Chin. Ann. Math.
35B(3), 2014, 319–336
DOI: 10.1007/s11401-014-0838-8

Chinese Annals of Mathematics, Series B

On the Error Estimate of the Harmonic B_z Algorithm in MREIT from Noisy Magnetic Flux Field*

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Abstract Magnetic resonance electrical impedance tomography (MREIT, for short) is a new medical imaging technique developed recently to visualize the cross-section conductivity of biologic tissues. A new MREIT image reconstruction method called harmonic B_z algorithm was proposed in 2002 with the measurement of B_z that is a single component of an induced magnetic flux density subject to an injection current. The key idea is to solve a nonlinear integral equation by some iteration process. This paper deals with the convergence analysis as well as the error estimate for noisy input data B_z, which is the practical situation for MREIT. By analyzing the iteration process containing the Laplacian operation on the input magnetic field rigorously, the authors give the error estimate for the iterative solution in terms of the noisy level δ and the regularizing scheme for determining ΔB_z approximately from the noisy input data. The regularizing scheme for computing the Laplacian from noisy input data is proposed with error analysis. Our results provide both the theoretical basis and the implementable scheme for evaluating the reconstruction accuracy using harmonic B_z algorithm with practical measurement data containing noise.

Keywords MREIT, Image reconstruction, Iteration, Error estimate
2000 MR Subject Classification 35R30, 35J05, 65N12

1 Introduction

Magnetic resonance electrical impedance tomography (MREIT, for short) is a new electrical conductivity imaging technique to visualize the cross-sectional images of a conductivity distribution σ of biologic tissues. In contrast to the traditional electrical impedance tomography (EIT, for short) technique (see [1, 7, 18]), this new technique applies essentially the internal electrical current distribution to recover the conductivity, which weakens the ill-posedness of EIT problem and provides a higher resolution of conductivity image.

In MREIT, we place a subject inside a magnetic resonance imaging (MRI, for short) scanner and inject a current I between two electrodes attached on its boundary. Then there exists the internal current $\mathbf{J} = (J_x, J_y, J_z)$ inside the subject, generating a magnetic flux density $\mathbf{B} = (B_x, B_y, B_z)$. Here z-axis is the direction of the main magnetic field of the scanner. The B_z data can be measured by using the MRI scanner, from which we try to reconstruct the bio-tissue conductivity, see Figure 1 for the configuration of this system.

Recently, some reconstruction schemes using B_z data as inversion input have been proposed, such as harmonic B_z method, gradient B_z method and variational gradient B_z method (see

Manuscript received October 23, 2013.
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*This work was supported by the National Natural Science Foundation of China (No. 91330109) and the Research Found for the Doctoral Program of Higher Education of China (No. 20110092110018).

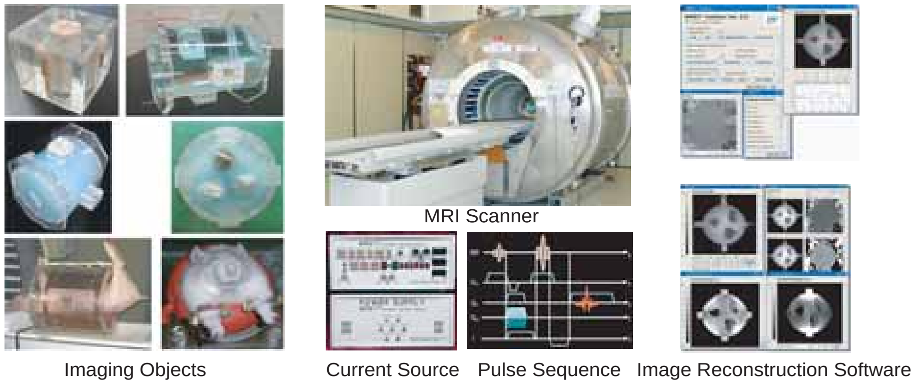

Figure 1 MREIT system at impedance imaging research center and harmonic B_z algorithm mathlab toolkit

[2, 12, 17]). It has been proven that the measurements $B_{z,j}$ corresponding to two incoherent injection currents I_j with $j = 1, 2$ can determine the conductivity distribution σ uniquely in 2-dimensional case (see [3, 14]) under some a priori assumptions.

The harmonic B_z algorithm was the first constructive MREIT imaging method based on B_z data (see [17]). From the Ampere law

$$\mu_0 \nabla \times \mathbf{J} = -\nabla^2 \mathbf{B} \tag{1.1}$$

and

$$\mathbf{J} = -\sigma \nabla u[\sigma], \tag{1.2}$$

we have

$$\nabla^2 \mathbf{B} = \mu_0 \nabla \times (\sigma \nabla u[\sigma]) = \mu_0 \nabla \sigma \times \nabla u[\sigma], \tag{1.3}$$

where μ_0 is the magnetic permeability of the free space, $u[\sigma]$ as a nonlinear function of σ is the induced electrical potential satisfying a nonstandard PDE problem specified in Section 2.

Taking the z-component of (1.3), it follows that

$$\frac{1}{\mu_0} \nabla^2 B_z = \begin{pmatrix} \dfrac{\partial u[\sigma]}{\partial y} & -\dfrac{\partial u[\sigma]}{\partial x} \end{pmatrix} \begin{pmatrix} \dfrac{\partial \sigma}{\partial x} \\ \dfrac{\partial \sigma}{\partial y} \end{pmatrix}. \tag{1.4}$$

Corresponding to two incoherent injected currents $I_j, j = 1, 2$ through two pairs of surface electrodes $\varepsilon_1^{\pm}$ and $\varepsilon_2^{\pm}$, it follows from (1.4) that

$$\begin{bmatrix} \dfrac{\partial \sigma}{\partial x} \\ \dfrac{\partial \sigma}{\partial y} \end{bmatrix} = \frac{1}{\mu_0} \mathbf{A}[\sigma]^{-1} \begin{bmatrix} \nabla^2 B_{z,1} \\ \nabla^2 B_{z,2} \end{bmatrix}, \tag{1.5}$$

where

$$\mathbf{A}[\sigma] =: \begin{bmatrix} \dfrac{\partial u_1[\sigma]}{\partial y} & -\dfrac{\partial u_1[\sigma]}{\partial x} \\ \dfrac{\partial u_2[\sigma]}{\partial y} & -\dfrac{\partial u_2[\sigma]}{\partial x} \end{bmatrix} \tag{1.6}$$

and $u_j[\sigma]$, $B_{z,j}$ are the potential and the magnetic flux density, respectively, corresponding to I_j with $j = 1, 2$.

The harmonic B_z algorithm is an explicit iteration scheme to approximate σ at each 2-dimensional slice $\Omega_{z_0} = \Omega \cap \{z = z_0\} \subset \mathbb{R}^2$ based on the relation (1.5). Since the harmonic B_z algorithm was proposed, it has been improved rapidly in various numerical simulations and phantom experiments (see [8–11, 15–16]). In [4], the authors proved that, for a relatively small contrast of the target conductivity, the iterative harmonic B_z algorithm based on (1.5) with a good initial guess is stable and convergent in the continuous norm. In [5], the author improved the convergence result based on the following equivalent equality of (1.5):

$$\begin{bmatrix} \dfrac{\partial \ln \sigma}{\partial x} \\ \dfrac{\partial \ln \sigma}{\partial y} \end{bmatrix} = \frac{1}{\mu_0} (\sigma \mathbf{A}[\sigma])^{-1} \begin{bmatrix} \nabla^2 B_{z,1} \\ \nabla^2 B_{z,2} \end{bmatrix}$$

and derived a posteriori error estimate of $\| \ln \sigma^n - \ln \sigma^* \|$, where σ^* is the true conductivity. However, these two convergence results are considered only for exact magnetic flux field B_z.

In practical situations, we can only acquire the noisy data B_z^δ of B_z using MRI equipment. However, the harmonic B_z algorithm applies in fact the Laplacian of B_z as inversion input, the noise contained in B_z will be amplified by such an operation and therefore has essential influence on the approximation accuracy of the iteration solution. Such an influence depends not only on the error level of noisy input data, but also on the regularizing strategy computing the Laplacian from the noisy data B_z^δ. So it is necessary to give an error estimate on the iterative harmonic B_z algorithm for the noisy input data B_z^δ corresponding to some regularization scheme for the practical applications of harmonic B_z algorithm, which is the purpose of this paper.

This paper is organized as follows. In Section 2 we state the mathematical formulation of the harmonic B_z algorithm. Then for a relatively small contrast of the target conductivity, the iteration error of this algorithm is established in Section 3 for noisy input data, under the assumption that a stable numerical differentiation process has been applied. In Section 4, we propose a numerical regularizing scheme for the computation of Laplacian from the noisy measurement data B_z^δ with error estimate, which provides the basis on computing the error of iterative solution of conductivity.

2 Mathematical Model

Let $\Omega \subset \mathbb{R}^3$ be an electrically conducting subject with its smooth connected boundary $\partial\Omega$. In MREIT, we inject a current I through a pair of surface electrodes $\varepsilon^\pm$, then it produces an internal current density $\mathbf{J} = (J_x, J_y, J_z)$ inside the subject Ω satisfying the following problem:

$$\begin{cases} \nabla \cdot \mathbf{J} = 0, & \mathbf{r} \in \Omega, \\ I = -\displaystyle\int_{\varepsilon^+} \mathbf{J} \cdot \mathbf{n} \mathrm{d}s = \int_{\varepsilon^-} \mathbf{J} \cdot \mathbf{n} \mathrm{d}s, & \\ \mathbf{J} \times \mathbf{n} = 0, & \mathbf{r} \in \varepsilon^+ \cup \varepsilon^- \\ \mathbf{J} \cdot \mathbf{n} = 0, & \mathbf{r} \in \partial\Omega \backslash \overline{\varepsilon^+ \cup \varepsilon^-}, \end{cases} \tag{2.1}$$

where $\mathbf{n}$ is the outward unit normal vector on $\partial\Omega$ and $\mathrm{d}s$ is the surface area element.

Since $\mathbf{J} = -\sigma\nabla u[\sigma]$, (2.1) can be converted to

$$\begin{cases} \nabla\cdot(\sigma\nabla u[\sigma]) = 0, & \mathbf{r}\in\Omega, \\ I = \displaystyle\int_{\varepsilon^+}\sigma\frac{\partial u}{\partial\mathbf{n}}\mathrm{d}s = -\int_{\varepsilon^-}\sigma\frac{\partial u}{\partial\mathbf{n}}\mathrm{d}s, & \\ \nabla u\times\mathbf{n} = 0, & \mathbf{r}\in\varepsilon^+\cup\varepsilon^-, \\ -\sigma\dfrac{\partial u[\sigma]}{\partial\mathbf{n}} = 0, & \mathbf{r}\in\partial\Omega\backslash\overline{\varepsilon^+\cup\varepsilon^-}. \end{cases} \tag{2.2}$$

This exact model (2.2) can be solved in terms of the following standard problem (see [4]):

$$\begin{cases} \nabla\cdot(\sigma\nabla\widetilde{u}) = 0, & \mathbf{r}\in\Omega, \\ \widetilde{u}|_{\varepsilon^+} = 1, \widetilde{u}|_{\varepsilon^-} = 0, & \\ -\sigma\dfrac{\partial\widetilde{u}}{\partial\mathbf{n}} = 0, & \mathbf{r}\in\partial\Omega\backslash\overline{\varepsilon^+\cup\varepsilon^-}. \end{cases} \tag{2.3}$$

More precisely, if $u[\sigma]$ and $\widetilde{u}[\sigma]$ are the solution of problems (2.2)–(2.3), respectively, then

$$u[\sigma] = \frac{I}{\int_{\varepsilon^+}\sigma\frac{\partial\widetilde{u}[\sigma]}{\partial\mathbf{n}}\mathrm{d}s}\widetilde{u}[\sigma] + C \quad \text{in } \Omega,$$

where C is a constant decided by the electric potential specified at one point.

We now consider the magnetic field produced by the injection current I. From the Biot-Savart law, it follows that

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_\Omega \mathbf{J}(\mathbf{r}')\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\mathrm{d}\mathbf{r}', \tag{2.4}$$

which generates the following relation between B_z and σ from (1.2):

$$B_z(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_\Omega \frac{\sigma(\mathbf{r}')\Big[(x-x')\dfrac{\partial u[\sigma(\mathbf{r}')]}{\partial y} - (y-y')\dfrac{\partial u[\sigma(\mathbf{r}')]}{\partial x}\Big]}{|\mathbf{r}-\mathbf{r}'|^3}\mathrm{d}\mathbf{r}', \quad \mathbf{r} = (x,y,z)\in\Omega. \tag{2.5}$$

Recently, a new iteration scheme based on the nonlinear integral equation (2.5) was proposed in [6], which applies the B_z data as the inversion data directly in the algorithm, without the computation of Laplacian on the magnetic flux.

The harmonic B_z algorithm is an iterative scheme at each 2-dimensional slice $\Omega_{z_0} = \Omega\cap\{z = z_0\}$ based on the identity (1.5). To give the complete iteration scheme, we introduce the fundamental solution of 2-dimensional Laplace operator $\Phi(\mathbf{r},\mathbf{r}') := -\frac{1}{2\pi}\ln\frac{1}{|\mathbf{r}-\mathbf{r}'|}$ satisfying $\Delta_{\mathbf{r}'}\Phi(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')$ for $\mathbf{r}\in\mathbb{R}^2$, then at each 2-dimensional slice, it holds that

$$\sigma(x,y,z_0) = \frac{1}{2\pi\mu_0}\int_{\Omega_{z_0}}\frac{(x-x',y-y')}{|x-x'|^2+|y-y'|^2}\cdot\nabla\sigma(x',y',z_0)\mathrm{d}x'\mathrm{d}y' - H(\sigma), \tag{2.6}$$

where $\nabla = (\partial_{x'},\partial_{y'})$, $H(\sigma) := \frac{1}{2\pi}\int_{\partial\Omega_{z_0}}\frac{(x-x',y-y')\cdot n}{|x-x'|^2+|y-y'|^2}\cdot\sigma(x',y',z_0)\mathrm{d}l$.

It has been noticed that $\mathbf{A}[\sigma]^{-1}(x,y,z_0)$ may be large near $\partial\Omega_{z_0}$ due to the fact that two induced currents $\sigma\nabla u_1[\sigma], \sigma\nabla u_2[\sigma]$ are probably almost parallel for some configuration. This phenomena may lead to the unconvergence of the iteration scheme. To avoid this difficulty, we assume as usual that the unknown true conductivity is constant in $\Omega_{z_0}\backslash\widetilde{\Omega}_{z_0}$ for some interior domain $\widetilde{\Omega}_{z_0}$. Then it is easy to see from (1.4) that $\nabla^2 B_{z,1}\equiv\nabla^2 B_{z,2}\equiv 0$ in $\Omega_{z_0}\backslash\widetilde{\Omega}_{z_0}$.

We denote by σ^* the true unknown conductivity and assume that its value on $\partial\Omega_{z_0}$, still denoted as σ^*, is known. Let $B_{z,j}, j=1,2$ be the exact magnetic flux density corresponding to σ^* for two inject currents. For given initial guess $\sigma^0(x,y,z_0)$ in Ω_{z_0} with exact values in $\Omega_{z_0}\backslash\widetilde{\Omega}_{z_0}$, the harmonic B_z iteration algorithm constructs an approximation sequence $\{\sigma^n(x,y,z_0): n=0,1,2,\cdots\}$ from

$$\begin{cases}\nabla\sigma^{n+1}(x,y,z_0) := \dfrac{1}{\mu_0}\mathbf{A}[\sigma^n]^{-1}\begin{bmatrix}\nabla^2 B_{z,1}\\ \nabla^2 B_{z,2}\end{bmatrix}, \\ \sigma^{n+1}(x,y,z_0) = \dfrac{1}{2\pi}\displaystyle\int_{\widetilde{\Omega}_{z_0}}\dfrac{(x-x',y-y')}{|x-x'|^2+|y-y'|^2}\cdot\nabla\sigma^{n+1}(x',y',z_0)\mathrm{d}x'\mathrm{d}y' - \widetilde{H}(\sigma^*)\end{cases} \tag{2.7}$$

for $(x,y)\in\widetilde{\Omega}_{z_0}$ based on the relations (1.5) and (2.6), where $\widetilde{H}(\sigma^*)$ is $H(\sigma^*)$ with $\partial\Omega_{z_0}$ replaced by $\partial\widetilde{\Omega}_{z_0}$. For $(x,y)\in\Omega_{z_0}\backslash\widetilde{\Omega}_{z_0}$, it is obvious that $\sigma^n(x,y,z_0)\equiv\sigma^*(x,y,z_0)$ from the first equation in (2.7) since $\nabla^2 B_z^1\equiv\nabla^2 B_z^2\equiv 0$ in $\Omega_{z_0}\backslash\widetilde{\Omega}_{z_0}$.

To give the error estimate for the iteration solution with noisy input data in the next section, we need two regularity results for direct problems.

Lemma 2.1 *Denote by $\mathcal{E}$ the regular open subsurface of the boundary $\partial\Omega$ of $\Omega\subset\mathbb{R}^2$. Then for the boundary value problem*

$$\begin{cases}\nabla\cdot(\sigma\nabla u) = \nabla\cdot f, & \mathbf{r}\in\Omega,\\ u|_{\mathcal{E}} = h, & \mathbf{r}\in\mathcal{E},\\ -\sigma\nabla u\cdot\mathbf{n} = g, & \mathbf{r}\in\partial\Omega\backslash\overline{\mathcal{E}}\end{cases}$$

with $\sigma\in L^\infty(\Omega)$ satisfying $\inf\limits_{\Omega}\sigma>0$, $h\in H^{\frac12}(\mathcal{E})$ and $g\in H^{-\frac12}(\partial\Omega\backslash\overline{\mathcal{E}})$, the following estimates hold:

If $f\in(L^2(\Omega))^2$ and $\sigma\in C(\Omega)$, then $u\in H^1(\Omega)$ and

$$\|u\|_{H^1(\Omega)} \le C_1(\sigma)[\|f\|_{L^2(\Omega)} + \|h\|_{H^{\frac12}(\mathcal{E})} + \|g\|_{H^{-\frac12}(\partial\Omega\backslash\overline{\mathcal{E}})}]; \tag{2.8}$$

if $f\in(H^1(\Omega))^2$ and $\sigma\in C^1(\Omega)$, then $u\in H^2(\widetilde{\Omega})$ and

$$\|u\|_{H^2(\widetilde{\Omega})} \le C_2(\sigma)[\|u\|_{H^1(\Omega)} + \|\nabla\cdot f\|_{L^2(\Omega)}]; \tag{2.9}$$

if $f\in(C^{0,\alpha}(\Omega))^2$ with $\alpha\in(0,1)$ and $\sigma\in C^1(\Omega)$, then $u\in C^{1,\alpha}(\widetilde{\widetilde{\Omega}})$ and

$$\|\nabla u\|_{C^{0,\alpha}(\widetilde{\Omega})} \le C_3(\sigma)[\|u\|_{C^{0,\alpha}(\widetilde{\widetilde{\Omega}})} + \|f\|_{C^{0,\alpha}(\widetilde{\widetilde{\Omega}})}]; \tag{2.10}$$

if $f\in(L^p(\Omega))^2$ with $p>1$and $\sigma\in C(\Omega)$, then $u\in W^{1,p}(\widetilde{\widetilde{\Omega}})$ and

$$\|\nabla u\|_{L^p(\widetilde{\widetilde{\Omega}})} \le C_4(\sigma)[\|u\|_{L^p(\widetilde{\widetilde{\Omega}})} + \|f\|_{L^p(\widetilde{\widetilde{\Omega}})}], \tag{2.11}$$

where $\widetilde{\Omega}\subset\subset\widetilde{\widetilde{\Omega}}\subset\subset\Omega$ are regular domains, and $C_i(\Omega)$ have the following forms:

$$C_i(\sigma) = F_i\Big(\|\sigma\|_{C(\Omega)}, \|\nabla\sigma\|_{C(\Omega)}, \frac{1}{\inf\limits_{\Omega}\sigma}\Big), \quad i=2,3, \tag{2.12}$$

$$C_i(\sigma) = F_i\Big(\|\sigma\|_{C(\Omega)}, \frac{1}{\inf\limits_{\Omega}\sigma}\Big), \quad i=1,4. \tag{2.13}$$

The functions F_i $(i=1,2,3,4)$ are known bounded continuous functions with respect to the arguments.

This result can be found in [4].

Lemma 2.2 *Let $\widetilde{u}$ be the solution of the following problem:*

$$\begin{cases} \nabla\cdot(\sigma\nabla\widetilde{u})=0, & \mathbf{r}\in\Omega,\\ \widetilde{u}|_{\varepsilon^+}=1, \widetilde{u}|_{\varepsilon^-}=0, & \\ -\sigma\dfrac{\partial\widetilde{u}}{\partial\mathbf{n}}=0, & \mathbf{r}\in\partial\Omega\backslash\overline{\varepsilon^+\cup\varepsilon^-}. \end{cases}$$

Then there exists a constant $C(\sigma)$ such that

$$\|\nabla\widetilde{u}\|_{C(\widetilde{\Omega})}+\|\widetilde{u}\|_{H^2(\widetilde{\Omega})}\le C(\sigma), \tag{2.14}$$

where $C(\sigma)=(C_sC_3(\sigma)+1)C_1(\sigma)C_2(\sigma)$, $\widetilde{\Omega}\subset\subset\Omega$.

Proof It follows from (2.8)–(2.9) that

$$\|\widetilde{u}\|_{H^2(\widetilde{\widetilde{\Omega}})}\le C_2(\sigma)\|\widetilde{u}\|_{H^1(\Omega)}\le C_2(\sigma)C_1(\sigma),$$

where $\widetilde{\Omega}\subset\subset\widetilde{\widetilde{\Omega}}\subset\subset\Omega$. By the Sobolev imbedding theorem, we can obtain

$$\|\widetilde{u}\|_{C^{0,\alpha}(\widetilde{\widetilde{\Omega}})}\le C_s\|\widetilde{u}\|_{H^2(\widetilde{\widetilde{\Omega}})}\le C_sC_2(\sigma)C_1(\sigma)$$

for every $\alpha\in(0,1)$.

Finally, combining these two estimates with (2.10), we have

$$\|\nabla\widetilde{u}\|_{C(\widetilde{\Omega})}+\|\widetilde{u}\|_{H^2(\widetilde{\Omega})}\le(C_sC_3(\sigma)+1)C_1(\sigma)C_2(\sigma):=C(\sigma),$$

which leads to (2.14).

3 Error Estimate for Noisy Input Data

We consider the error estimate of harmonic B_z iteration algorithm in axially symmetric cylindrical sections. Let Ω be a cylinder along the z direction with infinite length and the electrode pair be parallel to the z direction. Moreover we assume that the conductivity σ^* in Ω does not change along z direction. Then the conductivity is actually reconstructed in the 2-dimensional domain. To unify the notations, we still use Ω in the sequel to represent the 2-dimensional domain Ω_{z_0}.

In this section, we consider the error estimate of harmonic B_z algorithm for noisy input data B_z^δ. In this noise input data situation, for given initial guess $\sigma^0(x,y)$ in Ω with exact values in $\Omega\backslash\widetilde{\Omega}$, where $\widetilde{\Omega}\subset\subset\Omega$, the iterative sequence $\{\sigma^{n,\delta}:n=1,2,\cdots\}$ is generated from

$$\begin{cases} \nabla\sigma^{1,\delta}:=\dfrac{1}{\mu_0}\mathbf{A}[\sigma^0]^{-1}\begin{bmatrix}\nabla^2B^\delta_{z,1}\\ \nabla^2B^\delta_{z,2}\end{bmatrix}, \\ \sigma^{1,\delta}(x,y)=\dfrac{1}{2\pi}\displaystyle\int_{\widetilde{\Omega}}\frac{(x-x',y-y')}{|x-x'|^2+|y-y'|^2}\cdot\nabla\sigma^{1,\delta}(x',y')\mathrm{d}x'\mathrm{d}y'-\widetilde{H}(\sigma^*), \end{cases} \tag{3.1}$$

and then for $n=1,2,\cdots$,

$$\begin{cases} \nabla\sigma^{n+1,\delta}:=\dfrac{1}{\mu_0}\mathbf{A}[\sigma^{n,\delta}]^{-1}\begin{bmatrix}\nabla^2B^\delta_{z,1}\\ \nabla^2B^\delta_{z,2}\end{bmatrix}, \\ \sigma^{n+1,\delta}(x,y)=\dfrac{1}{2\pi}\displaystyle\int_{\widetilde{\Omega}}\frac{(x-x',y-y')}{|x-x'|^2+|y-y'|^2}\cdot\nabla\sigma^{n+1,\delta}(x',y')\mathrm{d}x'\mathrm{d}y'-\widetilde{H}(\sigma^*), \end{cases} \tag{3.2}$$

where the notation $\nabla^2 B_z^\delta$ is the approximation to $\nabla^2 B_z$, for which we will propose a regularizing scheme in Section 4.

We firstly give two known results related to the convergence for the exact magnetic field, which will be applied in our error estimate.

Lemma 3.1 *For $\widetilde{\Omega} \subset\subset \Omega$, if σ lies in the set*

$$\Xi[\sigma^0, \lambda, \epsilon_0] =: \Big\{\sigma \in C^1(\overline{\Omega}) : \frac{1}{\lambda} < \sigma < \lambda,\ \|\nabla\sigma\|_{C(\Omega)} < \epsilon_0,\ \sigma|_{\Omega\backslash\widetilde{\Omega}} = \sigma^0\Big\},$$

where $\sigma^0, \lambda, \epsilon_0$ are positive constants, then there exists a constant d_-^* depending only on $\lambda, \epsilon_0, \Omega$, $\mathrm{dist}(\partial\Omega, \widetilde{\Omega})$ and $\varepsilon_j^\pm$, such that

$$\inf_{\widetilde{\Omega}} |\det \mathbf{A}[\sigma]| \geq d_-^* > 0. \tag{3.3}$$

Lemma 3.2 *Assume that the target conductivity $\sigma^*(x,y) \in C^1(\overline{\Omega})$ meets the following conditions:*

(H1) $0 < \sigma_-^* \leq \sigma^* \leq \sigma_+^*$ *for known constants $\sigma_\pm^*$;*

(H2) *there exists $\widetilde{\Omega} \subset\subset \Omega$ such that σ^* is a known constant in $\Omega\backslash\widetilde{\Omega}$;*

(H3) $|\det A[\sigma^*](x,y)| \geq d_-^* > 0$ *in $\widetilde{\Omega}$, where d_-^* is a known constant.*

Under these hypotheses, there exist constants $\epsilon = \epsilon(\sigma_\pm^*, d_-^*) > 0$ small enough and $\theta = \theta(\epsilon, \sigma_\pm^*, d_-^*) \in (0,1)$, such that if we take the initial guess σ^0 as the constant $\sigma^*|_{\Omega\backslash\widetilde{\Omega}}$, then the sequence σ^n given by the harmonic B_z iteration algorithm using exact input data holds for $\|\nabla\sigma^*\|_{C(\widetilde{\Omega})} \leq \epsilon$ that

$$\sigma^n \equiv \sigma^* \text{ in } \Omega\backslash\widetilde{\Omega}, \quad \|\sigma^n - \sigma^*\|_{C^1(\widetilde{\Omega})} \leq K\theta^n\epsilon,\ n = 1, 2, \cdots,$$

where $K := \mathrm{diam}(\Omega) + 1$.

These two results can be found in [4–5], respectively.

From Lemma 3.2, for true conductivity σ^* lying in the set

$$\mathbb{S}_1 := \{\sigma(x,y) : 0 < \sigma_-^* \leq \sigma \leq \sigma_+^*,\ \|\nabla\sigma\|_{C(\widetilde{\Omega})} \leq \epsilon,\ (x,y) \in \Omega\}, \tag{3.4}$$

the iterative sequence $\{\sigma^n : n = 1, 2, \cdots\}$ using exact input data lies in

$$\mathbb{S}_2 := \Big\{\sigma(x,y) : \frac{1}{2}\sigma_-^* \leq \sigma \leq \frac{1}{2}\sigma_-^* + \sigma_+^*,\ \|\nabla\sigma\|_{C(\widetilde{\Omega})} \leq \frac{K+1}{2K}\sigma_-^*,\ (x,y) \in \Omega\Big\} \tag{3.5}$$

for any $\epsilon \in (0, \frac{1}{2K}\sigma_-^*)$.

In fact, for any $\epsilon \in (0, \frac{1}{2K}\sigma_-^*)$ and $\theta \in (0,1)$, it follows from Lemma 3.2 that

$$|\sigma^n - \sigma^*| \leq \|\sigma^n - \sigma^*\|_{C(\widetilde{\Omega})} \leq \|\sigma^n - \sigma^*\|_{C^1(\widetilde{\Omega})} \leq K\theta^n\epsilon \leq \frac{1}{2}\sigma_-^*$$

and $\|\nabla(\sigma^n - \sigma^*)\|_{C(\widetilde{\Omega})} \leq K\theta^n\epsilon \leq \frac{1}{2}\sigma_-^*$. Then we have

$$\frac{1}{2}\sigma_-^* \leq \sigma^* - \frac{1}{2}\sigma_-^* \leq \sigma^n \leq \sigma^* + \frac{1}{2}\sigma_-^* \leq \sigma_+^* + \frac{1}{2}\sigma_-^*$$

in $\widetilde{\Omega}$ and

$$\|\nabla\sigma^n\|_{C(\widetilde{\Omega})} \leq \|\nabla\sigma^*\|_{C(\widetilde{\Omega})} + \frac{1}{2}\sigma_-^* \leq \frac{K+1}{2K}\sigma_-^*.$$

Noticing that $\sigma^n \equiv \sigma^*$ in $\Omega\backslash\widetilde{\Omega}$, we get (3.5).

For practical measurement data with noise, the input data for the iteration scheme is in fact the Laplacian operation $\nabla^2 B_z^\delta$ from (3.1)–(3.2). When presenting our error estimate, we must firstly analyze the error $\rho(\delta)$ of computing $\nabla^2 B_z$ from the noisy measurement data B_z^δ, which depends on the regularizing scheme. Since we generally measure the error of magnetic field itself in L^2-norm, while we need the error estimate of Laplacian in C-norm in our iteration, we give the following approximation for our computation on Laplacian:

(H4) For the noisy data $B_{z,j}^\delta$ satisfying

$$\|B_{z,j}^\delta - B_{z,j}\|_{L^2(\Omega)} \le \delta,$$

a stable differentiation scheme is used to compute $\nabla^2 B_{z,j}^\delta$ such that

$$\left\| \begin{bmatrix} \nabla^2 e_1 \\ \nabla^2 e_2 \end{bmatrix} \right\|_{H_0^2(\widetilde{\Omega})} \le \rho(\delta) \to 0 \quad as \quad \delta \to 0, \tag{3.6}$$

where $\nabla^2 e_j := \nabla^2 B_{z,j}^\delta - \nabla^2 B_{z,j},\ j = 1, 2$.

(H5) $\nabla^2 B_{z,j}^\delta$ $(j = 1, 2)$ is understood such that $\nabla^2 B_{z,1}^\delta = \nabla^2 B_{z,2}^\delta = 0$ in $\Omega\backslash\widetilde{\Omega}$, which is a natural condition if the conductivity is assumed to be a known constant in the domain $\Omega\backslash\widetilde{\Omega}$ implying $\nabla^2 B_{z,j} \equiv 0$ in $\Omega\backslash\widetilde{\Omega}$.

An implementable regularizing scheme to compute ΔB_z approximately from B_z^δ to reach (3.6) with $\rho(\delta) = \sqrt{\delta}$ as well as the regularity requirement on the target conductivity will be given in Section 4.

Now we can state the main result of our work as follows.

Theorem 3.1 *Assume that the target conductivity $\sigma^*(x, y) \in C_1(\overline{\Omega})$ meets the three hypotheses in Lemma* 3.2 *and* (H4)–(H5). *Then there exist constants $\epsilon = \epsilon(\sigma_\pm^*, d_-^*) > 0$ small enough and $\theta = \theta(\epsilon, \sigma_\pm^*, d_-^*) \in (0, 1)$, $M = M(\sigma_\pm^*, d_-^*)$ such that if we take the initial guess σ^0 as the constant $\sigma^*|_{\Omega\backslash\widetilde{\Omega}}$, the sequence $\{\sigma^{n,\delta}\}$ given by* (3.1)–(3.2) *with noisy input data holds for $\|\nabla\sigma^*\|_{C(\widetilde{\Omega})} \le \epsilon$ and δ small enough that*

$$\sigma^{n,\delta} \equiv \sigma^* \text{ in } \Omega\backslash\widetilde{\Omega}, \quad \|\sigma^{n,\delta} - \sigma^*\|_{C^1(\widetilde{\Omega})} \le M\rho(\delta) + K\theta^n\epsilon, \quad n = 1, 2, \cdots.$$

Proof Let us take $\epsilon \in (0, \frac{1}{2K}\sigma_-^*)$. Denote by u_j^* and u_j^n the solutions of the direct problem

$$\begin{cases} \nabla\cdot(\sigma\nabla u_j) = 0, & \mathbf{r} \in \Omega, \\ u_j|_{\varepsilon_j^+} = 1, \quad u_j|_{\varepsilon_j^-} = 0, & \\ -\sigma\dfrac{\partial u_j}{\partial \mathbf{n}} = 0, & \mathbf{r} \in \partial\Omega\backslash\overline{\varepsilon_j^+ \cup \varepsilon_j^-} \end{cases} \tag{3.7}$$

with $\sigma = \sigma^*$ and $\sigma = \sigma^n$, respectively. It follows from Lemma 2.2 that

$$\|\nabla u_j^*\|_{C(\widetilde{\Omega})} + \|u_j^*\|_{H^2(\widetilde{\Omega})} \le C(\sigma^*) \tag{3.8}$$

and

$$\|\nabla u_j^n\|_{C(\widetilde{\Omega})} + \|u_j^n\|_{H^2(\widetilde{\Omega})} \le C(\sigma^n). \tag{3.9}$$

However, Lemma 3.2 says that $\{\sigma^n : n = 1, 2, \cdots\} \subset \mathbb{S}_2$. So it follows from the expressions of $C(\sigma^*)$ and $C(\sigma^n)$ in Lemma 2.2 that the constants are of a uniform upper bound:

$$C(\sigma^*), C(\sigma^n) \le \overline{C}_* := \sup_{(t_1,t_2,t_3)\in\mathbf{S}} [C_s F_3(t_1, t_2, t_3) + 1] F_1(t_1, t_3) F_2(t_1, t_2, t_3)$$

with

$$\mathbf{S}=\Big[\frac{1}{2}\sigma_-^*,\frac{1}{2}\sigma_-^*+\sigma_+^*\Big]\times\Big[0,\frac{2K+1}{2K}\sigma_-^*\Big]\times\Big[\frac{1}{\frac{1}{2}\sigma_-^*+\sigma_+^*},\frac{1}{\frac{1}{2}\sigma_-^*}\Big].$$

Step 1 Estimate $\|\sigma^{1,\delta}-\sigma^1\|_{C^1(\widetilde{\Omega})}$.

Firstly, expand the initial guess σ^0 at σ^* as $\sigma^0=\sigma^*+e^0$. Since $\|\nabla\sigma^*\|_{C(\widetilde{\Omega})}\le\epsilon$ and $\sigma^0=\sigma^*$ in $\Omega\backslash\widetilde{\Omega}$, it follows that

$$\|e^0\|_{C(\Omega)}\le\mathrm{diam}(\Omega)\|\nabla e^0\|_{C(\Omega)}\le\mathrm{diam}(\Omega)\epsilon.$$

Hence, $\|e^0\|_{C^1(\Omega)}=\|e^0\|_{C(\Omega)}+\|\nabla e^0\|_{C(\Omega)}\le(\mathrm{diam}(\Omega)+1)\epsilon=:K\epsilon$.

We expand u_j^0 at u_j^* as

$$u_j^0=u_j^*+w_j^0. \tag{3.10}$$

Noticing that $\sigma^0=\sigma^*$ in $\Omega\backslash\widetilde{\Omega}$, w_j^0 meets

$$\begin{cases}\nabla\cdot(\sigma^0\nabla w_j^0)=-\nabla\cdot(e^0\nabla u_j^*), & \mathbf{r}\in\Omega,\\ w_j^0|_{\varepsilon_j^+}=0, w_j^0|_{\varepsilon_j^-}=0, & \\ -\sigma^0\nabla w_j^0\cdot\mathbf{n}=e^0\nabla u_j^0\cdot\mathbf{n}=0, & \mathbf{r}\in\partial\Omega\backslash\overline{\varepsilon^+\cup\varepsilon^-}.\end{cases} \tag{3.11}$$

Since $\|e^0\|_{C^1(\Omega)}\le K\epsilon$ and $e^0=0$ in $\Omega\backslash\widetilde{\Omega}$, it follows from (3.8) that the right-hand side of the first equation in (3.11) satisfies

$$\|\nabla\cdot(e^0\nabla u_j^*)\|_{L^2(\Omega)}\le\overline{C}_*\|e^0\|_{C^1(\widetilde{\Omega})}.$$

Therefore it follows from Lemma 2.1 and the Sobolev imbedding theorem that

$$\begin{aligned}\|w_j^0\|_{C^{0,\alpha}(\widetilde{\widetilde{\Omega}})}&\le C_s\|w_j^0\|_{H^2(\widetilde{\widetilde{\Omega}})}\le C_sC_2(\sigma^0)[\|w_j^0\|_{H^1(\widetilde{\Omega})}+\|\nabla\cdot(e^0\nabla u_j^*)\|_{L^2(\widetilde{\Omega})}]\\&\le C_sC_2(\sigma^0)[C_1(\sigma^0)\|e^0\nabla u_j^*\|_{L^2(\Omega)}+\overline{C}_*\|e^0\|_{C^1(\widetilde{\Omega})}]\\&\le C_sC_2(\sigma^0)[C_1(\sigma^0)C_1(\sigma^*)+\overline{C}_*]\|e^0\|_{C^1(\widetilde{\Omega})}.\end{aligned}$$

According to the above estimate and (2.10), we have for $\widetilde{\Omega}\subset\subset\widetilde{\widetilde{\Omega}}\subset\subset\Omega$ that

$$\begin{aligned}\|\nabla w_j^0\|_{C(\widetilde{\Omega})}&\le C_3(\sigma^0)[\|w_j^0\|_{C^\alpha(\widetilde{\widetilde{\Omega}})}+\|e^0\nabla u_j^*\|_{C(\widetilde{\widetilde{\Omega}})}]\\&\le C_3(\sigma^0)\{C_sC_2(\sigma^0)[C_1(\sigma^0)C_1(\sigma^*)+\overline{C}_*]+\|\nabla u_j^*\|_{C(\widetilde{\widetilde{\Omega}})}\}\|e^0\|_{C^1(\widetilde{\Omega})}.\end{aligned}$$

Using the same arguments as those in deriving (3.8), we can get

$$\|\nabla u_j^*\|_{C(\widetilde{\widetilde{\Omega}})}\le\widetilde{C}_*=\widetilde{C}_*(\sigma_\pm^*,K,\widetilde{\widetilde{\Omega}}).$$

Therefore we have

$$\|\nabla w_j^0\|_{C(\widetilde{\Omega})}\le C_3(\sigma^0)\{C_sC_2(\sigma^0)[C_1(\sigma^0)C_1(\sigma^*)+\overline{C}_*]+\widetilde{C}_*\}\|e^0\|_{C^1(\widetilde{\Omega})}. \tag{3.12}$$

Denote by

$$\overline{F}(\sigma):=C_3(\sigma)\Big\{C_sC_2(\sigma)\Big[C_1(\sigma)\sup_{[\sigma_-^*,\sigma_+^*]\times[\frac{1}{\sigma_+^*},\frac{1}{\sigma_-^*}]}C_1(\sigma^*)+\overline{C}_*\Big]+\widetilde{C}_*\Big\} \tag{3.13}$$

a known function due to Lemma 2.1. For $\epsilon \in (0, \frac{1}{2K}\sigma^*_-)$, we introduce the constant

$$\overline{C}_\epsilon(\sigma^*) := \sup_{\|\sigma-\sigma^*\|_{C^1(\Omega)} \leq K\epsilon} \overline{F}(\sigma), \tag{3.14}$$

which is well defined. Noticing that$\|\sigma - \sigma^*\|_{C^1(\Omega)} \leq K\epsilon$, we have $\sigma > \frac{1}{2}\sigma^*_- > 0$ for $0 < \epsilon < \frac{1}{2K}\sigma^*_-$ due to (H1) in Lemma 3.2. Moreover, this constant can be estimated by a known constant as

$$\overline{C}_\epsilon(\sigma^*) \leq \sup_{\mathbb{S}_2} \overline{F}(\sigma) =: \overline{G}(\sigma^*_\pm). \tag{3.15}$$

Now it follows from (3.12)–(3.15) that

$$\|\nabla w^0_j\|_{C(\tilde{\Omega})} \leq \overline{G}(\sigma^*_\pm)\|e^0\|_{C^1(\tilde{\Omega})}. \tag{3.16}$$

Since $\|e^0\|_{C^1(\Omega)} \leq K\epsilon$, it follows that

$$\|\nabla w^0_j\|_{C(\tilde{\Omega})} \leq \overline{G}(\sigma^*_\pm)K\epsilon. \tag{3.17}$$

On the other hand, it follows from (2.7) and (3.1) that

$$\mathbf{A}[\sigma^0]\nabla(\sigma^{1,\delta} - \sigma^1) = \frac{1}{\mu_0}\begin{bmatrix} \nabla^2 e_1 \\ \nabla^2 e_2 \end{bmatrix}, \tag{3.18}$$

which can be written as

$$\left(I + \mathbf{A}[\sigma^*]^{-1}\begin{bmatrix} \dfrac{\partial w^0_1}{\partial y} & -\dfrac{\partial w^0_1}{\partial x} \\ \dfrac{\partial w^0_2}{\partial y} & -\dfrac{\partial w^0_2}{\partial x} \end{bmatrix}\right)\nabla(\sigma^{1,\delta} - \nabla\sigma^1) = \frac{1}{\mu_0}\mathbf{A}[\sigma^*]^{-1}\begin{pmatrix} \nabla^2 e_1 \\ \nabla^2 e_2 \end{pmatrix} \tag{3.19}$$

due to the definition of the matrix $\mathbf{A}[\sigma^{\mathbf{0}}]$ and (3.10).

However, it is obvious from (3.17) that

$$\begin{aligned} \left\|\mathbf{A}[\sigma^*]^{-1}\begin{bmatrix} \dfrac{\partial w^0_1}{\partial y} & -\dfrac{\partial w^0_1}{\partial x} \\ \dfrac{\partial w^0_2}{\partial y} & -\dfrac{\partial w^0_2}{\partial x} \end{bmatrix}\right\|_{C(\tilde{\Omega})} &\leq \|\mathbf{A}[\sigma^*]^{-1}\|_{C(\tilde{\Omega})} \max_{j=1,2} \|\nabla w^0_j\|_{C(\tilde{\Omega})} \\ &\leq \|\mathbf{A}[\sigma^*]^{-1}\|_{C(\tilde{\Omega})}\overline{G}(\sigma^*_\pm)K\epsilon. \end{aligned} \tag{3.20}$$

A direct computation leads to $\|\mathbf{A}[\sigma^*]^{-1}\|_{C(\tilde{\Omega})} \leq \frac{\max_{j=1,2}\|\nabla u^*_j\|_{C(\bar{\Omega})}}{|\det \mathbf{A}[\sigma^*]|}$, from which we deduce

$$\|\mathbf{A}[\sigma^*]^{-1}\|_{C(\tilde{\Omega})} \leq \frac{\overline{C}_*}{d^*_-} \tag{3.21}$$

due to (3.8) and (H3).

Now we take $\epsilon \in (0, \frac{1}{2K}\sigma^*_-)$ small enough such that

$$\frac{\overline{C}_*}{d^*_-}\overline{G}(\sigma^*_\pm)K\epsilon < \frac{1}{2},$$

which implies from (3.20) that

$$\left\| \mathbf{A}[\sigma^*]^{-1} \begin{bmatrix} \dfrac{\partial w_1^0}{\partial y} & -\dfrac{\partial w_1^0}{\partial x} \\ \dfrac{\partial w_2^0}{\partial y} & -\dfrac{\partial w_2^0}{\partial x} \end{bmatrix} \right\|_{C(\widetilde{\Omega})} < \frac{1}{2}. \tag{3.22}$$

Now it follows from (3.19), (3.21)–(3.22) that

$$\|\nabla(\sigma^{1,\delta} - \sigma^1)\|_{C(\widetilde{\Omega})} \leq \frac{2}{\mu_0}\frac{\overline{C}_*}{d_-^*}\rho(\delta),$$

where the Sobolev embedding theorem $H_0^2(\widetilde{\Omega}) \hookrightarrow C^\beta(\widetilde{\Omega})$ with $\beta \in [0,1)$ on $\nabla^2 e_j$ based on (3.6) is applied. This last estimate generates

$$\|\sigma^{1,\delta} - \sigma^1\|_{C^1(\widetilde{\Omega})} \leq K\|\nabla(\sigma^{1,\delta} - \sigma^1)\|_{C(\widetilde{\Omega})} \leq \frac{2K}{\mu_0}\frac{\overline{C}_*}{d_-^*}\rho(\delta). \tag{3.23}$$

Introduce a new constant

$$M = M(\sigma_\pm^*, d_-^*) =: \frac{2K}{\mu_0}\frac{\overline{C}_*}{d_-^*},$$

then the estimate (3.23) becomes

$$\|\sigma^{1,\delta} - \sigma^1\|_{C^1(\widetilde{\Omega})} \leq M\rho(\delta). \tag{3.24}$$

On the other hand, it follows from (3.1) and (H5) that $\nabla\sigma^{1,\delta} = 0$ in $\Omega\backslash\widetilde{\Omega}$, and therefore $\sigma^{1,\delta} = \sigma^*$ in $\Omega\backslash\widetilde{\Omega}$.

Take δ small enough such that $M\rho(\delta) \leq \frac{1}{4}\sigma_-^*$. Then it follows from the above estimate and Lemma 3.2 that

$$\|\sigma^{1,\delta} - \sigma^*\|_{C^1(\widetilde{\Omega})} \leq \|\sigma^{1,\delta} - \sigma^1\|_{C^1(\widetilde{\Omega})} + \|\sigma^1 - \sigma^*\|_{C^1(\widetilde{\Omega})} \leq \frac{1}{4}\sigma_-^* + K\epsilon \leq \frac{3}{4}\sigma_-^*$$

for any $\epsilon \in (0, \frac{1}{2K}\sigma_-^*)$. So $\sigma^{1,\delta}$ lies in the set

$$\mathbb{S}_3 := \left\{\sigma(x,y) : \frac{1}{4}\sigma_-^* < \sigma < \sigma_+^* + \frac{3}{4}\sigma_-^*, \|\nabla\sigma\|_{C(\widetilde{\Omega})} \leq \frac{3K+2}{4K}\sigma_-^*,\ (x,y) \in \Omega\right\}. \tag{3.25}$$

Now we can apply the induction argument to prove the theorem. That is, assume that the following properties

$$\sigma^{k,\delta} = \sigma^* \ \text{ in } \Omega\backslash\widetilde{\Omega} \quad \text{and} \quad \|\sigma^{k,\delta} - \sigma^k\|_{C^1(\widetilde{\Omega})} \leq M\rho(\delta) \tag{3.26}$$

hold for $k = n$, which specially yields that

$$\sigma^{n,\delta} \in \mathbb{S}_3 \ \text{ and } \quad \sigma^{n,\delta} = \sigma^* \quad \text{in } \Omega\backslash\widetilde{\Omega}, \tag{3.27}$$

noticing $\sigma^n \in \mathbb{S}_2$. Then we need to prove that these properties are also true for $k = n+1$.

Step 2 Expand $\sigma^{n,\delta}$ at σ^n.

Let $u_j^{n,\delta}$ $(j = 1,2)$ be the solutions of the problem (3.7) with $\sigma = \sigma^{n,\delta}$. It follows from Lemma 2.2 that

$$\|\nabla u_j^{n,\delta}\|_{C(\widetilde{\Omega})} + \|u_j^{n,\delta}\|_{H_0^2(\widetilde{\Omega})} \leq C(\sigma^{n,\delta}).$$

Since $\sigma^{n,\delta} \in \mathbb{S}_3$, similarly to the estimate of $C(\sigma^n)$ in (3.9), there exists a constant $\overline{C}_{*,3}$ depending only on the upper and lower bounds $\sigma^*_\pm$ of σ^*, K and domain $\widetilde{\Omega}$, still denoted by $\overline{C}_*$ for the simplicity of notation, such that

$$\|\nabla u_j^{n,\delta}\|_{C(\widetilde{\Omega})} + \|u_j^{n,\delta}\|_{H_0^2(\widetilde{\Omega})} \le \overline{C}_*. \tag{3.28}$$

We expand $\sigma^{n,\delta}$ at σ^n as $\sigma^{n,\delta} = \sigma^n + e^{n,\delta}$ and $u_j^{n,\delta}$ at u_j^n as $u_j^{n,\delta} = u_j^n + w_j^{n,\delta}$.

Since $\sigma^{n,\delta} = \sigma^n = \sigma^*$ in $\Omega\backslash\widetilde{\Omega}$, $w_j^{n,\delta}$ satisfies the following problem:

$$\begin{cases} \nabla\cdot(\sigma^{n,\delta}\nabla w_j^{n,\delta}) = -\nabla\cdot(e^{n,\delta}\nabla u_j^n), & \mathbf{r}\in\Omega, \\ w_j^{n,\delta}|_{\varepsilon^+} = 0, w_j^{n,\delta}|_{\varepsilon^-} = 0, \\ -\sigma^{n,\delta}\nabla w_j^{n,\delta}\cdot\mathbf{n} = e^{n,\delta}\nabla u_j^n\cdot\mathbf{n} = 0, & \mathbf{r}\in\partial\Omega\backslash\overline{\varepsilon^+\cup\varepsilon^-}. \end{cases}$$

Similarly to the derivation of (3.16), we have

$$\|\nabla(u_j^{n,\delta} - u_j^n)\|_{C(\widetilde{\Omega})} = \|\nabla w_j^{n,\delta}\|_{C(\widetilde{\Omega})} \le \overline{G}_1(\sigma^*_\pm)\|\sigma^{n,\delta} - \sigma^n\|_{C^1(\widetilde{\Omega})}, \tag{3.29}$$

where the definition of $\overline{G}_1(\sigma^*_\pm)$ is similar to $\overline{G}(\sigma^*_\pm)$ and $\overline{G}(\sigma^*_\pm) \le \overline{G}_1(\sigma^*_\pm)$ due to $\mathbb{S}_2 \subset \mathbb{S}_3$.

On the other hand, it follows from $\sigma^n \in \mathbb{S}_2$, $\sigma^{n,\delta} \in \mathbb{S}_3$ and Lemma 3.1 that

$$\inf|\det\mathbf{A}[\sigma^n]| \ge d^*_- > 0, \quad \inf|\det\mathbf{A}[\sigma^{n,\delta}]| \ge d^*_- > 0, \tag{3.30}$$

where d^*_- is a constant depending only on $\sigma^*_\pm$, ϵ, Ω, $\mathrm{dist}(\partial\Omega,\widetilde{\Omega})$ and $\varepsilon_j^\pm$.

Step 3 Estimate $\|\sigma^{n,\delta} - \sigma^n\|_{C^1(\widetilde{\Omega})}$.

It follows from (3.2) and (2.7) that

$$\begin{aligned} &\nabla(\sigma^{n+1,\delta} - \sigma^{n+1}) \\ =& \frac{1}{\mu_0}(\mathbf{A}[\sigma^{n,\delta}]^{-1} - \mathbf{A}[\sigma^n]^{-1})\begin{bmatrix}\nabla^2 B_{z,1}\\ \nabla^2 B_{z,2}\end{bmatrix} + \frac{1}{\mu_0}\mathbf{A}[\sigma^{n,\delta}]^{-1}\begin{bmatrix}\nabla^2 e^1\\ \nabla^2 e^2\end{bmatrix}. \end{aligned} \tag{3.31}$$

Hence we have

$$\begin{aligned} \|\nabla(\sigma^{n+1,\delta} - \sigma^{n+1})\|_{C(\widetilde{\Omega})} \le& \frac{1}{\mu_0}\|\mathbf{A}[\sigma^{n,\delta}]^{-1} - \mathbf{A}[\sigma^n]^{-1}\|_{C(\widetilde{\Omega})}\left\|\begin{bmatrix}\nabla^2 B_{z,1}\\ \nabla^2 B_{z,2}\end{bmatrix}\right\|_{C(\widetilde{\Omega})} + \\ & \frac{1}{\mu_0}\|\mathbf{A}[\sigma^{n,\delta}]^{-1}\|_{C(\widetilde{\Omega})}\left\|\begin{bmatrix}\nabla^2 e_1\\ \nabla^2 e_2\end{bmatrix}\right\|_{C(\widetilde{\Omega})} \\ =:& \mathrm{I} + \mathrm{II}. \end{aligned} \tag{3.32}$$

Firstly, we estimate II. From the definition of $\mathbf{A}[\sigma]$ in (1.6), we have

$$\|\mathbf{A}[\sigma^{n,\delta}]^{-1}\|_{C(\widetilde{\Omega})} \le \frac{1}{|\det\mathbf{A}[\sigma^{n,\delta}]|}\max_{j=1,2}\|\nabla u_j^{n,\delta}\|_{C(\widetilde{\Omega})}. \tag{3.33}$$

On the other hand, the Sobolev imbedding theorem and (H4) yield

$$\left\|\begin{bmatrix}\nabla^2 e_1\\ \nabla^2 e_2\end{bmatrix}\right\|_{C(\widetilde{\Omega})} \le C_s\left\|\begin{bmatrix}\nabla^2 e_1\\ \nabla^2 e_2\end{bmatrix}\right\|_{H_0^2(\widetilde{\Omega})} \le C_s\rho(\delta) =: \rho(\delta) \tag{3.34}$$

for the simplicity of notation. So it follows from (3.28), (3.30) and (3.34) that

$$\mathrm{II} \le \frac{1}{\mu_0}\frac{\overline{C}_*}{d^*_-}\rho(\delta). \tag{3.35}$$

Then we estimate I in (3.32). Again using the definition of $\mathbf{A}[\sigma]$, we get that

$$
\begin{aligned}
&\|\mathbf{A}[\sigma^{n,\delta}]^{-1} - \mathbf{A}[\sigma^n]^{-1}\|_{C(\tilde{\Omega})} \\
&\leq \Big\|\frac{1}{\det \mathbf{A}[\sigma^{n,\delta}]} - \frac{1}{\det \mathbf{A}[\sigma^n]}\Big\|_{C(\tilde{\Omega})} \left\| \begin{pmatrix} -\dfrac{\partial u_2^{n,\delta}}{\partial x} & \dfrac{\partial u_1^{n,\delta}}{\partial x} \\ -\dfrac{\partial u_2^{n,\delta}}{\partial y} & \dfrac{\partial u_1^{n,\delta}}{\partial y} \end{pmatrix} \right\|_{C(\tilde{\Omega})} \\
&\quad + \Big\|\frac{1}{\det \mathbf{A}[\sigma^n]}\Big\|_{C(\tilde{\Omega})} \left\| \begin{pmatrix} -\dfrac{\partial (u_2^{n,\delta} - u_2^n)}{\partial x} & \dfrac{\partial (u_1^{n,\delta} - u_1^n)}{\partial x} \\ -\dfrac{\partial (u_2^{n,\delta} - u_2^n)}{\partial y} & \dfrac{\partial (u_1^{n,\delta} - u_1^n)}{\partial y} \end{pmatrix} \right\|_{C(\tilde{\Omega})} \\
&\leq \Big\|\frac{1}{\det \mathbf{A}[\sigma^{n,\delta}]} - \frac{1}{\det \mathbf{A}[\sigma^n]}\Big\|_{C(\tilde{\Omega})} \max_{j=1,2} \|\nabla u_i^{n,\delta}\|_{C(\tilde{\Omega})} \\
&\quad + \Big\|\frac{1}{\det \mathbf{A}[\sigma^n]}\Big\|_{C(\tilde{\Omega})} \max_{j=1,2} \|\nabla u_i^{n,\delta} - u_i^n\|_{C(\tilde{\Omega})} \\
&=: \mathrm{III} + \mathrm{IV}.
\end{aligned} \tag{3.36}
$$

A direct computation leads to

$$
\begin{aligned}
\det \mathbf{A}[\sigma^n] - \det \mathbf{A}[\sigma^{n,\delta}] = {}& \frac{\partial u_2^{n,\delta}}{\partial x} \frac{\partial (u_1^{n,\delta} - u_1^n)}{\partial y} + \frac{\partial (u_2^{n,\delta} - u_2^n)}{\partial x} \frac{\partial u_1^n}{\partial y} \\
&+ \frac{\partial u_1^n}{\partial x} \frac{\partial (u_2^n - u_2^{n,\delta})}{\partial y} + \frac{\partial (u_1^n - u_1^{n,\delta})}{\partial x} \frac{\partial u_2^{n,\delta}}{\partial y},
\end{aligned}
$$

which yields

$$
\begin{aligned}
&\Big\|\frac{1}{\det \mathbf{A}[\sigma^{n,\delta}]} - \frac{1}{\det \mathbf{A}[\sigma^n]}\Big\|_{C(\tilde{\Omega})} \\
&\leq \Big\|\frac{1}{\det \mathbf{A}[\sigma^{n,\delta}] \det \mathbf{A}[\sigma^n]}\Big\|_{C(\tilde{\Omega})} \|\det \mathbf{A}[\sigma^n] - \det \mathbf{A}[\sigma^{n,\delta}]\|_{C(\tilde{\Omega})} \\
&\leq \frac{4\overline{C}_*}{(d_-^*)^2} \max_{j=1,2} \|\nabla (u_j^{n,\delta} - u_j^n)\|_{C(\tilde{\Omega})}
\end{aligned} \tag{3.37}
$$

from (3.9), (3.28) and (3.30). Therefore it follows from (3.28) and (3.37) that

$$
\mathrm{III} \leq 4\Big(\frac{\overline{C}_*}{d_-^*}\Big)^2 \max_{j=1,2} \|\nabla (u_j^{n,\delta} - u_j^n)\|_{C(\tilde{\Omega})}. \tag{3.38}
$$

Again from (3.28), we have

$$
\mathrm{IV} \leq \frac{1}{d_-^*} \max_{j=1,2} \|\nabla (u_j^{n,\delta} - u_j^n)\|_{C(\tilde{\Omega})}. \tag{3.39}
$$

By (3.36), (3.38)–(3.39), we obtain that

$$
\|\mathbf{A}[\sigma^{n,\delta}]^{-1} - \mathbf{A}[\sigma^n]^{-1}\|_{C(\tilde{\Omega})} \leq \Big[4\Big(\frac{\overline{C}_*}{d_-^*}\Big)^2 + \frac{1}{d_-^*}\Big] \max_{j=1,2} \|\nabla (u_j^{n,\delta} - u_j^n)\|_{C(\tilde{\Omega})}. \tag{3.40}
$$

So it follows from (3.29) and (3.40) that

$$
\|\mathbf{A}[\sigma^{n,\delta}]^{-1} - \mathbf{A}[\sigma^n]^{-1}\|_{C(\tilde{\Omega})} \leq \Big[4\Big(\frac{\overline{C}_*}{d_-^*}\Big)^2 + \frac{1}{d_-^*}\Big] \overline{G}_1(\sigma_\pm^*) \|\sigma^{n,\delta} - \sigma^n\|_{C^1(\tilde{\Omega})}. \tag{3.41}
$$

On the other hand, (1.5) yields

$$\left\| \begin{pmatrix} \nabla^2 B_{z,1} \\ \nabla^2 B_{z,2} \end{pmatrix} \right\|_{C(\widetilde{\Omega})} \le \mu_0 \left\| \begin{pmatrix} \dfrac{\partial u_1^*}{\partial y} & -\dfrac{\partial u_1^*}{\partial x} \\ \dfrac{\partial u_2^*}{\partial y} & -\dfrac{\partial u_2^*}{\partial x} \end{pmatrix} \right\|_{C(\widetilde{\Omega})} \|\nabla \sigma^*\|_{C(\widetilde{\Omega})}$$

$$\le \mu_0 \max_{j=1,2} \|\nabla u_j^*\|_{C(\Omega)} \|\nabla \sigma^*\|_{C(\widetilde{\Omega})} \le \mu_0 \overline{C}_* \epsilon \tag{3.42}$$

from (3.8) and the condition $\|\nabla \sigma^*\|_{C(\widetilde{\Omega})} \le \epsilon$.

Finally combining (3.41) and (3.42) together yields

$$\mathrm{I} \le \overline{G}_1(\sigma_\pm^*)\epsilon \Big[4 \frac{(\overline{C}_*)^3}{(d_-^*)^2} + \frac{\overline{C}_*}{d_-^*} \Big] \|\sigma^{n,\delta} - \sigma^n\|_{C^1(\widetilde{\Omega})}. \tag{3.43}$$

Inserting (3.35) and (3.43) into (3.32), we get

$$\|\nabla(\sigma^{n+1,\delta} - \sigma^{n+1})\|_{C(\widetilde{\Omega})} \le \epsilon \overline{G}_1(\sigma_\pm^*) \Big[\frac{4(\overline{C}_*)^3}{(d_-^*)^2} + \frac{\overline{C}_*}{d_-^*} \Big] \|\sigma^{n,\delta} - \sigma^n\|_{C^1(\widetilde{\Omega})} + \frac{1}{\mu_0} \frac{\overline{C}_*}{d_-^*} \rho(\delta).$$

This estimate together with $\|\sigma^{n,\delta} - \sigma^n\|_{C^1(\widetilde{\Omega})} \le K \|\nabla(\sigma^{n,\delta} - \sigma^n)\|_{C(\widetilde{\Omega})}$ leads to

$$\|\sigma^{n+1,\delta} - \sigma^{n+1}\|_{C^1(\widetilde{\Omega})} \le K\epsilon \overline{G}_1(\sigma_\pm^*) \Big[\frac{4(\overline{C}_*)^3}{(d_-^*)^2} + \frac{\overline{C}_*}{d_-^*} \Big] \|\sigma^{n,\delta} - \sigma^n\|_{C^1(\widetilde{\Omega})} + \frac{K}{\mu_0} \frac{\overline{C}_*}{d_-^*} \rho(\delta). \tag{3.44}$$

Now we take $\epsilon \in (0, \frac{1}{2K}\sigma_-^*)$ small enough such that

$$K\epsilon \overline{G}_1(\sigma_\pm^*) \Big[\frac{4(\overline{C}_*)^3}{(d_-^*)^2} + \frac{\overline{C}_*}{d_-^*} \Big] < \frac{1}{2}$$

and then it follows from (3.44) that

$$\|\sigma^{n+1,\delta} - \sigma^{n+1}\|_{C^1(\widetilde{\Omega})} \le \frac{1}{2} \|\sigma^{n,\delta} - \sigma^n\|_{C^1(\widetilde{\Omega})} + \frac{M}{2} \rho(\delta). \tag{3.45}$$

Inserting (3.26) for $k = n$ into (3.45), we can get

$$\|\sigma^{n+1,\delta} - \sigma^{n+1}\|_{C^1(\widetilde{\Omega})} \le \frac{M}{2} \rho(\delta) + \frac{M}{2} \rho(\delta) = M\rho(\delta). \tag{3.46}$$

Moreover, we conclude that $\nabla \sigma^{n+1,\delta} = 0$ in $\Omega \backslash \widetilde{\Omega}$ from (3.2) and (H5), and therefore $\sigma^{n+1,\delta} = \sigma^*$ in $\Omega \backslash \widetilde{\Omega}$.

Then it follows from Lemma 3.2, (3.46) and the triangle inequality that

$$\sigma^{n+1,\delta} \equiv \sigma^* \text{ in } \Omega \backslash \widetilde{\Omega}, \quad \|\sigma^{n+1,\delta} - \sigma^*\|_{C^1(\widetilde{\Omega})} \le M\rho(\delta) + K\theta^{n+1}\epsilon.$$

The proof is complete.

The important conclusion derived from Theorem 3.3 is that, different from using the exact input magnetic field, the iteration solution $\sigma^{n,\delta}$ of harmonic B_z algorithm using noisy magnetic field B_z^δ can only approximate the exact conductivity σ^* up to a finite accuracy. More precisely, it follows from Theorem 3.3 that

$$\lim_{n \to \infty} \|\sigma^{n,\delta} - \sigma^*\|_{C^1(\widetilde{\Omega})} \le M\rho(\delta) \tag{3.47}$$

for any fixed error level $\delta > 0$. This is reasonable from the general iteration scheme based on nonlinear integral equation of the second kind that the accuracy of the kernel determines the accuracy of solution, which can not be improved by increasing the iteration times.

To have $\|\sigma^{n,\delta} - \sigma^*\|_{C^1(\widetilde{\Omega})} \to 0$, we must choose $n \to \infty$ and $\rho(\delta) \to 0$ simultaneously. The total error $\|\sigma^{n,\delta} - \sigma^*\|_{C^1(\widetilde{\Omega})}$ constitutes of two parts: $M\rho(\delta)$ and $K\theta^n\varepsilon$. The former depends on the noise level δ and the strategy computing the Laplacian such that $\rho(\delta) \to 0$; while the later describes the iteration error which can be improved by increasing n. Notice that we need to distinguish two different input errors: Input data error δ for our MREIT problem and input error $\rho(\delta)$ for the harmonic B_z algorithm of MREIT problem. The efficient realization of harmonic B_z algorithm depends on decreasing both the iteration error $K\theta^n$ and the input error $\rho(\delta)$ for the algorithm. In the next section, we will analyze the input data error $\rho(\delta)$.

4 Stable Computation for Laplacian of B_z^δ

We propose a regularizing scheme for computing 2-dimensional $\nabla^2 B_z$ approximately from the noisy input data B_z^δ in $\widetilde{\Omega}$. Noticing that we can take $\widetilde{\Omega} \subset\subset \Omega$ such that $\partial\widetilde{\Omega}$ locates in the domain, where σ^* is a known constant, we can assume that $B_z(x)$ is exactly specified near $\partial\widetilde{\Omega}$, which means

$$B_z^\delta(x) = B_z(x) \quad \text{in the neighbourhood of } \partial\widetilde{\Omega}. \tag{4.1}$$

Since we need the $H_0^2(\widetilde{\Omega})$ $(:= W_0^{2,2}(\widetilde{\Omega}))$ estimate for the Laplacian computation in Theorem 3.3, we assume that the exact magnetic field is approximated by its noisy measurement data in the sense

$$\|B_z^\delta - B_z\|_{L^2(\widetilde{\Omega})} \le \delta \tag{4.2}$$

due to (4.1). Notice that the space $H_0^k(\widetilde{\Omega})$ can be characterized in terms of boundary conditions for $\partial\widetilde{\Omega} \in C^k$ (see [13, Theorem 7.41]):

$$H_0^k(\widetilde{\Omega}) \equiv \Big\{ u \in H^k(\widetilde{\Omega}),\ u = \frac{\partial u}{\partial \mathbf{n}} = \cdots = \frac{\partial^{k-1} u}{\partial \mathbf{n}^{k-1}} = 0 \text{ on } \partial\widetilde{\Omega} \Big\}. \tag{4.3}$$

Denote by $G(x,y) := \frac{1}{2\pi} \ln \frac{1}{|x-y|}$ the fundamental solution of $-\Delta$ operator, i.e.,

$$-\Delta G(x,y) = \delta(x-y), \quad x, y \in \widetilde{\Omega}.$$

Then for exact $B_z(x)$, its Laplacian $\Delta B_z(x) := f(x)$ in $\widetilde{\Omega}$ meets

$$\begin{aligned}\int_{\widetilde{\Omega}} f(y) G(x,y) \mathrm{d}y = \int_{\partial\widetilde{\Omega}} G(x,y) \frac{\partial B_z(y)}{\partial \mathbf{n}(y)} \mathrm{d}s(y) - \int_{\partial\widetilde{\Omega}} B_z(y) \frac{\partial G(x,y)}{\partial \mathbf{n}(y)} \mathrm{d}s(y) - B_z(x) \\ := \mathcal{F}[B_z](x), \quad x \in \widetilde{\Omega}. \end{aligned} \tag{4.4}$$

Moreover, by the Newtonian potential method for Poisson's equation, we know that $B_z(x) \in H^4(\widetilde{\Omega})$ for $f(x) \in H^2(\widetilde{\Omega})$, noticing $f(x) \equiv 0$ near $\partial\widetilde{\Omega}$ from (1.4) and the choice of $\partial\widetilde{\Omega}$.

Now let us define a linear bounded map $\mathbf{K} : H_0^2(\widetilde{\Omega}) \to L^2(\widetilde{\Omega})$ by

$$\mathbf{K}[g](x) := \int_{\widetilde{\Omega}} g(y) G(x,y) \mathrm{d}y, \quad \forall g(x) \in H_0^2(\widetilde{\Omega}). \tag{4.5}$$

Then (4.4) can be written as

$$\mathbf{K}[f](x) = \mathcal{F}[B_z](x), \quad x \in \widetilde{\Omega} \tag{4.6}$$

from (4.3) and $f(x) \equiv 0$ near $\partial\widetilde{\Omega}$.

For noisy input data B_z^δ satisfying (4.1)–(4.2), we define $\Delta B_z^\delta := f^\delta = g^{\alpha(\delta),\delta}$ as the solution to the following integral equation of the second kind:

$$\alpha g^{\alpha,\delta} + \widetilde{\mathbf{K}}^*\mathbf{K}[g^{\alpha,\delta}] = \widetilde{\mathbf{K}}^*[\mathcal{F}[B_z^\delta](x)] \tag{4.7}$$

for some suitable regularizing parameter $\alpha = \alpha(\delta) > 0$, where $\widetilde{\mathbf{K}}^* : L^2(\widetilde{\Omega}) \to H_0^2(\widetilde{\Omega})$ is the adjoint operator of $\mathbf{K} : H_0^2(\widetilde{\Omega}) \to L^2(\widetilde{\Omega})$. Obviously, (4.7) is the Tikhonov regularizing equation to the following integral equation of the first kind:

$$\mathbf{K}[g](x) = \mathcal{F}[B_z^\delta](x), \quad x \in \widetilde{\Omega}. \tag{4.8}$$

The unique solvability of (4.7) follows from the standard Tikhonov regularizing theory. Notice that $\mathbf{K} : H_0^2(\widetilde{\Omega}) \to L^2(\widetilde{\Omega})$ is neither self-adjoint nor nonnegative.

The first result in this section is the error estimate on $f^\delta - f$.

Theorem 4.1 *Assume that $f \in \widetilde{\mathbf{K}}^*(L^2(\widetilde{\Omega}))$. If we choose the regularizing $\alpha = \delta$, then we have for the noisy input data B_z^δ satisfying* (4.2) *that*

$$\|f^\delta - f\|_{H_0^2(\widetilde{\Omega})} \le C\delta^{\frac{1}{2}}. \tag{4.9}$$

Proof It follows from (4.6) and (4.7) that

$$\alpha(g^{\alpha,\delta} - f) + \widetilde{\mathbf{K}}^*\mathbf{K}[g^{\alpha,\delta} - f] = \widetilde{\mathbf{K}}^*[B_z - B_z^\delta] - \alpha f,$$

noticing (4.1). Since $f = \widetilde{\mathbf{K}}^*\phi_0$ for some $\phi_0 \in L^2(\widetilde{\Omega})$ due to $f \in \widetilde{\mathbf{K}}^*(L^2(\widetilde{\Omega}))$, the above equation becomes

$$g^{\alpha,\delta} - f = (\alpha I + \widetilde{\mathbf{K}}^*\mathbf{K})^{-1}\widetilde{\mathbf{K}}^*[(B_z - B_z^\delta) - \alpha\phi_0]. \tag{4.10}$$

Then we have

$$\begin{aligned} \|g^{\alpha,\delta} - f\|_{H_0^2(\widetilde{\Omega})} &\le \|(\alpha I + \widetilde{\mathbf{K}}^*\mathbf{K})^{-1}\widetilde{\mathbf{K}}^*\|_{\mathcal{L}(L^2,H_0^2)}(\|B_z - B_z^\delta\|_{L^2(\widetilde{\Omega})} + \alpha\|\phi_0\|_{L^2(\widetilde{\Omega})}) \\ &\le \frac{C}{\sqrt{\alpha}}(\delta + \alpha) \end{aligned} \tag{4.11}$$

from the standard estimate on the Tikhonov regularizing operator. The proof is complete by taking $\alpha = \delta$.

Remark 4.1 This result is a classical a priori choice strategy for Tikhonov regularization. The source condition $\Delta B_z = f \in \widetilde{\mathbf{K}}^*(L^2(\widetilde{\Omega}))$ implies the requirement that B_z should be very smooth, not necessarily in $L^2(\Omega)$. The other a posterior strategies such as discrepancy principle can also be considered under the framework of Tikhonov regularizing. However, since $\mathbf{K} : H_0^2(\widetilde{\Omega}) \to L^2(\widetilde{\Omega})$ is not nonnegative, the scheme applying the Lavrentive regularizing for computing the Laplacian (see [19]) does not work.

To avoid the explicit expression of the adjoint operator $\widetilde{\mathbf{K}}^* : L^2(\widetilde{\Omega}) \to H_0^2(\widetilde{\Omega})$, we consider (4.7) directly. Using the property of definition of $H_0^2(\widetilde{\Omega})$ inner product, it follows for all $\chi \in L^2(\widetilde{\Omega})$ and $\psi \in H_0^2(\widetilde{\Omega})$ that

$$\langle \widetilde{\mathbf{K}}^*\chi, \psi\rangle_{H_0^2(\widetilde{\Omega})} = \langle \chi, \mathbf{K}\psi\rangle_{L^2(\widetilde{\Omega})} = \langle \mathbf{K}^*\chi, \psi\rangle_{L^2(\widetilde{\Omega})},$$

where $\mathbf{K}^*$ is the adjoint operator of $\mathbf{K}$ in the sense mapping $L^2(\widetilde{\Omega})$ to itself. Therefore the regularizing equation (4.7) in $H_0^2(\widetilde{\Omega})$ has the following equivalent weak form:

$$\alpha\langle g^{\alpha,\delta},\psi\rangle_{H_0^2(\widetilde{\Omega})}+\langle \mathbf{K}^*\mathbf{K}g^{\alpha,\delta},\psi\rangle_{L^2(\widetilde{\Omega})}=\langle \mathbf{K}^*\mathcal{L}[B_z^\delta],\psi\rangle_{L^2(\widetilde{\Omega})},\quad \forall\psi\in H_0^2(\widetilde{\Omega}). \tag{4.12}$$

However, it is easy to verify that $\mathbf{K}:L^2(\widetilde{\Omega})\to L^2(\widetilde{\Omega})$ is self-adjoint. In fact, for any $h,g\in L^2(\widetilde{\Omega})$, we have

$$\begin{aligned}\langle \mathbf{K}h,g\rangle_{L^2(\widetilde{\Omega})}&=\int_{\widetilde{\Omega}}\int_{\widetilde{\Omega}}G(x,y)h(y)\mathrm{d}yg(x)\mathrm{d}x=\int_{\widetilde{\Omega}}\int_{\widetilde{\Omega}}G(x,y)g(x)\mathrm{d}xh(y)\mathrm{d}y\\&=\int_{\widetilde{\Omega}}\int_{\widetilde{\Omega}}G(y,x)g(x)\mathrm{d}xh(y)\mathrm{d}y=\langle h,\mathbf{K}g\rangle_{L^2(\widetilde{\Omega})}.\end{aligned}$$

Therefore (4.12) is equivalent to

$$\alpha\langle g^{\alpha,\delta},\psi\rangle_{H_0^2(\widetilde{\Omega})}+\langle \mathbf{K}^2g^{\alpha,\delta},\psi\rangle_{L^2(\widetilde{\Omega})}=\langle \mathbf{K}\mathcal{L}[B_z^\delta],\psi\rangle_{L^2(\widetilde{\Omega})},\quad \forall\psi\in H_0^2(\widetilde{\Omega}) \tag{4.13}$$

with $g^{\alpha,\delta},B_z^\delta\in H_0^2(\Omega)$.

Now we consider how to solve (4.13). We define the inner product in Hilbert space $H_0^2(\widetilde{\Omega})$ by

$$\langle h,g\rangle_{H_0^2(\widetilde{\Omega})}:=\int_{\widetilde{\Omega}}(\Delta h(x)\ \Delta g(x)+\nabla h(x)\cdot\nabla g(x)+h(x)g(x))\mathrm{d}x,\quad \forall h,g\in H_0^2(\widetilde{\Omega}),$$

which yields the equivalent norm to $\|h\|_{H_0^2(\widetilde{\Omega})}$. Integrating by parts yields

$$\langle h,g\rangle_{H_0^2(\widetilde{\Omega})}=\langle \Delta^2h-\Delta h+h,g\rangle_{L^2(\widetilde{\Omega})}:=\langle \mathbb{L}h,g\rangle_{L^2(\widetilde{\Omega})},\quad \forall h,g\in H_0^2(\widetilde{\Omega}).$$

Therefore (4.13) is equivalent to the following integral-differential system:

$$\begin{cases}\alpha\mathbb{L}g^{\alpha,\delta}+\mathbf{K}^2g^{\alpha,\delta}=\mathbf{K}\mathcal{L}[B_z^\delta] & \text{in } \widetilde{\Omega},\\ g^{\alpha,\delta}=\dfrac{\partial g^{\alpha,\delta}}{\partial\mathbf{n}}=0 & \text{on } \partial\widetilde{\Omega},\end{cases} \tag{4.14}$$

noticing that $H_0^2(\widetilde{\Omega})$ is dense in $L^2(\widetilde{\Omega})$.

We can solve this well-posed system to get the approximation of ∇^2B_z from noisy data B_z^δ. Once we generate ∇^2B_z from this system, which is a good approximation to ∇^2B_z, the harmonic B_z algorithm can be implemented efficiently.

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