Chin. Ann. Math.
35B(3), 2014, 355–382
DOI: 10.1007/s11401-014-0836-x

Chinese Annals of Mathematics, Series B

An Inverse Problem of Identifying the Radiative Coefficient in a Degenerate Parabolic Equation*

Zuicha DENG[1] Liu YANG[2]

Abstract The authors investigate an inverse problem of determining the radiative coefficient in a degenerate parabolic equation from the final overspecified data. Being different from other inverse coefficient problems in which the principle coefficients are assumed to be strictly positive definite, the mathematical model discussed in this paper belongs to the second order parabolic equations with non-negative characteristic form, namely, there exists a degeneracy on the lateral boundaries of the domain. Based on the optimal control framework, the problem is transformed into an optimization problem and the existence of the minimizer is established. After the necessary conditions which must be satisfied by the minimizer are deduced, the uniqueness and stability of the minimizer are proved. By minor modification of the cost functional and some a priori regularity conditions imposed on the forward operator, the convergence of the minimizer for the noisy input data is obtained in this paper. The results can be extended to more general degenerate parabolic equations.

Keywords Inverse problem, Degenerate parabolic equation, Optimal control, Existence, Uniqueness, Stability, Convergence
2000 MR Subject Classification 35R30, 49J20

1 Introduction

In this paper, we study an inverse problem of identifying the radiative coefficient in a degenerate parabolic equation from the final overspecified data. Problems of this type have important applications in several fields of applied science and engineering. The problem can be stated in the following form.

Problem 1.1 Consider the following parabolic equation:

$$\begin{cases} u_t - (a(x)u_x)_x + q(x)u = 0, & (x,t) \in Q = (0,l) \times (0,T], \\ u|_{t=0} = \phi(x), & x \in (0,l), \end{cases} \tag{1.1}$$

where a and ϕ are two given smooth functions, which satisfy

$$a(0) = a(l) = 0, \quad a(x) > 0, \quad x \in (0,l) \tag{1.2}$$

Manuscript received October 18, 2013. Revised November 14, 2013.
[1]Corresponding author. Department of Mathematics, Lanzhou Jiaotong University, Lanzhou 730070, China. E-mail: zc_deng78@hotmail.com
[2]School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China.
E-mail: l_yang218@163.com
*This work was supported by the National Natural Science Foundation of China (Nos. 11061018, 11261029), the Youth Foundation of Lanzhou Jiaotong University (No. 2011028), the Long Yuan Young Creative Talents Support Program (No. 252003) and the Joint Funds of the Gansu Provincial Natural Science Foundation of China (No. 1212RJZA043).

and

$$\phi(x) \geq 0, \quad \phi(x) \not\equiv 0, \quad x \in (0, l), \tag{1.3}$$

respectively, and $q(x)$ is an unknown coefficient in (1.1). In this paper, we always assume that $a(x)$ is at least C^1 continuous, i.e., $a(x) \in C^1[0, l]$. Assume that an additional condition is given as follows:

$$u(x, T) = g(x), \quad x \in [0, l], \tag{1.4}$$

where g is a known function. We shall determine the functions u and q satisfying (1.1) and (1.4), respectively.

If the principle coefficient $a(x)$ is required to be strictly positive, i.e.,

$$a(x) \geq a_0 > 0, \quad x \in [0, l],$$

then the equation should be rewritten as an initial-boundary value problem, e.g., the homogeneous Dirichlet boundary value problem as follows:

$$\begin{cases} u_t - (a(x)u_x)_x + q(x)u = 0, \quad (x, t) \in Q, \\ u|_{x=0} = u|_{x=l} = 0, \\ u(x, 0) = \phi(x), \end{cases} \tag{1.5}$$

which is often referred as the classical parabolic equation. The mathematical model (1.5) arises in various physical and engineering settings. If (1.5) is used to describe the heat transfer system, the coefficient $q(x)$ is called the radiative coefficient which is often dependent on the medium property.

Being different from the ordinary parabolic equation (1.5), (1.1) belongs to the second order differential equations with non-negative characteristic form. The main character of such kinds of equations is degeneracy. It can be easily seen that at $x = 0$ and $x = l$, (1.1) degenerates into two hyperbolic equations

$$\frac{\partial u}{\partial t} - a'(0)\frac{\partial u}{\partial x} + q(0)u = 0,$$
$$\frac{\partial u}{\partial t} - a'(l)\frac{\partial u}{\partial x} + q(l)u = 0.$$

By the well-known Fichera's theory (see [33]) for degenerate parabolic equations, we know that whether or not boundary conditions should be given at the degenerate, boundaries are determined by the sign of the Fichera function. By simple calculations, one can easily check that boundary conditions for (1.1) on the lateral boundaries $x = 0$, $x = l$ and the terminal boundary $t = T$ should not be given, while on $t = 0$ they are indispensable. In other words, the parabolic problem (1.1) is well-defined.

In general, most physical and industrial phenomenons can be described by the classical parabolic model, such as (1.5). However, with the development of the modern financial mathematics, more and more degenerate elliptic or parabolic equations arising in derivatives pricing have to be taken into account. For example, the well-known Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2(S)S^2\frac{\partial^2 V}{\partial S^2} + (r - q)S\frac{\partial V}{\partial S} - rV = 0, \quad (S, t) \in [0, \infty) \times [0, T) \tag{1.6}$$

is such the case, where the degenerate parabolic boundary is $S = 0$.

For a given coefficient $q(x)$, the degenerate parabolic equation (1.1) which is referred as a direct problem consists of the determination of the solution from the given initial condition. It is well-known that in all cases, the inverse problem is ill-posed or improperly posed in the sense of Hadamard, while the direct problem is well-posed (see [23, 30, 32]). The ill-posedness, particularly the numerical instability, is the main difficulty for Problem 1.1. Since data errors in the extra condition $g(x)$ are inevitable, arbitrarily small changes in $g(x)$ may lead to arbitrarily large changes in $q(x)$, which may make the obtained results meaningless (see, e.g., [20, 36]).

Inverse coefficient problems for parabolic equations are well studied in the literature. However, most of these inverse problems are governed by classical parabolic equations in which the principle coefficients are assumed to be strictly positive definite. The inverse problem of identifying the diffusion coefficient $a(x)$ in the following parabolic equation:

$$u_t - \nabla \cdot (a(x)\nabla u) = f(x,t), \quad (x,t) \in \Omega \times (0,T),$$

from some additional conditions was investigated by several authors (see, e.g., [14, 21, 24, 29]). In [29, 21], the output least-squares method with Tikhonov regularization is applied to the inverse problem and the numerical solution is obtained by the finite element method. The determination of $a(x)$ with two Neumann measured data

$$a(0)u_x(0,t) = k(t), \quad a(1)u_x(1,t) = h(t), \quad t \in [0,T]$$

has been considered carefully in [14] by the semigroup approach. In [24], the inverse problem is reduced to a nonlinear equation and the uniqueness, as well as the conditional stability of the solution is proved.

The inverse problem of identification of the radiative coefficient $q(x)$ in the following heat conduction equation:

$$u_t - \Delta u + q(x)u = 0, \quad (x,t) \in Q,$$

from the final overdetermination data $u(x,T)$ was considered by several authors (see, e.g., in [8, 10–11, 34, 39]). Moreover, treatments on the case of purely time dependent $q = q(t)$ can be found in [6–7, 12–13]. For the general case in which the unknown coefficient(s) depend(s) on both spatial and temporal variables, we refer the readers to the references, e.g., in [16, 27–28, 35].

Compared with classical parabolic equations, the main difficulty for degenerate equations lies in the degeneracy of the principle coefficients which may lead to the corresponding solution has no sufficient regularity, even if the initial value and the coefficients are sufficiently smooth functions. Many effective tools, e.g., the Schauder's type a priori estimate which was extensively applied in classical parabolic equations, are no longer applicable for the degenerate parabolic equations. The documents concerned with inverse degenerate problems are quite few in contrast with those dealt with non-degenerate problems. In [2], the authors investigated an inverse problem of determining the source term g in the following degenerate parabolic equation:

$$u_t - (x^\alpha u_x)_x = g, \quad (x,t) \in (0,1) \times (0,T),$$

where $\alpha \in [0, 2)$. The uniqueness and Lipschitz stability of the solution are obtained by the global Carleman estimates, which was introduced in [22] in 1998. Recently, in [37], analogous methods were applied to a nonlinear inverse coefficient problem arising in the field of climate evolution, where the diffusion coefficient is assumed to vanish at both extremities of the domain. For other topics of degenerate parabolic equations, e.g., the null controllability, we may refer the reader to [3–5] and the references therein.

The most important inverse problem in which the underlying model is degenerate may be the reconstruction of local volatility in the Black-Scholes equation (1.6). In [25–26], the inverse problem of identifying the implied volatility $\sigma = \sigma(S)$ from current market prices of options was considered carefully. Based on the optimal control framework, the existence, the uniqueness of $\sigma(S)$ and a well-posed algorithm are obtained. Similar results were derived in [15], where a new extra condition, i.e., the average option premium, was assumed to be known. In [19], on the basis of the parameter-to-solution mapping, the stability and convergence of approximations for $\sigma(S)$ are gained by Tikhonov regularization.

It should be mentioned that the degeneracy in the Black-Scholes equation can be removed by some change of variable (see [19]). However, the degeneracy in our problem can not be removed by any method, which is also the main difficulty in this paper.

To our knowledge, this paper is the first one concerning uniqueness, stability and convergence of optimal solution in inverse problem for degenerate parabolic equations such as (1.1). In this paper, we use an optimal control framework (see, e.g., [16–17, 25, 39]) to discuss Problem 1.1 mainly from the theoretical analysis angle. The outline of the manuscript is as follows: In Section 2, the inverse Problem 1.1 is transformed into an optimal control Problem 2.1 and the existence of minimizer of the cost functional is proved. The necessary condition of the minimizer is established in Section 3. By assuming that T is relatively small, the local uniqueness and stability of the minimizer are shown in Section 4. The convergence of the minimizer with noisy input data is obtained in Section 5 by some a priori regularity conditions imposed on the forward operator. In Section 6, we complete this paper with concluding remarks.

2 Optimal Control Problem

In general, uniqueness is very important for the inverse problems. It illustrates if the extra condition is sufficient to identify the unknown information. There are many mathematical tools can be used to derive the uniqueness, such as maximum principle, energy estimate, unique continuation, integral equation, Carleman estimate, and so on. It should be mentioned that the Carleman estimate is an effective tool to derive uniqueness and conditional stability for inverse problems (see [22]). But unfortunately, it fails in treating the terminal control problems such as inverse Problem 1.1. We have obtained a uniqueness results of the inverse Problem 1.1 in a sense of partial order. It seems that the partial order imposed on the uniqueness is rather disgusting, but until now we do not know how to remove it due to the coefficient degeneration on the lateral boundaries. The details can be found in [18].

Since the original problem is ill-posed, we would like to discuss the regularization of Problem 1.1. Before this, let us to discuss the forward problem (2.1) and give some basic definitions, lemmas and estimations. We would like to consider the more general equation:

$$\begin{cases} u_t - (a(x)u_x)_x + q(x)u = f(x,t), & (x,t) \in Q = (0,l) \times (0,T], \\ u|_{t=0} = \phi(x), & x \in (0,l). \end{cases} \tag{2.1}$$

Definition 2.1 *Define $\mathcal{B}$ to be the closure of $C_0^\infty(Q)$ under the following norm:*

$$\|u\|_{\mathcal{B}}^2 = \iint_Q a(x)(|u|^2 + |\nabla u|^2)\mathrm{d}x\mathrm{d}t, \quad u \in \mathcal{B}.$$

Definition 2.2 *A function $u(x,t)$ is called the weak solution to* (2.1)*, if $u \in C([0,T]; L^2(0,l)) \cap \mathcal{B}$, and for any $\psi \in L^\infty((0,T); L^2(0,l)) \cap \mathcal{B}$, $\frac{\partial \psi}{\partial t} \in L^2(Q)$, $\psi(\cdot,T) = 0$, the following integration identity holds:*

$$\iint_Q \Big(-u\frac{\partial \psi}{\partial t} + a\nabla u \cdot \nabla\psi + qu\psi\Big)\mathrm{d}x\mathrm{d}t - \int_0^l \phi(x)\psi(x,0)\mathrm{d}x = \iint_Q f\psi \mathrm{d}x\mathrm{d}t. \tag{2.2}$$

Remark 2.1 Assume $u \in C([0,T]; L^2(0,l)) \cap \mathcal{B}$ and $\frac{\partial u}{\partial t} \in L^2(Q)$. Then (2.2) can be rewritten as

$$\iint_Q \Big(\frac{\partial u}{\partial t}\psi + a\nabla u \cdot \nabla \psi + qu\psi\Big)\mathrm{d}x\mathrm{d}t = \iint_Q f\psi\mathrm{d}x\mathrm{d}t,$$

where u satisfies $u|_{t=0} = \phi(x)$ in the sense of trace.

Theorem 2.1 *For any given $f \in L^\infty(Q)$, $\phi \in L^\infty(0,l)$, there exists a unique weak solution to* (2.1)*, which satisfies the following estimate:*

$$\|u\|_{L^\infty((0,T),L^2(0,l))} + \|a|\nabla u|^2\|_{L^1(Q)} \le C(\|f\|_{L^2(Q)}^2 + \|\phi\|_{L^2(0,l)}^2).$$

Furthermore, if $a|\nabla\phi|^2 \in L^1(0,l)$, then $\frac{\partial u}{\partial t} \in L^2(Q)$ and

$$\Big\|\frac{\partial u}{\partial t}\Big\|_{L^2(Q)} \le C(\|f\|_{L^2(Q)} + \|\phi\|_{L^2(0,l)} + \|a|\nabla\phi|^2\|_{L^1(0,l)}).$$

Proof Firstly, we prove the existence. For any given $0 < \varepsilon < 1$, we consider the following regularized problem:

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - (a_\varepsilon(x)u_{\varepsilon,x})_x + q(x)u_\varepsilon = f(x,t), & (x,t) \in Q, \\ u_\varepsilon(0,t) = u_\varepsilon(l,t) = 0, \\ u_\varepsilon(x,0) = \phi(x), \end{cases} \tag{2.3}$$

where

$$a_\varepsilon(x) = a(x) + \varepsilon, \quad x \in [0,l].$$

From the well-known theory for parabolic equations (see [31]), there exists a unique weak solution $u_\varepsilon(x,t)$ to (2.3).

Then, we will give some a priori estimates for $u_\varepsilon(x,t)$. Without loss of generality, we assume that $u_\varepsilon(x,t)$ is the classical solution to (2.3). Otherwise, one can smooth the coefficients of (2.3), and then consider the solution to the approximation problem.

Multiplying both sides of (2.3) by u_ε and integrating on $Q_t=[0,l]\times(0,t)$, we have

$$\iint_{Q_t}\frac{\partial u_\varepsilon}{\partial t}u_\varepsilon\mathrm{d}x\mathrm{d}t-\iint_{Q_t}(a_\varepsilon u_{\varepsilon,x})_x u_\varepsilon\mathrm{d}x\mathrm{d}t+\iint_{Q_t}qu_\varepsilon^2\mathrm{d}x\mathrm{d}t=\iint_{Q_t}fu_\varepsilon\mathrm{d}x\mathrm{d}t.$$

Integration by parts, we get

$$\begin{aligned}&\int_0^l\frac{1}{2}u_\varepsilon^2\mathrm{d}x+\int_0^t\int_0^l a_\varepsilon|u_{\varepsilon,x}|^2\mathrm{d}x\mathrm{d}t+\int_0^t\int_0^l qu_\varepsilon^2\mathrm{d}x\mathrm{d}t\\ \leq&\int_0^l\frac{1}{2}\phi^2\mathrm{d}x+\frac{1}{2}\int_0^t\int_0^l|u_\varepsilon|^2\mathrm{d}x\mathrm{d}t+\frac{1}{2}\int_0^t\int_0^l f^2\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{2.4}$$

From (2.4) and the Gronwall's inequality, we have

$$\max_{0<t\leq T}\int_0^l u_\varepsilon^2\mathrm{d}x+\iint_{Q_t}a_\varepsilon|u_{\varepsilon,x}|^2\mathrm{d}x\mathrm{d}t\leq C\Big(\int_0^l\phi^2\mathrm{d}x+\iint_{Q_t}f^2\mathrm{d}x\mathrm{d}t\Big).$$

On the other hand, if $a|\nabla\phi|^2\in L^1(0,l)$, then by multiplying $\frac{\partial u_\varepsilon}{\partial t}$ on both sides of (2.3) and integrating on Q_t, we obtain

$$\begin{aligned}&\iint_{Q_t}\Big|\frac{\partial u_\varepsilon}{\partial t}\Big|^2\mathrm{d}x\mathrm{d}t-\iint_{Q_t}(a_\varepsilon u_{\varepsilon,x})_x\cdot\frac{\partial u_\varepsilon}{\partial t}\mathrm{d}x\mathrm{d}t+\iint_{Q_t}qu_\varepsilon\frac{\partial u_\varepsilon}{\partial t}\mathrm{d}x\mathrm{d}t\\ =&\iint_{Q_t}f\frac{\partial u_\varepsilon}{\partial t}\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}&\iint_{Q_t}\Big|\frac{\partial u_\varepsilon}{\partial t}\Big|^2\mathrm{d}x\mathrm{d}t+\iint_{Q_t}\frac{q}{2}\frac{\partial}{\partial t}(u_\varepsilon^2)\mathrm{d}x\mathrm{d}t\\ &-\iint_{Q_t}\Big[\frac{\partial}{\partial x}\Big(a_\varepsilon\frac{\partial u_\varepsilon}{\partial x}\cdot\frac{\partial u_\varepsilon}{\partial t}\Big)-a_\varepsilon\frac{\partial u_\varepsilon}{\partial x}\cdot\frac{\partial^2 u_\varepsilon}{\partial x\partial t}\Big]\mathrm{d}x\mathrm{d}t\\ =&\iint_{Q_t}\Big|\frac{\partial u_\varepsilon}{\partial t}\Big|^2\mathrm{d}x\mathrm{d}t+\iint_{Q_t}\frac{q}{2}\frac{\partial}{\partial t}(u_\varepsilon^2)\mathrm{d}x\mathrm{d}t+\iint_{Q_t}\frac{a_\varepsilon}{2}\frac{\partial}{\partial t}\Big|\frac{\partial u_\varepsilon}{\partial x}\Big|^2\mathrm{d}x\mathrm{d}t\\ =&\iint_{Q_t}f\frac{\partial u_\varepsilon}{\partial t}\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{2.5}$$

From (2.5), we get

$$\begin{aligned}&\iint_{Q_t}\Big|\frac{\partial u_\varepsilon}{\partial t}\Big|^2\mathrm{d}x\mathrm{d}t+\int_0^l a_\varepsilon\Big|\frac{\partial u_\varepsilon}{\partial x}(x,t)\Big|^2\mathrm{d}x+\int_0^l\frac{q}{2}u_\varepsilon^2(x,t)\mathrm{d}x\\ \leq&\int_0^l a_\varepsilon\phi_x^2\mathrm{d}x+\frac{1}{2}\int_0^l q\phi^2\mathrm{d}x+\frac{1}{2}\iint_{Q_t}f^2\mathrm{d}x\mathrm{d}t+\frac{1}{2}\iint_{Q_t}\Big|\frac{\partial u_\varepsilon}{\partial t}\Big|^2\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{2.6}$$

From (2.6), we have

$$\Big\|\frac{\partial u_\varepsilon}{\partial t}\Big\|_{L^2(Q)}\leq C(\|f\|_{L^2(Q)}+\|\phi\|_{L^2(0,l)}+\|a_\varepsilon|\nabla\phi|^2\|_{L^1(0,l)}).$$

Moreover, it follows from the maximum principle that

$$\|u_\varepsilon\|_{L^\infty(Q)}\leq C.$$

From the estimations above, it can be derived that there exists a subsequence of $\{u_\varepsilon\}$ (denoted by itself) and

$$u \in C([0,T];L^2(0,l)), \quad \frac{\partial u}{\partial t} \in L^2(Q),$$

such that

$$\begin{aligned} u_\varepsilon &\to u && \text{in } L^2(Q), \\ \nabla u_\varepsilon &\rightharpoonup \nabla u && \text{in } L^2_{\text{loc}}(Q), \\ \frac{\partial u_\varepsilon}{\partial t} &\rightharpoonup \frac{\partial u}{\partial t} && \text{in } L^2(Q), \\ a_\varepsilon \nabla u_\varepsilon &\rightharpoonup a\nabla u && \text{in } L^2(Q). \end{aligned}$$

Letting $u = u_\varepsilon$ in (2.2), we have

$$\iint_Q \Big(-u_\varepsilon \frac{\partial \psi}{\partial t} + a\nabla u_\varepsilon \cdot \nabla\psi + qu_\varepsilon\psi\Big)\mathrm{d}x\mathrm{d}t - \int_0^l \phi(x)\psi(x,0)\mathrm{d}x = \iint_Q f\psi \mathrm{d}x\mathrm{d}t.$$

Letting $\varepsilon \to 0$, one can immediately obtain

$$\iint_Q \Big(-u \frac{\partial \psi}{\partial t} + a\nabla u \cdot \nabla\psi + qu\psi\Big)\mathrm{d}x\mathrm{d}t - \int_0^l \phi(x)\psi(x,0)\mathrm{d}x = \iint_Q f\psi \mathrm{d}x\mathrm{d}t,$$

which implies the existence of weak solutions.

Next, we prove the uniqueness of weak solutions. Suppose that $u_1,\ u_2$ be two solutions to (2.1), and let

$$U(x,t) = u_1(x,t) - u_2(x,t), \quad (x,t) \in Q.$$

It can be easily seen that $U \in C([0,T];L^2(0,l)) \cap \mathcal{B}$, and for any $\psi \in L^\infty((0,T);L^2(0,l)) \cap \mathcal{B}$, $\frac{\partial \psi}{\partial t} \in L^2(Q)$, $\psi(\cdot,T) = 0$, the following integration identity holds:

$$\iint_Q \Big(-U \frac{\partial \psi}{\partial t} + a\nabla U \cdot \nabla\psi + qU\psi\Big)\mathrm{d}x\mathrm{d}t = 0. \tag{2.7}$$

For any given $g \in C_0^\infty(Q)$, by the existence obtained above, we know that there exists a weak solution $v \in L^\infty((0,T);L^2(0,l)) \cap \mathcal{B}$ and $\frac{\partial v}{\partial t} \in L^2(Q)$ for the following equation:

$$\begin{aligned} &-\frac{\partial v}{\partial t} - (a(x)v_x)_x + q(x)v = g(x,t), \quad (x,t) \in Q, \\ &v(x,T) = 0, \quad x \in (0,l). \end{aligned}$$

Letting $\psi = v$ in (2.7), we obtain

$$\iint_Q Ug\mathrm{d}x\mathrm{d}t = 0.$$

Noting the arbitrariness of g, we have

$$U(x,t) = 0 \quad \text{a.e. } (x,t) \in Q,$$

i.e.,

$$u_1(x,t) = u_2(x,t) \quad \text{a.e. } (x,t) \in Q.$$

This completes the proof of Theorem 2.1.

Remark 2.2 The weak solution defined above is on the whole domain Q. If we only consider the spatial case, we can modify the Definition 2.1 as follows.

Definition 2.1′ *Define $\mathscr{H}^1(0,l)$ to be the closure of $C_0^\infty(0,l)$ under the following norm:*

$$\|v\|_{\mathscr{H}^1}^2 = \int_0^l a(x)(|v|^2 + |\nabla v|^2)\mathrm{d}x\mathrm{d}t, \quad v \in \mathscr{H}^1(0,l).$$

For the case of $f \equiv 0$, the Definition 2.2 can also be rewritten as follows.

Definition 2.2′ *A function $u \in H^1((0,T);L^2(0,l)) \cap L^2((0,T);\mathscr{H}^1(0,l))$ is called the weak solution to (2.1), if u satisfies*

$$u(x,0) = \phi(x), \quad x \in (0,l) \tag{2.8}$$

and the following integration identity

$$\int_0^l u_t\psi\mathrm{d}x + \int_0^l a\nabla u\cdot\nabla\psi\mathrm{d}x + \int_0^l qu\psi\mathrm{d}x = 0, \quad \forall\psi \in L^2(0,l)\cap\mathscr{H}^1(0,l) \tag{2.9}$$

holds for a.e. $t \in (0,T]$. Then, by analogously arguments, one can establish the existence, the uniqueness and the regularity for such kind of weak solution, which are similar to those of Theorem 2.1.

Remark 2.3. We recall that the principle coefficient $a(x) \in C^1[0,1]$. Due to the degeneracy at $x = 0$ and $x = l$, from $u \in \mathscr{H}^1(0,l)$, one can only derive $u \in H^1_{\mathrm{loc}}(0,l)$ rather than $u \in H^1(0,l)$, which is different from the case of non-degenerate. However, we may derive

$$au_x \to 0 \quad \text{as } x \to 0. \tag{2.10}$$

In fact, if (2.10) is not true, i.e., $au_x \to k,\ k \neq 0$, then we have $u_x \sim \frac{k}{a(x)}$ in $B_\delta(0)\cap[0,l]$, where $B_\delta(0)$ is a ball with δ-radius centered at $x = 0$. Note that

$$a(x) = a(0) + a'(\xi)x = a'(\xi)x, \quad \xi \in [0,x],\ x \in B_\delta(0)\cap[0,l].$$

Hence,

$$a|u_x|^2 \sim \frac{k^2}{a(x)} \sim \frac{k^2}{a'(\xi)x},$$

which is contradicts with $a|u_x|^2 \in L^1(0,l)$. By analogous arguments, we have

$$au_x \to 0 \quad \text{as } x \to l.$$

It should be mentioned that these conclusions are no longer valid for $a \notin C^1[0,l]$. For example, let

$$a(x) = x^\alpha(l-x)^\beta, \quad 0 < \alpha,\beta < 1. \tag{2.11}$$

It can be easily seen that $a|u_x|^2 \in L^1(0,l)$ can not guarantee $au_x \to 0$ as x tends to 0 or l. In some references (see, e.g., [3–4]), the case (2.11) is called the weak degeneracy and the boundary conditions are indispensable for corresponding mathematical model, e.g., we shall replace (1.1) by the following initial-boundary value problem:

$$\begin{cases} u_t - (a(x)u_x)_x + q(x)u = 0, \quad (x,t) \in Q, \\ u|_{x=0} = u|_{x=l} = 0, \\ u(x,0) = \phi(x). \end{cases}$$

Since the inverse Problem 1.1 is ill-posed, i.e., its solution depends unstably on the data, we turn to consider the following optimal control Problem 2.1.

Problem 2.1 Find $\overline{q}(x) \in \mathcal{A}$, such that

$$J(\overline{q}) = \min_{q \in \mathcal{A}} J(q), \tag{2.12}$$

where

$$J(q) = \frac{1}{2}\int_0^l |u(x,T;q) - g(x)|^2 \mathrm{d}x + \frac{N}{2}\int_0^l |\nabla q|^2 \mathrm{d}x, \tag{2.13}$$

$$\mathcal{A} = \{q(x) \mid 0 < \alpha \le q \le \beta, \ \|q\|_{H^1(0,l)} < \infty\}, \tag{2.14}$$

$u(x,t;q)$ is the solution to (1.1) for a given coefficient $q(x) \in \mathcal{A}$, N is the regularization parameter, and $\alpha, \ \beta$ are two given positive constants.

For the extra condition (1.4), we shall assume that

$$g(x) \in L^\infty(0,l). \tag{2.15}$$

From (2.15) and Theorem 2.1, it can be easily seen that the control functional (2.13) is well-defined for any $q \in \mathcal{A}$.

We are now going to show the existence of minimizers to the problem (2.12). Firstly, we assert that the functional $J(q)$ is of some continuous property in $\mathcal{A}$ in the following sense.

Lemma 2.1 *For any sequence $\{q_n\}$ in $\mathcal{A}$ which converges to some $q \in \mathcal{A}$ in $L^1(0,l)$ as $n \to \infty$, we have*

$$\lim_{n\to\infty} \int_0^l |u(q_n)(x,T) - g(x)|^2 \mathrm{d}x = \int_0^l |u(q)(x,T) - g(x)|^2 \mathrm{d}x. \tag{2.16}$$

Proof Step 1 By taking $q = q_n$ and choosing the test function as $u(q_n)(\cdot,t)$ in (2.9) and then integrating with respect to t, we derive that

$$\|u(q_n;t)\|^2_{L^2(0,l)} + \int_0^t \int_0^l a|\nabla u(q_n;t)|^2 \mathrm{d}x\mathrm{d}t + \int_0^t \int_0^l q_n |u(q_n;t)|^2 \mathrm{d}x\mathrm{d}t \le \|\phi\|^2_{L^2(0,l)} \tag{2.17}$$

for any $t \in (0,T]$.

From (2.17), we know that the sequence $\{u(q_n)\}$ is uniformly bounded in the space $L^2((0,T); \mathscr{H}^1(0,l))$. So we may extract a subsequence, still denoted by $\{u(q_n)\}$, such that

$$u(q_n)(x,t) \rightharpoonup u^*(x,t) \in L^2((0,T); \mathscr{H}^1(0,l)). \tag{2.18}$$

Step 2 Prove $u^*(x,t) = u(q)(x,t)$.

By taking $q = q_n$ in (2.9) and multiplying both sides by a function $\eta(t) \in C^1[0,T]$ with $\eta(T) = 0$, we have

$$\int_0^l u(q_n)_t \psi \eta(t) \mathrm{d}x + \int_0^l a \nabla u(q_n) \cdot \nabla \psi \eta(t) \mathrm{d}x + \int_0^l q_n u(q_n) \psi \eta(t) \mathrm{d}x = 0. \tag{2.19}$$

Then integrating with respect to t, we get

$$\begin{aligned} -\int_0^l \phi \eta(0) \psi \mathrm{d}x = &-\int_0^T \int_0^l u(q_n) \psi \eta_t(t) \mathrm{d}x\mathrm{d}t + \int_0^T \int_0^l \eta(t) a \nabla u(q_n) \cdot \nabla \psi \mathrm{d}x\mathrm{d}t \\ &+ \int_0^T \int_0^l \eta(t) q(x) u(q_n) \psi \mathrm{d}x\mathrm{d}t + \int_0^T \int_0^l \eta(t)(q_n - q) u(q_n) \psi \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{2.20}$$

Letting $n \to \infty$ in (2.20) and using (2.18), we obtain

$$\begin{aligned} -\int_0^l \phi \eta(0) \psi \mathrm{d}x = &-\int_0^T \int_0^l u^* \psi \eta_t(t) \mathrm{d}x\mathrm{d}t + \int_0^T \int_0^l \eta(t) a \nabla u^* \cdot \nabla \psi \mathrm{d}x\mathrm{d}t \\ &+ \int_0^T \int_0^l \eta(t) q(x) u^* \psi \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{2.21}$$

By noticing that (2.21) is valid for any $\eta(t) \in C^1[0,T]$ with $\eta(T) = 0$, we have

$$\int_0^l u_t^* \psi \mathrm{d}x + \int_0^l a \nabla u^* \cdot \nabla \psi \mathrm{d}x + \int_0^l q u^* \psi \mathrm{d}x = 0, \quad \forall \psi \in \mathscr{H}^1(0,l) \tag{2.22}$$

and $u^*(x,0) = \phi(x)$.

Therefore, $u^* = u(q)$ by the definition of $u(q)$.

Step 3 Prove $\|u(q_n)(\cdot,T) - g\|_{L^2(0,l)} \to \|u(q)(\cdot,T) - g\|_{L^2(0,l)}$ as $n \to \infty$.

We rewrite (2.9) for $q = q_n$ in the form

$$\begin{aligned} &\int_0^l (u(q_n) - g)_t \psi \mathrm{d}x + \int_0^l a \nabla (u(q_n) - g) \cdot \nabla \psi \mathrm{d}x + \int_0^l q_n (u(q_n) - g) \psi \mathrm{d}x \\ = &-\int_0^l a \nabla g \cdot \nabla \psi \mathrm{d}x - \int_0^l q_n g \psi dx. \end{aligned} \tag{2.23}$$

Taking $\psi = u(q_n) - g$ in (2.23), we have

$$\begin{aligned} &\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|u(q_n) - g\|_{L^2(0,l)}^2 + \int_0^l a |\nabla (u(q_n) - g)|^2 \mathrm{d}x + \int_0^l q_n |u(q_n) - g|^2 \mathrm{d}x \\ = &-\int_0^l a \nabla g \cdot \nabla (u(q_n) - g) \mathrm{d}x - \int_0^l q_n g (u(q_n) - g) \mathrm{d}x. \end{aligned} \tag{2.24}$$

Similar relations hold for $u(q)$, namely,

$$\begin{aligned} &\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|u(q) - g\|_{L^2(0,l)}^2 + \int_0^l a |\nabla (u(q) - g)|^2 \mathrm{d}x + \int_0^l q |u(q) - g|^2 \mathrm{d}x \\ = &-\int_0^l a \nabla g \cdot \nabla (u(q) - g) \mathrm{d}x - \int_0^l q g (u(q) - g) \mathrm{d}x. \end{aligned} \tag{2.25}$$

Subtracting (2.25) from (2.24), we obtain

$$\begin{aligned}&\Big\{\int_0^l q_n|u(q_n)-g|^2\mathrm{d}x-\int_0^l q|u(q)-g|^2\mathrm{d}x\Big\}+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(q_n)-u(q)\|^2_{L^2(0,l)}\\ =&\int_0^l a\nabla g\cdot\nabla(u(q)-u(q_n))\mathrm{d}x+\int_0^l qg(u(q)-u(q_n))\mathrm{d}x\\ &+\int_0^l(q-q_n)g(u(q_n)-g)\mathrm{d}x+\int_0^l a\nabla(u(q)-u(q_n))\cdot\nabla(u(q)+u(q_n)-2g)\mathrm{d}x\\ &-\int_0^l\frac{\mathrm{d}}{\mathrm{d}t}[(u(q)-g)(u(q_n)-u(q))]\mathrm{d}x. \end{aligned}\tag{2.26}$$

Taking $\psi=u(q_n)-u(q)$ in (2.9), we have

$$\begin{aligned}&\int_0^l u(q)_t(u(q_n)-u(q))\mathrm{d}x\\ =&\int_0^l a\nabla u(q)\cdot\nabla(u(q)-u(q_n))\mathrm{d}x+\int_0^l qu(q)(u(q)-u(q_n))\mathrm{d}x.\end{aligned}\tag{2.27}$$

Similarly, for $(u(q_n)-u(q))_t(u(q)-g)$, we have

$$\begin{aligned}&\int_0^l(u(q_n)-u(q))_t(u(q)-g)\mathrm{d}x\\ =&\int_0^l a\nabla(u(q_n)-u(q))\cdot\nabla(g-u(q))\mathrm{d}x+\int_0^l q(u(q_n)-u(q))(g-u(q))\mathrm{d}x\\ &+\int_0^l(q_n-q)u(q_n)(g-u(q))\mathrm{d}x.\end{aligned}\tag{2.28}$$

Substituting (2.27)–(2.28) into (2.26), and after some manipulations, we derive

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(q_n)-u(q))\|^2_{L^2(0,l)}+\int_0^l a|\nabla(u(q_n)-u(q))|^2\mathrm{d}x\\ &+\Big\{\int_0^l q_n|u(q_n)-g|^2\mathrm{d}x-\int_0^l q|u(q)-g|^2\mathrm{d}x\Big\}\\ =&\,2\int_0^l q(u(q_n)-u(q))(u(q)-g)\mathrm{d}x+\int_0^l(q-q_n)g(u(q_n)-g)\mathrm{d}x\\ &+\int_0^l(q-q_n)u(q_n)(g-u(q))\mathrm{d}x:=A_n.\end{aligned}\tag{2.29}$$

Then by rewriting the third term on the left-hand side of (2.29), we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(q_n)-u(q)\|^2_{L^2(0,l)}+\int_0^l a|\nabla(u(q_n)-u(q)|^2\mathrm{d}x+\int_0^l q_n|u(q_n)-u(q)|^2\mathrm{d}x\\ =&\,A_n+\Big\{\int_0^l(q-q_n)|u(q)-g|^2\mathrm{d}x-2\int_0^l q_n(u(q_n)-u(q))(u(q)-g)\mathrm{d}x\Big\}\\ :=&\,A_n+B_n.\end{aligned}\tag{2.30}$$

Integrating over the interval $(0,t)$ for any $t\le T$, we get

$$\frac{1}{2}\|u(q_n;t)-u(q;t)\|^2_{L^2(0,l)}\le\int_0^T|A_n+B_n|\mathrm{d}t.\tag{2.31}$$

By the convergence of $\{q_n\}$ and the weak convergence of $\{u(q_n)\}$, one can easily get

$$\int_0^T |A_n + B_n| \mathrm{d}t \to 0 \quad \text{as } n \to \infty. \tag{2.32}$$

Combining (2.31) and (2.32), we have

$$\max_{t\in[0,T]} \|u(q_n;t) - u(q;t)\|_{L^2(0,l)} \to 0 \quad \text{as } n \to \infty. \tag{2.33}$$

On the other hand, from the Hölder's inequality, we have

$$\begin{aligned}
&\Big| \int_0^l |u(q_n)(\cdot,T) - g|^2 \mathrm{d}x - \int_0^l |u(q)(\cdot,T) - g|^2 \mathrm{d}x \Big| \\
&\le \int_0^l |u(q_n)(\cdot,T) - u(q)(\cdot,T)| \cdot |u(q_n)(\cdot,T) + u(q)(\cdot,T) - 2g| \mathrm{d}x \\
&\le \|u(q_n)(\cdot,T) - u(q)(\cdot,T)\|_{L^2(0,l)} \cdot \|u(q_n)(\cdot,T) + u(q)(\cdot,T) - 2g\|_{L^2(0,l)}.
\end{aligned} \tag{2.34}$$

From (2.15), (2.17) and (2.33)–(2.34), we obtain

$$\lim_{n\to\infty} \int_0^l |u(q_n)(x,T) - g(x)|^2 \mathrm{d}x = \int_0^l |u(q)(x,T) - g(x)|^2 \mathrm{d}x.$$

This completes the proof of Lemma 2.1.

Theorem 2.2 *There exists a minimizer $\overline{q} \in \mathcal{A}$ of $J(q)$, i.e.,*

$$J(\overline{q}) = \min_{q\in\mathcal{A}} J(q).$$

Proof It is obvious that $J(q)$ is non-negative, and thus $J(q)$ has the greatest lower bound $\inf\limits_{q\in\mathcal{A}} J(q)$. Let $\{q_n\}$ be a minimizing sequence, i.e.,

$$\inf_{q\in\mathcal{A}} J(q) \le J(q_n) \le \inf_{q\in\mathcal{A}} J(q) + \frac{1}{n}, \quad n = 1, 2, \cdots.$$

By noticing that $J(q_n) \le C$, we deduce

$$\|\nabla q_n\|_{L^2(0,l)} \le C, \tag{2.35}$$

where C is independent of n. Noticing the boundedness of $\{q_n\}$ and (2.35), we also have

$$\|q_n\|_{H^1(0,l)} \le C. \tag{2.36}$$

So we can extract a subsequence, still denoted by $\{q_n\}$, such that

$$q_n(x) \rightharpoonup \overline{q}(x) \in H^1(0,l) \quad \text{as } n \to \infty. \tag{2.37}$$

By the Sobolev imbedding theorem (see [1]), we obtain

$$\|q_n(x) - \overline{q}(x)\|_{L^1(0,l)} \to 0 \quad \text{as } n \to \infty. \tag{2.38}$$

It can be easily seen that $\{q_n(x)\} \in \mathcal{A}$. So we get as $n \to \infty$ that

$$q_n(x) \to \overline{q}(x) \in \mathcal{A} \tag{2.39}$$

in $L^1(0,l)$.

Moreover, from (2.37), we have

$$\int_0^l |\nabla \overline{q}|^2 \mathrm{d}x = \lim_{n\to\infty} \int_0^l \nabla q_n \cdot \nabla \overline{q} \mathrm{d}x \le \lim_{n\to\infty} \sqrt{\int_0^l |\nabla q_n|^2 \mathrm{d}x \cdot \int_0^l |\nabla \overline{q}|^2 \mathrm{d}x}. \tag{2.40}$$

From Lemma 2.1 and the convergence of $\{q_n\}$, we know that there exists a subsequence of $\{q_n\}$, still denoted by $\{q_n\}$, such that

$$\lim_{n\to\infty} \int_0^l |u(q_n)(x,T) - g(x)|^2 \mathrm{d}x = \int_0^l |u(\overline{q})(x,T) - g(x)|^2 \mathrm{d}x. \tag{2.41}$$

From (2.39)–(2.41), we get

$$\begin{aligned} J(\overline{q}) &= \lim_{n\to\infty} \int_0^l |u(q_n)(x,T) - g(x)|^2 \mathrm{d}x + \int_0^l |\nabla \overline{q}|^2 \mathrm{d}x \\ &\le \lim_{n\to\infty} J(q_n) = \inf_{q\in\mathcal{A}} J(q). \end{aligned} \tag{2.42}$$

Hence, $J(\overline{q}) = \min\limits_{q\in\mathcal{A}} J(q)$.

This completes the proof of Theorem 2.2.

3 Necessary Condition

Theorem 3.1 *Let q be the solution to the optimal control problem* (2.12). *Then there exists a triple of functions $(u, v; q)$ satisfying the following system:*

$$\begin{cases} u_t - (au_x)_x + qu = 0, & (x,t) \in Q, \\ u|_{t=0} = \phi(x), & x \in (0,l), \end{cases} \tag{3.1}$$

$$\begin{cases} -v_t - (av_x)_x + qv = 0, & (x,t) \in Q, \\ v|_{t=T} = u(x,T) - g(x), & x \in (0,l) \end{cases} \tag{3.2}$$

and

$$\int_0^T \int_0^l uv(q-h)\mathrm{d}x\mathrm{d}t - N\int_0^l \nabla q \cdot \nabla (q-h)\mathrm{d}x \ge 0 \tag{3.3}$$

for any $h \in \mathcal{A}$.

Proof For any $h \in \mathcal{A},\ 0 \le \delta \le 1$, we have

$$q_\delta \equiv (1-\delta)q + \delta h \in \mathcal{A}.$$

Then

$$J_\delta \equiv J(q_\delta) = \frac{1}{2}\int_0^l |u(x,T;q_\delta) - g(x)|^2 \mathrm{d}x + \frac{N}{2}\int_0^l |\nabla q_\delta|^2 \mathrm{d}x. \tag{3.4}$$

Let u_δ be the solution to (1.1) with given $q = q_\delta$. Since q is an optimal solution, we have

$$\frac{\mathrm{d}J_\delta}{\mathrm{d}\delta}\Big|_{\delta=0} = \int_0^l [u(x,T;q) - g(x)]\frac{\partial u_\delta}{\partial \delta}\Big|_{\delta=0}\mathrm{d}x + N\int_0^l \nabla q\cdot\nabla(h-q)\mathrm{d}x \geq 0. \tag{3.5}$$

Let $\widetilde{u}_\delta \equiv \frac{\partial u_\delta}{\partial \delta}$. Direct calculations lead to the following equation:

$$\begin{cases} \dfrac{\partial}{\partial t}(\widetilde{u}_\delta) - \dfrac{\partial}{\partial x}\Big(a\dfrac{\partial \widetilde{u}_\delta}{\partial x}\Big) + q_\delta \widetilde{u}_\delta = (q-h)u_\delta, \\ \widetilde{u}_\delta|_{t=0} = 0. \end{cases} \tag{3.6}$$

Let $\xi = \widetilde{u}_\delta|_{\delta=0}$. Then ξ satisfies

$$\begin{cases} \xi_t - (a\xi_x)_x + q\xi = (q-h)u, \\ \xi|_{t=0} = 0. \end{cases} \tag{3.7}$$

From (3.5), we have

$$\int_0^l [u(x,T;q) - g(x)]\xi(x,T)\mathrm{d}x + N\int_0^l \nabla q\cdot\nabla(h-q)\mathrm{d}x \geq 0. \tag{3.8}$$

Let $\mathcal{L}\xi = \xi_t - (a\xi_x)_x + q\xi$, and suppose that v is the solution to the following problem:

$$\begin{cases} \mathcal{L}^* v \equiv -v_t - (av_x)_x + qv = 0, \\ v(x,T) = u(x,T;q) - g(x), \end{cases} \tag{3.9}$$

where $\mathcal{L}^*$ is the adjoint operator of the operator $\mathcal{L}$.

By the well-known Green's formula, we have

$$\begin{aligned} &\int_0^T\int_0^l (v\mathcal{L}\xi - \xi\mathcal{L}^*v)\mathrm{d}x\mathrm{d}t \\ =& \int_0^T\int_0^l (v\xi_t + \xi v_t)\mathrm{d}x\mathrm{d}t + \int_0^T\int_0^l [\xi(av_x)_x - v(a\xi_x)_x]\mathrm{d}x\mathrm{d}t \\ =& \int_0^l \xi v|_{t=0}^{t=T}\mathrm{d}x + \int_0^T\int_0^l (a\xi v_x - av\xi_x)_x\mathrm{d}x\mathrm{d}t \\ =& \int_0^l \xi(x,T)[u(x,T) - g(x)]\mathrm{d}x, \end{aligned} \tag{3.10}$$

which implies

$$\int_0^T\int_0^l v\mathcal{L}\xi\mathrm{d}x\mathrm{d}t = \int_0^l \xi(x,T)[u(x,T) - g(x)]\mathrm{d}x. \tag{3.11}$$

Combining (3.8) and (3.11), one can easily obtain that

$$\int_0^T\int_0^l uv(q-h)\mathrm{d}x\mathrm{d}t - N\int_0^l \nabla q\cdot\nabla(q-h)\mathrm{d}x \geq 0.$$

This completes the proof of Theorem 3.1.

4 Uniqueness and Stability

The optimal control Problem 1.1 is non-convex. So, in general one may not expect a unique solution. In fact, it is well-known that the optimization technique is a classical tool to yield "general solution" for inverse problems without unique solution. However, we find that if the terminal time T is relatively small, the minimizer of the cost functional can be proved to be local unique and stable.

Throughout this paper, if there is no specific illustration, C will be denoted the different constants.

Lemma 4.1 *Supposing $u \in \mathscr{H}^1(0,l)$, we have that for any $k \geq 0$,*

$$(u-k)^+ = \sup(u-k, 0) \in \mathscr{H}^1,$$
$$(u+k)^- = \sup(-(u+k), 0) \in \mathscr{H}^1.$$

Moreover, for a.e. $x \in (0,l)$, we have

$$\frac{\partial (u-k)^+}{\partial x} = \begin{cases} \dfrac{\partial u}{\partial x}, & \text{if } u > k, \\ 0, & \text{if } u \leq k \end{cases}$$

and

$$\frac{\partial (u+k)^-}{\partial x} = \begin{cases} 0, & \text{if } u > -k, \\ -\dfrac{\partial u}{\partial x}, & \text{if } u \leq -k. \end{cases}$$

Proof For $u \in \mathscr{H}^1$, we know

$$\int_0^l a(|u|^2 + |\nabla u|^2)\mathrm{d}x < +\infty.$$

Noting $a(x) > 0,\ x \in (0,l)$, we have that for all $\delta > 0$,

$$u \in \mathscr{H}^1(\delta, l-\delta).$$

By the definition of weak derivative (see [38]), it can be easily seen that

$$(u-k)^+ \in \mathscr{H}^1(\delta, l-\delta)$$

and for a.e. $x \in (\delta, l-\delta)$,

$$\frac{\partial (u-k)^+}{\partial x} = \begin{cases} \dfrac{\partial u}{\partial x}, & \text{if } u > k, \\ 0, & \text{if } u \leq k. \end{cases}$$

Then we have

$$\int_\delta^{l-\delta} a|((u-k)^+)_x|^2\mathrm{d}x = \int_{E_\delta} a|u_x|^2\mathrm{d}x,$$

where $E_\delta = \{x \in (\delta, l-\delta) \mid u(x) > k\}$. Since the quantity $\int_{E_\delta} a|u_x|^2\mathrm{d}x$ is bounded from the above $\int_0^l a|u_x|^2\mathrm{d}x$, which does not depend on δ, by passing to the limit as $\delta \to 0$, we get

$$\int_0^l a|((u-k)^+)_x|^2\mathrm{d}x \leq \int_0^l a|u_x|^2\mathrm{d}x < +\infty.$$

Moreover, the following inequality

$$\int_0^l a|(u-k)^+|^2 \mathrm{d}x \le \int_0^l a|u|^2 \mathrm{d}x < +\infty$$

is obvious. Hence, $(u-k)^+ \in \mathscr{H}^1$. Similar arguments can be used to treat the case of $(u+k)^-$.

This completes the proof of Lemma 4.1.

Now, we can give a weak maximum principle for the weak solution to (1.1).

Lemma 4.2 *Supposing* $\phi \in L^\infty(0,l) \cap \mathscr{H}^1(0,l)$, *we have for* u *the following estimate:*

$$\|u\|_\infty \le \|\phi\|_\infty. \tag{4.1}$$

Proof Let $k = \|\phi\|_\infty$. Multiplying (1.1) by $(u-k)^+$, we get from Lemma 5.1 that

$$\int_0^l u_t(u-k)^+ \mathrm{d}x + \int_0^l a|((u-k)^+)_x|^2 \mathrm{d}x = -\int_0^l qu(u-k)^+ \mathrm{d}x. \tag{4.2}$$

Denoting $E = \{x \in (0,l) \mid u(x) > k\}$, one has

$$-\int_0^l qu(u-k)^+ \mathrm{d}x = -\int_E qu(u-k)^+ \mathrm{d}x \le 0. \tag{4.3}$$

From (4.2)–(4.3), we have that for all $t \in [0,T]$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^l |(u-k)^+|^2 \mathrm{d}x = \int_0^l u_t(u-k)^+ \mathrm{d}x \le 0,$$

which implies that $t \mapsto \|(u-k)^+(t)\|_{L^2}^2$ is decreasing on $[0,T]$. Since $(\phi-k)^+ \equiv 0$, we deduce that for all $t \in [0,T]$ and for a.e. $x \in (0,l)$, $u(x,t) \le k$.

By analogous arguments for $(u+k)^-$, we can obtain that for all $t \in [0,T]$ and for a.e. $x \in (0,l)$, $u(x,t) \ge -k$.

This completes the proof of Lemma 4.2.

Lemma 4.3 *For* (3.2), *we have the following estimate:*

$$\|v\|_\infty \le \|u(x,T) - g(x)\|_\infty. \tag{4.4}$$

Proof Let $\tau = T - t$. Then (3.2) is reduced to

$$\begin{cases} v_\tau - (av_x)_x + qv = 0, & (x,t) \in Q, \\ v|_{\tau=0} = u(x,T) - g(x). \end{cases}$$

The rest of the proof is similar to that of Lemma 4.2.

Suppose that $g_1(x)$ and $g_2(x)$ are two given functions which satisfy the condition (2.15). Let $q_1(x)$ and $q_2(x)$ be the minimizers of Problem 2.1 corresponding to $g = g_i$ $(i = 1,2)$, respectively, and let $\{u_i, v_i\}$ $(i = 1,2)$ be solutions to (3.1)–(3.2) in which $q = q_i$ $(i = 1,2)$, respectively.

Set

$$u_1 - u_2 = U, \quad v_1 - v_2 = V, \quad q_1 - q_2 = \mathcal{Q}.$$

Then U and V satisfy

$$\begin{cases} U_t - (aU_x)_x + q_1 U = -\mathcal{Q}u_2, \\ U|_{t=0} = 0, \end{cases} \tag{4.5}$$

$$\begin{cases} -V_t - (aV_x)_x + q_1 V = -\mathcal{Q}v_2, \\ V|_{t=T} = U(x,T) - (g_1 - g_2). \end{cases} \tag{4.6}$$

Lemma 4.4 *For any bounded continuous function $k(x) \in C(0,l)$, we have*

$$\|k\|_\infty \le |k(x_0)| + \sqrt{l}\,\|\nabla k\|_{L^2(0,l)},$$

where x_0 is a fixed point in $(0,l)$.

Proof For $0 < x < l$, we have

$$|k(x)| \le |k(x_0)| + \Big|\int_{x_0}^x k' \mathrm{d}x\Big| \le |k(x_0)| + \Big(\int_0^l \mathrm{d}x\Big)^{\frac{1}{2}} \Big(\int_0^l |\nabla k|^2 \mathrm{d}x\Big)^{\frac{1}{2}}.$$

This completes the proof of Lemma 4.4.

Lemma 4.5 *For* (4.5)*, we have the following estimate:*

$$\max_{0\le t\le T} \int_0^l U^2 \mathrm{d}x\mathrm{d}t \le C(\max|\mathcal{Q}|)^2 \int_0^T \int_0^l |u_2|^2 \mathrm{d}x\mathrm{d}t, \tag{4.7}$$

where C is independent of T.

Proof From (4.5), we have that for $0 < t \le T$,

$$\begin{aligned} &\int_0^l \int_0^t \Big(\frac{U^2}{2}\Big)_t \mathrm{d}x\mathrm{d}t - \int_0^t \int_0^l (aU_x)_x U \mathrm{d}x\mathrm{d}t + \int_0^t \int_0^l q_1 U^2 \mathrm{d}x\mathrm{d}t \\ = &-\int_0^t \int_0^l u_2 \mathcal{Q} U \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{4.8}$$

Integrating by parts, we obtain

$$\begin{aligned} &\int_0^l \frac{U^2}{2}\Big|_{(x,t)} \mathrm{d}x + \int_0^t \int_0^l aU_x^2 \mathrm{d}x\mathrm{d}t - \int_0^t aU_x U|_{x=0}^{x=l} \mathrm{d}t + \int_0^t \int_0^l q_1 U^2 \mathrm{d}x\mathrm{d}t \\ \le &\int_0^t \int_0^l U^2 \mathrm{d}x\mathrm{d}t + (\max|Q|)^2 \int_0^t \int_0^l |u_2|^2 \mathrm{d}x\mathrm{d}t, \end{aligned} \tag{4.9}$$

which implies

$$\begin{aligned} &\int_0^l \frac{U^2}{2}\Big|_{(x,t)} \mathrm{d}x + \int_0^t \int_0^l aU_x^2 \mathrm{d}x\mathrm{d}t \\ \le &\int_0^t \int_0^l U^2 \mathrm{d}x\mathrm{d}t + (\max|Q|)^2 \int_0^t \int_0^l |u_2|^2 \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{4.10}$$

From the Gronwall's inequality and (4.10), we have

$$\int_0^l U^2 \mathrm{d}x\mathrm{d}t + \int_0^T \int_0^l aU_x^2 \mathrm{d}x\mathrm{d}t \le C(\max|\mathcal{Q}|)^2 \int_0^T \int_0^l |u_2|^2 \mathrm{d}x\mathrm{d}t.$$

This completes the proof of Lemma 4.5.

Lemma 4.6 *For* (4.6), *we have the following estimate:*

$$\max_{0\le t\le T}\int_0^l V^2\mathrm{d}x+\int_0^T\int_0^l a|V_x|^2\mathrm{d}x\mathrm{d}t$$
$$\le C(\max|\mathcal{Q}|)^2\int_0^T\int_0^l(|u_2|^2+|v_2|^2)\mathrm{d}x\mathrm{d}t+C\int_0^l|g_1-g_2|^2\mathrm{d}x, \tag{4.11}$$

where C is independent of T.

Proof From (4.6), we have

$$\int_t^T\int_0^l-\Big(\frac{V^2}{2}\Big)_t\mathrm{d}x\mathrm{d}t-\int_t^T\int_0^l(aV_x)_xV\mathrm{d}x\mathrm{d}t+\int_t^T\int_0^l q_1V^2\mathrm{d}x\mathrm{d}t$$
$$=-\int_t^T\int_0^l v_2\mathcal{Q}V\mathrm{d}x\mathrm{d}t.$$

Integrating by parts, we obtain

$$\int_0^l\frac{V^2}{2}\Big|_{(x,t)}\mathrm{d}x+\int_t^T\int_0^l a|V_x|^2\mathrm{d}x\mathrm{d}t+\int_t^T\int_0^l q_1V^2\mathrm{d}x\mathrm{d}t$$
$$\le\int_0^l|U(x,T)|^2\mathrm{d}x+\int_0^l|g_1-g_2|^2\mathrm{d}x-\int_t^T\int_0^l v_2\mathcal{Q}V\mathrm{d}x\mathrm{d}t$$
$$\le\int_0^l|U(x,T)|^2\mathrm{d}x+\int_0^l|g_1-g_2|^2\mathrm{d}x+\int_t^T\int_0^l\frac{V^2}{2}\mathrm{d}x\mathrm{d}t$$
$$+\frac12(\max|\mathcal{Q}|)^2\int_t^T\int_0^l|v_2|^2\mathrm{d}x\mathrm{d}t. \tag{4.12}$$

From Lemma 4.5 and (4.12), we have

$$\int_0^l\frac{V^2}{2}\Big|_{(x,t)}\mathrm{d}x+\int_t^T\int_0^l a|V_x|^2\mathrm{d}x\mathrm{d}t$$
$$\le\int_t^T\int_0^l\frac{V^2}{2}\mathrm{d}x\mathrm{d}t+\int_0^l|g_1-g_2|^2\mathrm{d}x$$
$$+C(\max|\mathcal{Q}|)^2\int_0^T\int_0^l(|u_2|^2+|v_2|^2)\mathrm{d}x\mathrm{d}t. \tag{4.13}$$

From the Gronwall's inequality, we have

$$\max_{0\le t\le T}\int_0^l V^2\mathrm{d}x+\int_0^T\int_0^l a|V_x|^2\mathrm{d}x\mathrm{d}t$$
$$\le C(\max|\mathcal{Q}|)^2\int_0^T\int_0^l(|u_2|^2+|v_2|^2)\mathrm{d}x\mathrm{d}t+C\int_0^l|g_1-g_2|^2\mathrm{d}x.$$

This completes the proof of Lemma 4.6.

Theorem 4.1 *Let $q_1(x)$, $q_2(x)$ be the minimizers of the optimal control Problem* 2.1 *corresponding to $g_1(x)$, $g_2(x)$, respectively. If there exists a point $x_0\in(0,l)$, such that*

$$q_1(x_0)=q_2(x_0),$$

then for relatively small T, we have

$$\max_{x\in(0,l)}|q_1-q_2|\le\frac{Cl^{\frac{1}{3}}}{N^{\frac{1}{3}}}\|g_1-g_2\|_{L^2(0,l)},$$

where the constant C is independent of T, l and N.

Proof By taking $h=q_2$ when $q=q_1$, and taking $h=q_1$ when $q=q_2$ in (3.3), we have

$$\int_0^T\int_0^l(q_1-q_2)u_1v_1\mathrm{d}x\mathrm{d}t-N\int_0^l\nabla q_1\cdot\nabla(q_1-q_2)\mathrm{d}x\ge 0, \tag{4.14}$$

$$\int_0^T\int_0^l(q_2-q_1)u_2v_2\mathrm{d}x\mathrm{d}t-N\int_0^l\nabla q_2\cdot\nabla(q_2-q_1)\mathrm{d}x\ge 0, \tag{4.15}$$

where $\{u_i,v_i\}$ $(i=1,2)$ are solutions to (3.1)–(3.2) with $q=q_i$ $(i=1,2)$, respectively.

From (4.14)–(4.15), we have

$$\begin{aligned}N\int_0^l|\nabla(q_1-q_2)|^2\mathrm{d}x&\le\int_0^T\int_0^l(u_1v_1-u_2v_2)(q_1-q_2)\mathrm{d}x\mathrm{d}t\\&\le\int_0^T\int_0^l(u_1v_1-u_2v_1+u_2v_1-u_2v_2)(q_1-q_2)\mathrm{d}x\mathrm{d}t\\&\le\int_0^T\int_0^l\mathcal{Q}v_1U\mathrm{d}x\mathrm{d}t+\int_0^T\int_0^l\mathcal{Q}u_2V\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{4.16}$$

From the assumption of Theorem 4.1, there exists a point $x_0\in(0,l)$, such that

$$\mathcal{Q}(x_0)=q_1(x_0)-q_2(x_0)=0. \tag{4.17}$$

From Lemma 4.4 and (4.17), we have

$$\max_{x\in(0,l)}|\mathcal{Q}(x)|\le\sqrt{l}\Big(\int_0^l|\nabla\mathcal{Q}|^2\mathrm{d}x\Big)^{\frac{1}{2}}. \tag{4.18}$$

From (4.16), (4.18) and the Young's inequality, we obtain that

$$\begin{aligned}\max|\mathcal{Q}|^2&\le l\int_0^l|\nabla\mathcal{Q}|^2\mathrm{d}x\\&\le\frac{l}{N}\int_0^T\int_0^l\mathcal{Q}(Uv_1+Vu_2)\mathrm{d}x\mathrm{d}t\\&\le\frac{1}{2l}\int_0^l|\mathcal{Q}|^2\mathrm{d}x+\frac{Tl^2}{2N^2}\int_0^T\int_0^l|Uv_1+Vu_2|^2\mathrm{d}x\mathrm{d}t\\&\le\frac{1}{2}\max|\mathcal{Q}|^2+\frac{Tl^2}{N^2}\|v_1\|_\infty^2\int_0^T\int_0^lU^2\mathrm{d}x\mathrm{d}t+\frac{Tl^2}{N^2}\|u_2\|_\infty^2\int_0^T\int_0^lV^2\mathrm{d}x\mathrm{d}t\\&\le\frac{1}{2}\max|\mathcal{Q}|^2+C\frac{T^2l^2}{N^2}\|v_1\|_\infty^2\cdot\Big(\int_0^T\int_0^l|u_2|^2\mathrm{d}x\mathrm{d}t\Big)\cdot\max|\mathcal{Q}|^2\\&\quad+C\frac{T^2l^2}{N^2}\|u_2\|_\infty^2\cdot\Big(\int_0^T\int_0^l(|u_2|^2+|v_2|^2)\mathrm{d}x\mathrm{d}t\Big)\cdot\max|\mathcal{Q}|^2\\&\quad+C\frac{T^2l^2}{N^2}\int_0^l|g_1-g_2|^2\mathrm{d}x,\end{aligned}\tag{4.19}$$

where we have used estimates (4.7) and (4.11).

From Lemmas 4.2–4.3, we have

$$\|v_1\|_\infty,\ \|v_2\|_\infty,\ \|u_2\|_\infty \le C. \tag{4.20}$$

From (4.19)–(4.20), we have

$$\max|\mathcal{Q}|^2 \le C\frac{T^3l^2}{N^2}\max|\mathcal{Q}|^2 + C\frac{T^2l^2}{N^2}\int_0^l |g_1-g_2|^2\mathrm{d}x. \tag{4.21}$$

Choose $T \ll 1$, such that

$$C\frac{T^3l^2}{N^2} = \frac{1}{2}. \tag{4.22}$$

Combining (4.21) and (4.22), one can easily get

$$\max_{x\in(0,l)}|q_1-q_2| \le \frac{Cl^{\frac{1}{3}}}{N^{\frac{1}{3}}}\|g_1-g_2\|_{L^2(0,l)}. \tag{4.23}$$

This completes the proof of Theorem 4.1.

Remark 4.1 It should be mentioned that the regularization parameter plays a major role in the numerical simulation of ill-posed problems. From Theorem 4.1, we can obtain that if there exists a constant δ, such that

$$\|g_1-g_2\| \le \delta \quad \text{and} \quad \frac{\delta^2}{N^{\frac{2}{3}}} \to 0,$$

then the reconstructed optimal solution is unique and stable, which is consistent with the existed results (see, e.g., [20]). Note that the estimate (4.23) is based on (4.22), from which we can see $T = O(N^{\frac{2}{3}})$. Since the parameter N is often taken to be very small, particularly in numerical computations, Theorem 4.1 is indeed the local well-posedness of the optimal solution. For more detailed discussion on the regularization parameter, we refer the readers to the references (see, e.g., [9, 20]).

5 Convergence Analysis

In this section, we would like to discuss the convergence of the optimal solution. It has been shown in previous section that the optimal solution is stable and unique, which is very important in numerical process. However, the optimization problem is just a "modified problem" rather than the original one. Therefore, it is necessary to investigate what about the difference between the optimal solution to the optimization problem and the exact solution to the original problem.

We assume that the "real solution" $g(x)$ is attainable, i.e., there exists a $q^*(x) \in H^1(0,l)$, such that

$$u(x,T;q^*) = g(x), \tag{5.1}$$

and that an upper bound δ for the noisy level

$$\|g^\delta - g\|_{L^2(0,l)} \le \delta \tag{5.2}$$

of the observation is known a priori.

It should be mentioned that for terminal control problems, it is rather difficult to derive the convergence. To the authors' knowledge, there is no convergence result for the optimal control problem with the cost functional whose form is similar to (2.13).

In this paper, we introduce the following auxiliary control problems with observations averaged over the given terminal time interval $[T-\sigma, T]$:

$$J_\sigma(q) = \frac{1}{2\sigma}\int_{T-\sigma}^{T}\int_0^l |u(x,t;q) - g(x)|^2 \mathrm{d}x\mathrm{d}t + \frac{N}{2}\int_0^l |\nabla q|^2 \mathrm{d}x. \tag{5.3}$$

Note that as $\sigma \to 0^+$,

$$\frac{1}{2\sigma}\int_{T-\sigma}^{T}\int_0^l |u(x,t;q) - g(x)|^2 \mathrm{d}x\mathrm{d}t \to \int_0^l \frac{1}{2}|u(x,T;q) - g(x)|^2 \mathrm{d}x,$$

which implies $J_\sigma(q) \to J(q)$. Analogously, instead of (5.2), we assume that for the real solution $q^*(x)$, we have

$$\frac{1}{2\sigma}\int_{T-\sigma}^{T}\int_0^l |u(x,t;q^*) - g^\delta(x)|^2 \mathrm{d}x\mathrm{d}t \le \frac{1}{2}\delta^2. \tag{5.4}$$

Define the following forward operator $u(q)$:

$$\begin{aligned} u(q):\quad & \mathcal{A} \to H^1((0,T);L^2(0,l)) \cap L^2((0,T);\mathscr{H}^1(0,l)),\\ & u(q)(x,t) = u(x,t;q(x)), \end{aligned}$$

where $u(x,t;q(x))$ is the solution to the variational problem (2.9) for $q \in \mathcal{A}$. For any $q \in \mathcal{A}$ and $p \in H^1(0,l)$, one can easily deduce that the Gâteaux directional differential $u'(q)p$ satisfies a homogeneous initial condition and solves

$$\int_0^l (u'(q)p)_t \varphi \mathrm{d}x + \int_0^l a\nabla(u'(q)p)\cdot\nabla\varphi \mathrm{d}x + \int_0^l q u'(q)p\varphi \mathrm{d}x = -\int_0^l p u(q)\varphi \mathrm{d}x \tag{5.5}$$

for any $\varphi \in L^2(0,l) \cap \mathscr{H}^1(0,l)$. For the remainder term $R(q) = u(p+q) - u(q) - u'(q)p$, we have the following variational characterization.

Lemma 5.1 *For any $q \in \mathcal{A}$ and $p \in H^1(0,l)$, such that $p + q \in \mathcal{A}$, the remainder $R(q) = u(p+q) - u(q) - u'(q)p$ solves*

$$\int_0^l (R(q))_t \varphi \mathrm{d}x + \int_0^l a\nabla(R(q))\cdot\nabla\varphi \mathrm{d}x + \int_0^l qR(q)\varphi \mathrm{d}x = \int_0^l p\varphi(u(q) - u(q+p))\mathrm{d}x \tag{5.6}$$

for any $\varphi \in L^2(0,l) \cap \mathscr{H}^1(0,l)$.

Proof Note that $u(q+p)$ satisfies

$$\int_0^l (u(q+p))_t \varphi \mathrm{d}x + \int_0^l a\nabla(u(q+p))\cdot\nabla\varphi \mathrm{d}x + \int_0^l (q+p)u(q+p)\varphi \mathrm{d}x = 0. \tag{5.7}$$

Subtracting (5.7) from (2.9) and denoting $W = u(q+p) - u(q)$, we obtain

$$\int_0^l \varphi W_t \mathrm{d}x + \int_0^l a\nabla W\cdot\nabla\varphi \mathrm{d}x + \int_0^l qW\varphi \mathrm{d}x = -\int_0^l p u(q+p)\varphi \mathrm{d}x. \tag{5.8}$$

Now (5.6) follows by subtracting (5.5) from (5.8).

This completes the proof of Lemma 5.1.

To obtain the convergence, we shall require some source conditions. We introduce the following linear operator $F(q)$:

$$\begin{aligned} F(q): \quad & L^2((0,T);L^2(0,l)) \to L^2(0,l) \\ & F(q)\Phi = -\frac{1}{\sigma}\int_{T-\sigma}^{T} u(q)\Phi \mathrm{d}t, \quad \forall \Phi \in L^2((0,T);L^2(0,l)), \end{aligned} \tag{5.9}$$

where $u(q)$ is the solution to (2.9). Using (5.5), we immediately see that for any $p \in H^1(0,l)$ and any $\varphi \in L^2(0,l) \cap \mathscr{H}^1(0,l)$, the following holds:

$$\begin{aligned} \langle F(q)\varphi,\ p\rangle &= -\frac{1}{\sigma}\int_{T-\sigma}^{T}\int_0^l pu(q)\varphi \mathrm{d}x\mathrm{d}t \\ &= \frac{1}{\sigma}\int_{T-\sigma}^{T}\int_0^l [(u'(q)p)_t\varphi + a\nabla(u'(q)p)\cdot\nabla\varphi + qu'(q)p\varphi]\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{5.10}$$

where $\langle\cdot\,,\,\cdot\rangle$ denote the scalar product in $L^2(0,l)$. Since ∇ is a linear operator, we can define its adjoint operator ∇^* by

$$\langle \nabla^*\omega,\ \varphi\rangle_{L^2(0,l)} = \langle \omega,\ \nabla\varphi\rangle_{L^2(0,l)}, \quad \forall \omega \in H^1(0,l),\ \varphi \in H^1(0,l). \tag{5.11}$$

It can be easily seen that if $\varphi \in H_0^1(0,l)$, then ∇^* is equivalent to ∇. In this paper, we will only need a weak form of $\nabla^*\nabla$.

Theorem 5.1 *Assume that there exists a function*

$$\varphi \in H_0^1((T-\sigma,T);L^2(0,l)) \cap L^2((T-\sigma,T);\mathscr{H}^1(0,l)),$$

such that the following source condition holds in the weak sense:

$$F(q^*)\varphi = \nabla^*\nabla q^* \tag{5.12}$$

with $F(q^*)$ *defined by* (5.9)*, i.e., for any* $p \in H^1(0,l)$*,*

$$\langle F(q^*)\varphi,\ p\rangle = \langle \nabla^*\nabla q^*,\ p\rangle = \langle \nabla q^*,\ \nabla p\rangle. \tag{5.13}$$

Furthermore, assume that

$$\nabla\cdot(a\nabla\varphi) \in L^2((T-\sigma,T);L^2(0,l)), \tag{5.14}$$

and q_N^δ *satisfies*

$$q_N^\delta(0) = q^*(0), \quad q_N^\delta(l) = q^*(l). \tag{5.15}$$

Then, with $N \sim \delta$*, we have*

$$\int_0^l |q_N^\delta - q^*|^2 \mathrm{d}x \le C\delta \tag{5.16}$$

and

$$\frac{1}{\sigma}\int_{T-\sigma}^{T}\int_0^l |u(q_N^\delta)-u(q^*)|^2\mathrm{d}x\mathrm{d}t \le C\delta^2, \tag{5.17}$$

where q_N^δ *is a minimizer of* (5.3) *with* g *replaced by* g^δ, $u(q_N^\delta)$ *is the solution to the variational problem* (2.9) *with* $q = q_N^\delta$, *and* C *is a positive constant independent of* δ, N *and* T.

Proof Noting that q_N^δ is a minimizer of (5.3), we have

$$J_\sigma(q_N^\delta) \le J_\sigma(q^*),$$

which implies

$$\frac{1}{2\sigma}\int_{T-\sigma}^{T}\int_0^l |u(q_N^\delta)-g^\delta|^2\mathrm{d}x\mathrm{d}t + \frac{N}{2}\int_0^l |\nabla q_N^\delta|^2\mathrm{d}x \le \frac{1}{2}\delta^2 + \frac{N}{2}\int_0^l |\nabla q^*|^2\mathrm{d}x. \tag{5.18}$$

From (5.18), we can derive

$$\begin{aligned}
&\frac{1}{2\sigma}\int_{T-\sigma}^{T}\int_0^l |u(q_N^\delta)-g^\delta|^2\mathrm{d}x\mathrm{d}t + \frac{N}{2}\int_0^l |\nabla q_N^\delta - \nabla q^*|^2\mathrm{d}x \\
&\le \frac{1}{2}\delta^2 + \frac{N}{2}\int_0^l |\nabla q^*|^2\mathrm{d}x - \frac{N}{2}\int_0^l |\nabla q_N^\delta|^2\mathrm{d}x + \frac{N}{2}\int_0^l |\nabla q_N^\delta - \nabla q^*|^2\mathrm{d}x \\
&= \frac{1}{2}\delta^2 + N\int_0^l \nabla q^* \cdot \nabla(q^* - q_N^\delta)\mathrm{d}x \\
&= \frac{1}{2}\delta^2 + N\langle \nabla q^*,\ \nabla(q^* - q_N^\delta)\rangle.
\end{aligned} \tag{5.19}$$

Using (5.10) and (5.13), we have for the last term in (5.19) that

$$\begin{aligned}
\langle \nabla q^*,\ \nabla(q^* - q_N^\delta)\rangle &= \langle F(q^*)\varphi,\ q^* - q_N^\delta\rangle \\
&= -\frac{1}{\sigma}\int_{T-\sigma}^{T}\int_0^l (q^* - q_N^\delta)u(q^*)\varphi\mathrm{d}x\mathrm{d}t \\
&= \frac{1}{\sigma}\int_{T-\sigma}^{T}\int_0^l [(u'(q^*)(q^* - q_N^\delta))_t\varphi + a\nabla(u'(q^*)(q^* - q_N^\delta))\cdot\nabla\varphi \\
&\quad + q^*u'(q^*)(q^* - q_N^\delta)\varphi]\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{5.20}$$

Let

$$R_N^\delta := u(q_N^\delta) - u(q^*) - u'(q^*)(q_N^\delta - q^*). \tag{5.21}$$

Using this notation, we obtain

$$\begin{aligned}
&N\langle \nabla q^*,\ \nabla(q^* - q_N^\delta)\rangle \\
&= \frac{N}{\sigma}\int_{T-\sigma}^{T}\int_0^l [(R_N^\delta)_t\varphi + a\nabla R_N^\delta\cdot\nabla\varphi + q^*R_N^\delta\varphi]\mathrm{d}x\mathrm{d}t \\
&\quad - \frac{N}{\sigma}\int_{T-\sigma}^{T}\int_0^l [u(q_N^\delta) - u(q^*)]_t\varphi\mathrm{d}x\mathrm{d}t - \frac{N}{\sigma}\int_{T-\sigma}^{T}\int_0^l a\nabla(u(q_N^\delta) - u(q^*))\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&-\frac{N}{\sigma}\int_{T-\sigma}^{T}\int_0^l q^*[u(q_N^\delta)-u(q^*)]\varphi \mathrm{d}x\mathrm{d}t\\
&=\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3+\mathrm{I}_4. \qquad (5.22)
\end{aligned}
$$

Now, we need to estimate I_1–I_4. The main idea is to control I_1–I_4 by the left-hand side item of inequality (5.19).

For I_1, we use (5.6) to get

$$
\mathrm{I}_1=\frac{N}{\sigma}\int_{T-\sigma}^{T}\int_0^l (q_N^\delta-q^*)[u(q^*)-u(q_N^\delta)]\varphi \mathrm{d}x\mathrm{d}t. \qquad (5.23)
$$

From (5.23) and the Hölder's inequality, we have

$$
\begin{aligned}
|\mathrm{I}_1|&\le \frac{N}{\sigma}\int_{T-\sigma}^{T}\int_0^l |(q_N^\delta-q^*)\varphi|\cdot|u(q^*)-g^\delta|\mathrm{d}x\mathrm{d}t\\
&\quad+\frac{N}{\sigma}\int_{T-\sigma}^{T}\int_0^l |(q_N^\delta-q^*)\varphi|\cdot|g^\delta-u(q_N^\delta)|\mathrm{d}x\mathrm{d}t\\
&\le \frac{N}{\sigma}\int_{T-\sigma}^{T}\|(q_N^\delta-q^*)\varphi\|_{L^2(0,l)}\cdot\|u(q^*)-g^\delta\|_{L^2(0,l)}\mathrm{d}t\\
&\quad+\frac{N}{\sigma}\int_{T-\sigma}^{T}\|(q_N^\delta-q^*)\varphi\|_{L^2(0,l)}\cdot\|g^\delta-u(q_N^\delta)\|_{L^2(0,l)}\mathrm{d}t. \qquad (5.24)
\end{aligned}
$$

Using (2.14) and the Young's inequality, we obtain

$$
\begin{aligned}
|\mathrm{I}_1|&\le \frac{1}{8\sigma}\int_{T-\sigma}^{T}\int_0^l |u(q^*)-g^\delta|^2\mathrm{d}x\mathrm{d}t+CN^2\int_{T-\sigma}^{T}\int_0^l |(q_N^\delta-q^*)\varphi|^2\mathrm{d}x\mathrm{d}t\\
&\quad+\frac{1}{16\sigma}\int_{T-\sigma}^{T}\int_0^l |g^\delta-u(q_N^\delta)|^2\mathrm{d}x\mathrm{d}t+CN^2\int_{T-\sigma}^{T}\int_0^l |(q_N^\delta-q^*)\varphi|^2\mathrm{d}x\mathrm{d}t\\
&\le \frac{1}{8}\delta^2+\frac{1}{16\sigma}\int_{T-\sigma}^{T}\int_0^l |g^\delta-u(q_N^\delta)|^2\mathrm{d}x\mathrm{d}t+CN^2\int_{T-\sigma}^{T}\int_0^l |\varphi|^2\mathrm{d}x\mathrm{d}t, \qquad (5.25)
\end{aligned}
$$

where we have used the assumption (5.4).

For I_2, using integration by parts with respect to t and noticing $\varphi\in H_0^1((T-\sigma,T);L^2(0,l))$, we derive

$$
\begin{aligned}
|\mathrm{I}_2|&=\frac{N}{\sigma}\Big|\int_{T-\sigma}^{T}\int_0^l (u(q_N^\delta)-u(q^*))\varphi_t\mathrm{d}x\mathrm{d}t\Big|\\
&\le \frac{N}{\sigma}\int_{T-\sigma}^{T}\int_0^l |(u(q_N^\delta)-u(q^*))\varphi_t|\mathrm{d}x\mathrm{d}t\\
&\le \frac{N}{\sigma}\int_{T-\sigma}^{T}\int_0^l |(u(q_N^\delta)-g^\delta)\varphi_t|\mathrm{d}x\mathrm{d}t+\frac{N}{\sigma}\int_{T-\sigma}^{T}\int_0^l |(g^\delta-u(q^*))\varphi_t|\mathrm{d}x\mathrm{d}t\\
&\le \frac{1}{8}\delta^2+\frac{1}{16\sigma}\int_{T-\sigma}^{T}\int_0^l |g^\delta-u(q_N^\delta)|^2\mathrm{d}x\mathrm{d}t+CN^2\int_{T-\sigma}^{T}\int_0^l |\varphi_t|^2\mathrm{d}x\mathrm{d}t. \qquad (5.26)
\end{aligned}
$$

For I_3, using integration by parts with respect to x and noticing $a(0)=a(l)=0$, we obtain

$$
|\mathrm{I}_3|=\frac{N}{\sigma}\Big|\int_{T-\sigma}^{T}\int_0^l a\nabla(u(q_N^\delta)-u(q^*))\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t\Big|
$$

$$\begin{aligned}&= \frac{N}{\sigma}\Big|\int_{T-\sigma}^{T}\Big\{a(x)(u(q_N^\delta)-u(q^*))\frac{\mathrm{d}\varphi}{\mathrm{d}x}\Big|_{x=0}^{x=l}\\&\quad -\int_0^l (u(q_N^\delta)-u(q^*))\nabla\cdot(a\nabla\varphi)\mathrm{d}x\Big\}\mathrm{d}t\Big|\\&\le \frac{N}{\sigma}\int_{T-\sigma}^{T}\int_0^l |u(q_N^\delta)-u(q^*)|\cdot|\nabla\cdot(a\nabla\varphi)|\mathrm{d}x\mathrm{d}t\\&\le \frac{N}{\sigma}\int_{T-\sigma}^{T}\int_0^l |u(q_N^\delta)-g^\delta|\cdot|\nabla\cdot(a\nabla\varphi)|\mathrm{d}x\mathrm{d}t\\&\quad +\frac{N}{\sigma}\int_{T-\sigma}^{T}\int_0^l |g^\delta-u(q^*)|\cdot|\nabla\cdot(a\nabla\varphi)|\mathrm{d}x\mathrm{d}t\\&\le \frac{1}{8}\delta^2+\frac{1}{16\sigma}\int_{T-\sigma}^{T}\int_0^l |u(q_N^\delta)-g^\delta|^2\mathrm{d}x\mathrm{d}t\\&\quad + CN^2\int_{T-\sigma}^{T}\int_0^l |\nabla\cdot(a\nabla\varphi)|^2\mathrm{d}x\mathrm{d}t. \end{aligned}\tag{5.27}$$

The last term I_4 can be estimated similarly as follows by using the Young's inequality:

$$\begin{aligned}|\mathrm{I}_4| &\le \frac{N}{\sigma}\int_{T-\sigma}^{T}\int_0^l |q^*||u(q_N^\delta)-g^\delta||\varphi|\mathrm{d}x\mathrm{d}t\\&\quad +\frac{N}{\sigma}\int_{T-\sigma}^{T}\int_0^l |q^*||g^\delta-u(q^*)||\varphi|\mathrm{d}x\mathrm{d}t\\&\le \frac{1}{8}\delta^2+\frac{1}{16\sigma}\int_{T-\sigma}^{T}\int_0^l |u(q_N^\delta)-g^\delta|^2\mathrm{d}x\mathrm{d}t + CN^2\int_{T-\sigma}^{T}\int_0^l |\varphi|^2\mathrm{d}x\mathrm{d}t,\end{aligned}\tag{5.28}$$

where we have used the bound of q^*.

Combining (5.19), (5.22) and (5.25)–(5.28), we obtain

$$\begin{aligned}&\frac{1}{2\sigma}\int_{T-\sigma}^{T}\int_0^l |u(q_N^\delta)-g^\delta|^2\mathrm{d}x\mathrm{d}t+\frac{N}{2}\int_0^l |\nabla q_N^\delta-\nabla q^*|^2\mathrm{d}x\\&\le \frac{1}{2}\delta^2+N\langle\nabla q^*,\ \nabla(q^*-q_N^\delta)\rangle\\&\le \frac{1}{2}\delta^2+\sum_{j=1}^{4}|\mathrm{I}_j|\\&\le \delta lta^2+\frac{1}{4\sigma}\int_{T-\sigma}^{T}\int_0^l |u(q_N^\delta)-g^\delta|^2\mathrm{d}x\mathrm{d}t\\&\quad + CN^2\int_{T-\sigma}^{T}\int_0^l (|\varphi|^2+|\varphi_t|^2+|\nabla\cdot(a\nabla\varphi)|^2)\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{5.29}$$

From (5.29) and noticing the regularity of φ, we have

$$\frac{1}{4\sigma}\int_{T-\sigma}^{T}\int_0^l |u(q_N^\delta)-g^\delta|^2\mathrm{d}x\mathrm{d}t+\frac{N}{2}\int_0^l |\nabla q_N^\delta-\nabla q^*|^2\mathrm{d}x \le \delta^2+CN^2. \tag{5.30}$$

By choosing $N\sim\delta$, one can easily get

$$\frac{1}{\sigma}\int_{T-\sigma}^{T}\int_0^l |u(q_N^\delta)-g^\delta|^2\mathrm{d}x\mathrm{d}t+N\int_0^l |\nabla q_N^\delta-\nabla q^*|^2\mathrm{d}x \le C\delta^2. \tag{5.31}$$

The estimate (5.16) follows immediately from (5.31) and the Poincaré's inequality.

This completes the proof of Theorem 5.1.

Remark 5.1 The motivation of replacing the cost functional (2.13) by (5.3) mainly lies in the difficulty in treating the second integration term in (5.22). In fact, if we choose the functional form (2.13), then we can deduce the second term in (5.22) (denoted by $\widetilde{\mathrm{I}}_2$) to be

$$\widetilde{\mathrm{I}}_2 = -\frac{N}{\sigma}\int_0^l (u(q_N^\delta) - u(q^*))_t(\cdot, T)\varphi \mathrm{d}x.$$

Since we have no information regarding to the t-derivative of the real and approximate solution, it is quite difficult, even impossible, to control the term $\widetilde{\mathrm{I}}_2$ by the left-hand side of (5.19), and thus we can not obtain any convergence.

6 Concluding Remarks

The inverse problem of identifying the coefficient in parabolic equations from some extra conditions is very important in some engineering texts and many industrial applications. Classical parabolic models are plentifully discussed and developed well, while documents dealt with degenerate parabolic models are quite few.

In this paper, we solve the inverse Problem 1.1 of recovering the radiative coefficient $q(x)$ in the following degenerate parabolic equation:

$$u_t - (au_x)_x + q(x)u = 0$$

in an optimal control framework. Being different from other works (see, e.g., [24, 29]), which also treat with inverse radiative coefficient problems, the mathematical model discussed in this paper contains degeneracy on the lateral boundaries. Furthermore, unlike the well-known Black-Scholes equation whose degeneracy can be removed by some change of variable, the degeneracy in our problem can not be removed by any method. On the basis of the optimal control framework, the existence, the uniqueness, the stability and the convergence of the minimizer for the cost functional are established.

This paper focuses on the theoretical analysis of the 1-D inverse problem. For the multi-dimensional case, i.e., the determination of $q(x)$ in the following equation:

$$u_t - \nabla\cdot(a(x)\nabla u) + q(x)u = 0, \quad (x,t)\in Q = \Omega\times(0,T],$$

where the principle coefficient $a(x)$ satisfies

$$a(x) \ge 0, \quad x\in\overline{\Omega}$$

and $\Omega\subset\mathbb{R}^m$ $(m\ge 1)$ is a given bounded domain, the method proposed in this paper is also applicable.

Acknowledgements The authors would like to thank professor Jin Cheng of School of Mathematical Sciences at Fudan University for his valuable comments and helpful suggestions to improve the earlier version of the paper.

References

[1] Adams, R. A., Sobolev Spaces, Academic Press, New York, 1975.

[2] Cannarsa, P., Tort, J. and Yamamoto, M., Determination of source terms in a degenerate parabolic equation, *Inverse Problems*, **26**, 2010, 105003.

[3] Cannarsa, P., Martinez, P. and Vancostenoble, J., Carleman estimates for a class of degenerate parabolic operators, *SIAM J. Control Optim.*, **47**, 2008, 1–19.

[4] Cannarsa, P., Martinez, P. and Vancostenoble, J., Null controllability of degenerate heat equations, *Adv. Differ. Equ.*, **10**, 2005, 153–190.

[5] Cannarsa, P., Martinez, P. and Vancostenoble, J., Persistent regional null controllability for a class of degenerate parabolic equations, *Commun. Pure Appl. Anal.*, **3**, 2004, 607–635.

[6] Cannon, J. R., Lin, Y. and Xu, S., Numerical procedure for the determination of an unknown coefficient in semilinear parabolic partial differential equations, *Inverse Problems*, **10**, 1994, 227–243.

[7] Cannon, J. R. and Lin, Y., An inverse problem of finding a parameter in a semilinear heat equation, *J. Math. Anal. Appl.*, **145**, 1990, 470–484.

[8] Chen, Q. and Liu, J. J., Solving an inverse parabolic problem by optimization from final measurement data, *J. Comput. Appl. Math.*, **193**, 2006, 183–203.

[9] Cheng, J. and Yamamoto, M., One new strategy for a priori choice of regularizing parameters in Tikhonov's regularization, *Inverse problems*, **16**(4), 2000, 31–36.

[10] Choulli, M. and Yamamoto, M., Generic well-posedness of an inverse parabolic problem — the Hölder space approach, *Inverse Problems*, **12**(3), 1996, 195–205.

[11] Choulli, M. and Yamamoto, M., An inverse parabolic problem with non-zero initial condition, *Inverse Problems*, **13**, 1997, 19–27.

[12] Dehghan, M., Identification of a time-dependent coefficient in a partial differential equation subject to an extra measurement, *Numer. Meth. Part. Diff. Equ.*, **21**, 2005, 611–622.

[13] Dehghan, M., Determination of a control function in three-dimensional parabolic equations, *Math. Comput. Simul.*, **61**, 2003, 89–100.

[14] Demir, A. and Hasanov A., Identification of the unkonwn diffusion coefficient in a linear parabolic equation by the semigroup approach, *J. Math. Anal. Appl.*, **340**, 2008, 5–15.

[15] Deng, Z. C., Yu, J. N. and Yang L., An inverse problem of determining the implied volatility in option pricing, *J. Math. Anal. Appl.*, **340**(1), 2008, 16–31.

[16] Deng, Z. C., Yu, J. N. and Yang L., Optimization method for an evolutional type inverse heat conduction problem, *J. Phys. A: Math. Theor.*, **41**, 2008, 035201.

[17] Deng, Z. C., Yu, J. N. and Yang L., Identifying the coefficient of first-order in parabolic equation from final measurement data, *Math. Comput. Simul.*, **77**, 2008, 421–435.

[18] Deng, Z. C. and Yang, L., An inverse problem of identifying the radiative coefficient in a degenerate parabolic equation, 2013. arXiv: 1309.7421

[19] Egger, H. and Engl, H. W., Tikhonov regularization applied to the inverse problem of option pricing: convergence analysis and rates, *Inverse Problems*, **21**, 2005, 1027–1045.

[20] Engl, H. W., Hanke, M. and Neubauer, A., Regularization of Inverse Problems, Kluwer Academic Publishers, Dordrecht, 1996.

[21] Engl, H. W. and Zou, J., A new approach to convergence rate analysis of Tikhonov regularization for parameter identification in heat conduction, *Inverse Problems*, **16**, 2000, 1907–1923.

[22] Imanuvilov, O. Y. and Yamamoto, M., Lipschitz stability in inverse parabolic problems by the Carleman estimates, *Inverse Problems*, **14**, 1998, 1229–1245.

[23] Isakov, V., Inverse Problems for Partial Differential Equations, Springer-Verlag, New York, 1998.

[24] Isakov, V. and Kindermann, S., Identification of the diffusion coefficient in a one-dimensional parabolic equation, *Inverse Problems*, **16**, 2000, 665–680.

[25] Jiang, L. S. and Tao, Y. S., Identifying the volatility of underlying assets from option prices, *Inverse Problems*, **17**, 2001, 137–155.

[26] Jiang, L. S., Chen, Q. H., Wang, L. J. and Zhang, J. E., A new well-posed algorithm to recover implied local volatility, *Quantitative Finance*, **3**, 2003, 451–457.

[27] Jiang, L. S. and Bian, B. J., Identifying the principal coefficient of parabolic equations with non-divergent form, *Journal of Physics: Conference Series*, **12**, 2005, 58–65.

[28] Kaltenbacher, B. and Klibanov, M. V., An inverse problem for a nonlinear parabolic equation with applications in population dynamics and magnetics, *SIAM J. Math. Anal.*, **39**(6), 2008, 1863–1889.

[29] Keung, Y. L. and Zou, J., Numerical identifications of parameters in parabolic systems, *Inverse Problems*, **14**, 1998, 83–100.

[30] Kirsch. A., An Introduction to the Mathematical Theory of Inverse Problem, Springer-Verlag, New York, 1999.

[31] Ladyzenskaya, O., Solonnikov, V. and Ural'Ceva, N., Linear and Quasilinear Equations of Parabolic Type, Vol. **23**, A. M. S., Providence, RI, 1968.

[32] Liu, J. J., Regularization Method and Application for the Ill-posed Problem, Science Press, Beijing, 2005.

[33] Oleǐnik, O. A. and Radkevič, E. V., Second Order Differential Equations with Non-negative Characteristic Form, A. M. S., Rhode Island and Plenum Press, New York, 1973.

[34] Rundell, W., The determination of a parabolic equation from initial and final data, *Proc. Am. Math. Soc.*, **99**, 1987, 637–642.

[35] Samarskii, A. A. and Vabishchevich, P. N., Numerical Methods for Solving Inverse Problems of Mathematical Physics, Walter de Gruyter GmbH & Co. KG, Berlin, 2007.

[36] Tikhonov, A. and Arsenin, V., Solutions of Ill-posed Problems, Geology Press, Beijing, 1979.

[37] Tort, J. and Vancostenoble, J., Determination of the insolation function in the nonlinear Sellers climate model, *Ann. I. H. Poincaré-AN*, **29**, 2012, 683–713.

[38] Wu, Z. Q., Yin, J. X. and Wang, C. P., Elliptic and Parabolic Equations, Science Press, Beijing, 2003.

[39] Yang, L., Yu, J. N. and Deng, Z. C., An inverse problem of identifying the coefficient of parabolic equation, *Appl. Math. Model.*, **32**(10), 2008, 1984–1995.