha,r} G$. Different from the C^*-algebra case, the completion of $C_c(G, A)$ depends on the faithful regular representation. To avoid this trouble, we use the supreme form to define the reduced norm. Then a natural question will be asked: When does $A \rtimes_\alpha G$ coincide with $A \rtimes_{\alpha,r} G$ for a Banach algebra dynamical (A, G, α)? It is natural to think that a more strict condition may be required for the group G, since A is weakened to be a Banach algebra. We find a sufficient condition, and thus when G is a compact group, $A \rtimes_\alpha G$ coincides with $A \rtimes_{\alpha,r} G$ for a Banach algebra dynamical system (A, G, α).

2 Preliminaries

In this section, we introduce the basic definitions and notations, and establish some preliminary results. If X is a normed space, we denote by $B(X)$ the normed algebra of bounded linear operators on X. We let $B(X)^\times$ denote the group of the invertible operators in $B(X)$. If A is a normed algebra, we write $\mathrm{Aut}(A)$ for the group of bounded automorphisms. A representation U of a group G on a normed space X is a group homomorphism $U : G \to B(X)^\times$. A representation π of a normed algebra A on a normed space X is an algebra homomorphism $\pi : A \to B(X)$. The representation π is non-degenerate if $\pi(A)X := \mathrm{span}\{\pi(a)x : a \in A, x \in X\}$ is dense in X.

In order to make sense of the integrals, where the integrand is a function taking values in a Banach algebra or a Banach space, we need a workable theory of what is referred to in the literature as vector-valued integration. Fortunately, the theory simplifies significantly when it is possible to restrict to the Haar measure on a locally compact group G, and to integrands which are continuous with compact support on G taking values in a Banach space. We base ourselves on an integral defined by duality. The definition, as well as the existence is contained in the next result, for the proof, we refer to [7, Lemma 1.91].

Lemma 2.1 *Suppose that X is a Banach space and G is a locally compact group with the left Haar measure. Then there exists a linear map $f \mapsto \int_G f(s)\mathrm{d}s$ from $C_c(G, X)$ to X, which* *is characterized by*

$$\varphi\Big(\int_G f(s)\mathrm{d}s\Big) = \int_G \varphi(f(s))\mathrm{d}s, \quad \forall \varphi \in X^*.$$

The integral from Lemma 2.1 enables us to integrate compactly supported and strongly continuous operator-valued functions. We summarize the results in the next proposition without any proofs.

Proposition 2.1 *Let X be a Banach space, G be a locally compact group, and $\psi : G \to B(X)$ be compactly supported and strongly continuous. Define*

$$\int_G \psi(s)\mathrm{d}s := \Big[x \mapsto \int_G \psi(s)x\mathrm{d}s\Big],$$

where the integral on the right-hand side is the integral from Lemma 2.1*. Then $\int_G \psi(s)\mathrm{d}s \in B(X)$, and*

$$\Big\| \int_G \psi(s)\mathrm{d}s\Big\| \le \int_G \|\psi(s)\|\mathrm{d}s.$$

If $T, R \in B(X)$, then

$$T\int_G \psi(s)\mathrm{d}sR = \int_G T\psi(s)R\mathrm{d}s.$$

Definition 2.1 (Banach Algebra Dynamical System) *A Banach algebra dynamical system is a triple (A, G, α), where A is a Banach algebra, G is a locally compact group, and $\alpha : G \to$ $\mathrm{Aut}(A)$ is a strongly continuous representation of G on A with each α_t being an isometric automorphism on A.*

Example 2.1 Let A be a Banach algebra, $\sigma : A \to A$ be an isometric automorphism and $\alpha : Z \to \mathrm{Aut}(A)$ be defined by $\alpha_n(a) = \sigma^n(a)$. Then (A, Z, α) is a Banach algebra dynamical system.

Example 2.2 Let $A = K(L^p(G))$ be the compact operator algebra on $L^p(G)$ $(1 < p < \infty)$, where G is a local compact group. Let $U : G \to B(L^p(G))$ be defined by

$$U_s f(t) = f(s^{-1}t)$$

and $\alpha : G \to \mathrm{Aut}(A)$ be defined by

$$\alpha_s(a) = U_s a U_s^{-1}.$$

Then (A, G, α) becomes a Banach algebra dynamical system.

In fact, fix $f \in C_c(G)$ and let $K := \operatorname{supp} f$. If $\epsilon > 0$, there exists a neighbourhood B of e in G such that $B^{-1} = B$ and $\mid f(s) - f(t) \mid < \epsilon$ if $st^{-1} \in B$. Then for fixed $s_0 \in G$, if $ss_0^{-1} \in B$,

$$\begin{aligned}\|U_s(f) - U_{s_0}(f)\|^p &= \int_G \mid U_s(f)(t) - U_{s_0}(f)(t) \mid^p \mathrm{d}\mu(t)\\ &= \int_G \mid f(s^{-1}t) - f(s_0^{-1}t) \mid^P \mathrm{d}\mu(t)\\ &\le \int_{Bs_0K} \epsilon^p \mathrm{d}\mu(t).\end{aligned}$$

Therefore, U is strongly continuous.

To show $s \to \alpha_s(a)$ is norm continuous, we can assume that $a \in K(L^p(G))$ is a finite-rank operator, since $L^p(G)$ has an approximate property. Now we assume $a = \mid k)(\theta \mid$, and thus

$$al(t) = k(t) \int_G l(s)\theta(s)\mathrm{d}\mu(s)$$

for $l, k \in L^p(G);\ \theta \in L^q(G), \frac{1}{p} + \frac{1}{q} = 1$. Since $U_s a U_s^{-1} = U_s \mid k)(\theta \mid U_s^{-1} = \mid U_s k)(U_s \theta \mid$, we have

$$\|U_s a U_s^{-1} - U_{s_0} a U_{s_0}^{-1}\| \to 0,$$

if $s \to s_0$ by the strong continuity of U. Then we are done.

Definition 2.2 (Covariant Representation) *Let (A, G, α) be a Banach algebra dynamical system, and let X be a Banach space. Then a covariant representation of (A, G, α) on X is a pair (π, U), where π is a representation of A on X and U is a representation of G on X, such that for all $a \in A$ and $s \in G$,*

$$\pi(\alpha_s(a)) = U_s \pi(a) U_s^{-1}.$$

The covariant representation (π, U) is called non-degenerate if (π, X) is a non-degenerate representation of A.

Example 2.3 Give a Banach algebra dynamical system (A, G, α), where A is a Banach algebra, unital or nonunital. Let $\widetilde{A}$ be A^+, the unitalization, if A is nonunital; and let $\widetilde{A}$ be A itself if A is unital. Let $\pi : A \to B(\widetilde{A})$ be the natural representation defined by

$$\pi(a)(b, t) = ab + ta,$$

if A is nonunital; and

$$\pi(a)b = ab,$$

if A is unital, where $a, b \in A,\ t \in \mathbb{C}$.

Define $\widetilde{\pi} : A \to B(L^p(G, \widetilde{A}))$ $(p \geq 1)$ by

$$(\widetilde{\pi}(x)\xi)(t) = \pi(\alpha_{t^{-1}}(x))\xi(t),$$

and $\lambda : G \to B(L^p(G, \widetilde{A}))$ by

$$(\lambda_s \xi)(t) = \xi(s^{-1}t).$$

Then $(\widetilde{\pi}, \lambda, L^p(G, \widetilde{A}))$ is a covariant representation of the Banach algebra dynamical system (A, G, α).

In fact, it is easy to check that $\widetilde{\pi}$ is a homomorphism. To see the contraction proposition, we have

$$\begin{aligned}\|\widetilde{\pi}(x)\xi\| &= \Big(\int_G \|(\widetilde{\pi}(x)\xi)(t)\|^p \mathrm{d}t\Big)^{\frac{1}{p}} \\ &= \Big(\int_G \|\pi(\alpha_{t^{-1}}(x))\xi(t)\|^p \mathrm{d}t\Big)^{\frac{1}{p}} \\ &\leq \|x\|\|\xi\|,\end{aligned}$$

by the isometry of α_t, so $\|\widetilde{\pi}(x)\| \leq \|x\|$. Therefore, $\widetilde{\pi}$ is a contractive representation, and similarly we can check that λ is also a representation. For the covariant proposition,

$$(\widetilde{\pi}(\alpha_t(x))\xi)(s) = \pi(\alpha_{s^{-1}t}(x))\xi(s)$$

and

$$(\lambda_t \widetilde{\pi}(x) \lambda_t^{-1} \xi)(s) = (\widetilde{\pi}(x)\lambda_t^{-1}\xi)(t^{-1}s) = \pi(\alpha_{s^{-1}t}(x))(\lambda_{t^{-1}}\xi)(t^{-1}s) = \pi(\alpha_{s^{-1}t}(x))\xi(s).$$

Therefore, $\widetilde{\pi}(\alpha_t(x)) = \lambda_t \widetilde{\pi}(x) \lambda_t^{-1}$.

3 Crossed Product Banach Algebras

Suppose that (A, G, α) is a Banach algebra dynamical system. We define convolution on the linear space $C_c(G, A)$ of a continuous function from G to A with compact supports by

$$(f * g)(t) = \int_G f(s)\alpha_s(g(s^{-1}t))\mathrm{d}s$$

for all $f, g \in C_c(G, A)$. It is well-defined, as shown in [7 Lemma 1.102]. Straightforward computations show that $C_c(G, A)$ becomes an algebra with convolution as a product. For each $f \in C_c(G, A)$, define

$$\|f\|_1 = \int_G \|f(t)\|\mathrm{d}t.$$

Then $C_c(G, A)$ is a normed algebra, and we denote by $L^1(G, A)$ its completion.

If (π, U, X) is a covariant representation of a given Banach algebra dynamical system (A, G, α), for $f \in C_c(G, A)$, the function $s \mapsto \pi(f(s))U_s$ is strongly continuous from G to $B(X)$ by continuity of multiplication in the strong operator topology on uniformly bounded subsets. Therefore we can define

$$\pi \rtimes U(f) := \int_G \pi(f(s))U_s\mathrm{d}s,$$

where the integral on the right-hand side is as in Proposition 2.1. We call $\pi \rtimes U$ the integrated form of (π, U).

Differently from the case of C^*-algebra, the norms of π and U_t may be bigger than 1, which will pose difficulties for our definition of crossed products. To solve this problem, we may restrict the covariant representation in a proper way.

Definition 3.1 *Given a Banach algebra dynamical system (A, G, α), let $\Omega_{(A,G,\alpha)}$ denote the set of covariant representation (π, U, X) of (A, G, α), such that $\|\pi\| \le 1$ and $\|U_t\| \le 1$ for each $t \in G$.*

Remark 3.1 $\Omega_{(A,G,\alpha)}$ is non-empty since the covariant representation $(\widetilde{\pi}, \lambda, L^p(G, \widetilde{A}))$ given by Example 2.3 is in $\Omega_{(A,G,\alpha)}$. For the homomorphism $U : G \to B(x)^\times$, since $\|U_t\| \le 1$ for each $t \in G$, then $\|x\| = \|U_{t^{-1}}U_t x\| \le \|U_t x\| \le \|x\|$, and hence U_t is an isometric automorphism for each $t \in G$.

Proposition 3.1 *Let (A, G, α) be a Banach algebra dynamical system and $(\pi, U, X) \in \Omega_{(A,G,\alpha)}$. Then the integrated form $\pi \rtimes U$ is a contractive representation of $C_c(G, A)$ on X.*

Proof For $f, g \in C_c(G, A)$, we have

$$\begin{aligned}
\pi \rtimes U(f * g) &= \int_G \pi(f * g(t))u_t\mathrm{d}\mu(t) \\
&= \int_G \int_G \pi(f(s)\alpha_s(g(s^{-1}t)))\mathrm{d}\mu(s)u_t\mathrm{d}\mu(t) \\
&= \int_G \int_G \pi(f(s))\pi(\alpha_s(g(s^{-1}t)))u_t\mathrm{d}\mu(s)\mathrm{d}\mu(t) \\
&= \int_G \int_G \pi(f(s))u_s\pi(g(s^{-1}t))u_s^{-1}u_t\mathrm{d}\mu(s)\mathrm{d}\mu(t) \\
&= \int_G \int_G \pi(f(s))u_s\pi(g(s^{-1}t))u_{s^{-1}t}\mathrm{d}\mu(s)\mathrm{d}\mu(t)
\end{aligned}$$

$$= \int_G \pi(f(s))u_s \mathrm{d}\mu(s) \int_G \pi(g(z))u_z \mathrm{d}\mu(z)$$
$$= (\pi \rtimes u(f))(\pi \rtimes u(g))$$

and

$$\|\pi \rtimes u(f)\| \leq \int_G \|\pi(f(s))\| \|u_s\| \mathrm{d}\mu(s) \leq \|f\|_1.$$

Therefore, $\pi \rtimes U$ is a contractive representation of $C_c(G, A)$ on X.

By virtue of Proposition 3.1, we get a bounded set of non-negative numbers

$$\{\|\pi \rtimes u(f)\| : (\pi, u) \in \Omega_{(A,G,\alpha)}\}.$$

Then we can define a semi-norm on $C_c(G, A)$ by

$$\|f\| := \sup\{\|\pi \rtimes u(f)\| : (\pi, u) \in \Omega_{(A,G,\alpha)}\}. \tag{3.1}$$

In fact, it is a norm on $C_c(G, A)$, and to prove that, it is necessary to find a faithful integrated representation. Example 2.3 will give a perfect one. Let $(\widetilde{\pi}, \lambda, L^p(G, \widetilde{A}))$ be the covariant representation of (A, G, α) given by Example 2.3. For $f \in C_c(G, A)$, if $\widetilde{\pi} \rtimes \lambda(f) = \int_G \widetilde{\pi}(f(s))\lambda_s \mathrm{d}s = 0$, then for any $\xi \otimes 1 \in L^p(G) \otimes \widetilde{A} \subseteq L^p(G, \widetilde{A})$ and $t \in G$,

$$\begin{aligned}
(\widetilde{\pi} \rtimes \lambda(f)\xi)(t) &= \int_G (\widetilde{\pi}(f(s))\lambda_s \xi)(t) \mathrm{d}s \\
&= \int_G \pi(\alpha_{t^{-1}}(f(s)))\xi(s^{-1}t) \mathrm{d}s \\
&= \int_G \alpha_{t^{-1}}(f(s))\xi(s^{-1}t) \mathrm{d}s \\
&= \int_G \alpha_{t^{-1}}(\xi(s^{-1}t) f(s)) \mathrm{d}s \\
&= \alpha_{t^{-1}}\Big(\int_G \xi(s^{-1}t) f(s) \mathrm{d}s\Big) \\
&= 0.
\end{aligned}$$

Hence, $\int_G (\xi(s^{-1}t) f(s)) \mathrm{d}s = 0$. Then for any $\varphi \in A^*$, we have

$$\varphi\Big(\int_G \xi(s^{-1}t) f(s) \mathrm{d}s\Big) = \int_G \xi(s^{-1}t)\varphi(f(s)) \mathrm{d}s = 0. \tag{3.2}$$

Let $\eta(s) = \varphi(f(s))$. Then $\eta \in L^p(G)$, since f is compactly supported. Then by (3.2), we have $\eta * \xi = 0$. If e_n is an approximate unit of $L^p(G)$, then $0 = \eta * e_n \to \eta$, and hence $\eta \doteq 0$. Then $\eta = 0$ by the continuity of η. Therefore, $f = 0$ since A^* separates the points of A. It follows that $\widetilde{\pi} \rtimes \lambda$ is a faithful representation of $C_c(G, A)$, and hence (3.1) defines a norm of $C_c(G, A)$.

Definition 3.2 (Crossed Product Banach Algebra) *Given a Banach algebra dynamical system (A, G, α), the crossed product Banach algebra of this dynamical system is the completion of $C_c(G, A)$ with respect to the norm defined by* (3.1)*, and denoted by $A \rtimes_\alpha G$.*

4 Representations of the Crossed Product Banach Algebras

In this section, by extending the representation of a given Banach algebra to its left multiplier algebra, we generalize the method in [7]. Hence we show that, roughly speaking, all bounded representations of the crossed product are integrated forms of the given covariant representations of the original dynamical system. Thus we totally describe the representations of the crossed product $A \rtimes_\alpha G$.

Proposition 4.1 *Given a Banach algebra dynamical system (A, G, α), suppose that A has an m-bounded left approximate identity (u_i). Let Λ be a neighbourhood basis of e, of which all elements are contained in a fixed compact set K. For each $V \in \Lambda$, take a positive $z_V \in C_c(G)$ with support contained in V and an integral equal to one. Then the set $(f_{(V,i)})$, where*

$$f_{(V,i)} := z_V \otimes u_i,$$

directed by $(V, i) \leq (W, j)$ if and only if $W \subseteq V$ and $i \leq j$, is a left approximate identity of $A \rtimes_\alpha G$.

Proof We first show $f_{(V,i)} * f(s) \to f(s)$ for each $f \in C_c(G) \otimes A$ and $s \in G$, where $C_c(G) \otimes A$ is an algebraic tensor product.

For this, it is sufficient to consider elementary tensors, so let $f = z \otimes a$ with $z (\neq 0) \in C_c(G)$ and $a \in A$.

$$\begin{aligned}
\|(f_{(V,i)} * f)(s) - f(s)\| &= \left\| \int_G (z_V(r) z(r^{-1}s) u_i \alpha_r(a) - z_V(r) z(s) a) \mathrm{d}r \right\| \\
&\leq \int_G z_V(r) \| z(r^{-1}s) u_i \alpha_r(a) - z(s) a \| \mathrm{d}r \\
&\leq \int_G z_V(r) \| z(r^{-1}s) u_i \alpha_r(a) - z(s) u_i \alpha_r(a) \| \mathrm{d}r \\
&\quad + \int_G z_V(r) \| z(s) u_i \alpha_r(a) - z(s) u_i a \| \mathrm{d}r \\
&\quad + \int_G z_V(r) \| z(s) u_i a - z(s) a \| \mathrm{d}r \\
&\leq m \|a\| \int_{\mathrm{supp}(z_V)} z_V(r) | z(r^{-1}s) - z(s) | \mathrm{d}r \\
&\quad + \|z\|_\infty m \int_{\mathrm{supp}(z_V)} z_V(r) \| \alpha_r(a) - a \| \mathrm{d}r \\
&\quad + \|z\|_\infty \int_{\mathrm{supp}(z_V)} z_V(r) \| u_i a - a \| \mathrm{d}r.
\end{aligned}$$

Given $\varepsilon > 0$, by the uniform continuity of z, there exists a neighbourhood U_1 of e, such that $|z(r^{-1}s - z(s)| < \frac{\varepsilon}{3m\|a\|}$ for all $r \in U_1$ and $s \in G$. By the strong continuity of α, there exists a neighbourhood U_2 of e, such that $\|\alpha_r(a) - a\| < \varepsilon(3m\|z\|_\infty)$ for all $r \in U_2$. There exists an index i_0 such that $\|u_i a - a\| < \frac{\varepsilon}{3\|z\|_\infty}$. Choose $V_0 \in \omega$ such that $V_0 \subseteq U_1 \cap U_2$. Then if $(V, i) \geq (V_0, i_0)$, we have $\|f_{(V,i)} * f(s) - f(s)\| < \varepsilon$ for all $s \in G$ as required.

Since $C_c(G) \otimes A$ is dense in $C_c(G, A)$ in the inductive limit topology (see [7]), we have $f_{(V,i)} * f(s) \to f(s)$ for each $f \in C_c(G, A)$, $s \in G$. It follows that $f_{(V,i)} * f \to f$ in the norm defined by (3.1). To show it, let (π, U) be any covariant representation in $\Omega_{(A,G,\alpha)}$ and $I = K\mathrm{supp}(f)$, so then

$$\begin{aligned}
\|\pi \rtimes U(f_{(V,i)} * f - f)\| &= \left\| \int_I \pi(f_{(V,i)} * f(s) - f(s)) U_s \mathrm{d}s \right\| \\
&\leq \int_I \|\pi(f_{(V,i)} * f(s) - f(s)) U_s\| \mathrm{d}s
\end{aligned}$$

$$\leq \sup_{s\in I} \|\pi(f_{(V,i)} * f(s) - f(s))\|\mu(I)$$
$$\leq \sup_{s\in I} \|f_{(V,i)} * f(s) - f(s)\|\mu(I).$$

Since $f_{(V,i)} * f \to f$ uniformly on I, we have that $f_{(V,i)} * f \to f$ in the norm defined by (3.1).

If (π, U) is any covariant representation in $\Omega_{(A,G,\alpha)}$, we have

$$\begin{aligned}\|\pi \rtimes U(f_{(V,i)})\| &= \Big\| \int_K z_V(s)\pi(u_i)U_s \mathrm{d}s \Big\| \\ &\leq \int_K \|z_V(s)\pi(u_i)U_s\| \mathrm{d}s \\ &\leq m \int_K z_V(s)\mathrm{d}s \\ &= m.\end{aligned}$$

It follows that $(f_{(V,i)})$ is uniformly bounded by m in $A \rtimes_\alpha G$. Since $C_c(G, A)$ is dense in $A \rtimes_\alpha G$, a 3ε-argument shows that $(f_{(V,i)})$ is indeed a bounded left approximate identity of $A \rtimes_\alpha G$.

Next, we will extend the representation (π, X) of Banach algebra A to the left multiplier algebra $M_l(A)$, where A has a left approximate identity. It is the key step to construct the one-to-one correspondences between the covariant representations and the integrated representations. Recall that the left multiplier algebra $M_l(A)$ for the Banach algebra A is defined by

$$M_l(A) = \{L \in B(A) : L(ab) = L(a)b \text{ for all } a, b \in A\}.$$

There exists a canonical contractive homomorphism $\lambda : A \to M_l(A)$, defined by $\lambda(a)b = ab$ for $a, b \in A$. Since $L \circ \lambda(a) = \lambda(L(a))$, for $a \in A$ and $L \in M_l(A)$, $\lambda(A)$ is a left ideal in $M_l(A)$.

We begin with the general case, and for this we collect some results from [9]. It is not hard to prove these results.

Theorem 4.1 (see [9, Theorem 4.1]) *Let A be a Banach algebra with an m-bounded approximate left identity. If (π, X) is a non-degenerate representation of A, then there exists a unique representation $\widetilde{\pi} : M_l(A) \to B(X)$ such that*

$$\widetilde{\pi}(\lambda(a)) = \pi(a).$$

Besides, $\widetilde{\pi}$ is non-degenerate, unital, and $\|\widetilde{\pi}\| \leq m\|\pi\|$. Moreover, for $a \in A$ and $L \in M_l(A)$, $\widetilde{\pi}(L)\pi(a) = \pi(L(a))$.

Definition 4.1 (Integrable Covariant Representation) *Let (π, U, X) be a covariant representation of a given Banach algebra dynamical system (A, G, α), and call (π, U, X) an integrable covariant representation if $(\pi \rtimes U, X)$ is a bounded representation of $C_c(G, A)$ with respect to the crossed product norm.*

Lemma 4.1 *Suppose that (A, G, α) is a Banach algebra dynamical system. Then there exists a non-degenerate bounded homomorphism*

$$i_A : A \to M_l(A \rtimes_\alpha G)$$

and a bounded strongly continuous homomorphism

$$i_G : G \to M_l(A \rtimes_\alpha G),$$

such that for $f \in C_c(G, A)$, $r, s \in G$ *and* $a \in A$,

$$i_A(a)f(s) = af(s), \quad i_G(r)f(s) = \alpha_r(f(r^{-1}s)). \tag{4.1}$$

Moreover, (i_A, i_G) *is covariant, and thus*

$$i_A(\alpha_r(a)) = i_G(r)i_A(a)i_G(r)^{-1}.$$

If, in addition, A *has a bounded left approximate identity, then* (i_A, i_G) *is non-degenerate, and moreover if* (π, U) *is a non-degenerate integrable covariant representation, then*

$$(\widetilde{\pi \rtimes U})(i_A(a)) = \pi(a) \text{ and } (\widetilde{\pi \rtimes U})(i_G(s)) = U_s.$$

Proof Firstly we consider $i_A : A \to \mathrm{End}(C_c(G, A))$ and $i_G : G \to \mathrm{End}(C_c(G, A))$, defined by (4.1). It is easy to check that the two maps are homomorphisms. If (π, U) is a covariant representation of (A, G, α), then for $a \in A$, $r \in G$, $f \in C_c(G, A)$,

$$\begin{aligned}
\pi \rtimes u(i_G(r)f) &= \int_G \pi(i_G(r)f(s))u_s \mathrm{d}\mu(s) \\
&= \int_G \pi(\alpha_r(f(r^{-1}s)))u_s \mathrm{d}\mu(s) \\
&= \int_G \pi(\alpha_r(f(s)))u_{rs} \mathrm{d}\mu(s) \\
&= u_r \circ \pi \rtimes u(f).
\end{aligned}$$

Therefore

$$\pi \rtimes U(i_G(r)f) = U_r \circ \pi \rtimes U(f).$$

Similarly,

$$\pi \rtimes U(i_A(a)f) = \pi(a) \circ \pi \rtimes U(f).$$

Then

$$\begin{aligned}
\|i_A(a)f\| &= \sup\{\|\pi \rtimes U(i_A(f))\| : (\pi, U) \in \Omega_{(A,G,\alpha)}\} \\
&= \sup\{\|\pi(a) \circ \pi \rtimes U(f)\| : (\pi, U) \in \Omega_{(A,G,\alpha)}\} \\
&\le \sup\{\|\pi(a)\|\|\pi \rtimes U(f)\| : (\pi, U) \in \Omega_{(A,G,\alpha)}\} \\
&\le \sup\{\|a\|\|\pi \rtimes U(f)\| : (\pi, U) \in \Omega_{(A,G,\alpha)}\} \\
&= \|a\|\|f\|.
\end{aligned}$$

Hence $\|i_A(a)\| \le \|a\|$ for $a \in A$. Similarly, for all $r \in G$, $f \in C_c(G, A)$, since U_r is isometric, we have

$$\|i_G(r)f\| = \|f\|.$$

Thus we can extend $i_A(a)$ and $i_G(r)$ to the maps of $A \rtimes_\alpha G$ to itself, and for $f, g \in C_c(G, A)$, we have

$$\begin{aligned}
i_G(r)(f \times g)(s) &= \alpha_r(f \times g(r^{-1}s)) \\
&= \alpha_r \int_G f(z)\alpha_z(g(z^{-1}r^{-1}s))\mathrm{d}\mu(z) \\
&= \int_G \alpha_r(f(z))\alpha_{rz}(g(z^{-1}r^{-1}s))\mathrm{d}\mu(z)
\end{aligned}$$

$$= \int_G \alpha_r(f(r^{-1}t))\alpha_t(g(t^{-1}s))\mathrm{d}\mu(t)$$
$$= \int_G i_G(r)(f(t))\alpha_t(g(t^{-1}s))\mathrm{d}\mu(t)$$
$$= (i_G(r)f) \times g(s).$$

Hence $i_G(r)$ is a well-defined element of $M_l(A \rtimes_\alpha G)$, and the case of $i_A(a)$ is similar.

To check that $(i_A(a), i_G(r))$ is covariant, we only have to check the actions on $C_c(G, A)$ as the boundedness of $i_A(a)$ and $i_G(r)$. For $f \in C_c(G, A),\ r, s \in G$ and $a \in A$,

$$\begin{aligned}(i_G(r)i_A(a)i_G(r)^{-1}f)(s) &= \alpha_r((i_A(a)i_G(r)^{-1}f)(r^{-1}s))\\ &= \alpha_r(ai_G(r)^{-1}f(r^{-1}s))\\ &= \alpha_r(a\alpha_{r^{-1}}(f(s)))\\ &= \alpha_r(a)(f(s))\\ &= (i_A(\alpha_r(a))f)(s).\end{aligned}$$

To show the last assertion, we claim that if (π, U, X) is a non-degenerate integrable covariant representation, then $\pi \rtimes U$ is a non-degenerate representation of $A \rtimes_\alpha G$.

In fact, give $x \in X$ which is of the form $x = \pi(a)y$, and $\epsilon \geq 0$. Then there exists a neighbourhood V of e such that for $s \in V$, $\|U_s y - y\| \leq \frac{\epsilon}{\|\pi(a)\|}$. Let $z \in C_c(G)$ be nonnegative with compact support contained in V and with an integral equal to 1. Then a simple computation shows that

$$\|\pi \rtimes U(z \otimes a)y - x\| \leq \epsilon$$

and it suffices to show the non-degenerateness of $\pi \rtimes U$.

By Proposition 4.1, we can use Theorem 4.1 to the Banach algebra $A \rtimes_\alpha G$, since

$$\pi \rtimes U(i_G(r)f) = U_r \circ \pi \rtimes U(f) \quad \text{and} \quad \widetilde{\pi \rtimes U}(i_G(r)) \circ \pi \rtimes U(f) = \pi \rtimes U(i_G(r)f),$$

and $\pi \rtimes U$ is non-degenerate, so we have

$$(\widetilde{\pi \rtimes U})(i_G(s)) = U_s.$$

Similarly,

$$(\widetilde{\pi \rtimes U})(i_A(a)) = \pi(a).$$

To show that i_G is strongly continuous, fix $f \in C_c(G, A)$, a compact neighbourhood W of e in G, and let $K = \operatorname{supp} f$. Notice that as long as $r \in W$,

$$\operatorname{supp}(i_G(r)f) \subset WK.$$

If $\epsilon > 0$, then the uniform continuity of f implies that we can choose $V \subset W$, such that $r \in V$ implies

$$\|f(r^{-1}s) - f(s)\| < \frac{\epsilon}{2\mu(WK)} \quad \text{for all } s \in G.$$

Since f has compact support, we can shrink V if needed so that $r \in V$ also implies

$$\|\alpha_r(f(s)) - f(s)\| < \frac{\epsilon}{2\mu(WK)} \quad \text{for all } s \in G.$$

Since

$$\|i_G(r)f(s) - f(s)\| \leq \|\alpha_r(f(r^{-1}s) - f(s))\| + \|\alpha_r(f(s)) - f(s)\|,$$

it follows that

$$\begin{aligned}\|i_G(r)f - f\| &= \sup\{\|\pi \rtimes U(i_G(r)f - f)\| : (\pi, U) \in \Omega_{(A,G,\alpha)}\} \\ &= \sup\left\{\left\| \int_{WK} \pi(i_G(r)f - f)(s)U_s \mathrm{d}s \right\| : (\pi, U) \in \Omega_{(A,G,\alpha)}\right\} \\ &\leq \sup\left\{ \int_{WK} \|\pi(i_G(r)f - f)(s)U_s\| \mathrm{d}s : (\pi, U) \in \Omega_{(A,G,\alpha)}\right\} \\ &\leq \sup\{\|i_G(r)f - f\|_\infty \mu(WK) : (\pi, U) \in \Omega_{(A,G,\alpha)}\} \\ &\leq \epsilon.\end{aligned}$$

Therefore $r \to i_G(r)$ is strongly continuous.

If, in addition, A has a bounded left approximate identity (u_i), then i_A is non-degenerate as $i_A(u_i)$ converges to I strongly.

Theorem 4.2 *Suppose that (A, G, α) is a Banach algebra dynamical system, where A has a bounded left approximate identity. Then $(\pi, u, X) \to (\pi \rtimes u, X)$ is a one-to-one correspondence between the non-degenerate integrable covariant representations of (A, G, α) and the non-degenerate bounded representations of $A \rtimes_\alpha G$.*

Proof In Lemma 4.1, we have already proved that a non-degenerate integrable covariant representation (π, U) of (A, G, α) induces a non-degenerate integrated representation $\pi \rtimes U$ of $A \rtimes_\alpha G$, and that

$$\widetilde{(\pi \rtimes U)}(i_A(a)) = \pi(a) \quad \text{and} \quad \widetilde{(\pi \rtimes U)}(i_G(s)) = U_s.$$

To prove the one-to-one correspondence, suppose that (L, X) is a bounded non-degenerate representation of $A \rtimes_\alpha G$, and let

$$\pi(a) := \widetilde{L}(i_A(a)) \quad \text{and} \quad u_s := \widetilde{L}(i_G(s)).$$

We first show that π is non-degenerate. Given $f \in C_c(G, A)$ and $x \in X$, if e_i is a bounded left approximate identity in A, then by Theorem 4.1,

$$\begin{aligned}\pi(e_i)L(f)x &= \widetilde{L}(i_A(e_i))L(f)x \\ &= L(i_A(e_i)f)x \\ &\to L(f)x.\end{aligned}$$

Then by the non-degenerateness of L and as $C_c(G, A)$ is dense in $A \rtimes_\alpha G$, we have that π is non-degenerate. Since i_G is strongly continuous, it is similar to check that U is strongly continuous. Since the covariance condition is straightforward to check, it follows that (π, U) is a non-degenerate covariant representation. For $f, g \in C_c(G, A)$, $x \in X$,

$$\begin{aligned}\pi \rtimes u(f)(L(g)x) &= \int_G \pi(f(s))U_s \mathrm{d}s(L(g)x) \\ &= \int_G \widetilde{L}(i_A(f(s)))\widetilde{L}(i_G(s))\mathrm{d}s(L(g)x) \\ &= \int_G \widetilde{L}(i_A(f(s))i_G(s))L(g)\mathrm{d}s(x) \\ &= \int_G L(i_A(f(s))i_G(s)(g))\mathrm{d}s(x)\end{aligned}$$

$$= L\Big(\int_G (i_A(f(s))i_G(s)(g)\mathrm{d}s\Big)(x)$$
$$= L(i_A \rtimes i_G(f)(g))(x).$$

Let us compute $i_A \rtimes i_G(f)(g)$ as follows:

$$\begin{aligned} i_A \rtimes i_G(f)(g)(t) &= \int_G (i_A(f(s)) \circ i_G(s)(g))(t)\mathrm{d}s \\ &= \int_G f(s)(i_G(s)g)(t)\mathrm{d}s \\ &= \int_G f(s)\alpha_s(g(s^{-1}t))\mathrm{d}s \\ &= f * g(t). \end{aligned}$$

Thus $i_A \rtimes i_G(f)(g) = f * g$. It follows that

$$\pi \rtimes u(f)(L(g)x) = L(f * g)(x) = L(f)L(g)(x).$$

Therefore, $\pi \rtimes U = L$ on $L(C_c(G, A))$. Since $C_c(G, A)$ is dense in $A \rtimes_\alpha G$ and L is non-degenerate, we have that $\pi \rtimes U = L$ on $A \rtimes_\alpha G$.

In Theorem 4.2, if A has an m-bounded left approximate identity, then L is a non-degenerate bounded representation of $A \rtimes_\alpha G$, and

$$\pi(a) := \widetilde{L}(i_A(a)) \quad \text{and} \quad u_s := \widetilde{L}(i_G(s)).$$

Then by Theorem 4.1,

$$\|\pi(a)\| \le m\|L\|\|i_A(a)\| \le m\|L\|\|a\|$$

and

$$\|U_s\| \le \|m\|L\|\|i_G(s)\| \le m\|L\|.$$

On the other hand, if $(\pi, U) \in \Omega_{(A,G,\alpha)}$, then $\pi \rtimes U$ is contractive. Therefore, we get the following corresponding theorem, which is more similar to the C^*-algebra case.

Theorem 4.3 *Suppose that (A, G, α) is a Banach algebra dynamical system, where A has a 1-bounded left approximate identity. Then $(\pi, u, X) \to (\pi \rtimes u, X)$ is a one-to-one correspondence between the non-degenerate covariant representations in $\Omega_{(A,G,\alpha)}$ and the non-degenerate contractive representations of $A \rtimes_\alpha G$.*

5 Regular Representations

In the case of C^*-algebra, if (A, G, α) is a C^*-dynamical system, then a representation (π, H) of A determines a regular representation of $C_c(G, A)$. This representation induces a reduced norm on $C_c(G, A)$, and the completion of this norm, denoted by $A \rtimes_{\alpha r} G$, is called the reduced crossed product, which is independent of the choice of the faithful representation (π, H). The regular representations are very important, since they are concrete and very easy to get. Moreover, Landstad [4] constructed an important theorem which implies that $A \rtimes_{\alpha,r} G$ is equal to $A \rtimes_\alpha G$ when G is amenable. This theorem ensures the further development of the theory of crossed products. For example, it is a crucial step to construct the duality theory for crossed products, Connes' Thom isomorphism for $A \rtimes_\alpha \mathbb{R}$ (see [5]), and Pimsner and Voiculescu's six-term exact sequence for the K-group of certain crossed product C^*-algebras (see [6]).

We will generalize this theory to the case of Banach algebra dynamical systems. In this case, things are very different. Let us start with the definition of the regular representation.

Give a Banach algebra dynamical system (A, G, α) and a representation (π, X) of A. Since the representation space of a Banach algebra is a Banach space, we will let the covariant representation space be $L^p(G, X)$ with $p \geq 1$, not just as $L^2(G, X)$. We define a covariant representation $(\widetilde{\pi}, \lambda, L^p(G, X))$ by

$$(\widetilde{\pi}(x)\xi)(t) = \pi(\alpha_{t^{-1}}(x))\xi(t), \quad (\lambda_s\xi)(t) = \xi(s^{-1}t)$$

for every x in A, s in G and ξ in $L^p(G, X)$,

$$\begin{aligned}
\|\widetilde{\pi}(x)\xi\| &= \Big(\int_G \|\widetilde{\pi}(x)\xi(t)\|^p \mathrm{d}t\Big)^{\frac{1}{p}} \\
&= \Big(\int_G \|\pi(\alpha_{t^{-1}}(x))\xi(t)\|^p \mathrm{d}t\Big)^{\frac{1}{p}} \\
&\leq \Big(\int_G \|\pi(\alpha_{t^{-1}}(x))\|^p \|\xi(t)\|^p \mathrm{d}t\Big)^{\frac{1}{p}} \\
&\leq \|\pi\|\|x\|\|\xi\|_p,
\end{aligned}$$

and hence $\|\widetilde{\pi}(x)\| \leq \|\pi\|\|x\|$. So $\widetilde{\pi}$ is bounded by $\|\pi\|$. λ_s is a natural isometry for any $s \in G$, and it is easy to check that $\widetilde{\pi}$ and λ are homomorphisms and that

$$\widetilde{\pi}(\alpha_s(x)) = \lambda_s\widetilde{\pi}(x)\lambda_s^{-1} \quad \text{for } x \in A, s \in G.$$

Therefore, $(\widetilde{\pi}, \lambda, L^p(G, X))$ is a well-defined covariant representation of (A, G, α). Then we can define the integral representation of $C_c(G, A)$ on $L^p(G, X)$. To determine the reduced norm, let us consider the semi-norm $\|f\|_{p,X} = \|\widetilde{\pi} \rtimes \lambda(f)\|$. For the C^*-dynamical system, where $p = 2$ and X is a Hilbert space, it is a norm if (π, H) is a faithful representation, and the norm is independent of the choice of the faithful representation (π, H). But in the case of Banach algebra dynamical system, things are different. We give some examples to show that.

We firstly fix p, and let (A, G, α) be a Banach algebra dynamical system, where $G = \{e\}$ and A be any Banach algebra. Then

$$\begin{aligned}
\|f\|_{p,X} &= \|\widetilde{\pi} \rtimes \lambda(f)\| \\
&= \sup_{\|x\|\leq 1} \|\widetilde{\pi} \rtimes \lambda(f)x\| \\
&= \sup_{\|x\|\leq 1} \|\pi(f(e))x\| \\
&= \|\pi(f(e))\|.
\end{aligned}$$

Hence the semi-norm depends on the choice of the representation of (π, X). But if A is a C^*-algebra, the semi-norm is independent of the choice of the representation (π, H) since the monomorphism between C^*-algebras is an isometry (see [10, Chapter VIII, Theorem 4.8]). But in the case of Banach algebras, this property does not hold.

Next, let (A, G, α) be a Banach algebra dynamical system, where $A = \mathbb{C}$ and α_t is trivial. Define a representation $\pi : \mathbb{C} \to B(\mathbb{C})$ by $\pi(x)y = xy$. Then

$$\begin{aligned}
\|f\|_{p,X} &= \|\widetilde{\pi} \rtimes \lambda(f)\| \\
&= \sup_{\|\xi\|\leq 1} \Big(\int \Big|\int f(s)\xi(s^{-1}t)\mathrm{d}s\Big|^p \mathrm{d}t\Big)^{\frac{1}{p}}
\end{aligned}$$

$$= \sup_{\|\xi\|\le 1}\Big(\int |f*\xi(t)|^p \mathrm{d}t\Big)^{\frac{1}{p}}$$
$$= \sup_{\|\xi\|\le 1}\|f*\xi\|_p.$$

We see that this semi-norm depends on the number p.

Summing up the two examples above, we see that $\|f\|_{p,X}$ depends on the choice of both p and (π, X). But we can use the supreme to define the reduced norm. We give the semi-norm on $C_c(G,A)$ below, for $f\in C_c(G,A)$, and let

$$\|f\|_r := \sup\{\|\widetilde{\pi}\rtimes\lambda(f)\|_{p,X} : (\pi,X) \text{ is a contractive representation of } A,\ p\ge 1\}.$$

It is easy to check that this gives a semi-norm on $C_c(G,A)$, but in fact, this is a norm. To check it, we only have to find a representation (π,X), such that $(\widetilde{\pi}\rtimes\lambda, L^p(G,X))$ is a faithful representation on $C_c(G,A)$, and Example 2.3 gives a wonderful example (see the procedure of the definition of crossed products in Section 3). Then we can give the definition of the reduced crossed products Banach algebras.

Definition 5.1 (Reduced Crossed Product Banach Algebras) *Given a Banach algebra dynamical system (A,G,α), the reduced crossed product Banach algebra of this system is the completion of $C_c(G,A)$ by the norm of $\|\cdot\|_r$, and is denoted by $A\rtimes_{\alpha,r}G$.*

In the following, we will discuss when $A\rtimes_{\alpha,r}G$ equals $A\rtimes_\alpha G$ for a given Banach algebra dynamical system (A,G,α). We give a result as a theorem.

Theorem 5.1 *If (A,G,α) is a Banach algebra dynamical system, then $A\rtimes_{\alpha,r}G$ equals $A\rtimes_\alpha G$ if G is a compact group.*

Proof Let (π,U,X) be a covariant representation of (A,G,α), and (π,U,X) lie in $\Omega_{(A,G,\alpha)}$. Since the Haar measure of a compact group is finite, $\mu(G)$ is a finite positive number. Define a map $\nu : X\to L^p(G,X)$ by $\nu(x)(t) = \frac{U_{t^{-1}}(x)}{(\mu(G))^{\frac{1}{p}}}$. Then we have

$$\begin{aligned}\|\nu(x)\|_p &= \Big(\int_G \|\nu(x)(t)\|^p \mathrm{d}t\Big)^{\frac{1}{p}}\\ &= \Big(\int_G \Big\|\frac{U_{t^{-1}}(x)}{\mu(G)^{\frac{1}{p}}}\Big\|^p \mathrm{d}t\Big)^{\frac{1}{p}}\\ &= \Big(\int_G \Big(\frac{\|x\|}{\mu(G)^{\frac{1}{p}}}\Big)^p \mathrm{d}t\Big)^{\frac{1}{p}}\\ &= \|x\|.\end{aligned}$$

Therefore, ν is well-defined and is an isometry, so ν is an isometric embedding. Recall that the representation $\widetilde{\pi} : A\to B(L^p(G,X))$ is defined by

$$\widetilde{\pi}(a)\xi(t) = \pi(\alpha_{t^{-1}}(a))\xi(t)$$

for $a\in A,\ t\in G$ and $\xi\in L^p(G,A)$.

For each $x\in X,\ t\in G$, $\nu\pi(a)x(t) = \frac{U_{t^{-1}}(\pi(a)x)}{(\mu(G))^{\frac{1}{p}}}$ and

$$\widetilde{\pi}(a)\nu(x)(t) = \pi(\alpha_{t^{-1}}(a))(\nu(x)(t)) = \pi(\alpha_{t^{-1}}(a))\frac{U_{t^{-1}}(x)}{(\mu(G))^{\frac{1}{p}}}.$$

Then by the covariant property of (π, U), we have that $\nu\pi(a)x(t) = \widetilde{\pi}(a)\nu(x)(t)$. It follows that

$$\nu\pi(a) = \widetilde{\pi}(a)\nu$$

for each $a \in A$. Then we get the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\pi(a)} & X \\ \nu\big\downarrow & & \big\downarrow\nu \\ L^p(G,X) & \xrightarrow{\widetilde{\pi}(a)} & L^p(G,X) \end{array}$$

On the other hand, for each $x \in X$ and $t \in G$,

$$(\nu U_t(x))(s) = \frac{U_{s^{-1}}(U_t(x))}{(\mu(G))^{\frac{1}{p}}} = \frac{U_{s^{-1}t}(x)}{\mu(G)^{\frac{1}{p}}}$$

and

$$(\lambda_t \nu(x))(s) = \nu(x)(t^{-1}s) = \frac{U_{s^{-1}t}(x)}{(\mu(G))^{\frac{1}{p}}}.$$

Thus $\nu U_t = \lambda_t \nu$ and the diagram

$$\begin{array}{ccc} X & \xrightarrow{U_t} & X \\ \nu\big\downarrow & & \big\downarrow\nu \\ L^p(G,X) & \xrightarrow{\lambda_t} & L^p(G,X) \end{array}$$

is commutative.

Then for each $f \in C_c(G, A)$,

$$\begin{aligned} \pi \rtimes U(f) &= \int_G \pi(f(s))U_s \mathrm{d}s \\ &= \int_G (\nu^{-1}\widetilde{\pi}(f(s))\nu)(\nu^{-1}\lambda_s\nu)\mathrm{d}s \\ &= \int_G \nu^{-1}\widetilde{\pi}(f(s))\lambda_s\nu\mathrm{d}s \\ &= \nu^{-1}\Big(\int_G \widetilde{\pi}(f(s))\lambda_s\mathrm{d}s\Big)\nu \\ &= \nu^{-1}(\widetilde{\pi} \rtimes \lambda(f))\nu. \end{aligned}$$

Since the diagrams above are commutative, ν^{-1} is well-defined as it only acts on $\mathrm{Ran}(\nu)$. Therefore,

$$\|\pi \rtimes U(f)\| \le \|\widetilde{\pi} \rtimes \lambda(f)\|.$$

This may not be an equation, since ν may not be an isometric isomorphism, so we can not get

$$\nu(\pi \rtimes U(f))\nu^{-1} = (\widetilde{\pi} \rtimes \lambda(f)),$$

but that is enough to get our result. By the definitions of crossed products and reduced crossed products, the reduced norm is equal to the universal norm on $C_c(G, A)$, and we have that $A \rtimes_{\alpha,r} G$ is equal to $A \rtimes_\alpha G$.

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