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Local Stability for an Inverse Coefficient Problem of a Fractional Diffusion Equation*

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Abstract Time-fractional diffusion equations are of great interest and importance on describing the power law decay for diffusion in porous media. In this paper, to identify the diffusion rate, i.e., the heterogeneity of medium, the authors consider an inverse coefficient problem utilizing finite measurements and obtain a local Hölder type conditional stability based upon two Carleman estimates for the corresponding differential equations of integer orders.

Keywords Carleman estimate, Conditional stability, Inverse coefficient problem, Fractional diffusion equation
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1 Introduction

Nowadays time-fractional diffusion equations are of practical interest and importance, since they describe the power law decay for the diffusion in porous media perfectly. For instance, we refer to Bisquert [1], Hatano [5] and Hilfer [6] where a concrete physical experiment is designed to study the decay behavior of free-carrier density in a semiconductor with an exponential distribution of traps, and the decay of ion-recombination isothermal luminescence. In mathematics, forward problems of time-fractional diffusion equations are well studied and considerable results have been obtained both theoretically and numerically, e.g., [2, 7, 9, 12–13] and references cited therein. Moreover, some inverse problems which arise in fractional diffusion equations are also of great interest and attract much attention, e.g. [4, 8, 10–11, 14, 16–17]. For example, Cheng et al [4] established a uniqueness result in determining the fractional order and the coefficient in the principal part simultaneously based upon the Gel'fand-Levitan theory. Xu et al [14] derived a Carleman estimate for a fractional diffusion equation with half order and obtained a Hölder type conditional stability for the Cauchy problem. Zhang et al [17] investigated an inverse source problem for a fractional diffusion equation and proved the uniqueness of the inverse problem by analytic continuation and Laplace transform. Yamamoto et al [16] applied the Carleman estimate in [14] to obtain conditional stability of identifying a lower order coefficient in a fractional diffusion equation from some additional data through the study of an inverse source problem.

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Throughout this paper, we consider the following fractional diffusion equation with half order

$$\begin{cases} \partial_t^{\frac{1}{2}} u(x,t) = \partial_x(p(x)\partial_x u(x,t)), & (x,t) \in Q, \\ u(x,0) = 0, & x \in \Omega, \\ u(0,t) = h_{u,1}(t), \quad \dfrac{\partial u}{\partial x}(0,t) = h_{u,2}(t), & t \in (0,T), \end{cases} \tag{1.1}$$

where $\Omega = (0,l)$, $Q = (0,l) \times (0,T)$ and $\partial_t^{\frac{1}{2}} u$ denotes the fractional Caputo derivative in time of order $\frac{1}{2}$, which is defined by

$$\frac{\mathrm{d}^\gamma}{\mathrm{d}t^\gamma} f(t) := \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{\mathrm{d}f(s)}{\mathrm{d}s} \frac{\mathrm{d}s}{(t-s)^\gamma}, \quad 0 < \gamma < 1.$$

Let $0 < t_0 < T$ be fixed arbitrarily. We discuss the following inverse coefficient problem.

Problem 1.1 Can we estimate the leading order coefficient $p(x)$, $x \in \Omega$ from a final observation $u(x,t_0)$ and Cauchy data $h_{u,1}(t)$, $h_{u,2}(t)$?

The positive answer is given in Cheng et al [4] where the authors proved that Cauchy data could uniquely determine $p(x)$ by utilizing the Gel'fand-Levitan theory. However there is no more stability indication which can be extracted from their proof. Therefore, this paper is aiming at establishing a local Hölder stability for this problem. The main idea is deriving a Carleman estimate with variable coefficients which is similar as the estimate with constant coefficients in Xu [14] and the methodology utilized in Yamamoto [15] for typical parabolic equations.

The outline of this paper is as follows. In Section 2, we reformulate the inverse coefficient problem by converting the governing equation into a fourth order equation and then present the main result. Section 3 is devoted to prove the main theorem in two steps which involves two kinds of Carleman estimates, respectively. Some concluding remarks are given in Section 4 to close the paper.

2 Formulation and the Main Theorem

In this section, we introduce some notations and present the main result, i.e., a local Hölder conditional stability for Problem 1.1.

Throughout this paper, we denote by $u(p)$ the solution of the initial-boundary value problem (1.1) and by $v(q)$ the solution of a similar problem as follows:

$$\begin{cases} \partial_t^{\frac{1}{2}} v(x,t) = \partial_x(q(x)\partial_x v(x,t)), & (x,t) \in Q, \\ v(x,0) = 0, & x \in \Omega, \\ v(0,t) = h_{v,1}(t), \quad \dfrac{\partial v}{\partial x}(0,t) = h_{v,2}(t), & t \in (0,T). \end{cases} \tag{2.1}$$

For simplicity we restrict on some special complete function spaces. Denote by $H^\alpha(\Omega)$ the normal Sobolev space with order $\alpha > 0$, i.e.,

$$H^\alpha(\Omega) := \left\{ u(x) : \sum_{j \le \alpha} \|D^j u\|_{L^2(\Omega)} < \infty \right\}$$

and

$$\begin{aligned}
&C^{4,3}(\overline{Q}) = C^2([\delta_0,T];C^4(\overline{\Omega}))\cap C^3([\delta_0,T]:C^2(\overline{\Omega})) \quad \text{for } \delta_0\in[0,T],\\
&C_0^{4,3}(Q)=\{u\in C^{4,3}(\overline{Q}):\mathrm{supp}u\subset Q\},\\
&H^{4,3}(Q)=H^2(\delta_0,T;H^4(\Omega)\cap H_0^1(\Omega))\cap H^3(\delta_0,T;H^2(\Omega)\cap H_0^1(\Omega)),\\
&H_0^{4,3}(Q)=\{u\in H^{4,3}(Q):\partial_x^k\partial_t^l u(\cdot,\delta_0)=\partial_x^k\partial_t^l u(\cdot,T)=0,\ k,l=0,1,2\}.
\end{aligned}$$

Moreover, to accurately describe a local stability, we need a sub-domain which is generally characterized by the weight function defined in (2.2) in Carleman estimate. Let $d(x)\in C^4(\overline{\Omega})$ and $\mu(x)=\partial_x d\neq 0$ on $\overline{\Omega}$ and set

$$\psi(x,t)=d(x)-\beta(t-t_0)^2,\quad \varphi(x,t)=\mathrm{e}^{\lambda\psi(x,t)},\tag{2.2}$$

where $t_0\in(0,T)$, $\beta>0$. Denote by Q_ε the local sub-domain defined as follows:

$$Q_\varepsilon=\{(x,t):\psi(x,t)>\varepsilon\},\quad \Omega_\varepsilon=Q_\varepsilon\cap\{t=t_0\}.\tag{2.3}$$

By choosing proper $\beta>0$ and $d(x)>0$ such that

$$\begin{aligned}
&\|d(x)\|_{C[0,l]}<l,\quad \beta>\max\Big\{\frac{\|d(x)\|_{C[0,l]}}{t_0^2},\frac{4\|d(x)\|_{C[0,l]}}{T^2}\Big\},\\
&\max\Big\{\sup_{(x,t)\in Q_0}\psi(x,t),\sup_{x\in\Omega_\varepsilon}\psi(x,t_0)\Big\}\le\Phi,
\end{aligned}$$

where $\Phi>0$ is a constant, we can easily verify that $Q_\varepsilon\subset Q_0\subset Q$ and $\Omega_\varepsilon\subset\Omega_0\subset\Omega$.

Now we proceed to the main result, i.e., the local conditional Hölder stability while determining the leading order coefficient $p(x)$.

Theorem 2.1 *Suppose that $u(p),v(q)\in H^{4,3}(Q)$ are solutions to (1.1) and (2.1), respectively, with positive $p(x),q(x)\in H^3(\Omega)$ and $\partial_x^j p(0)=\partial_x^j q(0)$ for $j=0,1,2$. Assume $\partial_x v(x,t_0)>0$ for $x\in\Omega$, $h_{u,1}(t),h_{u,2}(t),h_{v,1}(t),h_{v,2}(t)\in H^4(0,T)$ and*

$$F=\sum_{k=1}^{2}\|h_{u,k}(t)-h_{v,k}(t)\|_{H^3(0,T)},\quad \widetilde{F}=\|u(\cdot,t_0)-v(\cdot,t_0)\|^2_{H^4(\Omega\varepsilon)}+F.$$

Then there exists $\varepsilon_0>0$, such that for any $\varepsilon<\varepsilon_0$, we have $\theta\in(0,1)$ and constant $C(\varepsilon,\sigma_0)$ such that

$$\|p(x)-q(x)\|_{H^3(\Omega_{3\varepsilon})}\le C(\widetilde{F}+M^{1-\theta}\widetilde{F}^\theta).\tag{2.4}$$

Here $M=\|u(\cdot)\|_{H^{4,3}(Q)}$ is considered as an a priori given bound for solutions of fractional diffusion equations with the leading order coefficients under consideration. Estimation (2.4) is, in general, impossible without such a priori condition on solutions, and hence it is so-called a conditional stability estimate.

Remark 2.1 To well understand the theorem, we give some following remarks.

(1) For the forward problem (1.1), the regularity of the solution $u\in H^{4,3}(Q)$ can be achieved if the coefficient $p(x)$ and Cauchy data $h_{u,1}(t),h_{u,2}(t)$ are smooth enough, and satisfy sufficient compatibility conditions

$$h_{u,1}(0)=0,\quad h_{u,1}(t_0)=u(0,t_0),\quad h_{u,2}(t_0)=\partial_x u(0,t_0).$$

(2) It should be noticed that the assumption on the final observation, i.e., $\partial_x v(x,t_0)>0$ is also reasonable. Denoting $z=\frac{\partial v}{\partial x}(x,t)$ and taking derivatives with respect to x on both sides of (2.1), we have

$$\partial_t^{\frac{1}{2}} z(x,t)=\partial_x(q(x)\partial_x z(x,t))+\partial_x q(x)\partial_x z(x,t)+\partial_x^2 q(x)z(x,t).$$

If we add a positive boundary condition at $x=l$, i.e.,

$$z(l,t)=\frac{\partial v}{\partial x}(l,t)=\widetilde{h_{v,2}}(t)>0$$

for $0<t<T$. Moreover, let $z(0,t)=h_{v,2}(t)>0$. Since $v(x,0)=0$ which implies

$$z(x,0)=\frac{\partial v}{\partial x}(x,0)=0,$$

the maximum principle for fractional equations (see [9]) implies that the solution $z(x,t)>0$ for $t>0$, i.e., $\partial_x v(x,t_0)>0$.

(3) The θ is complicated and the exact formulation will come out in the following section (see (3.23)).

(4) The sub-domain $\Omega_{3\varepsilon}$ in (2.4) can not be replaced by Ω due to the cut-off function utilized in the Carleman estimate. Hence the estimate in (2.4) is so-called a local stability.

(5) The argument can be extended to be more general, i.e., for any sub-domain $\omega\subset\subset\Omega$, there exists $\varepsilon>0$, such that $\omega\subset\Omega_{3\varepsilon}$ and hence

$$\|p(x)-q(x)\|_{H^3(\omega)}\le\|p(x)-q(x)\|_{H^3(\Omega_{3\varepsilon})}\le C(\widetilde{F}+M^{1-\theta}\widetilde{F}^{\theta}).$$

3 Proof of the Main Theorem

In this section, we aim at the detailed proof of Theorem 2.1. The methodology relies on a classical Carleman estimate for the parabolic equations, where its applications on various inverse coefficient problems are reviewed in [15, Section 6].

To start with, we denote

$$y:=y(x,t)=u(x,t)-v(x,t),\quad f:=f(x)=p(x)-q(x)$$

in Q. Subtracting (2.1) from (1.1) yields the following system:

$$\begin{cases}\partial_t^{\frac{1}{2}} y=\partial_x(p\partial_x y)+\partial_x(f\partial_x v), & (x,t)\in Q,\\ y(x,0)=0, & x\in\Omega,\\ y(0,t)=h_1(t),\quad \dfrac{\partial y}{\partial x}(0,t)=h_2(t), & t\in(0,T),\end{cases}\tag{3.1}$$

where $h_1(t)=h_{u,1}(t)-h_{v,1}(t),\ h_2(t)=h_{u,2}(t)-h_{v,2}(t)$.

Notice that $y(x,0)=u(x,0)-v(x,0)=0$ and $v(x,0)=0$. We obtain

$$\partial_t^{\frac{1}{2}} y(x,0)=\partial_x(p\partial_x y(x,0))+\partial_x(f\partial_x v(x,0))=0.$$

It allows us to recall a technical lemma in [14].

Lemma 3.1 (see [14]) *Let $AC([a,b])$ be the space of absolutely continuous functions on $[a,b]$. Assume that $y \in AC([a,b])$ and satisfies*

$$y(a) = \partial^\gamma y(a) = 0,$$

then the following equality holds:

$$\partial^\alpha \partial^\gamma y = \partial^{\alpha+\gamma} y,$$

where $0 < \alpha, \gamma < 1$ and $0 < \alpha + \gamma \le 1$.

Direct calculation with Lemma 3.1 yields

$$\begin{aligned}
\partial_t y &= \partial_t^{\frac{1}{2}}(\partial_t^{\frac{1}{2}} y) = \partial_t^{\frac{1}{2}}(\partial_x(p\partial_x y) + \partial_x(f\partial_x v)) \\
&= \partial_x(p\partial_x \partial_t^{\frac{1}{2}} y) + \partial_x(f\partial_x \partial_t^{\frac{1}{2}} v) \\
&= \partial_x(p\partial_x(\partial_x(p\partial_x y) + \partial_x(f\partial_x v))) + \partial_x(f\partial_x(\partial_x(q\partial_x v))) \\
&= p^2\partial_x^4 y + 4p\partial_x p\partial_x^3 y + (3(\partial_x^2 p)p + 2(\partial_x p)^2)\partial_x^2 y + ((\partial_x^3 p)p + (\partial_x^2 p)(\partial_x p))\partial_x y \\
&\quad + p\partial_x v\partial_x^3 f + (3p\partial_x^2 v + \partial_x p\partial_x v)\partial_x^2 f \\
&\quad + ((3p+q)\partial_x^3 v + 2(\partial_x p + \partial_x q)\partial_x^2 v + \partial_x^2 q\partial_x v)\partial_x f \\
&\quad + ((p+q)\partial_x^4 v + (\partial_x p + 3\partial_x q)\partial_x^3 v + 3\partial_x^2 q\partial_x^2 v + \partial_x^3 q\partial_x v)f
\end{aligned} \tag{3.2}$$

with $y(x,0) = 0$ and $y(0,t) = h_1(t)$, $\frac{\partial y}{\partial x}(0,t) = h_2(t)$.

Fix a certain time $t = t_0$, (3.2) can be considered as a third order differential equation with respect to $f(x)$, for instance, let

$$a(x) = u(x,t_0) - v(x,t_0) = y(x,t_0), \quad b(x) = v(x,t_0), \quad x \in \Omega.$$

The equation (3.2) can be reduced into the following one:

$$\begin{aligned}
P_0 \circ f :\triangleq\ & p\partial_x b\partial_x^3 f + (3p\partial_x^2 b + \partial_x p\partial_x b)\partial_x^2 f \\
& + ((3p+q)\partial_x^3 b + 2(\partial_x p + \partial_x q)\partial_x^2 b + \partial_x^2 q\partial_x b)\partial_x f \\
& + ((p+q)\partial_x^4 b + (\partial_x p + 3\partial_x q)\partial_x^3 b + 3\partial_x^2 q\partial_x^2 b + \partial_x^3 q\partial_x b)f \\
=\ & \partial_t y(x,t_0) - p^2\partial_x^4 a - 4p\partial_x p\partial_x^3 a - (3p\partial_x^2 p + 2(\partial_x p)^2)\partial_x^2 a \\
& - (p\partial_x^3 p + \partial_x^2 p\partial_x p)\partial_x a.
\end{aligned} \tag{3.3}$$

For sake of simplicity, we set the following differential operators for the coefficients in (3.3) such that

$$\begin{cases}
\mathcal{C}_3(x) = p\partial_x, \\
\mathcal{C}_2(x) = 3p\partial_x^2 + \partial_x p\partial_x, \\
\mathcal{C}_1(x) = (3p+q)\partial_x^3 + 2(\partial_x p + \partial_x q)\partial_x^2 + \partial_x^2 q\partial_x, \\
\mathcal{C}_0(x) = (p+q)\partial_x^4 + (\partial_x p + 3\partial_x q)\partial_x^3 + 3\partial_x^2 q\partial_x^2 + \partial_x^3 q\partial_x.
\end{cases}$$

Thus we firstly establish the Carleman estimate with respect to f in some appropriate subdomain of Ω for the new equation (3.3).

Notice the a priori assumption on the regularity of p, q and u, v (consequently a, b). We observe that the coefficient $q\partial_x b$ near the leading term $\partial_x^3 f$ in (3.3) satisfies $q\partial_x b \in H^3(\Omega)$ while the rest coefficients near lower order terms are bounded. The Carleman estimate then is carried out for a general third order differential operator $\mathcal{L}$ such that

$$\mathcal{L}g = L_3(x)\partial_x^3 g + L_2(x)\partial_x^2 g + L_1(x)\partial_x g + L_0(x)g, \tag{3.4}$$

where $L_3(x) \in H^3(\Omega)$, $L_j(x) \in L^\infty(\Omega)$ with $j = 0, 1, 2$. In addition, recall the following notation of weight functions:

$$\varphi(x, t_0) = \mathrm{e}^{\lambda\psi(x,t_0)}, \quad \psi(x, t_0) = d(x), \quad \mu(x) = \partial_x d(x).$$

We establish a technical lemma for the Carleman estimate in the form of (3.4).

Lemma 3.2 *Suppose that* (3.4) *holds true with* $g(x) \in C_0^\infty(\Omega)$ *and* $\operatorname{supp} g \subset D \subset \Omega$. *Moreover, assume*

$$\mu(x) \neq 0, \quad L_3(x) \neq 0, \quad \forall x \in D,$$

and we can choose some $s_1 > 0$ *and constant* $C > 0$ *such that*

$$\int_D (|\partial_x^3 g|^2 + s^2|\partial_x^2 g|^2 + s^3|\partial_x g|^2 + s^4|g|^2)\mathrm{e}^{2s\varphi(x,t_0)}\mathrm{d}x \le C\int_D |\mathcal{L}g|^2\mathrm{e}^{2s\varphi(x,t_0)}\mathrm{d}x \tag{3.5}$$

for all $s \ge s_1$.

Proof The proof contains three steps.

Step 1 We first prove the Carleman estimate for a first-order differential operator $\mathcal{P}g = \partial_x g$. Set

$$w = \mathrm{e}^{s\varphi}g, \quad Pw = \mathrm{e}^{s\varphi}\partial_x g = \mathrm{e}^{s\varphi}\partial_x(\mathrm{e}^{-s\varphi}w) = -sw\partial_x\varphi + \partial_x w.$$

Then

$$\|Pw\|_{L^2(D)}^2 = \|-sw\partial_x\varphi\|_{L^2(D)}^2 + \|\partial_x w\|_{L^2(D)}^2 + 2\langle -sw\partial_x\varphi, \partial_x w\rangle_D.$$

Notice

$$\langle -sw\partial_x\varphi, \partial_x w\rangle_D = \frac{s}{2}\int_D |w|^2\partial_x^2\varphi\mathrm{d}x$$

and

$$\partial_x^2\varphi = \lambda^2\mathrm{e}^{\lambda\psi}(\partial_x\psi)^2 + \lambda\mathrm{e}^{\lambda\psi}\partial_x^2\psi. \tag{3.6}$$

By the assumption $\psi \in C^4(\overline{\Omega})$ and $\partial_x\psi \neq 0$, one can choose λ sufficiently large such that $\partial_x^2\varphi > 0$. We then obtain

$$\int_D \big(|\partial_x g|^2 + s^2|g|^2\big)\mathrm{e}^{2s\varphi}\mathrm{d}x \le C\int_D |\partial_x g|^2\mathrm{e}^{2s\varphi}\mathrm{d}x \tag{3.7}$$

and

$$\int_D \big(|\partial_x^2 g|^2 + s^2|\partial_x g|^2\big)\mathrm{e}^{2s\varphi}\mathrm{d}x \le C\int_D |\partial_x^2 g|^2\mathrm{e}^{2s\varphi}\mathrm{d}x \tag{3.8}$$

by applying the same claims to $\partial_x^2 g$.

Step 2 Now, we proceed to the principle term in (3.4) such that $\mathcal{L}_0 g = L_3(x)\partial_x^3 g$ with $L_3(x) \in H^3(\Omega) \hookrightarrow C^2(\overline{\Omega})$. Set $\widetilde{g} = \partial_x^2 g$, similar to previous step, we let $\widetilde{w} = \mathrm{e}^{s\varphi}\widetilde{g}$ and

$$\widetilde{P}\widetilde{w} = \mathrm{e}^{s\varphi}L_3(x)\partial_x\widetilde{g} = \mathrm{e}^{s\varphi}L_3(x)\partial_x(\mathrm{e}^{-s\varphi}\widetilde{w}) = -sL_3(x)\widetilde{w}\partial_x\varphi + L_3(x)\partial_x\widetilde{w}.$$

Standard calculation yields

$$\|\widetilde{P}\widetilde{w}\|^2_{L^2(D)} = \| - sL_3(x)\widetilde{w}\partial_x\varphi\|^2_{L^2(D)} + \|L_3(x)\partial_x\widetilde{w}\|^2_{L^2(D)} + 2\langle - sL_3(x)\widetilde{w}\partial_x\varphi, L_3(x)\partial_x\widetilde{w}\rangle_D$$

and

$$\langle - sL_3(x)\widetilde{w}\partial_x\varphi, L_3(x)\partial_x\widetilde{w}\rangle_D = \frac{s}{2}\Big(\int_D (L_3(x))^2\widetilde{w}^2\partial_x^2\varphi \mathrm{d}x + 2\int_D L_3(x)\partial_x L_3(x)\widetilde{w}^2\partial_x\varphi \mathrm{d}x\Big).$$

Notice $\partial_x\varphi = \lambda \mathrm{e}^{\lambda\psi}\partial_x\psi$ and (3.6). We thus conclude that, for sufficiently large λ, there exists

$$\int_D (|\partial_x^3 g|^2 + s^2|\partial_x^2 g|^2)\mathrm{e}^{2s\varphi}\mathrm{d}x \le C\int_D |\mathcal{L}_0 g|^2\mathrm{e}^{2s\varphi}\mathrm{d}x. \tag{3.9}$$

Consequently, choosing sufficiently large $s > 0$, we calculate (3.7) $\times s^2+$ (3.8) $\times s+$ (3.9) and obtain

$$\int_D (|\partial_x^3 g|^2 + s^2|\partial_x^2 g|^2 + s^3|\partial_x g|^2 + s^4|g|^2)\mathrm{e}^{2s\varphi}\mathrm{d}x \le C\int_D |\mathcal{L}_0 g|^2\mathrm{e}^{2s\varphi}\mathrm{d}x.$$

Step 3 Finally, noticing the fact that

$$|\mathcal{L}_0 g|^2 \le 2(|\mathcal{L}g|^2 + |L_2(x)\partial_x^2 g(x) + L_1(x)\partial_x g(x) + L_0(x)g(x)|^2),$$

one can prove (3.5) easily. Since $L_i(x) \in L^\infty(\Omega)$, $L_2(x)\partial_x^2 g(x) + L_1(x)\partial_x g(x) + L_0(x)g(x)$ can be absorbed by the left-hand side of (3.5) with sufficiently large parameter s.

We note that the order of the weight function with respect to s in (3.5) is different from a classic Carleman estimate which is derived directly for $L_3(x)\partial_x^3 g$. However it does not make too much difference towards the main result in Theorem 2.1.

Now we proceed to estimate $f(x)$ in (3.3). Since a compact support is necessary to apply the Carleman estimate in Lemma 3.2, we introduce a cut-off function. Without loss of generality, let $\chi(x,t)$ be a $C_0^\infty(Q)$ function satisfying $0 \le \chi \le 1$, and

$$\chi(x) = \begin{cases} 1, & x \in Q_{2\varepsilon}, \\ 0, & x \in Q\backslash Q_\varepsilon, \end{cases} \tag{3.10}$$

where Q_ε and $Q_{2\varepsilon}$ are defined, respectively, as in (2.3). Noticing

$$\Omega_\epsilon = Q_\epsilon \cap \{t = t_0\}$$

and substituting χf into the left-hand side of (3.3), we derive

$$\begin{aligned} &P_0 \circ (\chi f) \\ &= \chi P_0 \circ f + \underbrace{3\mathcal{C}_3 b\partial_x\chi\partial_x^2 f + (3\mathcal{C}_3 b\partial_x^2\chi + 2\mathcal{C}_2 b\partial_x\chi)\partial_x f + (\mathcal{C}_3 b\partial_x^3\chi + \mathcal{C}_2 b\partial_x^2\chi + \mathcal{C}_1 b\partial_x\chi)f}_{g_3(\partial_x^3\chi,\partial_x^2\chi,\partial_x\chi,\partial_x^2 f,\partial_x f,f,\partial_x p,p,\partial_x^2 q,\partial_x q,q,\partial_x^3 b,\partial_x^2 b,\partial_x b)}. \end{aligned} \tag{3.11}$$

In light of the definition on χ, we observe that g_3 vanishes in $\Omega_{2\varepsilon}$ and only survives in $\Omega_\varepsilon \setminus \Omega_{2\varepsilon}$. Moreover, the regularity of p, q, b in Theorem 2.1 and the Sobolev embedding theorem yield

$$|g_3(\partial_x^3\chi, \partial_x^2\chi, \partial_x\chi, \partial_x^2 f, \partial_x f, f, \partial_x p, p, \partial_x^2 q, \partial_x q, q, \partial_x^3 b, \partial_x^2 b, \partial_x b)| \le CM$$

with $x \in \Omega_\varepsilon \setminus \Omega_{2\varepsilon}$ and the constant M in Theorem 2.1. We thus apply the Carleman estimate in Lemma 3.2 to $P_0 \circ (\chi f)$ in (3.11) and obtain

$$\begin{aligned}
&\int_{\Omega_\varepsilon} (|\partial_x^3(\chi f)| + s^2|\partial_x^2(\chi f)| + s^3|\partial_x(\chi f)| + s^4|(\chi f)|^2)\mathrm{e}^{2s\varphi(x,t_0)}\mathrm{d}x \\
&\le C\int_{\Omega_\varepsilon} (\chi P_0 \circ f)^2 \mathrm{e}^{2s\varphi(x,t_0)}\mathrm{d}x + C\mathrm{e}^{2s\exp(2\lambda\varepsilon)}\int_{\Omega_\varepsilon\setminus\Omega_{2\varepsilon}} |g_3|^2\mathrm{d}x \\
&\le C\int_{\Omega_\varepsilon} \chi^2|\partial_t y(x,t_0) - p^2\partial_x^4 a - 4p\cdot\partial_x p\partial_x^3 a - 3(\partial_x^2 p\cdot p + 2(\partial_x p)^2)\partial_x^2 a \\
&\quad - (\partial_x^3 p\cdot p + \partial_x^2 p\cdot\partial_x p)\partial_x a|^2 \mathrm{e}^{2s\varphi(x,t_0)}\mathrm{d}x + C\mathrm{e}^{2s\exp(2\lambda\varepsilon)}\int_{\Omega_\varepsilon\setminus\Omega_{2\varepsilon}} |g_3|^2\mathrm{d}x \\
&\le C\int_{\Omega_\varepsilon} |\partial_t y(x,t_0)|^2\mathrm{e}^{2s\varphi(x,t_0)}\mathrm{d}x + C\mathrm{e}^{2s\exp(\lambda\Phi)}\|a\|^2_{H^4(\Omega_\varepsilon)} + C\mathrm{e}^{2s\exp(2\lambda\varepsilon)}M^2 \qquad (3.12)
\end{aligned}$$

for all $s \ge s_1$.

Now, a further estimate upon $\int_{\Omega_\varepsilon} |\partial_t y(x,t_0)|^2\mathrm{e}^{2s\varphi(x,t_0)}\mathrm{d}x$ in the equation (3.12) is necessary to complete the full estimate on χf. To this end, we turn to the Carleman estimate which is derived in [14]. However, the local Carleman estimate there is established only with a first order time derivative term, whereas in our situation, the Carleman estimate on a trace $t = t_0$ is required which means that a high order term $\partial_t^2 y(x,t)$ (see (3.13) for details) is necessarily included in the Carleman estimate. Therefore, to fit the current situation, we established another differential equation for the 2nd order time derivative $\partial_t^2 y(x,t)$.

Before embarking on the estimate on $\int_{\Omega_\varepsilon} |\partial_t y(x,t_0)|^2\mathrm{e}^{2s\varphi(x,t_0)}\mathrm{d}x$, we present the second Carleman estimate for (1.1) in Lemma 3.3. The derivation is lengthy and different from Xu et al [14], for the sake of compactness of the proof structure, we give the detailed proof in Appendix A.

Denote by $\mathcal{T}$ the transformed operator of (1.1) with integer order, i.e.,

$$\mathcal{T} = \partial_t - p^2\partial_x^4 - 4p\partial_x p\partial_x^3 - (3(\partial_x^2 p)p + 2(\partial_x p)^2)\partial_x^2 - ((\partial_x^3 p)p + (\partial_x^2 p)(\partial_x p))\partial_x.$$

Lemma 3.3 (Carleman Estimate for $\mathcal{T}u$) *There exists λ_0, such that for any $\lambda \ge \lambda_0$, we can choose s_0 and C such that*

$$\begin{aligned}
&\int_Q \Big(\frac{1}{s}|\partial_t u|^2 + s\lambda^2\varphi|\partial_x^3 u|^2 + s^3\lambda^4\varphi^3|\partial_x^2 u|^2 + s^5\lambda^6\varphi^5 u^6|\partial_x y|^2 + s^7\lambda^8\varphi^7|u|^2\Big)\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t \\
&\le C\int_Q |\mathcal{T}u|^2\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t
\end{aligned}$$

for all $s \ge s_0$ and $u \in C_0^{4,2}(Q)$.

Similarly, in order to use the local Carleman estimate in Lemma 3.3 without involving boundary integrals, we again imply the cut-off function $\chi(x,t) \in C_0^\infty(Q)$, $0 \le \chi \le 1$ defined in (3.10). Since $Q_0 \subset\subset Q$ based upon the definition of Q_ε with $\varepsilon = 0$, we verify $\chi(x,0) = 0$, and hence

$$\begin{aligned}
\int_{\Omega_\varepsilon} |\partial_t y(x,t_0)|^2\mathrm{e}^{2s\varphi(x,t_0)}\mathrm{d}x &= \int_{\Omega_\varepsilon} \chi^2(x,t_0)|\partial_t y(x,t_0)|^2\mathrm{e}^{2s\varphi(x,t_0)}\mathrm{d}x \\
&= \int_0^{t_0} \frac{\partial}{\partial t}\Big(\int_{\Omega_\varepsilon} \chi^2(x,t)|\partial_t y(x,t)|^2\mathrm{e}^{2s\varphi(x,t)}\mathrm{d}x\Big)\mathrm{d}t \\
&\le C\int_{Q_\varepsilon} (2\chi\partial_t\chi|\partial_t y(x,t)|^2 + 2\chi^2\partial_t y(x,t)\cdot\partial_t^2 y(x,t)
\end{aligned}$$

$$
\begin{aligned}
&- 4s\lambda\beta(t-t_0)\varphi\chi^2|\partial_t y(x,t)|^2)\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t \\
&\le C\int_{Q_\varepsilon}(s|m_1|^2+|m_2|^2)\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t, \\
&\le C\int_{Q_\varepsilon}(s|m_1|^2+s|m_2|^2)\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t \qquad (3.13)
\end{aligned}
$$

with $m_1 = \partial_t y(x,t)$, $m_2 = \partial_t^2 y(x,t)$, $s \ge 1$ and a fixed constant λ. The additional large parameter s near the $|m_2|^2$ term makes the rest proof more consistent though one can prove the same result without it. Substituting (3.13) into (3.12) yields

$$
\begin{aligned}
&\int_{\Omega_\varepsilon}(|\partial_x^3(\chi f)|^2+s^2|\partial_x^2(\chi f)|+s^3|\partial_x(\chi f)|^2+s^4|(\chi f)|^2)\mathrm{e}^{2s\varphi(x,t_0)}\mathrm{d}x \\
&\le C\int_{Q_\varepsilon}(s|m_1|^2+s|m_2|^2)\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t + C\mathrm{e}^{2s\exp(2\lambda\varepsilon)}M^2 + C\mathrm{e}^{2s\exp(\lambda\Phi)}\|a\|^2_{H^4(\Omega_\varepsilon)}. \qquad (3.14)
\end{aligned}
$$

In order to estimate the term $\int_{Q_\varepsilon}(s|m_1|^2+s|m_2|^2)\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t$, we consider the following two equations of m_1 and m_2 which can be obtained by taking time derivative on both side of (3.2):

$$
\begin{aligned}
&\partial_t m_i - p^2\partial_x^4 m_i - 4p\partial_x p\partial_x^3 m_i - (3(\partial_x^2 p)p + 2(\partial_x p)^2)\partial_x^2 m_i - ((\partial_x^3 p)p + (\partial_x^2 p)(\partial_x p))\partial_x m_i \\
&= \mathcal{C}_3\partial_t^i v\partial_x^3 f + \mathcal{C}_2\partial_t^i v\partial_x^2 f + \mathcal{C}_1\partial_t^i v\partial_x f + \mathcal{C}_0\partial_t^i v f \\
&= P_i \circ f, \quad i=1,2. \qquad (3.15)
\end{aligned}
$$

The equation (3.15) is a similar fourth order differential equation as [14] with respect to m_i. In order to establish Carleman estimate for m_i, we need a further transformation as follows:

$$
n_1(x,t) = \chi(x,t)(m_1(x,t) - m_{1,0}(x,t)), \quad n_2(x,t) = \chi(x,t)(m_2(x,t) - m_{2,0}(x,t)),
$$

where $\{m_{i,0}(x,t)\}_{i=1}^2$ are chosen such that $\partial_x^j(m_i - m_{i,0})|_{x=0} = 0$ for $i=1,2$ and $j=0,1,2,3$. Thus $\chi(m_i - m_{i,0}) \in C_0^{4,2}(Q)$ and the boundary integral will vanish automatically during the integration by part while deriving the Carleman estimate in Lemma 3.3. Simple calculation gives

$$
m_{i,0}(x,t) = m_i(0,t) + \partial_x m_i(0,t)x + \partial_x^2 m_i(0,t)\frac{x^2}{2} + \partial_x^3 m_i(0,t)\frac{x^3}{6}, \quad i=1,2.
$$

More precisely, based on the governing equation for y, i.e., equation (3.1), and the assumption that $\partial_x^j p(0) = \partial_x^j q(0)$ for $j=0,1,2,3$, further calculation gives the coefficients as follows:

$$
\begin{cases}
m_1(0,t) = h_1'(t), \quad \partial_x^2 m_1(0,t) = \dfrac{1}{p(0)}\partial_t^{\frac12}h_1'(t) + \dfrac{\partial_x p(0)}{p(0)}h_2'(t), \quad \partial_x m_1(0,t) = h_2'(t), \\
\partial_x^3 m_1(0,t) = -\dfrac{2\partial_x p(0)}{(p(0))^2}\partial_t^{\frac12}h_1'(t) - \Big(\dfrac{\partial_x^2 p(0)}{p(0)} + \dfrac{2(\partial_x p(0))^2}{(p(0))^2}\Big)h_2'(t) + \dfrac{1}{p(0)}\partial_t^{\frac12}h_2'(t),
\end{cases}
$$

$$
\begin{cases}
m_2(0,t) = h_1''(t), \quad \partial_x^2 m_2(0,t) = \dfrac{1}{p(0)}\partial_t^{\frac12}h_1''(t) + \dfrac{\partial_x p(0)}{p(0)}h_2''(t), \quad \partial_x m_2(0,t) = h_2''(t), \\
\partial_x^3 m_2(0,t) = -\dfrac{2\partial_x p(0)}{(p(0))^2}\partial_t^{\frac12}h_1''(t) - \Big(\dfrac{\partial_x^2 p(0)}{p(0)} + \dfrac{2(\partial_x p(0))^2}{(p(0))^2}\Big)h_2''(t) + \dfrac{1}{p(0)}\partial_t^{\frac12}h_2''(t).
\end{cases}
$$

Based on a priori bound assumption for $h_{u,i}(t)$, $h_{v,i}(t)$ and $\partial_x^i p(0)$, $i=0,1,2$, we have

$$
\sum_{i=1}^2\|m_{i,0}(x,t)\|_{H^1(0,T;H^3(\Omega))} \le C\sum_{i=1}^2\sum_{j=1}^4\|\partial_t^j h_i(t)\|_{L^2(0,T)} \le C\sum_{i=1}^2\|h_i(t)\|_{H^4(0,T)}.
$$

Replacing m_i by n_i in equation (3.15) gives

$$\begin{aligned}
&\partial_t n_i - p^2\partial_x^4 n_i - 4p\partial_x p\partial_x^3 n_i - 3((\partial_x^2 p)p + 2(\partial_x p)^2)\partial_x^2 n_i - ((\partial_x^3 p)p + (\partial_x^2 p)(\partial_x p))\partial_x n_i\\
=\ &\chi(P_i\circ f) + (m_i - m_{i,0})\partial_t\chi - \chi\partial_t m_{i,0}\\
&- p^2((m_i - m_{i,0})\partial_x^4\chi + 4\partial_x^3\chi(\partial_x m_i - \partial_x m_{i,0}) + 6\partial_x^2\chi(\partial_x^2 m_i - \partial_x^2 m_{i,0})\\
&+ 4\partial_x\chi(\partial_x^3 m_i - \partial_x^3 m_{i,0}))\\
&- 4p\partial_x p((m_i - m_{i,0})\partial_x^3\chi + 3\partial_x^2\chi(\partial_x m_i - \partial_x m_{i,0}) + 3\partial_x\chi(\partial_x^2 m_i - \partial_x^2 m_{i,0}) - \chi\partial_x^3 m_{i,0})\\
&- 3((\partial_x^2 p)p + 2(\partial_x p)^2)((m_i - m_{i,0})\partial_x^2\chi + 2\partial_x\chi(\partial_x m_i - \partial_x m_{i,0}) - \chi\partial_x^2 m_{i,0})\\
&- ((\partial_x^3 p)p + (\partial_x^2 p)\partial_x p)((m_i - m_{i,0})\partial_x\chi - \chi\partial_x m_{i,0}) \qquad (3.16)
\end{aligned}$$

and

$$\begin{aligned}
\chi(P_i\circ f) = P_i\circ(\chi f) &- 3\mathcal{C}_3\partial_t^i v\partial_x\chi\partial_x^2 f - (3\mathcal{C}_3\partial_t^i v\partial_x^2\chi + 2\mathcal{C}_2\partial_t^i v\partial_x\chi)\partial_x f\\
&- (\mathcal{C}_3\partial_t^i v\partial_x^3\chi + \mathcal{C}_2\partial_t^i v\partial_x^2\chi + \mathcal{C}_1\partial_t^i v\partial_x\chi)f \qquad (3.17)
\end{aligned}$$

for $i = 1, 2$ and $(x,t)\in Q$. For simplicity, we define, for $i = 1, 2$,

$$\begin{aligned}
g_i(x,t) = 3\mathcal{C}_3\partial_t^i v\partial_x\chi\partial_x^2 f &+ (3\mathcal{C}_3\partial_t^i v\partial_x^2\chi + 2\mathcal{C}_2\partial_t^i v\partial_x\chi)\partial_x f\\
&+ (\mathcal{C}_3\partial_t^i v\partial_x^3\chi + \mathcal{C}_2\partial_t^i v\partial_x^2\chi + \mathcal{C}_1\partial_t^i v\partial_x\chi)f.
\end{aligned}$$

Noting that $\partial_x^i\chi,\ \partial_t\chi,\ i = 1,\cdots,4$ only survives while $\varepsilon < \psi(x,t)\le 2\varepsilon$, by (2.3), for some fixed sufficiently large $\lambda > 0$ applying Lemma 3.3 to $\{n_i\}_{i=1}^2$ on Q_ε implies

$$\begin{aligned}
&\int_{Q_\varepsilon} s^7(|n_1|^2 + |n_2|^2)\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t\\
\le\ & C\|\mathrm{e}^{s\varphi}(P_1\circ(\chi f) + P_2\circ(\chi f))\|_{L^2(Q_\varepsilon)}^2 + C\sum_{i=1}^2\|\mathrm{e}^{s\varphi}\partial_t\chi m_i(x,t)\|_{L^2(Q_\varepsilon)}^2\\
&+ C\sum_{i=1}^2\sum_{j=1}^4\sum_{k=1}^j\|\mathrm{e}^{s\varphi}\partial_x^k\chi\partial_x^{j-k}m_i(x,t)\|_{L^2(Q_\varepsilon)}^2 + C\sum_{i=1}^2\|\mathrm{e}^{s\varphi}g_i(x,t)\|_{L^2(Q_\epsilon)}^2\\
&+ C\mathrm{e}^{2s\exp(\lambda\Phi)}\sum_{i=1}^2\|h_i\|_{H^4(0,T)}^2\\
\le\ & C\int_{Q_\varepsilon}\mathrm{e}^{2s\varphi}\sum_{j=0}^3|\partial_x^j(\chi f)|^2\mathrm{d}x\mathrm{d}t + C\mathrm{e}^{2s\exp(2\lambda\varepsilon)}M^2 + C\mathrm{e}^{2s\exp(\lambda\Phi)}F^2, \qquad (3.18)
\end{aligned}$$

where $M,\ F$ are defined in Theorem 2.1.

Since $n_i = m_i - m_{i,0}$ on Q_ε as the definition of $\chi(x,t)$, we have

$$\begin{aligned}
&\int_{Q_\varepsilon} s^7(|m_1|^2 + |m_2|^2)\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t \le C\int_{Q_\varepsilon} s^7\Big(\sum_{i=1}^2(|n_i|^2 + |m_{i,0}|^2)\Big)\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t\\
\le\ & C\int_{Q_\varepsilon} s^7(|n_1|^2 + |n_2|^2)\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t + C\mathrm{e}^{2s\exp(\lambda\Phi)}\sum_{k=1}^2\|h_i(t)\|_{H^4(0,T)}^2\\
\le\ & C\int_{Q_\varepsilon}\mathrm{e}^{2s\varphi}\sum_{j=0}^3|\partial_x^j(\chi f)|^2\mathrm{d}x\mathrm{d}t + C\mathrm{e}^{2s\exp(2\lambda\varepsilon)}M^2 + C\mathrm{e}^{2s\exp(\lambda\Phi)}F^2 \qquad (3.19)
\end{aligned}$$

for all $s > s_0$.

Substituting (3.19) into (3.14), we have

$$\begin{aligned}
&\int_{\Omega_\varepsilon}(|\partial_x^3(\chi f)|^2 + s^2|\partial_x^2(\chi f)|^2 + s^3|\partial_x(\chi f)|^2 + s^4|(\chi f)|^2)\mathrm{e}^{2s\varphi(x,t_0)}\mathrm{d}x\\
&\le \frac{C}{s^6}\int_{Q_\varepsilon}\mathrm{e}^{2s\varphi}\sum_{j=0}^{3}|\partial_x^j(\chi f)|^2\mathrm{d}x\mathrm{d}t + \frac{C}{s^6}\mathrm{e}^{2s\exp(2\lambda\varepsilon)}M^2 + C\mathrm{e}^{2s\exp(2\lambda\varepsilon)}M^2\\
&\quad + \frac{C}{s^6}\mathrm{e}^{2s\exp(\lambda\Phi)}F^2 + C\mathrm{e}^{2s\exp(\lambda\Phi)}\|a\|^2_{H^4(\Omega\varepsilon)}\\
&\le \frac{CT}{s^6}\int_{\Omega_\varepsilon}\mathrm{e}^{2s\varphi(x,t_0)}\sum_{j=0}^{3}|\partial_x^j(\chi f)|^2\mathrm{d}x + \frac{C}{s^6}\mathrm{e}^{2s\exp(2\lambda\varepsilon)}M^2 + C\mathrm{e}^{2s\exp(2\lambda\varepsilon)}M^2\\
&\quad + \frac{C}{s^6}\mathrm{e}^{2s\exp(\lambda\Phi)}F^2 + C\mathrm{e}^{2s\exp(\lambda\Phi)}\|a\|^2_{H^4(\Omega\varepsilon)} \qquad (3.20)
\end{aligned}$$

for all $s > \overline{s} \triangleq \max\{s_0, s_1\}$, where the first terms on the right-hand side can be absorbed by the left-hand side.

Moreover, since $\Omega_{3\varepsilon} \subset \Omega_\varepsilon$ and $\psi(x,t_0) > 3\varepsilon$ on $\Omega_{3\varepsilon}$, we can estimate the left-hand side of (3.20) by

$$\begin{aligned}
&\int_{\Omega_\varepsilon}(|\partial_x^3(\chi f)|^2 + s^2|\partial_x^2(\chi f)|^2 + s^3|\partial_x(\chi f)|^2 + s^4|(\chi f)|^2)\mathrm{e}^{2s\varphi}\mathrm{d}x\\
&\ge \mathrm{e}^{2s\exp(3\lambda\varepsilon)}\int_{\Omega_{3\varepsilon}}(|\partial_x^3(\chi f)|^2 + s^2|\partial_x^2(\chi f)|^2 + s^3|\partial_x(\chi f)|^2 + s^4|(\chi f)|^2)\mathrm{d}x. \qquad (3.21)
\end{aligned}$$

Combining (3.20) and (3.21) and dividing $\mathrm{e}^{2s\exp(3\lambda\varepsilon)}$ on both sides, we can obtain that for all $s > \overline{s}$ with $\overline{s} > 0$ sufficiently large,

$$\begin{aligned}
\|f(x)\|^2_{H^3(\Omega_{3\varepsilon})} &\le \frac{CM^2}{s^6\mathrm{e}^{2s\exp(\lambda\varepsilon)}} + \frac{CM^2}{\mathrm{e}^{2s\exp(\lambda\varepsilon)}} + \frac{CF^2}{s^6}\mathrm{e}^{2s\exp(\lambda(\Phi-3\varepsilon))}\\
&\quad + C\|a\|^2_{H^4(\Omega\varepsilon)}\mathrm{e}^{2s\exp(\lambda(\Phi-3\varepsilon))}\\
&\le \frac{CM^2}{\mathrm{e}^{2s\exp(\lambda\varepsilon)}} + C\widetilde{F}^2\mathrm{e}^{2s\exp(\lambda(\Phi-3\varepsilon))}\\
&\triangleq C\mathrm{e}^{-2s\nu}M^2 + C\mathrm{e}^{C_0 s}\widetilde{F}^2, \qquad (3.22)
\end{aligned}$$

where $\widetilde{F} = \|a\|^2_{H^4(\Omega\varepsilon)} + F$, $\nu = \exp(\lambda\varepsilon)$ and $C_0 = 2\exp(\lambda(\Phi - 3\varepsilon))$. One can choose ε small enough such that $\Phi - 3\varepsilon > 0$.

Assuming $\widetilde{F} = 0$, by letting $s \to \infty$ in (3.22), we can derive $p(x) = q(x)$ on $\Omega_{3\varepsilon}$ which verifies the local uniqueness of the inverse coefficient problem. If there exists $\widetilde{F} \ge M$, (3.22) immediately implies

$$\|f(x)\|_{H^3(\Omega_{3\varepsilon})} \le C\widetilde{F}$$

by fixing a certain $s > 0$. At the same time, if $\widetilde{F} < M$, we can choose s appropriately to balance the right-hand side of (3.22) to derive that

$$\|f(x)\|_{H^3(\Omega_{3\varepsilon})} \le \sqrt{2}M^{\frac{C_0}{2\nu+C_0}}\widetilde{F}^{\frac{2\nu}{2\mu+C_0}} \qquad (3.23)$$

with the choice

$$s = \frac{2}{2\nu + C_0}\log\frac{M}{\widetilde{F}} > 0.$$

Thus, the proof of Theorem 2.1 is completed.

4 Conclusion

Fractional diffusion equations as well as the inverse nature of these equations have attracted considerable interests in view of their potential use in physical and chemical processes and in engineering during last decades. In this paper, we investigated an inverse coefficient problem with respect to a half-order time-fractional diffusion equation. After generalizing the Carleman estimate to fractional diffusion equations with spatially varying conductivity, we implement the methodology developed in [3] (see also the review paper [15]) to establish a Hölder type conditional stability estimation for an inverse problem identifying the coefficient in the principal part. As for the results identifying the coefficient near the lower order term, we refer to a very recent work [16]. Future works will be emphasized on deriving global Carleman estimates and Lipschitz stabilities on these inverse problems.

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Appendix A

Proof of Lemma 3.3 Consider a following general fourth order differential equation with variable coefficients $\mathcal{T}u$ where

$$\mathcal{T}u = \partial_t u - a_4(x)\partial_x^4 u - a_3(x)\partial_x^3 u - a_2(x)\partial_x^2 u - a_1(x)\partial_x u - a_0(x)u$$
$$a_4(x) \in C^3[0,1] \ \text{ and } \inf a_4(x) > 0, \quad a_3(x), a_2(x), a_1(x), a_0(x) \in L^\infty(0,1).$$

Obviously, for Lemma 3.3, we can take $a_4(x) = p^2$, $a_3(x) = 4p\partial_x p$, $a_2(x) = 3p\partial_x^2 p + 2(\partial_x p)^2$, $a_1(x) = p\partial_x^3 p + \partial_x p \partial_x^2 p$ and $a_0(x) = 0$. Denote the principal part of $\mathcal{T}$ by

$$\mathcal{T}_0 u = \partial_t u - a_4(x)\partial_x^4 u.$$

Note that to prove Lemma 3.3, it is sufficient to obtain a similar estimate for the principal part, i.e., $\mathcal{T}_0$, which takes the form as follows:

$$\begin{aligned}&\int_Q \Big(\frac{1}{s\varphi}|\partial_t u|^2 + \widetilde{s}\widetilde{\lambda}^2|\partial_x^3 u|^2 + \widetilde{s}^3\widetilde{\lambda}^4|\partial_x^2 u|^2 + \widetilde{s}^5\widetilde{\lambda}^6|\partial_x u|^2 + \widetilde{s}^7\widetilde{\lambda}^8|u|^2\Big)\mathrm{d}x\mathrm{d}t\\ &\le C\int_Q |\mathcal{T}_0 u|^2 \mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{A.1}$$

where $\widetilde{s} = s\varphi(x)$ and $\widetilde{\lambda} = \lambda\mu(x)$. The reason is same as the second step in the proof of Lemma 3.2, i.e., the lower order terms can be absorbed by choosing sufficiently large parameters s and λ. Here, for the sake of compactness, we omit the details.

Next throughout this section, we are aiming at proving the above estimate for $\mathcal{T}_0$. Let $w = \mathrm{e}^{s\varphi}u$ and $Pw = \mathrm{e}^{s\varphi}(\partial_t - a_4(x)\partial_x^4)(\mathrm{e}^{-s\varphi}w)$. Since $\partial_x\varphi = \lambda\varphi\mu, \partial_t\varphi = -2\beta(t-t_0)\lambda\varphi$, we have

$$\mathrm{e}^{s\varphi}\partial_t(\mathrm{e}^{-s\varphi}w) = 2s\beta(t-t_0)\lambda\varphi w + \partial_t w$$

and

$$\begin{aligned}\mathrm{e}^{s\varphi}(a_4(x)\partial_x^4(\mathrm{e}^{-s\varphi}w)) = a_4(x)(&\partial_x^4 w - 4\widetilde{s}\widetilde{\lambda}\partial_x^3 w + (6\widetilde{s}^2\widetilde{\lambda}^2 + O(s^{-1}\widetilde{s}^2\widetilde{\lambda}^2))\partial_x^2 w\\ &- (4\widetilde{s}^3\widetilde{\lambda}^3 + O(s^{-1}\widetilde{s}^3\widetilde{\lambda}^3))\partial_x w + (\widetilde{s}^4\widetilde{\lambda}^4 + O(s^{-1}\widetilde{s}^4\widetilde{\lambda}^4))w).\end{aligned}$$

Therefore, according to the order of $(\widetilde{s},\widetilde{\lambda})$, we can split Pw into two terms, i.e., $Pw = P_1 w + P_2 w$, where

$$\begin{aligned}P_1 w &= -a_4(x)\partial_x^4 w - a_4(x)(6\widetilde{s}^2\widetilde{\lambda}^2 + O(s^{-1}\widetilde{s}^2\widetilde{\lambda}^2))\partial_x^2 w - a_4(x)(\widetilde{s}^4\widetilde{\lambda}^4 + O(s^{-1}\widetilde{s}^4\widetilde{\lambda}^4))w,\\ P_2 w &= \partial_t w + 4a_4(x)\widetilde{s}\widetilde{\lambda}\partial_x^3 w + a_4(x)(4\widetilde{s}^3\widetilde{\lambda}^3 + O(s^{-1}\widetilde{s}^3\widetilde{\lambda}^3))\partial_x w.\end{aligned}$$

Denote the inner product on Q by

$$(f,g) = \int_Q f(x,t)g(x,t)\mathrm{d}x\mathrm{d}t.$$

We can compute

$$(P_1 w, P_2 w) = \sum_{k=1}^{9} \mathrm{J}_k.$$

In the following, we use $b_j(x,t), b(x,t)$ to represent bounded functions when (s,λ) are sufficiently large. For large $\lambda>1,\ s>1$, utilizing integration by parts and $u\in C_0^\infty(Q)$, we obtain

$$\begin{aligned}
\mathrm{J}_1 &= (-a_4\partial_x^4 w, \partial_t w) = \int_Q a_4\partial_x^3 w\partial_x\partial_t w\mathrm{d}x\mathrm{d}t + \int_Q \partial_x a_4\partial_x^3 w\partial_t w\mathrm{d}x\mathrm{d}t \\
&= \frac{1}{2}\int_Q \partial_t(a_4)|\partial_x^2 w|^2\mathrm{d}x\mathrm{d}t + \int_Q \partial_x^2 a_4\partial_x^2 w\partial_t w\mathrm{d}x\mathrm{d}t + 2\int_Q \partial_x a_4\partial_x^3 w\partial_t w\mathrm{d}x\mathrm{d}t \\
&= \int_Q \partial_x^2 a_4\partial_x^2 w\partial_t w\mathrm{d}x\mathrm{d}t + 2\int_Q \partial_x a_4\partial_x^3 w\partial_t w\mathrm{d}x\mathrm{d}t \\
&\triangleq \mathrm{J}_{11} + \mathrm{J}_{12}.
\end{aligned} \tag{A.2}$$

Since

$$\begin{aligned}
\partial_t w &= P_2 w - 4a_4\widetilde{s}\widetilde{\lambda}\partial_x^3 w - a_4(4\widetilde{s}^3\widetilde{\lambda}^3 + O(s^{-1}\widetilde{s}^3\widetilde{\lambda}^3))\partial_x w \\
&= P_2 w - 4a_4\widetilde{s}\widetilde{\lambda}\partial_x^3 w - 4\widetilde{s}^3\widetilde{\lambda}^3 a_4(1+s^{-1}b_1)\partial_x w,
\end{aligned}$$

we have

$$\begin{aligned}
\mathrm{J}_{11} &= \int_Q \partial_x^2 a_4\partial_x^2 w(P_2 w - 4a_4\widetilde{s}\widetilde{\lambda}\partial_x^3 w - 4\widetilde{s}^3\widetilde{\lambda}^3 a_4(1+s^{-1}b_1)\partial_x w)\mathrm{d}x\mathrm{d}t \\
&= \mathrm{J}_{11}^{(1)} + \mathrm{J}_{11}^{(2)} + \mathrm{J}_{11}^{(3)}.
\end{aligned}$$

Let ε be any small positive constant, we have

$$\begin{aligned}
|\mathrm{J}_{11}^{(1)}| &= \Big|\int_Q \partial_x^2 a_4\partial_x^2 w P_2 w\mathrm{d}x\mathrm{d}t\Big| \le C(a_4)\Big(\varepsilon\int_Q |P_2 w|^2\mathrm{d}x\mathrm{d}t + \frac{1}{4\varepsilon}\int_Q |\partial_x^2 w|^2\mathrm{d}x\mathrm{d}t\Big), \\
\mathrm{J}_{11}^{(2)} &= -4\int_Q \widetilde{s}\widetilde{\lambda}\partial_x^2 a_4\cdot a_4\partial_x^2 w\cdot\partial_x^3 w\mathrm{d}x\mathrm{d}t \\
&= 2\int_Q \widetilde{s}\widetilde{\lambda}^2\partial_x^2 a_4\cdot a_4|\partial_x^2 w|^2\mathrm{d}x\mathrm{d}t + 2\int_Q \widetilde{s}\partial_x(\widetilde{\lambda}\partial_x^2 a_4\cdot a_4)|\partial_x^2 w|^2\mathrm{d}x\mathrm{d}t, \\
\mathrm{J}_{11}^{(3)} &= -4\int_Q \widetilde{s}^3\widetilde{\lambda}^3 a_4\cdot\partial_x^2 a_4(1+s^{-1}b_1)\partial_x^2 w\cdot\partial_x w\mathrm{d}x\mathrm{d}t \\
&= 6\int_Q \widetilde{s}^4\widetilde{\lambda}^4 a_4\cdot\partial_x^2 a_4(1+s^{-1}b_1)|\partial_x w|^2\mathrm{d}x\mathrm{d}t \\
&\quad + 2\int_Q \widetilde{s}^3\partial_x(\widetilde{\lambda}^3 a_4\cdot\partial_x^2 a_4(1+s^{-1}b_1))|\partial_x w|^2\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Thus we can obtain

$$\begin{aligned}
\mathrm{J}_{11} \ge& -C(a_4)\Big(\varepsilon\int_Q |P_2 w|^2\mathrm{d}x\mathrm{d}t + \frac{1}{4\varepsilon}\int_Q |\partial_x^2 w|^2\mathrm{d}x\mathrm{d}t\Big) \\
&+ 2\int_Q \widetilde{s}\widetilde{\lambda}^2\partial_x^2 a_4\cdot a_4|\partial_x^2 w|^2\mathrm{d}x\mathrm{d}t + 2\int_Q \widetilde{s}\partial_x(\widetilde{\lambda}\partial_x^2 a_4\cdot a_4)|\partial_x^2 w|^2\mathrm{d}x\mathrm{d}t \\
&+ 6\int_Q \widetilde{s}^3\widetilde{\lambda}^4\partial_x^2 a_4\cdot a_4(1+s^{-1}b_1)|\partial_x w|^2\mathrm{d}x\mathrm{d}t \\
&+ 2\int_Q \widetilde{s}^3\partial_x(\widetilde{\lambda}^3 a_4\partial_x^2 a_4(1+s^{-1}b_1))|\partial_x w|^2\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Similar as J_{11}, for J_{12}, we have

$$\begin{aligned}\mathrm{J}_{12} &= 2\int_Q \partial_x a_4 \partial_x^3 w(P_2 w - 4\widetilde{s}\widetilde{\lambda} a_4 \partial_x^3 w - 4\widetilde{s}^3\widetilde{\lambda}^3 a_4(1+s^{-1}b_1)\partial_x w)\mathrm{d}x\mathrm{d}t \\ &= \mathrm{J}_{12}^{(1)} + \mathrm{J}_{12}^{(2)} + \mathrm{J}_{12}^{(3)},\end{aligned}$$

where

$$\begin{aligned}|\mathrm{J}_{12}^{(1)}| &= \Big|2\int_Q \partial_x a_4 \partial_x^3 w P_2 w \mathrm{d}x\mathrm{d}t\Big| \le C(a_4)\Big(\varepsilon\int_Q |P_2 w|^2 \mathrm{d}x\mathrm{d}t + \frac{1}{4\varepsilon}\int_Q |\partial_x^3 w|^2 \mathrm{d}x\mathrm{d}t\Big), \\ \mathrm{J}_{12}^{(2)} &= -8\int_Q \widetilde{s}\widetilde{\lambda}\partial_x a_4 \cdot a_4 |\partial_x^3 w|^2 \mathrm{d}x\mathrm{d}t, \\ \mathrm{J}_{12}^{(3)} &= -8\int_Q \widetilde{s}^3\widetilde{\lambda}^3 a_4 \cdot a_4(1+s^{-1}b_1)\partial_x^3 w \partial_x w \mathrm{d}x\mathrm{d}t \\ &= 8\int_Q \widetilde{s}^3\widetilde{\lambda}^3 \partial_x a_4 \cdot a_4(1+s^{-1}b_1)|\partial_x^2 w|^2 \mathrm{d}x\mathrm{d}t \\ &\quad - 36\int_Q \widetilde{s}^3\widetilde{\lambda}^5 \partial_x a_4 \cdot a_4(1+s^{-1}b_1)|\partial_x w|^2 \mathrm{d}x\mathrm{d}t \\ &\quad - 12\int_Q \widetilde{s}^3 \partial_x(\widetilde{\lambda}^4 \partial_x a_4 \cdot a_4(1+s^{-1}b_1))|\partial_x w|^2 \mathrm{d}x\mathrm{d}t \\ &\quad - 12\int_Q \widetilde{s}^3 \lambda\mu \partial_x(\widetilde{\lambda}^3 \partial_x a_4 \cdot a_4(1+s^{-1}b_1))|\partial_x w|^2 \mathrm{d}x\mathrm{d}t \\ &\quad - 4\int_Q \widetilde{s}^3 \partial_x^2(\widetilde{\lambda}^3 \partial_x a_4 \cdot a_4(1+s^{-1}b_1))|\partial_x w|^2 \mathrm{d}x\mathrm{d}t.\end{aligned}$$

Combining all components of J_1, we have

$$\begin{aligned}\mathrm{J}_1 &\ge -C\varepsilon\int_Q |P_2 w|^2 \mathrm{d}x\mathrm{d}t - \frac{C}{4\varepsilon}\int_Q |\partial_x^2 w|^2 \mathrm{d}x\mathrm{d}t - \frac{C}{4\varepsilon}\int_Q |\partial_x^3 w|^2 \mathrm{d}x\mathrm{d}t \\ &\quad - C\int_Q \widetilde{s}\widetilde{\lambda}|\partial_x^3 w|^2 \mathrm{d}x\mathrm{d}t - C\int_Q \widetilde{s}^3\widetilde{\lambda}^3|\partial_x^2 w|^2 \mathrm{d}x\mathrm{d}t - C\int_Q \widetilde{s}^3\lambda^5|\partial_x w|^2 \mathrm{d}x\mathrm{d}t. \qquad \text{(A.3)}\end{aligned}$$

Continuing to estimate the remainder J_2–J_9, we have

$$\mathrm{J}_2 = -4\int_Q \widetilde{s}\widetilde{\lambda} a_4^2 \partial_x^3 w \partial_x^4 w \mathrm{d}x\mathrm{d}t \ge \int_Q \widetilde{s}\widetilde{\lambda}^2 a_4^2 |\partial_x^3 w|^2 \mathrm{d}x\mathrm{d}t, \qquad \text{(A.4)}$$

$$\begin{aligned}\mathrm{J}_3 &= -18\int_Q \widetilde{s}^3\widetilde{\lambda}^4 a_4^2(1+s^{-1}b_1+\lambda^{-1}b)|\partial_x^2 w|^2 \mathrm{d}x\mathrm{d}t \\ &\quad + 54\int_Q \widetilde{s}^3\widetilde{\lambda}^6 a_4^2(1+s^{-1}b_1+\lambda^{-1}b)|\partial_x w|^2 \mathrm{d}x\mathrm{d}t, \qquad \text{(A.5)}\end{aligned}$$

$$\begin{aligned}\mathrm{J}_4 &\ge \mathrm{J}_4^0 + 12\int_Q \beta(t-t_0)\widetilde{s}^2\lambda\widetilde{\lambda}^2 a_4(1+s^{-1}b_2+\lambda^{-1}b)|\partial_x w|^2 \mathrm{d}x\mathrm{d}t \\ &\quad - 3\int_Q \widetilde{s}\widetilde{\lambda}^2 a_4 |\partial_x w|^2 \mathrm{d}x\mathrm{d}t, \qquad \text{(A.6)}\end{aligned}$$

where

$$\begin{aligned}\mathrm{J}_4^0 &= -\varepsilon\int_Q |P_2 w|^2 \mathrm{d}x\mathrm{d}t - \frac{1}{4\varepsilon}\int_Q 144\widetilde{s}^4\widetilde{\lambda}^6 a_4^2(1+s^{-1}b_2+\lambda^{-1}b)^2|\partial_x w|^2 \mathrm{d}x\mathrm{d}t \\ &\quad + 48\int_Q \widetilde{s}^3\widetilde{\lambda}^4 a_4^2|\partial_x^2 w|^2 \mathrm{d}x\mathrm{d}t - 48\int_Q \widetilde{s}^5\widetilde{\lambda}^6 a_4^2|\partial_x w|^2 \mathrm{d}x\mathrm{d}t - C\int_Q \widetilde{s}^3\widetilde{\lambda}^6 a_4^2|\partial_x w|^2 \mathrm{d}x\mathrm{d}t. \qquad \text{(A.7)}\end{aligned}$$

In addition,

$$\mathrm{J}_5 = 36\int_Q a_4^2 s^3\lambda^4\varphi^3\mu^4(1+s^{-1}b_2+\lambda^{-1}b)|\partial_x^2 w|^2\mathrm{d}x\mathrm{d}t, \tag{A.8}$$

$$\mathrm{J}_6 = 60\int_Q s^5\lambda^6\varphi^5\mu^6 a_4^2((1+s^{-1}b_1)(1+s^{-1}b_2)+\lambda^{-1}b)|\partial_x w|^2\mathrm{d}x\mathrm{d}t, \tag{A.9}$$

$$\mathrm{J}_7 \ge -C\int_Q \widetilde{s}^4\lambda\widetilde{\lambda}^4 a_4|w|^2\mathrm{d}x\mathrm{d}t, \tag{A.10}$$

$$\mathrm{J}_8 = -30\int_Q \widetilde{s}^5\widetilde{\lambda}^5 a_4^2|\partial_x w|^2\mathrm{d}x\mathrm{d}t + 250\int_Q \widetilde{s}^5\widetilde{\lambda}^8 a_4^2|w|^2\mathrm{d}x\mathrm{d}t, \tag{A.11}$$

$$\mathrm{J}_9 = 14\int_Q a_4^2\widetilde{s}^7\widetilde{\lambda}^8(1+s^{-1}b\lambda^{-1}b)|w|^2\mathrm{d}x\mathrm{d}t. \tag{A.12}$$

Combining J_1–J_9, we derive

$$\begin{aligned}(P_1w,P_2w) \ge{}& \int_Q \widetilde{s}\widetilde{\lambda}^2 a_4^2|\partial_x^3 w|^2\mathrm{d}x\mathrm{d}t + 18\int_Q \widetilde{s}^3\widetilde{\lambda}^4|\partial_x^2 w|^2\mathrm{d}x\mathrm{d}t + 30\int_Q \widetilde{s}^5\widetilde{\lambda}^6 a_4^2|\partial_x w|^2\mathrm{d}x\mathrm{d}t\\ &+ 54\int_Q \widetilde{s}^3\widetilde{\lambda}^6 a_4^2|\partial_x w|^2\mathrm{d}x\mathrm{d}t + 14\int_Q \widetilde{s}^7\widetilde{\lambda}^8 a_4^2|w|^2\mathrm{d}x\mathrm{d}t + 250\int_Q a_4^2\widetilde{s}^5\widetilde{\lambda}^8|w|^2\mathrm{d}x\mathrm{d}t\\ &- C\int_Q \widetilde{s}^4\lambda\widetilde{\lambda}^4 a_4|w|^2\mathrm{d}x\mathrm{d}t + 12\int_Q \beta(t-t_0)\widetilde{s}^2\lambda\widetilde{\lambda}^2 a_4|\partial_x w|^2\mathrm{d}x\mathrm{d}t\\ &- 3\int_Q \widetilde{s}^2\widetilde{\lambda}^2 a_4\frac{\partial_t b_3}{s}|\partial_x w|^2\mathrm{d}x\mathrm{d}t + \mathrm{J}_1 + \mathrm{J}_4^0.\end{aligned} \tag{A.13}$$

Moreover, based on (A.3) and (A.7), we have

$$\begin{aligned}\mathrm{J}_4^0 + \mathrm{J}_1 \ge{}& -C_0\varepsilon\int_Q |P_2w|^2\mathrm{d}x\mathrm{d}t - \frac{144}{4\varepsilon}\int_Q \widetilde{s}^4\widetilde{\lambda}^6 a_4^2|\partial_x w|^2\mathrm{d}x\mathrm{d}t - \frac{C}{4\varepsilon}\int_Q |\partial_x^2 w|^2\mathrm{d}x\mathrm{d}t\\ &- \frac{C}{4\varepsilon}\int_Q |\partial_x^3 w|^2\mathrm{d}x\mathrm{d}t - C\int_Q \widetilde{s}\lambda|\partial_x^3 w|^2\mathrm{d}x\mathrm{d}t + 47\int_Q \widetilde{s}^3\widetilde{\lambda}^4 a_4^2|\partial_x^2 w|^2\mathrm{d}x\mathrm{d}t\\ &- 48\int_Q \widetilde{s}^5\widetilde{\lambda}^6 a_4^2|\partial_x w|^2\mathrm{d}x\mathrm{d}t - C\int_Q \widetilde{s}^3\widetilde{\lambda}^6 a_4^2|\partial_x w|^2\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{A.14}$$

Inserting (A.14) into (A.13) yields

$$\begin{aligned}&\frac{1}{4}\int_Q \widetilde{s}\widetilde{\lambda}^2 a_4^2|\partial_x^3 w|^2\mathrm{d}x\mathrm{d}t + 63\int_Q \widetilde{s}^3\widetilde{\lambda}^4 a_4^2|\partial_x^2 w|^2\mathrm{d}x\mathrm{d}t - 17\int_Q \widetilde{s}^5\widetilde{\lambda}^5|\partial_x w|^2\mathrm{d}x\mathrm{d}t\\ &- C\int_Q \widetilde{s}^3\widetilde{\lambda}^6 a_4^2|\partial_x w|^2\mathrm{d}x\mathrm{d}t - \frac{144}{4\varepsilon}\int_Q \widetilde{s}^4\widetilde{\lambda}^6 a_4^2|\partial_x w|^2\mathrm{d}x\mathrm{d}t + 13\int_Q \widetilde{s}^7\widetilde{\lambda}^8 a_4^2|w|^2\mathrm{d}x\mathrm{d}t\\ \le{}& C_0\Big((P_1w,P_2w) + \varepsilon\int_Q |P_2w|^2\mathrm{d}x\mathrm{d}t\Big).\end{aligned} \tag{A.15}$$

Noticing that in (A.15), the sign of $|\partial_x w|^2$ is negative, hence we have to obtain another estimate. In the following, we will estimate $\int_Q (P_1w)\times\widetilde{s}^3\widetilde{\lambda}^4 a_4 w\mathrm{d}x\mathrm{d}t$ to further estimate $\int_Q \widetilde{s}^5\widetilde{\lambda}^6 a_4^2|\partial_x w|^2\mathrm{d}x\mathrm{d}t$, where

$$\int_Q (Pw)\widetilde{s}^3\widetilde{\lambda}^4 a_4 w\mathrm{d}x\mathrm{d}t = \sum_{k=1}^{3} I_k.$$

By integration by parts and $w \in C_0^\infty(Q)$, we have

$$\mathrm{I}_1 = -\int_Q \widetilde{s}^3\widetilde{\lambda}^4 a_4^2 |\partial_x^2 w|^2 \mathrm{d}x\mathrm{d}t + 18\int_Q \widetilde{s}^3\widetilde{\lambda}^6 a_4^2 |\partial_x w|^2 \mathrm{d}x\mathrm{d}t - 81\int_Q \widetilde{s}^3\widetilde{\lambda}^8 a_4^2 |w|^2 \mathrm{d}x\mathrm{d}t. \tag{A.16}$$

In addition,

$$\mathrm{I}_2 = 6\int_Q \widetilde{s}^5\widetilde{\lambda}^6 a_4^2 |\partial_x w|^2 - 75\int_Q \widetilde{s}^5\widetilde{\lambda}^8 a_4^2 |w|^2 \mathrm{d}x\mathrm{d}t. \tag{A.17}$$

Combining (A.16) and (A.17), we obtain

$$\begin{aligned}&5\int_Q s^5\lambda^6\varphi^5\mu^6 a_4^2 |\partial_x w|^2 \mathrm{d}x\mathrm{d}t\\ &\le \frac{1}{2}\|P_1 w\|^2 + \int_Q \widetilde{s}^7\widetilde{\lambda}^8 a_4^2 |w|^2 \mathrm{d}x\mathrm{d}t + \int_Q \widetilde{s}^3\widetilde{\lambda}^4 a_4^2 |\partial_x^2 w|^2 \mathrm{d}x\mathrm{d}t.\end{aligned} \tag{A.18}$$

Thus (A.18) $\times$ 5 + (A.15) gives

$$\begin{aligned}&\frac{1}{4}\int_Q \widetilde{s}\widetilde{\lambda}^2 a_4^2 |\partial_x^3 w|^2 \mathrm{d}x\mathrm{d}t + 58\int_Q \widetilde{s}^3\widetilde{\lambda}^4 a_4^2 |\partial_x^2 w|^2 \mathrm{d}x\mathrm{d}t\\ &+ 8\int_Q \widetilde{s}^5\widetilde{\lambda}^6 a_4^2 |\partial_x w|^2 \mathrm{d}x\mathrm{d}t + 7\int_Q \widetilde{s}^7\widetilde{\lambda}^7 a_4^2 |w|^2 \mathrm{d}x\mathrm{d}t\\ &\le \frac{1}{2}\|P_1 w\|^2 + C_0((P_1 w, P_2 w) + \varepsilon\|P_2 w\|^2).\end{aligned} \tag{A.19}$$

Letting $C_0 > 1$, by taking $0 < \varepsilon \le \frac{1}{2}$, we have

$$\begin{aligned}&\frac{1}{2}\|P_1 w\|^2 + C_0(P_1 w, P_2 w) + C_0\varepsilon\|P_2 w\|^2\\ &\le C_0\Big(\frac{1}{2}\|P_1 w\|^2 + \frac{1}{2}\|P_2 w\|^2 + (P_1 w, P_2 w)\Big) \le \frac{1}{2}C_0\int_Q |L_0 u|^2 \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t.\end{aligned} \tag{A.20}$$

Combining (A.19) and (A.20), we have

$$\begin{aligned}&\frac{1}{4}\int_Q \widetilde{s}\widetilde{\lambda}^2 a_4^2 |\partial_x^3 w|^2 \mathrm{d}x\mathrm{d}t + 58\int_Q \widetilde{s}^3\widetilde{\lambda}^4 a_4^2 |\partial_x^2 w|^2 \mathrm{d}x\mathrm{d}t\\ &+ 8\int_Q \widetilde{s}^5\widetilde{\lambda}^6 a_4^2 |\partial_x w|^2 \mathrm{d}x\mathrm{d}t + 7\int_Q \widetilde{s}^7\widetilde{\lambda}^7 a_4^2 |w|^2 \mathrm{d}x\mathrm{d}t\\ &\le \frac{1}{2}C\int_Q |L_0 u|^2 \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t.\end{aligned} \tag{A.21}$$

Finally, we proceed to estimating $\partial_t w$ and $\partial_x^4 w$. Since

$$|\partial_t w|^2 \le C_1|P_2 w|^2 + C_1\widetilde{s}^2\widetilde{\lambda}^2 a_4^2 |\partial_x^3 w|^2 + C_1\widetilde{s}^6\widetilde{\lambda}^6 a_4^2 |\partial_x w|^2, \tag{A.22}$$

by taking $\varepsilon = \frac{1}{4}$ and $s \geq \frac{4}{C_0}$, we have

$$\begin{aligned}\int_Q \frac{1}{\widetilde{s}}|\partial_t w|^2 \mathrm{d}x\mathrm{d}t &\leq C_1 \int_Q \frac{1}{s\varphi}|P_2 w|^2 \mathrm{d}x\mathrm{d}t + C_1 \int_Q s\lambda^2\varphi\mu^2 a_4^2 |\partial_x^3 w|^2 \mathrm{d}x\mathrm{d}t \\ &\quad + C_1 \int_Q s^5\lambda^6\varphi^5\mu^6 a_4^2 |\partial_x w|^2 \mathrm{d}x\mathrm{d}t \\ &\leq \frac{C_1}{s}\|P_2 w\|^2 + C_1\Big(\frac{1}{2}\|P_1 w\|^2 + C(P_1 w, P_2 w) + C\varepsilon\|P_2 w\|^2\Big) \\ &\leq C\cdot C_1\Big[\Big(\frac{1}{Cs}+\varepsilon\Big)\|P_2 w\|^2 + \frac{1}{2}\|P_1 w\|^2 + (P_1 w, P_2 w)\Big] \\ &\leq \frac{1}{2}C\cdot C_1 \int_Q |L_0 u|^2 \mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{A.23}$$

Finally, adding (A.23) to (A.21) gives

$$\begin{aligned}&\int_Q \Big(\frac{1}{\widetilde{s}}|\partial_t w|^2 \mathrm{d}x\mathrm{d}t + \widetilde{s}\widetilde{\lambda}^2 |\partial_x^3 w|^2 \mathrm{d}x\mathrm{d}t + \widetilde{s}^3\widetilde{\lambda}^4 |\partial_x^2 w|^2 \mathrm{d}x\mathrm{d}t + \widetilde{s}^5\widetilde{\lambda}^6 |\partial_x w|^2 \mathrm{d}x\mathrm{d}t + \widetilde{s}^7\widetilde{\lambda}^8 |w|^2\Big)\mathrm{d}x\mathrm{d}t \\ &\leq C\int_Q |L_0 u|^2 \mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{A.24}$$

Noticing $w = \mathrm{e}^{s\varphi}u$, we finally get the Carleman estimate of u in Lemma 3.3.