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Local Smooth Solutions to the 3-Dimensional Isentropic Relativistic Euler equations*

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Abstract The authors consider the local smooth solutions to the isentropic relativistic Euler equations in (3+1)-dimensional space-time for both non-vacuum and vacuum cases. The local existence is proved by symmetrizing the system and applying the Friedrichs-Lax-Kato theory of symmetric hyperbolic systems. For the non-vacuum case, according to Godunov, firstly a strictly convex entropy function is solved out, then a suitable symmetrizer to symmetrize the system is constructed. For the vacuum case, since the coefficient matrix blows-up near the vacuum, the authors use another symmetrization which is based on the generalized Riemann invariants and the normalized velocity.

Keywords Isentropic relativistic Euler equations, local-in-time smooth solutions, Strictly convex entropy, Generalized Riemann invariants
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1 Introduction

The Euler system of conservation laws for a perfect fluid in special relativity can be written as follows (see, e.g., [1, 18, 26–27, 33, 39, 41–47]):

$$\begin{cases} \partial_t\Big(\dfrac{n}{\sqrt{1-\frac{v^2}{c^2}}}\Big)+\nabla\cdot\Big(\dfrac{n}{\sqrt{1-\frac{v^2}{c^2}}}\mathbf{v}\Big)=0,\\ \partial_t\Big(\dfrac{\frac{p}{c^2}+\rho}{1-\frac{v^2}{c^2}}\mathbf{v}\Big)+\nabla\cdot\Big(\dfrac{\frac{p}{c^2}+\rho}{1-\frac{v^2}{c^2}}\mathbf{v}\otimes\mathbf{v}\Big)+\nabla p=0,\\ \partial_t\Big(\dfrac{\frac{p}{c^2}+\rho}{1-\frac{v^2}{c^2}}-\dfrac{p}{c^2}\Big)+\nabla\cdot\Big(\dfrac{\frac{p}{c^2}+\rho}{1-\frac{v^2}{c^2}}\mathbf{v}\Big)=0, \end{cases} \tag{1.1}$$

where n and ρ are the rest mass density and the mass-energy density, respectively, satisfying

$$\rho=n\Big(1+\frac{e}{c^2}\Big) \tag{1.2}$$

with the specific internal energy e, and p represents the pressure. The constant c is the speed of light, $\mathbf{v}=(v_1,v_2,v_3)^{\mathrm{T}}$ denotes the particle speed, and $v=|\mathbf{v}|$ satisfies the relativistic constraint $v^2<c^2$. All these variables are the functions of $(t,\mathbf{x})\in\mathbb{R}^+\times\mathbb{R}^3$.

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For the classical Euler system which is the non-relativistic version of (1.1), Makino, Ukai and Kawashima [31] introduced a new symmetrization to deduce the local-in-time solution even for the case with vacuum states.

If the pressure p depends only on the mass-energy ρ, and the system of energy and momentum conservation laws is closed, (1.1) reduces to the following subsystem (see, e.g., [1, 18, 26–27, 33, 39, 41–47]):

$$\begin{cases}\partial_t\Big(\dfrac{\frac{p}{c^2}+\rho}{1-\frac{v^2}{c^2}}-\dfrac{p}{c^2}\Big)+\nabla\cdot\Big(\dfrac{\frac{p}{c^2}+\rho}{1-\frac{v^2}{c^2}}\mathbf{v}\Big)=0,\\ \partial_t\Big(\dfrac{\frac{p}{c^2}+\rho}{1-\frac{v^2}{c^2}}\mathbf{v}\Big)+\nabla\cdot\Big(\dfrac{\frac{p}{c^2}+\rho}{1-\frac{v^2}{c^2}}\mathbf{v}\otimes\mathbf{v}\Big)+\nabla p=0.\end{cases} \tag{1.3}$$

Great progress has been made with (1.3), yet mainly for the 1-dimensional or spherically symmetric 3-dimensional cases (see [2–4, 6, 11–13, 15–17, 20, 22–25, 32, 34, 37–38, 40, 48–49] and the references therein).

For general multi-dimensional cases of (1.3), Makino-Ukai [29–30] constructed a suitable symmetrizer if a strictly convex entropy exists, and then by applying Friedrichs-Lax-Kato's theory (see [14, 28]), the authors established the local existence of solutions with the data away from the vacuum. For the vacuum case of (1.3), since the coefficient matrix in [29–30] is degenerate near the vacuum, Lefloch-Ukai [19] introduced a different symmetrization based on the generalized Riemann invariants and the normalized velocity, and then established the local existence results of smooth solutions by also using Friedrichs-Lax-Kato's theory (see [14, 28]). Moreover, for (1.3), the singularity formation of smooth solutions is studied in [10, 35, 37].

In this paper, we consider the system of isentropic relativistic Euler equations, which corresponds to the conservation of the baryon numbers and momentum and reads as (see, e.g., [1, 18, 26–27, 33, 39, 41–47]):

$$\begin{cases}\partial_t\Big(\dfrac{n}{\sqrt{1-\frac{v^2}{c^2}}}\Big)+\nabla\cdot\Big(\dfrac{n}{\sqrt{1-\frac{v^2}{c^2}}}\mathbf{v}\Big)=0,\\ \partial_t\Big(\dfrac{\frac{p}{c^2}+\rho}{1-\frac{v^2}{c^2}}\mathbf{v}\Big)+\nabla\cdot\Big(\dfrac{\frac{p}{c^2}+\rho}{1-\frac{v^2}{c^2}}\mathbf{v}\otimes\mathbf{v}\Big)+\nabla p=0.\end{cases} \tag{1.4}$$

We know that, formally, the Newtonian limit of (1.4) is the following classical system of non-relativistic isentropic Euler equations (see [7, 26, 39]):

$$\begin{cases}\partial_t n+\nabla\cdot(n\mathbf{v})=0,\\ \partial_t(n\mathbf{v})+\nabla\cdot(n\mathbf{v}\otimes\mathbf{v}+p)=0,\end{cases} \tag{1.5}$$

which is one of the motivation for our study on (1.4). Another motivation for our study is that some special relativistic effects are revealed for 3-dimensional relativistic equations (see [8]), which do not appear in the corresponding non-relativistic case.

We consider (1.4) with the equation of the state

$$p=p(\rho), \tag{1.6}$$

satisfying

$$p(0)=0,\quad p(\rho)\ge 0,\quad 0<p_\rho<c^2,\quad p_{\rho\rho}\ge 0\quad \text{for } \rho\in(\rho_*,\rho^*),$$

where $0 \le \rho_* < \rho^* \le \infty$ are any non-negative constants subject to the subluminal condition $p_\rho(\rho^*) \le c^2$. Note that if $p(\rho) = a^2\rho^\gamma$, then $\rho_* = 0,\quad \rho^* = \infty$ for $\gamma = 1$ and $\rho^* = \left[\frac{c^2}{\gamma a^2}\right]^{\frac{1}{\gamma-1}}$ for $\gamma > 1$.

The first law of thermodynamics (Gibb's Equation) reads as

$$\theta \mathrm{d}S = \frac{\mathrm{d}\rho}{n} - \frac{\frac{p}{c^2}+\rho}{n^2}\mathrm{d}n \tag{1.7}$$

with temperature θ and the specific entropy S. For isentropic fluids ($S \equiv$ const.), we have

$$\frac{\mathrm{d}n}{n} = \frac{\mathrm{d}\rho}{\frac{p}{c^2}+\rho}.$$

Denote

$$\frac{\mathrm{d}\rho}{\mathrm{d}n} := \rho' = \frac{p+\rho c^2}{nc^2}. \tag{1.8}$$

For simplicity, we denote $' = \frac{\mathrm{d}}{\mathrm{d}n}$ in the sequel. From (1.8), we have

$$n = n(\rho) = C\mathrm{e}^{\int_{\rho_m}^{\rho} \frac{\mathrm{d}s}{s+\frac{p(s)}{c^2}}}, \tag{1.9}$$

with ρ_m being any fixed number in (ρ_*, ρ^*) and $C = n_m := n(\rho_m)$.

We consider the Cauchy problem (1.4) with initial data

$$t = 0: \quad n(0,\mathbf{x}) = n_0, \quad \mathbf{v}(0,\mathbf{x}) = \mathbf{v}_0. \tag{1.10}$$

Research results of (1.4) is not so rich as that of (1.3), and all results are about 1-dimensional case (see [21, 23, 36, 39, 48]). Naturally, we are interested in the local existence of smooth solutions to the Cauchy problem (1.4) and (1.10) for both vacuum and non-vacuum cases.

The main result of our paper is as follows.

Theorem 1.1 *Suppose that the initial data $(n_0, \mathbf{v}_0) \in H^l_{ul}(\mathbb{R}^3),\ l \ge \frac{5}{2}$, and there exists a positive constant δ_0 which is sufficiently small, such that*

$$n_* + \delta_0 \le n_0(\mathbf{x}) \le n^* - \delta_0, \quad v_0^2 = |\mathbf{v}_0|^2 \le (1-\delta_0)c^2 \quad \textit{for the non-vacuum case} \tag{1.11}$$

and

$$0 \le n_0(\mathbf{x}) \le n^* - \delta_0, \quad v_0^2 = |\mathbf{v}_0|^2 \le (1-\delta_0)c^2 \quad \textit{for the vacuum case,} \tag{1.12}$$

where $H^l_{ul}(\mathbb{R}^3)$ is the uniformly local Sobolev space defined in [14], *n_*, n^* are determined by* (1.9) *with $\rho(n_*) = \rho_*$ or $\rho(n^*) = \rho^*$. Then, the Cauchy problem* (1.4) *and* (1.10) *admits a unique solution $(n(t,\mathbf{x}), \mathbf{v}(t,\mathbf{x}))$ with $n_* \le n(t,\mathbf{x}) \le n^*,\ |\mathbf{v}(t,\mathbf{x})| < c$ for the non-vacuum case and $0 \le n(t,\mathbf{x}) \le n^*,\ |\mathbf{v}(t,\mathbf{x})| < c$ for the vacuum case, and*

$$(n(t,\mathbf{x}), \mathbf{v}(t,\mathbf{x})) \in L^\infty([0,T]; H^l_{ul}) \cap C([0,T]; H^l_{ul}) \cap C^1([0,T]; H^{l-1}_{ul}),$$

where T depends only on δ and the $H^l_{ul}(\mathbb{R}^3)$ norm $\|(n_0, \mathbf{v}_0)\|_{H^l_{ul}}$ of the initial data.

We shall prove the main theorem by symmetrizing (1.4) and applying the Friedrichs-Lax-Kato theory (see [14, 28]) of symmetric hyperbolic systems. Thus for both the non-vacuum and vacuum cases, the construction of a suitable symmetrizer is necessary and important. We borrow some ideas from [19, 29–30], which are more complicated due to the structure of the system itself, and different to some extent since we involve variables n and $\mathbf{v}$ instead of ρ and $\mathbf{v}$.

More precisely, in Section 2, we firstly solve out the strictly convex entropy function of (1.4) for the non-vacuum case according to [9], then we construct a symmetrizer as in [29–30]. In Section 3 for the vacuum case, due to the degeneracy of the symmetrized system near the vacuum, as in [19], we transform (1.4) into a symmetric form in terms of the generalized Riemann invariants and the normalized velocity.

2 Non-vacuum Case

In this section, we will establish the existence of local smooth solutions to the Cauchy problem (1.4) and (1.10) for the non-vacuum case in Theorem 1.1. For clarity, we will divide it into two subsections: Strictly convex entropy function and symmetrization.

2.1 Strictly convex entropy function

If an entropy-entropy flux pair of (1.4) exists, then we may construct a symmetrizer accordingly. According to [9], we first find the strictly convex entropy function of (1.4). To do this, we fit (1.4) into the following general form of conservation laws:

$$\frac{\partial \mathbf{u}}{\partial t}+\sum_{j=1}^{3}\frac{\partial \mathbf{F}^j(\mathbf{u})}{\partial x_j}=0, \tag{2.1}$$

where

$$\begin{aligned}\mathbf{u}&=(u_1,u_2,u_3,u_4)^{\mathrm{T}}\\&=\Big(\frac{nc}{\sqrt{c^2-v^2}},\frac{(p+\rho c^2)v_1}{c^2-v^2},\frac{(p+\rho c^2)v_2}{c^2-v^2},\frac{(p+\rho c^2)v_3}{c^2-v^2}\Big)^{\mathrm{T}},\\ \mathbf{F}^j(\mathbf{u})&=\big(F_0^j(\mathbf{u}),F_1^j(\mathbf{u}),F_2^j(\mathbf{u}),F_3^j(\mathbf{u})\big)^{\mathrm{T}}\end{aligned}$$

and

$$F_0^j(\mathbf{u})=\frac{ncv_j}{\sqrt{c^2-v^2}},\quad F_i^j(\mathbf{u})=\frac{p+\rho c^2}{c^2-v^2}v_iv_j+p\delta_{ij},\quad i,j=1,2,3,$$

where δ_{ij} is the Kronecker symbol.

A scalar function $\eta(\mathbf{u})$ is called the entropy to (2.1) if there exist scalar functions $q^j(\mathbf{u})$ ($j=1,2,3$) satisfying

$$\nabla_{\mathbf{u}}\eta(\mathbf{u})\nabla_{\mathbf{u}}\mathbf{F}^j(\mathbf{u})=\nabla_{\mathbf{u}}q^j(\mathbf{u}). \tag{2.2}$$

Let $\mathbf{z} = (n, v_1, v_2, v_3)^{\mathrm{T}}$. By direct but tedious computations, we have

$$\nabla_{\mathbf{z}}\mathbf{u} = \begin{pmatrix} \frac{c}{\sqrt{c^2-v^2}} & \frac{ncv_1}{(\sqrt{c^2-v^2})^3} & \frac{ncv_2}{(\sqrt{c^2-v^2})^3} & \frac{ncv_3}{(\sqrt{c^2-v^2})^3} \\ \frac{p'+\rho' c^2}{c^2-v^2}v_1 & \frac{p+\rho c^2}{c^2-v^2}\Big(1+\frac{2v_1^2}{c^2-v^2}\Big) & \frac{2(p+\rho c^2)v_1v_2}{(c^2-v^2)^2} & \frac{2(p+\rho c^2)v_1v_3}{(c^2-v^2)^2} \\ \frac{p'+\rho' c^2}{c^2-v^2}v_2 & \frac{2(p+\rho c^2)v_1v_2}{(c^2-v^2)^2} & \frac{p+\rho c^2}{c^2-v^2}\Big(1+\frac{2v_2^2}{c^2-v^2}\Big) & \frac{2(p+\rho c^2)v_2v_3}{(c^2-v^2)^2} \\ \frac{p'+\rho' c^2}{c^2-v^2}v_3 & \frac{2(p+\rho c^2)v_1v_3}{(c^2-v^2)^2} & \frac{2(p+\rho c^2)v_2v_3}{(c^2-v^2)^2} & \frac{p+\rho c^2}{c^2-v^2}\Big(1+\frac{2v_3^2}{c^2-v^2}\Big) \end{pmatrix} \tag{2.3}$$

and

$$(\nabla_{\mathbf{z}}\mathbf{u})^{-1} = \begin{pmatrix} \frac{\rho' c(c^2+v^2)\sqrt{c^2-v^2}}{\rho' c^4 - p'v^2} & \frac{-(c^2-v^2)}{\rho' c^4-p'v^2}\mathbf{v}^{\mathrm{T}} \\ \frac{-(p'+\rho' c^2)(\sqrt{c^2-v^2})^3}{nc(\rho' c^4-p'v^2)}\mathbf{v} & \frac{c^2-v^2}{n\rho' c^2}\Big(\mathbf{I}_3 + \frac{p'-\rho' c^2}{(\rho' c^4-p'v^2)}\mathbf{v}\mathbf{v}^{\mathrm{T}}\Big) \end{pmatrix}, \tag{2.4}$$

where $\mathbf{I}_3$ stands for the 3×3 identity matrix.

Similarly, we have

$$\nabla_{\mathbf{z}}\mathbf{F}^j(\mathbf{u}) = \begin{pmatrix} \frac{cv_j}{\sqrt{c^2-v^2}} & \frac{nc}{\sqrt{c^2-v^2}^3}v_j\mathbf{v}^{\mathrm{T}} + \frac{nc}{\sqrt{c^2-v^2}}\mathbf{e}_j^{\mathrm{T}} \\ \frac{(p'+\rho' c^2)}{c^2-v^2}v_j\mathbf{v} + p'\mathbf{e}_j & \frac{2(p+\rho c^2)}{(c^2-v^2)^2}v_j\mathbf{v}\mathbf{v}^{\mathrm{T}} + \frac{p+\rho c^2}{c^2-v^2}\mathbf{v}\mathbf{e}_j^{\mathrm{T}} + \frac{p+\rho c^2}{c^2-v^2}v_j\mathbf{I}_3 \end{pmatrix}, \tag{2.5}$$

where $\mathbf{e}_j = (\delta_{1j}, \delta_{2j}, \delta_{3j})^{\mathrm{T}}$.

Using (2.4) together with (2.5), we get

$$(\nabla_{\mathbf{z}}\mathbf{u})^{-1}\nabla_{\mathbf{z}}\mathbf{F}^j = \begin{pmatrix} B_1 v_j & B_2\mathbf{e}_j^{\mathrm{T}} \\ B_3\mathbf{e}_j + B_4\mathbf{v} & B_5\mathbf{v}\mathbf{e}_j^{\mathrm{T}} + v_j\mathbf{I}_3 \end{pmatrix}, \tag{2.6}$$

where

$$B_1 = \frac{(\rho' c^2 - p')c^2}{\rho' c^4 - p'v^2}, \quad B_2 = \frac{n\rho' c^4}{\rho' c^4-p'v^2}, \quad B_3 = \frac{p'(c^2-c^2)}{n\rho' c^2},$$
$$B_4 = \frac{p'(p'-\rho' c^2)(c^2-v^2)v_j}{n\rho' c^2(\rho' c^4-p'v^2)}, \quad B_5 = \frac{-p'(c^2-v^2)}{\rho' c^4-p'v^2}.$$

From (2.2), we have

$$\nabla_{\mathbf{z}}\eta(\nabla_{\mathbf{z}}\mathbf{u})^{-1}\nabla_{\mathbf{z}}\mathbf{F}^j = \nabla_{\mathbf{z}}q^j. \tag{2.7}$$

Setting $\eta = \eta(n, y)$ and $q^j = Q(n, y)v_j$, where $y = v^2 = v_1^2 + v_2^2 + v_3^2$, (2.7) becomes

$$(\eta_n, 2\mathbf{v}^{\mathrm{T}}\eta_y)\begin{pmatrix} B_1 v_j & B_2\mathbf{e}_j^{\mathrm{T}} \\ B_3\mathbf{e}_j + B_4\mathbf{v} & B_5\mathbf{v}\mathbf{e}_j^{\mathrm{T}} + v_j\mathbf{I}_3 \end{pmatrix} = (Q_n v_j, 2\mathbf{v}^{\mathrm{T}}Q_y v_j + Q\mathbf{e}_j^{\mathrm{T}}), \tag{2.8}$$

which yields

$$\begin{cases} \eta_y = Q_y, \\ B_1\eta_n + 2B_3'\eta_y = Q_n, \\ B_2\eta_n + 2B_5 y\eta_y = Q, \end{cases} \tag{2.9}$$

where $B_3' = \frac{p'(c^2-v^2)^2}{n(\rho' c^4 - p' v^2)}$. Furthermore, (2.9) leads to

$$Q_y = \frac{B_2 Q_n - B_1 Q}{2(B_2 B_3' - B_1 B_4 y)}. \tag{2.10}$$

It follows from the first equation of (2.9) that

$$\eta = Q(n, y) + G(n), \tag{2.11}$$

where G depends only on n.

Inserting this equation into the third equation of (2.9), we have

$$G'(n) = -\frac{B_2 B_3' + B_5 y(1 - B_1)}{B_2 B_3' - B_1 B_5 y} Q_n + \frac{B_3' Q}{B_2 B_3' - B_1 B_5 y}. \tag{2.12}$$

Furthermore, plugging B_1, B_2, B_3' and B_4 into (2.12), we obtain

$$G'(n) = -\frac{c^2 - y}{c^2} Q_n + \frac{c^2 - y}{c^2} \frac{1}{n} Q. \tag{2.13}$$

Assuming $q(n, y) = \frac{c^2 - y}{c^2} Q$, (2.13) becomes

$$G'(n) = -q_n + \frac{1}{n} q. \tag{2.14}$$

Integrating this equation with respect to the variable n, we have

$$q(n, y) = \frac{n}{n_m} \Big(\int_{n_m}^{n} \frac{G'(s)}{s} \mathrm{d}s + h(y) \Big) := \frac{n}{n_m} (g(n) + h(y)). \tag{2.15}$$

Substituting (2.15) into (2.13) and separating variables, we get

$$\frac{p + \rho c^2}{p'} g'(n) - g = 2(c^2 - y) h'(y) + h(y), \tag{2.16}$$

where the left-hand side of (2.16) depends only on n, and the right-hand side depends only on y. So both sides of (2.16) should be equal to the same constant, assumed as D, which implies

$$\begin{cases} \dfrac{p + \rho c^2}{p'} g'(n) - g = D, \\ 2(c^2 - y) h'(y) + h(y) = D. \end{cases} \tag{2.17}$$

Noting (1.8), we solve the first equation of (2.17) to have

$$\begin{aligned} g &= \widetilde{D}_1 \exp \int_{n_m}^{n} \frac{p'}{p + \rho c^2} \mathrm{d}n = \widetilde{D}_1 \exp \int_{n_m}^{n} \frac{p'}{\rho' n c^2} \mathrm{d}n \\ &= \widetilde{D}_1 \exp \int_{n_m}^{n} \frac{p_\rho}{n c^2} \mathrm{d}n = \widetilde{D}_1 \exp \int_{\rho_m}^{\rho} \frac{p_\rho}{p + \rho c^2} \mathrm{d}\rho, \end{aligned} \tag{2.18}$$

where $\widetilde{D}_1$ is a constant.

From the fact that

$$n = n_m \exp \int_{n_m}^{n} \frac{1}{n} \mathrm{d}n = n_m \exp \int_{\rho_m}^{\rho} \frac{c^2}{p + \rho c^2} \mathrm{d}\rho, \tag{2.19}$$

it holds that

$$ng = \widetilde{D}_1 n_m \exp \int_{\rho_m}^{\rho} \frac{p_\rho + c^2}{p + \rho c^2} \mathrm{d}\rho = \widetilde{D}_1 n_m (p + \rho c^2). \tag{2.20}$$

Solving the second ordinary differential equation of (2.17), we have

$$h(y) = -\widetilde{D}_2 \sqrt{c^2 - y}, \tag{2.21}$$

where $\widetilde{D}_2$ is a constant.

Inserting (2.20) and (2.21) into (2.15), we have

$$q = \widetilde{D}_1 (p + \rho c^2) - \widetilde{D}_2 \frac{n}{n_m} \sqrt{c^2 - y}, \tag{2.22}$$

where $n_m \in (n_*, n^*)$ is an integration constant. Noting $q(n, y) = \frac{c^2 - y}{c^2} Q$, we obtain

$$Q = D_1 \frac{p + \rho c^2}{c^2 - v^2} + D_2 \frac{n}{n_m \sqrt{c^2 - v^2}}. \tag{2.23}$$

Inserting (2.22) into (2.14) and using (1.8), we get

$$G'(n) = -\widetilde{D}_1 p'.$$

Integrating this equation yields

$$G = -\frac{D_1}{c^2} p + D_3. \tag{2.24}$$

Thus substituting (2.23)–(2.24) into (2.11) leads to

$$\eta = D_1 \frac{p + \rho c^2}{c^2 - v^2} + D_2 \frac{n}{n_m \sqrt{c^2 - v^2}} - \frac{D_1}{c^2} p + D_3, \tag{2.25}$$

where $D_1 = c^2 \widetilde{D}_1,\ D_2 = c^2 \widetilde{D}_2$, and D_3 is a constant.

Noting that

$$\int_{\rho_m}^{\rho} \frac{c^2}{p + \rho c^2} \mathrm{d}\rho = \ln \frac{p + \rho c^2}{p_m + \rho_m c^2} - \int_{\rho_m}^{\rho} \frac{p'_\rho}{p + \rho c^2} \mathrm{d}\rho, \tag{2.26}$$

we define

$$\frac{K \exp \int_{\rho_m}^{\rho} \frac{c^2}{p + \rho c^2} \mathrm{d}\rho}{p + \rho c^2} = \exp \int_{\rho_m}^{\rho} \frac{-p'_\rho}{p + \rho c^2} \mathrm{d}\rho := \Phi(\rho), \tag{2.27}$$

where $K := p_m + \rho_m c^2$, with $\rho_m = \rho(n_m) \in (\rho(n_*), \rho(n^*))$. Together with (2.19), we have

$$\frac{K}{p + \rho c^2} \frac{n}{n_m} = \Phi(\rho). \tag{2.28}$$

$\Phi(\rho)$ can be expanded with respect to $\frac{1}{c^2}$ at 0 as

$$\Phi(\rho) = 1 - \int_{\rho_m}^{\rho} \frac{p'_\rho}{\rho} \mathrm{d}\rho \frac{1}{c^2} + O\Big(\frac{1}{c^4}\Big). \tag{2.29}$$

Similarly, we also expand $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ with respect to $\frac{1}{c^2}$ at 0 as

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = 1 + \frac{v^2}{2}\frac{1}{c^2} + O\Big(\frac{1}{c^4}\Big). \tag{2.30}$$

From (2.25) and (2.28)–(2.9), we have

$$\eta = \frac{\frac{p}{c^2}+\rho}{\sqrt{1-\frac{v^2}{c^2}}}\Big(\frac{D_1}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{cD_2}{K} - \frac{D_2}{cK}\int_{\rho_m}^{\rho}\frac{p'_\rho}{\rho}\mathrm{d}\rho\Big) - \frac{D_1}{c^2}p + D_3. \tag{2.31}$$

To choose the constants D_1, D_2, D_3, we consider the entropy function of the corresponding non-relativistic fluid, which is

$$\eta^\infty = \frac{1}{2}nv^2 + n\int_{n_m}^{n}\frac{\mathrm{d}p}{n} - p, \tag{2.32}$$

which can be obtained in exactly the same way as (2.31).

Letting $c\to\infty$ in (2.31) and comparing with (2.32), we choose $D_1 = c^2,\ D_2 = -cK,\ D_3 = 0$. Then it holds that

$$\eta = c^2\Big(\frac{p+\rho c^2}{c^2-v^2} - \frac{p}{c^2}\Big) - \frac{cK\exp\int_{\rho_m}^{\rho}\frac{c^2\mathrm{d}\rho}{p+\rho c^2}}{\sqrt{c^2-v^2}}. \tag{2.33}$$

2.2 Symmetrization

In this subsection, we use the obtained strictly convex entropy function to construct a suitable symmetrization of (1.4) and verify the positive definiteness of the coefficient matrix.

Define

$$\Omega_{\mathbf{z}} = \{\mathbf{z} : n_* < n < n^*, v^2 < c^2\}.$$

The existence of a strictly convex entropy guarantees that classical solutions to the initial-value problem depend continuously on the initial data, even within the broader class of admissible bounded weak solutions (see [5, Theorem 5.2.1] or [35, Theorem B]). Thus, if the initial data $(n_0, \mathbf{v}_0)$ take values in any compact subset D of $\Omega_{\mathbf{z}} = \{\mathbf{z} : n_* < n < n^*,\ v^2 < c^2\}$, then there exists a classical solution $(n, \mathbf{v})$ taking values in $\Omega_{\mathbf{z}}$.

Let $\mathbf{w} = (\nabla_{\mathbf{u}}\eta)^{\mathrm{T}} = (w_0, w_1, w_2, w_3)^{\mathrm{T}}$. It holds that

$$\partial_\alpha\mathbf{u} = (\nabla_{\mathbf{u}}\mathbf{w})^{-1}\partial_\alpha\mathbf{w} = (\nabla^2_{\mathbf{u}}\eta)^{-1}\partial_\alpha\mathbf{w}, \tag{2.34}$$

where α stands for one of the arguments t, x_1, x_2, x_3. Then the system (2.1) reduces to

$$(\nabla^2_{\mathbf{u}}\eta)^{-1}\partial_t\mathbf{w} + \sum_{j=1}^{3}(\nabla_{\mathbf{u}}\mathbf{F}^j)(\nabla^2_{\mathbf{u}}\eta)^{-1}\partial_{x_j}\mathbf{w} = 0. \tag{2.35}$$

If we set

$$A^0(\mathbf{w}) = (\nabla^2_{\mathbf{u}}\eta)^{-1}, \quad A^j(\mathbf{w}) = (\nabla_{\mathbf{u}}\mathbf{F}^j)(\nabla^2_{\mathbf{u}}\eta)^{-1}, \tag{2.36}$$

(2.35) can be rewritten as

$$A^0(\mathbf{w})\partial_t \mathbf{w} + \sum_{j=1}^{3} A^j(\mathbf{w})\partial_{x_j}\mathbf{w} = 0, \tag{2.37}$$

whose coefficient matrices $A_\alpha(\mathbf{w})$ $(\alpha = 0, 1, 2, 3)$ satisfy the following:

$$\begin{aligned}&\text{(i) They are all real symmetric and smooth in } \mathbf{w},\\ &\text{(ii) } A^0(\mathbf{w}) \text{ is positively definite.}\end{aligned} \tag{2.38}$$

A hyperbolic system (2.37) satisfying (2.38) is called a symmetric hyperbolic system (see [15, 29]).

Now we figure out the expressions of $A^0(\mathbf{w})$ and $A^j(\mathbf{w})$ $(j = 1, 2, 3)$, and verify the positive definiteness of $A^0(\mathbf{w})$. $\mathbf{w}$ can be written as

$$\begin{aligned}\mathbf{w} &= (\nabla_{\mathbf{u}}\eta)^{\mathrm{T}} = (\nabla_{\mathbf{u}}\mathbf{z})^{\mathrm{T}}(\nabla_{\mathbf{z}}\eta) = ((\nabla_{\mathbf{z}}\mathbf{u})^{\mathrm{T}})^{-1}(\nabla_{\mathbf{z}}\eta)\\ &= \left(\rho' c\sqrt{c^2 - v^2} - \frac{K}{n_m}, \mathbf{v}^{\mathrm{T}}\right)^{\mathrm{T}}\\ &= \left(\frac{p + \rho c^2}{nc}\sqrt{c^2 - v^2} - \frac{K}{n_m}, \mathbf{v}^{\mathrm{T}}\right)^{\mathrm{T}},\end{aligned} \tag{2.39}$$

then we can compute that

$$\begin{aligned}&\nabla^2_{\mathbf{u}}\eta = \nabla_{\mathbf{u}}\mathbf{w} = (\nabla_{\mathbf{z}}\mathbf{w})(\nabla_{\mathbf{z}}\mathbf{u})^{-1}\\ &= \frac{c^2 - v^2}{nc(\rho' c^4 - p' v^2)}\begin{pmatrix}\rho' c(p' c^2 + 2p' v^2 + \rho' c^2 v^2) & -(p' + \rho' c^2)\sqrt{c^2 - v^2}\mathbf{v}^{\mathrm{T}}\\ -(p' + \rho' c^2)\sqrt{c^2 - v^2}\mathbf{v} & \dfrac{\rho' c^4 - p' v^2}{\rho' c}\mathbf{I}_3 + \dfrac{p' - \rho' c^2}{\rho' c}\mathbf{v}\mathbf{v}^{\mathrm{T}}\end{pmatrix}.\end{aligned} \tag{2.40}$$

It is not easy to show by direct calculation that

$$A^0(\mathbf{w}) = \begin{pmatrix}\dfrac{nc^2}{(c^2 - v^2)p'} & \dfrac{nc(p' + \rho' c^2)\mathbf{v}^{\mathrm{T}}}{p'(\sqrt{c^2 - v^2})^3}\\ \dfrac{nc(p' + \rho' c^2)\mathbf{v}}{p'(\sqrt{c^2 - v^2})^3} & \dfrac{n\rho' c^2}{c^2 - v^2}\left(\mathbf{I}_3 + \dfrac{(3p' + \rho' c^2)\mathbf{v}\mathbf{v}^{\mathrm{T}}}{p'(c^2 - v^2)}\right)\end{pmatrix} \tag{2.41}$$

and

$$\begin{aligned}&A^j(\mathbf{w})\\ &= \begin{pmatrix}\dfrac{nc^2 v^j}{p'(c^2 - v^2)} & \dfrac{nc(p' + \rho' c^2)v^j\mathbf{v}^{\mathrm{T}}}{p'\sqrt{c^2 - v^2}^3} + \dfrac{nc\mathbf{e}_j^{\mathrm{T}}}{\sqrt{c^2 - v^2}}\\ \dfrac{nc(p' + \rho' c^2)v^j\mathbf{v}}{p'\sqrt{c^2 - v^2}^3} + \dfrac{nc\mathbf{e}_j}{\sqrt{c^2 - v^2}} & \dfrac{n\rho' c^2(3p' + \rho' c^2)v_j\mathbf{v}\mathbf{v}^{\mathrm{T}}}{p'(c^2 - v^2)^2} + \dfrac{n\rho' c^2(\mathbf{v}\mathbf{e}_j^{\mathrm{T}} + \mathbf{e}_j\mathbf{v}^{\mathrm{T}} + v_j\mathbf{I}_3)}{c^2 - v^2}\end{pmatrix}.\end{aligned} \tag{2.42}$$

It is obvious that $A_j(\mathbf{w})$ $(j = 1, 2, 3)$ are symmetric forms. Now we prove that $(A_0(\mathbf{w}))^{-1}$ is uniformly bounded. Firstly, we verify that $A_0(\mathbf{w})$ has a strict lower bound. In fact, for any given four-dimensional vector $\mathbf{r} = (r_0, \widetilde{\mathbf{r}}^{\mathrm{T}}) = (r_0, r_1, r_2, r_3)$, it holds

$$\begin{aligned}\mathbf{r}^{\mathrm{T}} A_0(\mathbf{w})\mathbf{r} &= (r_0, \widetilde{\mathbf{r}}^{\mathrm{T}})\begin{pmatrix}a_1 & a_2\mathbf{v}^{\mathrm{T}}\\ a_2\mathbf{v} & a_3\mathbf{v}\mathbf{v}^{\mathrm{T}} + a_4\mathbf{I}_3\end{pmatrix}(r_0, \widetilde{\mathbf{r}})^{\mathrm{T}}\\ &= a_1 r_0^2 + 2a_2 r_0\mathbf{v}^{\mathrm{T}}\widetilde{\mathbf{r}} + a_3(\mathbf{v}^{\mathrm{T}}\widetilde{\mathbf{r}})^2 + a_4|\widetilde{\mathbf{r}}|^2,\end{aligned}$$

where

$$a_1 = \frac{nc^2}{(c^2 - v^2)p'}, \quad a_2 = \frac{nc(p' + \rho' c^2)}{p'\sqrt{c^2 - v^2}^3},$$

$$a_3 = \frac{n\rho' c^2(3p' + \rho' c^2)}{p'(c^2 - v^2)^2}, \quad a_4 = \frac{n\rho' c^2}{c^2 - v^2}.$$

Setting $\widetilde{a}_1 = (1-\delta)a_1$ with $0 < \delta < \frac{1}{2}$ to be determined later, it holds that

$$\begin{aligned}
\mathbf{r}^{\mathrm{T}} A_0(\mathbf{w})\mathbf{r} &= a_1 r_0^2 + 2a_2 r_0 \mathbf{v}^{\mathrm{T}}\widetilde{\mathbf{r}} + a_3(\mathbf{v}^{\mathrm{T}}\widetilde{\mathbf{r}})^2 + a_4|\widetilde{\mathbf{r}}|^2 \\
&= \widetilde{a}_1\Big(r_0 + \frac{a_2}{\widetilde{a}_1}\mathbf{v}^{\mathrm{T}}\widetilde{\mathbf{r}}\Big)^2 - \Big(\frac{1}{a_1}(a_2^2 - a_1 a_3) + \frac{\delta}{1-\delta}\frac{a_2^2}{a_1}\Big)(\mathbf{v}^{\mathrm{T}}\widetilde{\mathbf{r}})^2 + \delta a_1 r_0^2 + a_4|\widetilde{\mathbf{r}}|^2 \\
&\geq \Big(a_4 - \frac{1}{a_1}(a_2^2 - a_1 a_3)v^2 - \frac{\delta}{1-\delta}\frac{a_2^2}{a_1}v^2\Big)|\widetilde{\mathbf{r}}|^2 + \delta a_1 r_0^2 \\
&= \Big(\frac{n(\rho' c^4 + \rho' c^2 v^2 - p' v^2)}{(c^2 - v^2)^2} - 2\delta\frac{a_2^2}{a_1}v^2\Big)|\widetilde{\mathbf{r}}|^2 + \delta a_1 r_0^2 \\
&\geq \delta a_1 r_0^2 + \delta a_1|\widetilde{\mathbf{r}}|^2
\end{aligned}$$

under the condition that

$$0 < a_1 + 2\delta\frac{a_2^2}{a_1}v^2 < \frac{n(\rho' c^4 + \rho' c^2 v^2 - p' v^2)}{(c^2 - v^2)^2}, \tag{2.43}$$

i.e.,

$$0 < \delta < \frac{p'(\rho' c^4 + \rho' c^2 v^2 - p' v^2)}{c^2(c^2 - v^2) + 2(p' + \rho' c^2)^2 v^2}.$$

Since the right-hand side of the above inequality has a positive lower bound, denoted by δ_*, we can take $\delta < \min\left(\frac{1}{2}, \delta_*\right)$. Noting that

$$\delta a_1 \geq \delta\frac{n}{p'} = \delta\frac{n^2 c^2}{c_s^2(p + \rho c^2)} \geq \delta\frac{n_*^2}{p^* + \rho^* c^2} := \delta_{**},$$

we have

$$\mathbf{r}^{\mathrm{T}} A_0(\mathbf{w})\mathbf{r} \geq \delta_{**}|\mathbf{r}|^2.$$

Thanks to the fact that $(n, \mathbf{v}) \in \Omega_{\mathbf{z}}$, we get the upper bound of $(A_0(\mathbf{w}))^{-1}$.

Then the local existence of smooth solutions to the Cauchy problem (1.4) and (1.10) for the non-vacuum case follows from Friedrichs-Lax-Kato theory (see [14, 28]).

3 The Vacuum Case

In this section, when the initial data $(n_0, \mathbf{v}_0)$ are allowed to contain vacuum states, the coefficients $A^0(\mathbf{w})$ for (1.4) will blow-up near the vacuum. Thus the symmetric method to the non-vacuum case will not be valid any longer. To overcome this difficulty, we adopt Lefloch-Ukai's symmetrization (see [19]) for (1.3), however our transformation is about variables of n and $\mathbf{v}$ instead of variables of ρ and $\mathbf{v}$ in [19]. The coefficient matrix of the new system under this transformation is no longer degenerate near vacuum. Then we use Friedrichs-Lax-Kato theory (see [14, 28]) to prove the local existence of smooth solutions to (1.4) and (1.10) with the initial data of the vacuum case.

Now our initial data $n_0, \mathbf{v}_0$ satisfy the condition (1.12). Before proceeding, we first introduce some notations as in [19].

The modified mass density variable w is

$$w = w(\rho) := \int_0^\rho \frac{c_s}{q(s)} \mathrm{d}s, \tag{3.1}$$

where c_s is the local sound speed in the fluid.

The modified velocity scalar is

$$u = u(|\mathbf{v}|) = u(v) := \frac{c^2}{2} \ln\left(\frac{c+v}{c-v}\right), \tag{3.2}$$

and we refer to

$$z_\pm := w \pm u \tag{3.3}$$

as the generalized Riemann invariant variables.

We also introduce the normalized velocity $\widetilde{\mathbf{v}}$ and the associated projection operator $P(\mathbf{v})$ as follows:

$$\widetilde{\mathbf{v}} = (\widetilde{v}_1, \widetilde{v}_2, \widetilde{v}_3) := \frac{\mathbf{v}}{v}, \quad P(\mathbf{v}) := \mathbf{I}_3 - \widetilde{\mathbf{v}} \otimes \widetilde{\mathbf{v}}. \tag{3.4}$$

Here we present some useful identities in [19],

Proposition 3.1

$$\begin{aligned}
&(1)\ P(\mathbf{v})\mathbf{v} = 0,\\
&(2)\ P(\mathbf{v})\partial_t\mathbf{v} = v\partial_t\widetilde{\mathbf{v}},\\
&(3)\ P(\mathbf{v})((\mathbf{v}\cdot\nabla)\mathbf{v}) = v(\mathbf{v}\cdot\nabla)\widetilde{\mathbf{v}},\\
&(4)\ \nabla\cdot\mathbf{v} = \mathrm{tr}(\mathrm{P}(\mathbf{v})\nabla\widetilde{\mathbf{v}}) = \mathrm{tr}(\mathrm{P}(\widetilde{\mathbf{v}})\nabla\widetilde{\mathbf{v}}).
\end{aligned} \tag{3.5}$$

For the convenience of the reader, we give a simple proof here.

Proof of Proposition 3.1

(1)

$$P(\mathbf{v})\mathbf{v} = \begin{pmatrix} 1 - \dfrac{v_1^2}{v^2} & -\dfrac{v_1 v_2}{v^2} & -\dfrac{v_1 v_3}{v^2} \\ -\dfrac{v_1 v_2}{v^2} & 1 - \dfrac{v_2^2}{v^2} & -\dfrac{v_2 v_3}{v^2} \\ -\dfrac{v_1 v_3}{v^2} & -\dfrac{v_2 v_3}{v^2} & 1 - \dfrac{v_3^2}{v^2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0.$$

(2)

$$\begin{aligned}
P(\mathbf{v})\partial_t\mathbf{v} &= \begin{pmatrix} \partial_t v_1 - \dfrac{v_1^2\partial_t v_1}{v^2} - \dfrac{v_1 v_2 \partial_t v_1}{v^2} - \dfrac{v_1 v_3 \partial_t v_1}{v^2} \\ -\dfrac{v_1 v_2 \partial_t v_2}{v^2} + \partial_t v_2 - \dfrac{v_2^2 \partial_t v_2}{v^2} - \dfrac{v_2 v_3 \partial_t v_2}{v^2} \\ -\dfrac{v_1 v_3 \partial_t v_3}{v^2} - \dfrac{v_2 v_3 \partial_t v_3}{v^2} + \partial_t v_3 - \dfrac{v_3^2 \partial_t v_3}{v^2} \end{pmatrix} \\
&= \partial_t\mathbf{v} - \frac{\overline{\mathbf{v}}}{2}\frac{\partial_t v^2}{v} \\
&= \partial_t\mathbf{v} - \widetilde{\mathbf{v}}\partial_t v = v\partial_t\widetilde{\mathbf{v}},
\end{aligned}$$

where we used $\partial_t \mathbf{v} = \partial_t(\widetilde{\mathbf{v}} v) = v\partial_t \widetilde{\mathbf{v}} + \widetilde{\mathbf{v}}\partial_t v$.

(3) Similarly to (2), we get

$$P(\mathbf{v})\partial_t((\mathbf{v}\cdot\nabla)\mathbf{v}) = v(\mathbf{v}\cdot\nabla)\widetilde{\mathbf{v}}.$$

(4) On the one hand, noting that $|\widetilde{\mathbf{v}}| = 1$, we have

$$(\nabla\cdot\widetilde{\mathbf{v}}) = \nabla\cdot\widetilde{\mathbf{v}} - \widetilde{\mathbf{v}}\cdot\frac{\nabla|\widetilde{\mathbf{v}}|^2}{2}.$$

On the other hand, it holds that

$$\begin{aligned}
\mathrm{tr}(P(\mathbf{v})\nabla\widetilde{\mathbf{v}}) &= \mathrm{tr}(\nabla\widetilde{\mathbf{v}}\mathbf{I}_3) - \mathrm{tr}(\widetilde{\mathbf{v}}\otimes\widetilde{\mathbf{v}}\nabla\widetilde{\mathbf{v}}\mathbf{I}_3)\\
&= \nabla\cdot\widetilde{\mathbf{v}} - \mathrm{tr}\left(\begin{pmatrix}\widetilde{v}_1^2 & \widetilde{v}_1\widetilde{v}_2 & \widetilde{v}_1\widetilde{v}_3\\ \widetilde{v}_2\widetilde{v}_1 & \widetilde{v}_2^2 & \widetilde{v}_2\widetilde{v}_3\\ \widetilde{v}_3\widetilde{v}_1 & \widetilde{v}_3\widetilde{v}_2 & \widetilde{v}_3^2\end{pmatrix}\begin{pmatrix}\partial_{x_1}\widetilde{v}_1 & \partial_{x_2}\widetilde{v}_1 & \partial_{x_3}\widetilde{v}_1\\ \partial_{x_1}\widetilde{v}_2 & \partial_{x_2}\widetilde{v}_2 & \partial_{x_3}\widetilde{v}_2\\ \partial_{x_1}\widetilde{v}_3 & \partial_{x_2}\widetilde{v}_3 & \partial_{x_3}\widetilde{v}_3\end{pmatrix}\right)\\
&= \nabla\cdot\widetilde{\mathbf{v}} - \widetilde{\mathbf{v}}\cdot\frac{\nabla|\widetilde{\mathbf{v}}|^2}{2}.
\end{aligned}$$

From the definition of $P(\mathbf{v})$ in (3.4), (4) is proved.

3.1 Symmetric form of Euler equations

In this section, we will deduce a symmetric formulation of (1.4) with respect to the general Riemann invariants and the normalized velocity defined by (3.1)–(3.2). We conclude as follows.

Lemma 3.1 *In terms of the generalized Riemann invariant variables (z_+, z_-) and the normalized velocity $\widetilde{\mathbf{v}}$ defined by (3.3)–(3.4), respectively, the relativistic Euler equations reduce to the following symmetric form*:

$$\begin{aligned}
&\Big(1+\frac{vc_s}{c^2}\Big)\partial_t z_+ + \frac{1-\frac{c_s^2}{c^2}}{1-\frac{vc_s}{c^2}}(v+c_s)\widetilde{\mathbf{v}}\cdot\nabla z_+ + c_s v\,\mathrm{tr}(\mathrm{P}(\widetilde{\mathbf{v}})\nabla\widetilde{\mathbf{v}}) = 0,\\
&\Big(1-\frac{vc_s}{c^2}\Big)\partial_t z_- + \frac{1-\frac{c_s^2}{c^2}}{1+\frac{vc_s}{c^2}}(v-c_s)\widetilde{\mathbf{v}}\cdot\nabla z_- - c_s v\,\mathrm{tr}(\mathrm{P}(\widetilde{\mathbf{v}})\nabla\widetilde{\mathbf{v}}) = 0,\\
&\frac{2v^2}{1-\frac{v^2}{c^2}}(\partial_t\widetilde{\mathbf{v}} + \mathbf{v}\cdot\nabla\widetilde{\mathbf{v}}) + c_s v P(\widetilde{\mathbf{v}})\nabla z_+ - c_s v P(\widetilde{\mathbf{v}})\nabla z_- = 0,
\end{aligned}\tag{3.6}$$

where $z_\pm$ are real valued and $\widetilde{\mathbf{v}}$ is a unit vector satisfying $|\widetilde{\mathbf{v}}| = 1$.

Proof In Section 1, we know from (1.8) that

$$\frac{\mathrm{d}\rho}{\mathrm{d}n} = \frac{q}{n},\tag{3.7}$$

where q is defined as $q := \frac{p}{c^2} + \rho$.

Using (3.7), we expand the conservation equation of baryon numbers in (1.4) as follows:

$$\begin{aligned}
&\frac{n}{q\sqrt{1-\frac{v^2}{c^2}}}\partial_t\rho + \frac{n}{q\sqrt{1-\frac{v^2}{c^2}}}\mathbf{v}\cdot\nabla\rho\\
&+\frac{n}{2c^2\Big(\sqrt{1-\frac{v^2}{c^2}}\Big)^3}(\partial_t v^2 + \mathbf{v}\cdot\nabla v^2) + \frac{n\nabla\cdot\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}} = 0.
\end{aligned}\tag{3.8}$$

Multiplying (3.8) by $\frac{q\sqrt{1-\frac{v^2}{c^2}}}{n}$, we have

$$\partial_t\rho = -\mathbf{v}\cdot\nabla\rho - \frac{q}{2c^2(1-\frac{v^2}{c^2})}(\partial_t v^2 + \mathbf{v}\cdot\nabla v^2) - q\nabla\cdot\mathbf{v}. \tag{3.9}$$

Expanding the momentum conservation equation of (1.4) and using (3.7), we have

$$\begin{aligned}&\Big(\frac{1+\frac{c_s^2}{c^2}}{1-\frac{v^2}{c^2}}(\partial_t\rho + \mathbf{v}\cdot\nabla\rho) + \frac{q}{c^2(1-\frac{v^2}{c^2})^2}(\partial_t v^2 + \mathbf{v}\cdot\nabla v^2) + \frac{q\nabla\cdot\mathbf{v}}{1-\frac{v^2}{c^2}}\Big)\mathbf{v}\\ &+ \frac{q}{1-\frac{v^2}{c^2}}(\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}) + p'(\rho)\nabla\rho = 0,\end{aligned} \tag{3.10}$$

where we used

$$\sum_{k=1}^{3}\partial_{x_k}\Big(\frac{\frac{p}{c^2}+\rho}{1-\frac{v^2}{c^2}}v_k\mathbf{v}\Big) = \nabla\cdot\Big(\frac{\frac{p}{c^2}+\rho}{1-\frac{v^2}{c^2}}\mathbf{v}\Big)\mathbf{v} + \frac{\frac{p}{c^2}+\rho}{1-\frac{v^2}{c^2}}(\mathbf{v}\cdot\nabla)\mathbf{v}.$$

From (3.9)–(3.10), we have

$$\begin{aligned}&\frac{q(1-\frac{c_s^2}{c^2})}{2c^2(1-\frac{v^2}{c^2})^2}\mathbf{v}(\partial_t v^2 + \mathbf{v}\cdot\nabla v^2) + \frac{q}{1-\frac{v^2}{c^2}}(\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v})\\ &- \frac{qc_s^2}{c^2(1-\frac{v^2}{c^2})}\mathbf{v}\nabla\cdot\mathbf{v} + p'(\rho)\nabla\rho = 0.\end{aligned} \tag{3.11}$$

Moreover, multiplying (3.9) by $w'(\rho)$, (3.11) can be simplified as

$$\partial_t w + \mathbf{v}\cdot\nabla w + \frac{c_s}{2c^2(1-\frac{v^2}{c^2})}(\partial_t v^2 + \mathbf{v}\cdot\nabla v^2) + c_s\nabla\cdot\mathbf{v} = 0. \tag{3.12}$$

From the definition of u in (3.2), we have

$$\mathrm{d}u = \frac{1}{1-\frac{v^2}{c^2}}\mathrm{d}v, \tag{3.13}$$

then (3.12) reduces to

$$\partial_t w + \mathbf{v}\cdot\nabla w + \frac{c_s v}{c^2}(\partial_t u + \mathbf{v}\cdot\nabla u) + c_s\nabla\cdot\mathbf{v} = 0. \tag{3.14}$$

By $\cdot\frac{2\mathbf{v}}{q}$, (3.11) can be rewritten as

$$\frac{1-\frac{c_s^2v^2}{c^4}}{(1-\frac{v^2}{c^2})^2}(\partial_t v^2 + \mathbf{v}\cdot\nabla v^2) - \frac{2\frac{c_s^2v^2}{c^2}}{1-\frac{v^2}{c^2}}\nabla\cdot\mathbf{v} + 2c_s\mathbf{v}\cdot\nabla w = 0. \tag{3.15}$$

Multiplying this equation by $\frac{1}{2}v$ and using (3.13), (3.15) becomes

$$\frac{1-\frac{c_s^2v^2}{c^4}}{1-\frac{v^2}{c^2}}(\partial_t u + \mathbf{v}\cdot\nabla u) - \frac{\frac{c_s^2v^2}{c^2}}{(1-\frac{v^2}{c^2})v}\nabla\cdot\mathbf{v} + c_s\widetilde{\mathbf{v}}\cdot\nabla w = 0. \tag{3.16}$$

To obtain the expression of $\widetilde{\mathbf{v}}$, we multiply (3.11) by the projection $P(\widetilde{\mathbf{v}})$, and due to (3.5) and the definition of w, we get

$$\frac{|\mathbf{v}|}{1-\frac{v^2}{c^2}}(\partial_t\widetilde{\mathbf{v}} + (\mathbf{v}\cdot\nabla)\widetilde{\mathbf{v}}) + c_sP(\widetilde{\mathbf{v}})\nabla w = 0. \tag{3.17}$$

To obtain the expression of w, we combine (3.14) and (3.16) to give

$$\Big(1-\frac{c_s^2v^2}{c^4}\Big)\partial_t w+\Big(1-\frac{c_s^2}{c^2}\Big)\mathbf{v}\cdot\nabla w+c_s\nabla\cdot\mathbf{v}=0. \tag{3.18}$$

Using the definition of u in (3.2) again, we have

$$\nabla\cdot\mathbf{v}=\nabla\cdot(v\widetilde{\mathbf{v}})=v\nabla\cdot\widetilde{\mathbf{v}}+\widetilde{\mathbf{v}}\cdot\nabla v=v\nabla\cdot\widetilde{\mathbf{v}}+\Big(1-\frac{v^2}{c^2}\Big)\widetilde{\mathbf{v}}\cdot\nabla u. \tag{3.19}$$

Plugging this into (3.16), we have

$$\Big(1-\frac{c_s^2v^2}{c^4}\Big)\partial_t u+c_s\Big(1-\frac{v^2}{c^2}\Big)\widetilde{\mathbf{v}}\cdot\nabla w+\Big(1-\frac{c_s^2}{c^2}\Big)\mathbf{v}\cdot\nabla u-\frac{c_s^2v^2}{c^2}\nabla\cdot\widetilde{\mathbf{v}}=0. \tag{3.20}$$

Substituting (3.19) for (3.18) leads to

$$\Big(1-\frac{c_s^2v^2}{c^4}\Big)\partial_t w+\Big(1-\frac{c_s^2}{c^2}\Big)\mathbf{v}\cdot\nabla w+c_sv\nabla\cdot\widetilde{\mathbf{v}}+\Big(1-\frac{v^2}{c^2}\Big)\widetilde{\mathbf{v}}\cdot\nabla u=0. \tag{3.21}$$

To derive our desired symmetric form, $\nabla\cdot\widetilde{\mathbf{v}}$ needs to be transformed, and from (4) in (3.5), we have

$$\nabla\cdot\widetilde{\mathbf{v}}=\nabla\cdot\widetilde{\mathbf{v}}-\widetilde{\mathbf{v}}\cdot\nabla\frac{\widetilde{v}^2}{2}=\mathrm{tr}(E(\mathbf{v})\nabla\widetilde{\mathbf{v}}).$$

Plugging this identity into (3.20)–(3.21), and together with (3.17), we have

$$\begin{aligned}
&\Big(1-\frac{c_s^2v^2}{c^4}\Big)w_t+\Big(1-\frac{c_s^2}{c^2}\Big)\mathbf{v}\cdot\nabla w+c_s\Big(1-\frac{v^2}{c^2}\Big)\widetilde{\mathbf{v}}\cdot\nabla u+c_sv\,\mathrm{tr}(P(\widetilde{\mathbf{v}})\nabla\widetilde{\mathbf{v}})=0,\\
&\Big(1-\frac{c_s^2v^2}{c^4}\Big)u_t+\Big(1-\frac{c_s^2}{c^2}\Big)\mathbf{v}\cdot\nabla u+c_s\Big(1-\frac{v^2}{c^2}\Big)\widetilde{\mathbf{v}}\cdot\nabla w-\frac{c_s^2v^2}{c^2}\mathrm{tr}(P(\widetilde{\mathbf{v}})\nabla\widetilde{\mathbf{v}})=0,\\
&\frac{v}{1-\frac{v^2}{c^2}}(\widetilde{\mathbf{v}}_t+(\mathbf{v}\cdot\nabla)\widetilde{\mathbf{v}})+c_sP(\widetilde{\mathbf{v}})\nabla w=0.
\end{aligned} \tag{3.22}$$

Using the generalized Riemann invariant variables $z_\pm=u\pm w$, it is easy to obtain the symmetric formulation (3.6).

Moreover, (3.6) can be written as the symmetric form (2.37), in which $A^0(\mathbf{w})$ and $A^j(\mathbf{w})$ are, respectively,

$$A^0(\mathbf{w})=\begin{pmatrix}a_0&0&0\\0&b_0&0\\0&0&c_0\mathbf{I}_3\end{pmatrix},\quad A^j(\mathbf{w})=\begin{pmatrix}a_1\widetilde{v}_j&0&a_2vP_j\\0&b_1\widetilde{v}_j&-a_2vP_j\\a_2vP_j&-a_2vP_j&c_0v_j\mathbf{I}_3\end{pmatrix}, \tag{3.23}$$

where

$$\begin{aligned}
&a_0=1+\frac{vc_s}{c^2},\quad b_0=1-\frac{vc_s}{c^2},\quad c_0=\frac{2v^2}{1-\frac{v^2}{c^2}},\\
&a_1=\frac{1-\frac{c_s^2}{c^2}}{1-\frac{c_sv}{c^2}}(v+c_s),\quad b_1=\frac{1-\frac{c_s^2}{c^2}}{1+\frac{c_sv}{c^2}}(v-c_s),\quad a_2=c_s,\\
&P_j=(P_{j1}(\mathbf{v}),P_{j2}(\mathbf{v}),P_{j3}(\mathbf{v})),\quad j=1,2,3.
\end{aligned} \tag{3.24}$$

Remark 3.1 By this lemma, although (1.3)–(1.4) are different, they are symmetrized to the same form (3.6).

The following proof is the same as that in [19] and for the reader's convenience, here we briefly list the main steps.

Observe that the above matrix $A^0(\mathbf{w})$ allows the density to vanish, since the coefficients remain bounded as the density approaches to zero.

Moreover, from (3.23), we observe that

$$\langle A^0(\mathbf{w})\xi, \xi\rangle = a_0|\xi_1|^2 + b_0|\xi_2|^2 + c_0|\mathbf{v}|^2|\widehat{\xi}|^2, \tag{3.25}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the Eucilidian inner product in $\mathbb{R}^5$ and

$$\xi = (\xi_1, \xi_2, \cdots, \xi_5) = (\xi_1, \xi_2, \widehat{\xi}) \in \mathbb{R}^5, \quad \widehat{\xi} = (\xi_3, \xi_4, \xi_5) \in \mathbb{R}^3.$$

From (3.24), the matrix $A^0(\mathbf{w})$ is positively definite as long as the velocity $\mathbf{v}$ never vanishes. According to the Friedrichs-Lax-Kato theory (see [14, 28]), a local in-time solution exists.

However, the matrix $A^0(\mathbf{w})$ may lose its positive definiteness, since the coefficient c_0 of $\partial_t\widetilde{\mathbf{v}}$ in the third equation vanishes at $\mathbf{v} = \mathbf{0}$. On this occasion, we apply a well-chosen Lorentz transformation, which allows the Lorentz-transformed velocity not to exceed the light speed and remain bounded away from zero.

For the reader's convenience, we list some technical results about the Lorentz-transformed velocity (see [19]).

3.2 Lorentz transformation

Assume that K and $\overline{K}$ are two reference frames, in which $(t, \mathbf{x})$ and $(\bar{t}, \overline{\mathbf{x}})$ represent the space-time coordinates corresponding to K and $\overline{K}$, respectively. K moves with respect to $\overline{K}$ at the velocity $\mathbf{V}$. The transformation

$$\begin{cases} \bar{t} = \varpi\Big(t - \dfrac{\mathbf{V}\cdot\mathbf{x}}{c^2}\Big), \\ \overline{\mathbf{x}} = -\varpi\mathbf{V}t + \Big(\mathbf{I}_3 + (\varpi - 1)\dfrac{\mathbf{V}\otimes\mathbf{V}}{V^2}\Big)\mathbf{x} \end{cases} \tag{3.26}$$

is called a Lorentz transformation, where $\varpi = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ is the Lorentz factor.

From (3.26), there holds the velocity transformation law

$$\overline{\mathbf{v}} = \frac{\mathrm{d}\overline{\mathbf{x}}}{\mathrm{d}\bar{t}} = \frac{1}{1 - \mathbf{V}\cdot\frac{\mathbf{v}}{c^2}}(-\mathbf{V} + (\varpi\mathbf{I}_3 + (1-\varpi^{-1})\widetilde{\mathbf{V}}\otimes\widetilde{\mathbf{V}})\mathbf{v}), \tag{3.27}$$

where $\mathbf{v} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$. Denote

$$c\Phi\Big(\frac{\mathbf{v}}{c}, \frac{\mathbf{V}}{c}\Big) := \overline{\mathbf{v}} = \frac{\Big(-\frac{\frac{\mathbf{V}}{c}}{\frac{1}{c}} + \Big(\varpi\mathbf{I}_3 + (1-\varpi^{-1})\frac{\frac{\mathbf{V}}{c}}{\frac{|\mathbf{V}|}{c}}\otimes\frac{\frac{\mathbf{V}}{c}}{\frac{|\mathbf{V}|}{c}}\Big)\frac{\frac{\mathbf{v}}{c}}{\frac{1}{c}}\Big)}{1 - \mathbf{V}\cdot\frac{\mathbf{v}}{c^2}}. \tag{3.28}$$

Then we have the following lemma.

Lemma 3.2 (Uniform Bounds for the Velocity) (see [19]) *Given any $r_0 \in (0,1)$ and any vector $\mathbf{V} \in \mathbb{R}^3$ satisfying $r_0 < \frac{|\mathbf{V}|}{c} < 1$, there exist positive constants $0 < \delta_1 < \delta_2 < 1$ depending only on r_0 and $\frac{\mathbf{V}}{c}$, such that the Lorentz-transformed velocity* (3.27) *is uniform bound away from both the origin and the light speed, i.e.,*

$$\delta_1 \le \Big|\Phi\Big(\frac{\mathbf{v}}{c}, \frac{\mathbf{V}}{c}\Big)\Big| \le \delta_2 \tag{3.29}$$

holds for any $\frac{\mathbf{v}}{c} \in B_{r_0}$, where $B_{r_0} := \{\mathbf{y} \in \mathbb{R}^3 \mid |\mathbf{y}| \le r_0\}$.

Using the Lorentz invariance of relativistic Euler equations, (2.37) can also be expressed in the transformed coordinates $(\overline{\mathbf{x}}, \overline{t}\,)$ defined by (3.26), that is,

$$A^0(\overline{\mathbf{w}})\partial_{\overline{t}}\overline{\mathbf{w}} + \sum_{j=1}^{3} A^j(\overline{\mathbf{w}})\partial_{\overline{x}_j}\overline{\mathbf{w}} = 0, \tag{3.30}$$

and (3.25) becomes

$$\langle A^0(\overline{\mathbf{w}})\xi, \xi\rangle = \overline{a}_0|\xi_1|^2 + \overline{b}_0|\xi_2|^2 + \overline{c}_0|\overline{\mathbf{v}}|^2|\widehat{\xi}|^2. \tag{3.31}$$

In view of the upper and lower bounds (3.29), we conclude that the transformed matrix $A^0(\overline{\mathbf{w}})$ is positively definite in the coordinate system $(\overline{\mathbf{x}}, \overline{t})$. Hence the Friedrichs-Lax-Kato theory (see [14, 28]) applies to the initial-value problem (3.30), provided that the initial data are imposed on the initial hypersurface $\overline{t} = 0$. In the relativistic setting, the initial plane $H_0 : t = 0$ is not preserved under the transformation (3.26). However, in the new coordinate system $(\overline{t}, \overline{\mathbf{x}})$, the initial plane becomes

$$H_0' : \quad \overline{t} = -\frac{\mathbf{V}\cdot\overline{\mathbf{x}}}{c^2}.$$

In order to prove the local well-posedness of the oblique initial-value problem (3.30) with the data on H_0', it is convenient to introduce a further change of the coordinates

$$\overline{\overline{t}} = \overline{t} + \frac{\mathbf{V}\cdot\overline{\mathbf{x}}}{c^2}, \quad \overline{\overline{\mathbf{x}}} = \overline{\mathbf{x}}, \tag{3.32}$$

which maps the hyperplane H_0' to the hyperplane

$$H_0'' : \quad \overline{\overline{t}} = 0.$$

This transformation puts (3.31) into the following form:

$$B^0(\overline{\overline{\mathbf{w}}})\partial_{\overline{\overline{t}}}\overline{\overline{\mathbf{w}}} + \sum_{j=1}^{3} B^j(\overline{\overline{\mathbf{w}}})\partial_{\overline{\overline{x}}_j}\overline{\overline{\mathbf{w}}} = 0, \tag{3.33}$$

where the matrix $B^0(\overline{\overline{\mathbf{w}}})$ is still positively definite (see [19]).

Now Friedrichs-Lax-Kato theory (see [14, 28]) guarantees the existence of a solution defined in a small neighborhood of this hyperplane H_0''. Making the transformation back to the original variables, we obtain a solution in a small neighborhood of the initial line $t = 0$. This completes the proof of the main theorem for the vacuum case.

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