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On the Well-Posedness of Determination of Two Coefficients in a Fractional Integrodifferential Equation*

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Abstract The authors study an inverse problem for a fractional integrodifferential equation, which aims to determine simultaneously two time varying coefficients, a kernel function and a source function, from the additional integral overdetermination condition. By using the fixed point theorem in suitable Sobolev space, the global existence and uniqueness results of this inverse problem are obtained.

Keywords Inverse problem, Fractional integrodifferential equation, Existence, Uniqueness
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1 Introduction

In this paper, we discuss an inverse problem of determining two unknown coefficients, a kernel function and a source function, depending only on time in a fractional wave equation with memory effect. Let Ω be a bounded domain in $\mathbb{R}^N (N=1,2,3)$ with smooth boundary $\partial\Omega$, $Q_T := \Omega\times(0,T)$ and $\Sigma_T := \partial\Omega\times(0,T)$ for a given time $T>0$. We consider the following fractional integrodifferential equation:

$$\partial_t^\alpha u(x,t) - \mathscr{L}[u](x,t) + \int_0^t k(t-s)u(x,s)\mathrm{d}s = f(x)p(t), \quad (x,t)\in Q_T \tag{1.1}$$

with the Caputo time fractional derivative ∂_t^α of order $1<\alpha<2$, defined by

$$\partial_t^\alpha v(t) = \frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{1-\alpha} v''(s)\mathrm{d}s,$$

where Γ is the Gamma function, see [16] or [23] for details on the fractional derivative. Here the operator $\mathscr{L}$ is uniformly elliptic on $\overline{\Omega}$, defined by

$$\mathscr{L}[u](x) = \sum_{i=1}^N \frac{\partial}{\partial x_i}\Big(\sum_{j=1}^N A_{ij}(x)\frac{\partial}{\partial x_j}u(x)\Big), \quad x\in\Omega,$$

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where $A_{ij} = A_{ji}$, $1 \leq i, j \leq N$, are smooth functions, and there exist positive constants ν_1 and ν_2, such that

$$\nu_1 \sum_{i=1}^{N} |\xi_i|^2 \leq \sum_{i=1}^{N} A_{ij}(x)\xi_i\xi_j \leq \sum_{i=1}^{N} \nu_2 |\xi_i|^2, \quad x \in \overline{\Omega},\ \xi \in \mathbb{R}^N.$$

We supplement the above fractional wave equation with the following initial condition:

$$u(x,0) = a(x), \quad u_t(x,0) = b(x), \quad x \in \Omega \tag{1.2}$$

and the boundary condition

$$u(x,t) = 0, \quad (x,t) \in \Sigma_T. \tag{1.3}$$

Many modern science and engineering technology areas can be described very successfully by models using fractional differential equations (see [18, 27, 33, 35]). The direct problem was extensively studied by many authors (see, e.g., [2, 15, 17, 22, 26] and the references therein). In practical situations, the function k represents some physical property, which is very hard to be measured directly in advance. So we consider an inverse problem of determining convolution kernel function k from some additional measurements on u.

In this paper, we take the following additional conditions of integral overdetermination:

$$\mathscr{H}_i[u(\cdot,t)] = g_i(t), \quad i = 1, 2, \tag{1.4}$$

where $\mathscr{H}_i$ are defined by

$$\mathscr{H}_i[h] := \int_{\Omega} \phi_i(x) h(x,t) \mathrm{d}x, \quad i = 1, 2$$

with known functions ϕ_i. Here $g_i(t)$ are the measurement data representing the average temperature on a small part of Ω, because the weight functions $\phi_i(x)$ are usually chosen to satisfy $\mathrm{Supp}(\phi_i) \Subset \Omega$ in practice.

The inverse problem considered in this paper is stated as follows.

Inverse Problem Determine $u \in C([0,T]; H^2(\Omega)) \cap C^1([0,T]; L^2(\Omega))$, $p \in C^1[0,T]$ and $k \in C[0,T]$ from (1.1)–(1.3) and the additional measurements (1.4).

As for inverse convolution kernel problems for the integer order integrodifferential equation with $\alpha = 1$ or 2, Colombo and Guidetti gave an efficient strategy to prove global in time existence and uniqueness results based by using analytic semigroup theory (see [7]). Colombo and Guidetti showed that a semilinear integrodifferential parabolic inverse problem has a unique solution global in time under suitable growth conditions for the nonlinearity involved in the evolution equation (see [7]). Lorenzi and Rocca [21] proved the local existence and the global uniqueness of an identification problem, which focuses on recovering two unknown convolution kernels in a phase-field system coupling two hyperbolic integro-differential equations. For inverse problems related to other models, we refer to [4–6, 8–10, 13, 20].

Recently, the subject of inverse problem for the fractional differential equations received much attention. Cheng, Nakagawa and Yamamoto [3] obtained the uniqueness in determining α and a diffusion coefficient varying spatial variable on the basis of Gel'fand-Levitan theory.

Sakamoto and Yamamoto [25] proved the well-posedness in the Hadamard sense for an inverse problem of determining a spatially varying function of the source by final over-determined data. As for other kinds of inverse problems related to the fractional differential equations, we refer to [14, 19, 29–30, 32, 34]. However, all these papers focused on the cases without integral term, i.e., $k \equiv 0$. To the authors' knowledge, there are no works published concerning the identification of the kernel function and the source function simultaneously for the fractional integrodifferential equation, even the single source function identification problem.

In this paper, we investigate the global existence and uniqueness of our inverse problem. First, we prove a local existence of (u, p, k) in a suitable Sobolev space by using a fixed point argument. Then, we give the proof of global uniqueness result. Finally, with the aid of splitting process of convolution term, which was successfully used to prove the global existence for the strongly damped wave equation in [5], we prove the global existence of (u, p, k).

Next we give some notations which will be repeatedly used in the sequent sections.

For any integer m, we denote by $H^m(\Omega)$ the usual Sobolev spaces defined for spatial variable (see, e.g., [1]). For a given Banach space V on Ω, we use the notation $C^m([0,T];V)$ to denote the following space:

$$C^m([0,T];V) := \{u; \|D_t^\beta u(t)\|_V \text{ is continuous in } t \text{ on } [0,T] \ \text{ for all } 0 \le \beta \le m\}.$$

We endow $C^m([0,T];V)$ with the following norm making it be a Banach space:

$$\|u\|_{C^m([0,T];V)} = \sum_{\beta=0}^{m} \Big(\max_{0\le t\le T} \|D_t^\beta u(t)\|_V \Big).$$

For $r \in L^1(0,T)$ and $q : (0,T) \to V$, we define the convolution

$$(r * q)(x,t) := \int_0^t r(t-s)q(x,s)\mathrm{d}s,$$

whenever the integral has a meaning. We next define Banach space X_T by

$$X_T := C^1([0,T];L^2(\Omega)) \cap C([0,T];H^2(\Omega))$$

with the norm

$$\|u\|_{X_T} = \|u\|_{C^1([0,T];L^2(\Omega))} + \|u\|_{C([0,T];H^2(\Omega))}.$$

Furthermore, we set

$$\mathbf{Y}_T = X_T \times C^1[0,T] \times C[0,T]$$

endowed with the norm

$$\|(u,p,k)\|_{\mathbf{Y}_T} := \|u\|_{X_T} + \|p\|_{C^1[0,T]} + \|k\|_{C[0,T]}.$$

In order to discuss the uniformly elliptic operator $-\mathscr{L}$ conveniently, we denote the domain of $-\mathscr{L}$ by $\mathcal{D}(-\mathscr{L}) = H^2(\Omega) \cap H_0^1(\Omega)$. It is well-known that the operator $-\mathscr{L}$ has only real and simple eigenvalues λ_n, and with suitable numbering, we have $0 < \lambda_1 < \lambda_2 < \cdots$, $\lim_{n\to\infty} \lambda_n = \infty$. By φ_n, we denote the eigenfunction corresponding to λ_n, which satisfies $\|\varphi_n\|^2_{L^2(\Omega)} = (\varphi_n, \varphi_n) = 1$, where $(\,\cdot\,,\,\cdot\,)$ denotes the inner product in Hilbert space $L^2(\Omega)$. Then for $\gamma \in \mathbb{R}$, we define the function $\mathcal{D}((-\mathscr{L})^\gamma)$ by

$$\mathcal{D}((-\mathscr{L})^\gamma) = \Big\{ \psi \in L^2(\Omega); \sum_{n=1}^{\infty} \lambda_n^{2\gamma} |(\psi, \varphi_n)|^2 < \infty \Big\},$$

and that $\mathcal{D}((-\mathscr{L})^\gamma)$ is a Hilbert space with norm

$$\|\psi\|_{\mathcal{D}((-\mathscr{L})^\gamma)} = \Big\{ \sum_{n=1}^{\infty} \lambda_n^{2\gamma} |(\psi, \varphi_n)|^2 \Big\}^{\frac{1}{2}}.$$

Moreover, we introduce the Mittag-Leffler function in [23]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}$$

with $\alpha > 0$ and $\beta \in \mathbb{R}$. It is known that $E_{\alpha,\beta}(z)$ is an entire function in $z \in \mathbb{C}$.

We now give the definition of weak solution to (1.1)–(1.3), which is introduced by Sakamoto and Yamamoto in [24].

Definition 1.1 *We call u a weak solution to* (1.1)–(1.3) *if* (1.1) *holds in* $L^2(\Omega)$ *and* $u(\cdot, t) \in H_0^1(\Omega)$ *for almost all* $t \in (0, T)$, $u, \partial_t u \in C([0, T]; \mathcal{D}((-\mathscr{L})^\gamma))$ *and*

$$\lim_{t\to 0} \|u(\cdot, t) - a\|_{\mathcal{D}((-\mathscr{L})^{-\gamma})} = \lim_{t\to 0} \|\partial_t u(\cdot, t) - b\|_{\mathcal{D}((-\mathscr{L})^{-\gamma})} = 0 \tag{1.5}$$

with some $\gamma > \frac{N}{4} + 1$.

We make the following assumptions:
(H1) $\partial_t^\alpha g_1, \partial_t^\alpha g_2 \in C^1[0, T]$, $a \in H^2(\Omega) \cap H_0^1(\Omega)$, $b \in H_0^1(\Omega)$, $f \in \mathcal{D}((-\mathscr{L})^{\frac{1}{\alpha}})$;
(H2) $\partial_t^\alpha g_i(0) - \mathscr{H}_i[\mathscr{L}[a]] = \mathscr{H}_i[f]p(0)$, $i = 1, 2$;
(H3) $\mathscr{H}_i[a] = g_i(0)$, $\mathscr{H}_i[b] = g_i'(0)$, $i = 1, 2$;
(H4) $c_0 := -\mathscr{H}_1[f(x)]g_2(0) + \mathscr{H}_2[f(x)]g_1(0) \neq 0$;
(H5) $\phi_i(x) \in H_0^2(\Omega)$, $i = 1, 2$.

Remark 1.1 In (H1), $\partial_t^\alpha g_i \in C^1[0, T]$ implies $g_i \in H^1[0, T]$, which will be used in Lemma 3.4 below. Indeed, by $\partial_t^\alpha g_i(t) = D_t^{-(2-\alpha)} g_i''(t)$ and $D_t^{2-\alpha} D_t^{-(2-\alpha)} g_i''(t) = g_i''(t)$ (see [23, (2.114)]), we have

$$\begin{aligned} g_i''(t) &= D_t^{2-\alpha} D_t^{-(2-\alpha)} g_i''(t) = D_t^{2-\alpha} \partial_t^\alpha g_i(t) \\ &= \frac{1}{\Gamma(\alpha-1)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-s)^{\alpha-2} \partial_t^\alpha g_i(s) \mathrm{d}s \\ &= -\frac{1}{\Gamma(\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \partial_t^\alpha g_i(s) \mathrm{d}(t-s)^{\alpha-1} \\ &= \frac{1}{\Gamma(\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \Big(\int_0^t (t-s)^{\alpha-1} (\partial_t^\alpha g_i)'(s) \mathrm{d}s + \partial_t^\alpha g_i(0) t^{\alpha-1} \Big) \\ &= \frac{1}{\Gamma(\alpha)} \Big(-(\partial_t^\alpha g_i)'(0) t^{\alpha-1} + \frac{1}{\alpha-1} \int_0^t (t-s)^{\alpha-2} (\partial_t^\alpha g_i)'(s) \mathrm{d}s + \frac{\partial_t^\alpha g_i(0)}{\alpha-1} t^{\alpha-2} \Big), \end{aligned} \tag{1.6}$$

where $D_t^{2-\alpha}$ is the Riemann-Liouville fractional derivative, defined by

$$D_t^{2-\alpha} v(t) = \frac{1}{\Gamma(\alpha-1)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-s)^{\alpha-2} v(s) \mathrm{d}s$$

for $0 < 2-\alpha < 1$. Obviously, $\int_0^t (t-s)^{\alpha-2} \mathrm{d}s \leq C(\alpha, T)$, due to $0 < 2-\alpha < 1$. Therefore, from (1.6) and $\partial_t^\alpha g_i \in C^1[0, T]$, we conclude that $g_i \in W^{2,1}(0, T) \hookrightarrow H^1(0, T)$.

Remark 1.2 In order to guarantee $\partial_t^\alpha g_i \in C^1[0,T]$, we could give a usual regularity condition $g_i \in C^3[0,T]$, such that $g_i''(0) = 0$. In fact, by integration by parts and $g_i''(0) = 0$, we have

$$\partial_t^\alpha g_i(t) = \frac{1}{\Gamma(3-\alpha)} \int_0^t (t-s)^{2-\alpha} g_i'''(s)\mathrm{d}s.$$

This gives

$$\|(\partial_t^\alpha g_i)'\|_{C[0,T]} \le \max_{0\le t\le T} \left| \frac{1}{\Gamma(2-\alpha)} \int_0^t g_i'''(s)(t-s)^{1-\alpha}\mathrm{d}s \right| \le C(\alpha,T)\|g_i\|_{C^3[0,T]},$$

because of $\alpha - 1 \in (0,1)$.

Remark 1.3 (H2)–(H3) are the consistency conditions for our problem (1.1)–(1.4) when dealing with smooth solutions.

Remark 1.4 In engineering, $\phi_i(x)$ can be thought of as an internal (tiny) sensor (see [12, 28]) measuring the mean temperature in measurement area. Supp(ϕ) is always chosen small enough to make the measurement area very small. So hypothese (H5) is reasonable.

Our main result in this paper is the following global existence and uniqueness for our inverse problem.

Theorem 1.1 *Under hypotheses* (H1)–(H5)*, there exists a solution* $(u,p,k) \in X_T \times C^1[0,T] \times C[0,T]$ *to the inverse problem* (1.1)–(1.4) *for any* T*.*

In order to prove Theorem 1.1, we need the following two lemmas.

Lemma 1.1 *Under hypotheses* (H1)–(H5)*, there exists a sufficiently small* $\tau > 0$*, such that the inverse problem has a unique solution* $(u,p,k) \in X_\tau \times C^1[0,\tau] \times C[0,\tau]$*.*

Lemma 1.2 *Under hypotheses* (H1)–(H5)*, for given measurement data* $g_i(t)$ *for* $i = 1,2$ *in* (1.4)*, if the inverse problem* (1.1)–(1.4) *has two solutions* $(u_j,p_j,k_j) \in X_T \times C^1[0,T] \times C[0,T]$ $(j=1,2)$ *for any time* T*, then* $(u_1,p_1,k_1) = (u_2,p_2,k_2)$ *in* $[0,T]$*.*

The proofs of these two lemmas will be given in Sections 3–4.

2 Preliminary Results

In this section, we present some preliminary results, including well-posedness for a fractional differential equation, an equivalent lemma for our inverse problem and a technique result, which will be used in the proofs of our main results.

We first consider the following initial and boundary problem:

$$\begin{cases} \partial_t^\alpha u(x,t) - \mathscr{L}[u](x,t) = h(x,t), & (x,t) \in Q_T, \\ u(x,0) = a(x), \quad u_t(x,t) = b(x), & x \in \Omega, \\ u(x,t) = 0, & (x,t) \in \Sigma_T. \end{cases} \tag{2.1}$$

Based on the results of [24], we will prove the following well-posedness of (2.1). Different from [24], we need a better regularity to construct the fixed pointed operator in next section. Moreover, the constant C below in (2.2) and (2.4) should be taken apart from t, which is necessary to apply contraction mapping.

Lemma 2.1 *Let $a \in H^2(\Omega)\cap H_0^1(\Omega)$, $b \in H_0^1(\Omega)$ and $h \in C([0,T];\mathcal{D}((-\mathscr{L})^{\frac{1}{\alpha}}))$. Then there exists a unique weak solution $u \in X_T$ to (2.1), such that*

$$\|u(\cdot,t)\|_{H^2(\Omega)} + \|u_t(\cdot,t)\|_{L^2(\Omega)} \le C(1+t^{\alpha-1})\|a\|_{H^2(\Omega)} + C(1+t^{1-\frac{\alpha}{2}})\|b\|_{H^1(\Omega)} + C(t^{\alpha-1}+t)\|h\|_{C([0,t];\mathcal{D}((-\mathscr{L})^{\frac{1}{\alpha}}))} \tag{2.2}$$

for all $t \in [0,T]$, where the constant C is dependent on α,Ω and the coefficients of $\mathscr{L}$, but independent of T. Furthermore, we have

$$\begin{aligned} u(x,t) = &\sum_{n=1}^{\infty}[(a,\varphi_n)E_{\alpha,1}(-\lambda_n t^\alpha) + (b,\varphi_n)tE_{\alpha,2}(-\lambda_n t^\alpha)]\varphi_n(x) \\ &+ \sum_{n=1}^{\infty}\Big[\int_0^t (h(\cdot,\tau),\varphi_n)(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)\mathrm{d}s\Big]\varphi_n(x). \end{aligned} \tag{2.3}$$

For any $t \in (0,T]$, we also have the following estimate:

$$\|u_t(\cdot,t)\|_{H^1(\Omega)} \le Ct^{-1}\|a\|_{H^2(\Omega)} + C\|b\|_{H^1(\Omega)} + Ct^{\alpha-1}\|h\|_{C([0,t];\mathcal{D}((-\mathscr{L})^{\frac{1}{\alpha}}))}. \tag{2.4}$$

Remark 2.1 The estimate (2.2) will be used to construct the fixed pointed operator in the next section, which is the key ingredient to prove the local in time existence. In the proof of the global in time existence, we need (2.4) to extend repeatedly the local solution to a larger time interval.

To prove Lemma 2.1, we first give a property of the Mittag-Leffler function, i.e., the following Lemma 2.2.

Lemma 2.2 *Let $\beta \in \mathbb{R}$ be arbitrary and μ satisfy $\frac{\pi\alpha}{2} < \mu < \min\{\pi,\alpha\pi\}$. Then there exists a constant C depending on α,β and μ, such that*

$$|E_{\alpha,\beta}(z)| \le \frac{C}{1+|z|}, \quad \mu \le |\arg(z)| \le \pi. \tag{2.5}$$

The proof of this lemma can be found in [23], and we shall omit here.

Now we give the proof of Lemma 2.1.

Proof of Lemma 2.1 We first split (2.1) into the following two initial and boundary value problems:

$$\begin{cases} \partial_t^\alpha v(x,t) - \mathscr{L}[v](x,t) = 0, & (x,t)\in Q_T, \\ v(x,0) = a(x), \quad v_t(x,t) = b(x), & x \in \Omega, \\ v(x,t) = 0, & (x,t) \in \Sigma_T, \end{cases} \tag{2.6}$$

$$\begin{cases} \partial_t^\alpha w(x,t) - \mathscr{L}[w](x,t) = h(x,t), & (x,t)\in Q_T, \\ w(x,0) = 0, \quad w_t(x,t) = 0, & x \in \Omega, \\ w(x,t) = 0, & (x,t) \in \Sigma_T. \end{cases} \tag{2.7}$$

For $a \in H^2(\Omega)\cap H_0^1(\Omega)$, $b \in H_0^1(\Omega)$ and $h \in C([0,T];\mathcal{D}((-\mathscr{L})^{\frac{1}{\alpha}}))$, by Theorems 2.2–2.3 proved by Sakamoto and Yamamoto [24], there exist unique $v \in C([0,T];H^2(\Omega)\cap H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$ and $w \in C([0,T];H^2(\Omega)\cap H_0^1(\Omega))$ satisfying (2.6) and (2.7), respectively. And we have

$$v(x,t) = \sum_{n=1}^{\infty}[(a,\varphi_n)E_{\alpha,1}(-\lambda_n t^\alpha) + (b,\varphi_n)tE_{\alpha,2}(-\lambda_n t^\alpha)]\varphi_n(x), \tag{2.8}$$

$$w(x,t)=\sum_{n=1}^{\infty}\Big[\int_0^t (h(\cdot,s),\varphi_n)(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^{\alpha})\mathrm{d}s\Big]\varphi_n(x). \tag{2.9}$$

Therefore, (2.1) has a unique solution $u=v+w$ given by (2.3).

Next we prove (2.2). By [24, Corollary 2.7, Theorem 2.3], we have

$$\|v_t(\cdot,t)\|_{L^2(\Omega)}\le C(t^{\alpha-1}\|a\|_{H^2(\Omega)}+\|b\|_{L^2(\Omega)}),\quad t\in[0,T] \tag{2.10}$$

and

$$\|v(\cdot,t)\|_{H^2(\Omega)}\le C(\|a\|_{H^2(\Omega)}+t^{1-\frac{\alpha}{2}}\|b\|_{H^1(\Omega)}),\quad t\in[0,T]. \tag{2.11}$$

On the other hand, using [23, (1.83)], we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha}))=t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda t^{\alpha}), \tag{2.12}$$

from which it follows that

$$\frac{\partial}{\partial t}w(x,t)=\sum_{n=1}^{\infty}\Big[\int_0^t (h(\cdot,s),\varphi_n)(t-s)^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_n(t-s)^{\alpha})\mathrm{d}s\Big]\varphi_n(x). \tag{2.13}$$

So, combining the above result with Lemma 2.2 and $\mathcal{D}((-\mathscr{L})^{\frac{1}{\alpha}})\subset L^2(\Omega)$, we have

$$\begin{aligned}\|w_t(\cdot,t)\|^2_{L^2(\Omega)}&\le\sum_{n=1}^{\infty}\Big|\int_0^t (h(\cdot,s),\varphi_n)(t-s)^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_n(t-s)^{\alpha})\mathrm{d}s\Big|^2\\&\le\sum_{n=1}^{\infty}\max_{0\le s\le t}|(h(\cdot,s),\varphi_n)|^2\Big|\int_0^t s^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_n s^{\alpha})\mathrm{d}s\Big|^2\\&\le\|h\|^2_{C([0,t];L^2(\Omega))}\Big|\int_0^t s^{\alpha-2}\frac{1}{1+\lambda_n s^{\alpha}}\mathrm{d}s\Big|^2\\&\le C\|h\|_{C([0,t];\mathcal{D}((-\mathscr{L})^{\frac{1}{\alpha}}))}t^{2\alpha-2},\quad t\in[0,T].\end{aligned} \tag{2.14}$$

Similarly, we have

$$\begin{aligned}\|w(\cdot,t)\|^2_{H^2(\Omega)}&\le\sum_{n=1}^{\infty}\lambda_n^2\Big|\int_0^t (h(\cdot,s),\varphi_n)(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^{\alpha})\mathrm{d}s\Big|^2\\&\le\sum_{n=1}^{\infty}\max_{0\le s\le t}|((-\mathscr{L})^{\frac{1}{\alpha}}[h](\cdot,s),\varphi_n)|^2\Big|\int_0^t\frac{(\lambda_n(t-s)^{\alpha})^{\frac{\alpha-1}{\alpha}}}{1+\lambda_n(t-s)^{\alpha}}\mathrm{d}s\Big|^2\\&\le C\|h\|^2_{C([0,t];\mathcal{D}((-\mathscr{L})^{\frac{1}{\alpha}}))}t^2,\quad t\in[0,T].\end{aligned} \tag{2.15}$$

Here in the last inequality of (2.15), we have used

$$\max_{0\le y<+\infty}\frac{y^{\frac{\alpha-1}{\alpha}}}{1+y}=\frac{(\alpha-1)^{\frac{\alpha-1}{\alpha}}}{1+(\alpha-1)}.$$

From (2.10)–(2.11) and (2.14)–(2.15), we get the desired estimate (2.2).

Finally, we prove (3.4). Differentiating (2.8) with respect to t leads to

$$v_t(x,t)=\sum_{n=1}^{\infty}\{-\lambda_n t^{\alpha-1}(a,\varphi_n)E_{\alpha,\alpha}(-\lambda_n t^{\alpha})+(b,\varphi_n)E_{\alpha,1}(-\lambda_n t^{\alpha})\}\varphi_n(x). \tag{2.16}$$

By the estimate for λ_n in [11],

$$\lambda_n \geq C n^{\frac{2}{N}}, \quad \forall n \in \mathbb{N},$$

we can choose C_* sufficiently large, such that

$$C_* \lambda_n \geq C_* C n^{\frac{2}{N}} > 1, \quad \forall n \in \mathbb{N},$$

which implies

$$(C_*\lambda_n)^{\frac{1}{\beta}} \leq C_*\lambda_n, \quad \forall n \in \mathbb{N},\ \forall \beta > 1. \tag{2.17}$$

Then noticing that $\mathcal{D}((-\mathscr{L})^{\frac{1}{2}}) = H_0^1(\Omega)$, and applying (2.17) and Lemma 2.2, we obtain

$$\begin{aligned}
\|v_t(\cdot,t)\|^2_{H^1(\Omega)} &\leq \sum_{n=1}^{\infty} \lambda_n \big| -\lambda_n t^{\alpha-1}(a,\varphi_n) E_{\alpha,\alpha}(-\lambda_n t^\alpha) + (b,\varphi_n)E_{\alpha,1}(-\lambda_n t^\alpha)\big|^2 \\
&\leq C \sum_{n=1}^{\infty} \Big(\Big| C_*^{\frac{1}{2}} \lambda_n (a,\varphi_n) \frac{\lambda_n t^\alpha}{1+\lambda_n t^\alpha}\Big|^2 t^{-2} + \Big| \lambda_n^{\frac{1}{2}} (b,\varphi_n) \frac{1}{1+\lambda_n t^\alpha}\Big|^2 \Big) \\
&\leq C\|a\|^2_{H^2(\Omega)} t^{-2} + C\|b\|^2_{H^1(\Omega)}, \quad t \in (0,T].
\end{aligned} \tag{2.18}$$

On the other hand, using (2.13) and

$$\lambda_n^{\frac{1}{2}-\frac{1}{\alpha}} \leq \lambda_1^{\frac{1}{2}-\frac{1}{\alpha}}, \quad n = 1, 2, \cdots,$$

we have

$$\begin{aligned}
\|w_t(\cdot,t)\|^2_{H^1(\Omega)} &\leq \sum_{n=1}^{\infty} \lambda_n \Big| \int_0^t (h(\cdot,s),\varphi_n)(t-s)^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n (t-s)^\alpha) \mathrm{d}s \Big|^2 \\
&\leq \sum_{n=1}^{\infty} \max_{0\leq s\leq t} |\lambda_n^{\frac{1}{\alpha}} (h(\cdot,s),\varphi_n)|^2 \Big| \int_0^t \lambda_n^{\frac{1}{2}-\frac{1}{\alpha}} (t-s)^{\alpha-2} \frac{1}{1+\lambda_n(t-s)^\alpha} \mathrm{d}s \Big|^2 \\
&\leq C\lambda_1^{1-\frac{2}{\alpha}} \|h\|^2_{C([0,t];\mathcal{D}((-\mathscr{L})^{\frac{1}{\alpha}}))} t^{2\alpha-2}, \quad t \in (0,T].
\end{aligned} \tag{2.19}$$

Summing up (2.18)–(2.19) yields (2.4). This completes the proof of Lemma 2.1.

The next lemma aims to transfer the original inverse problem (1.1)–(1.4) to a new form including the explicit expression of $p(t)$ and $k(t)$.

Lemma 2.3 *Let*

$$l(t) = \int_0^t k(s)\mathrm{d}s.$$

If the inverse problem (1.1)–(1.4) *is solvable, then so is the following system:*

$$\begin{cases}
\partial_t^\alpha u(x,t) - \mathscr{L}[u](x,t) + \displaystyle\int_0^t k(s)u(x,s)\mathrm{d}s = f(x)p(t), & (x,t) \in Q_T, \\
u(x,0) = a(x), \quad u_t(x,t) = b(x), & x \in \Omega, \\
u(x,t) = 0, & (x,t) \in \Sigma_T
\end{cases} \tag{2.20}$$

with

$$p(t) = \frac{1}{c_0}[-g_2(0)\mathscr{N}_1[u,l](t) + g_1(0)\mathscr{N}_2[u,l](t)], \tag{2.21}$$

$$k(t)=\frac{1}{c_0}[\mathscr{H}_1[f]D_t\mathscr{N}_2[u,l](t)-\mathscr{H}_2[f]D_t\mathscr{N}_1[u,l](t)], \tag{2.22}$$

where c_0 *is the same one in* (H4) *and* $\mathscr{N}_i$ $(i=1,2)$ *are defined by* (2.25) *below. On the other hand, if* (2.20)–(2.22) *has a solution and the compatibility conditions* (H2), (H3) *and the technical condition* (H4) *hold, then there exists a solution to the inverse problem* (1.1)–(1.4).

Remark 2.2 From Lemma 2.3, we know that (2.20)–(2.22) is an equivalent form of the original inverse problem (1.1)–(1.4). So, in the following several sections, we turn to discuss (2.20)–(2.22), other than the original one.

Proof of Lemma 2.3 We split the proof into two steps.

Step 1 Assume that (1.1)–(1.4) has a solution $(u,p,k)\in\mathbf{Y}_T$. Applying $\mathscr{H}_i$ to both sides of (1.1) yields

$$\mathscr{H}_i[f]p(t)=\partial_t^\alpha g_i(t)-\mathscr{H}_i[\mathscr{L}[u]](t)+k(t)*g_i(t),\quad i=1,2. \tag{2.23}$$

We note that $l(t)=\int_0^t k(s)\mathrm{d}s$. Then by integration by parts, we get the following equality:

$$\int_0^t k(s)g_i(t-s)\mathrm{d}s=l(t)g_i(0)+\int_0^t l(t-s)g_i'(s)\mathrm{d}s. \tag{2.24}$$

With the help of (2.24), we can rewrite (2.23) as

$$\begin{aligned}\mathscr{H}_i[f]p(t)-g_i(0)l(t)&=\partial_t^\alpha g_i(t)-\mathscr{H}_i[\mathscr{L}[u]](t)+\int_0^t l(t-s)g_i'(s)\mathrm{d}s\\&:=\mathscr{N}_i[u,l],\quad i=1,2.\end{aligned} \tag{2.25}$$

Due to (H4), we can solve this system to get (2.21) and

$$l(t)=\frac{1}{c_0}[\mathscr{H}_1[f]\mathscr{N}_2[u,l](t)-\mathscr{H}_2[f]\mathscr{N}_1[u,l](t)]. \tag{2.26}$$

Furthermore, by differentiating (2.26) with respect to t, we get (2.22).

Step 2 Now we assume that (u,p,k) satisfies (2.20)–(2.22). In order to prove that (u,p,k) is the solution to the inverse problem (1.1)–(1.4), it suffices to show that (u,p,k) satisfies (1.4). Applying $\mathscr{H}_i$ to the equation in (2.20), we have

$$\partial_t^\alpha\mathscr{H}_i[u](t)-\mathscr{H}_i[\mathscr{L}[u]](t)+k(t)*\mathscr{H}_i[u](t)=\mathscr{H}_i[f]p(t). \tag{2.27}$$

On the other hand, from (H2), we easily see that

$$\frac{1}{c_0}[\mathscr{H}_1[f]\mathscr{N}_2[u,l](0)-\mathscr{H}_2[f]\mathscr{N}_1[u,l](0)]=0.$$

We get (2.26) by integrating (2.22) over $[0,t]$. From (2.21) and (2.26), we conclude that

$$\begin{aligned}\mathscr{H}_i[f]p(t)&=g_i(0)l(t)+\partial_t^\alpha g_i(t)-\mathscr{H}_i[\mathscr{L}[u]](t)+\int_0^t l(t-s)g_i'(s)\mathrm{d}s\\&=\partial_t^\alpha g_i(t)-\mathscr{H}_i[\mathscr{L}[u]](t)+k(t)*g_i(t).\end{aligned} \tag{2.28}$$

Then substituting (2.28) into (2.27), and using (H3), we have that $H_i(t) := \mathscr{H}_i[u](t) - g_i(t)$ $(i = 1, 2)$ satisfy

$$\begin{cases} \partial_t^\alpha H_i(t) + \int_0^t k(t-s)H_i(s)\mathrm{d}s = 0, \quad t > 0, \\ H_i(0) = H_i'(0) = 0. \end{cases} \tag{2.29}$$

By means of Laplace transform of the Caputo derivative (see [23]), we have that

$$s^\alpha \widehat{H}_i(s) + \widehat{k}(s)\widehat{H}_i(s) = 0, \tag{2.30}$$

where $\widehat{H}_i$ and $\widehat{k}$ are the Laplace transforms of H_i and k, respectively. This leads to $\widehat{H}_i(s) = 0$, which implies that $H_i(t) = 0$. So $\mathscr{H}_i[u] = g_i$ $(i = 1, 2)$. This completes the proof of Lemma 2.3.

At the end of this section, we give a technical lemma which will be used to estimate p, k in suitable Sobolev space in subsequent sections.

Lemma 2.4 *Let* (H1) *and* (H5) *hold. Then for all $u \in \mathbf{Y}_T$ and all $l \in C^1([0, T])$, there exists a constant $C > 0$ depending on g_i, ϕ_i, but independent of T, such that*

$$\|\mathscr{N}_i[u, l]\|_{C^1[0,T]} \le C(1 + \|u\|_{C([0,T];H^2(\Omega))} + \|u\|_{C^1([0,T];L^2(\Omega))} + T^{\frac{1}{2}}\|l\|_{C^1[0,T]}), \tag{2.31}$$

where $i = 1, 2$ and $\mathscr{N}_i$ are the same as those in (2.25).

Proof By the Hölder's inequality, we see that

$$\begin{aligned} \|\mathscr{N}_i[u, l]\|_{C[0,T]} &\le \|\partial_t^\alpha g_i\|_{C[0,T]} + \Big\| \int_\Omega \phi_i(x)\mathscr{L}[u](x, t)\mathrm{d}x \Big\|_{C[0,T]} + \|l * g_i'\|_{C[0,T]} \\ &\le \|\partial_t^\alpha g_i\|_{C[0,T]} + \|\phi\|_{L^2(\Omega)}\|u\|_{C([0,T];H^2(\Omega))} + T^{\frac{1}{2}}\|l\|_{C[0,T]}\|g_i'\|_{L^2(0,T)}. \end{aligned} \tag{2.32}$$

On the other hand, a direct calculation yields

$$D_t\mathscr{N}_i[u, l] = (\partial_t^\alpha g)' - \mathscr{H}_i[\mathscr{L}[u_t]] + l' * g_i'. \tag{2.33}$$

Here we note that $l(0) = 0$. By integration by parts, it follows from (H5) that

$$\mathscr{H}_i[\mathscr{L}[u_t]](t) = \int_\Omega \phi_i(x)\mathscr{L}[u_t](x, t)\mathrm{d}x = \int_\Omega \mathscr{L}[\phi_i](x)u_t(x, t)\mathrm{d}x, \tag{2.34}$$

which gives

$$\|\mathscr{H}_i[\mathscr{L}[u_t]]\|_{C[0,T]} \le C\|\phi_i\|_{H^2(\Omega)}\|u_t\|_{C([0,T];L^2(\Omega))}. \tag{2.35}$$

Hence, we have

$$\begin{aligned} &\|D_t\mathscr{N}_i[u, l]\|_{C[0,T]} \\ &\le \|\partial_t^\alpha g\|_{C^1[0,T]} + C\|\phi_i\|_{H^2(\Omega)}\|u_t\|_{C([0,T];L^2(\Omega))} + T^{\frac{1}{2}}\|l'\|_{C[0,T]}\|g_i'\|_{L^2(0,T)}. \end{aligned} \tag{2.36}$$

Combination (2.32) with (2.36) yields the desired estimate (2.31). This completes the proof of Lemma 2.4.

3 Proof of Lemma 1.1

We are now in a position to prove local in time existence, i.e., Lemma 1.1, which proceeds by the fixed point arguments. First, we define the function set

$$\begin{aligned}\mathscr{Z}_{T,M} = \{(\overline{u},\overline{p},\overline{k}) \in \mathbf{Y}_T : \overline{u}(x,0) = a(x),\ \overline{u}_t(x,0) = b(x),\ \overline{u}(x,t) = 0,\ (x,t) \in \Sigma_T,\\ \|\overline{u}\|_{X_T} + \|p\|_{C^1[0,T]} + \|\overline{k}\|_{C[0,T]} \le M\}.\end{aligned}$$

Here M is a large constant depending on the initial and boundary data a, b and measurement data g_i. For given $(\overline{u},\overline{p},\overline{k}) \in \mathscr{Z}_{T,M}$, we consider

$$\begin{cases}\partial_t^\alpha u(x,t) - \mathscr{L}[u](x,t) = -\overline{k}(t) * \overline{u}(x,t) + f(x)\overline{p}(t), & (x,t) \in Q_T,\\ u(x,0) = a(x),\quad u_t(x,0) = b(x), & x \in \Omega,\\ u(x,t) = 0, & (x,t) \in \Sigma_T\end{cases} \tag{3.1}$$

and

$$p(t) = \frac{1}{c_0}[-g_2(0)\mathscr{N}_1[u,\overline{l}\,](t) + g_1(0)\mathscr{N}_2[u,\overline{l}\,](t)], \tag{3.2}$$

$$k(t) = \frac{1}{c_0}[\mathscr{H}_1[f]D_t\mathscr{N}_2[u,\overline{l}\,](t) - \mathscr{H}_2[f]D_t\mathscr{N}_1[u,\overline{l}](t)] \tag{3.3}$$

to generate (u,p,k), where $\overline{l}$ is defined by $\overline{l}(t) = \int_0^t \overline{k}(s)\mathrm{d}s$, $\mathscr{N}_i$ $(i = 1,2)$ are the same as those in (2.25).

Remark 3.1 Usually, we use $\overline{u}$ on the right-hand sides of (3.2)–(3.3) to generate p and k. But, if we do so, we can not choose suitable T and R to prove $\|p\|_{C^1[0,T]} \le R$ and $\|k\|_{C[0,T]} \le R$, because there is a lack of T in the terms including u in (2.31). In this situation, the fixed point argument can not be applied to our problem. So we take the solution u to (3.1) to generate p and k. This process in principle is similar to the Gauss-Seidel iteration.

By (2.17) and the Hölder's inequality, we have

$$\begin{aligned}\|\overline{k} * \overline{u}(\cdot,t)\|^2_{\mathcal{D}((-\mathscr{L})^{\frac{1}{\alpha}})} &= \sum_{n=1}^\infty |\lambda_n^{\frac{1}{\alpha}}(\overline{k} * \overline{u},\varphi_n)|^2 \le \sum_{n=1}^\infty C_*^{2-\frac{2}{\alpha}}|\lambda_n(\overline{k} * \overline{u},\varphi_n)|^2\\ &\le C\|\overline{k} * \overline{u}(\cdot,t)\|^2_{H^2(\Omega)}\\ &\le C\int_0^t |k(t-s)|^2\mathrm{d}s \int_0^t \|u(\cdot,s)\|^2_{H^2(\Omega)}\mathrm{d}s\\ &\le Ct^2\|\overline{k}\|^2_{C[0,t]}\|\overline{u}\|^2_{C([0,t];H^2(\Omega))},\end{aligned} \tag{3.4}$$

which implies

$$\|\overline{k} * \overline{u}\|_{C([0,T];\mathcal{D}((-\mathscr{L})^{\frac{1}{\alpha}}))} \le C(M,T).$$

Using this result together with $\overline{p} \in C^1[0,T]$ and $f \in \mathcal{D}((-\mathscr{L})^{\frac{1}{\alpha}})$, we have

$$-\overline{k}(t) * \overline{u}(x,t) + \overline{p}(t)f(x) \in C([0,T];\mathcal{D}((-\mathscr{L})^{\frac{1}{\alpha}})).$$

Therefore, Lemma 2.1 ensures that there exists a unique solution $u \in X_T$ to (3.1). Then (3.2)–(3.3) define the functions $p(t)$ and $k(t)$ in terms of u. Furthermore, by Lemma 3.4, we have

$$\|p\|_{C^1[0,T]} + \|k\|_{C[0,T]} \le C(T)(1 + \|u\|_{X_T} + \|\overline{l}\|_{C^1[0,T]}). \tag{3.5}$$

Note that $\overline{l}(t)=\int_0^t\overline{k}(s)\mathrm{d}s$. We obtain

$$\|\overline{l}\|_{C^1[0,T]}=\Big\|\int_0^t\overline{k}(s)\mathrm{d}s\Big\|_{C[0,T]}+\|\overline{k}\|_{C[0,T]}\le(1+T)\|\overline{k}\|_{C[0,T]}. \tag{3.6}$$

Substituting (3.6) into (3.5) yields

$$\|p\|_{C^1[0,T]}+\|k\|_{C[0,T]}\le C(T)(1+\|u\|_{X_T}+\|\overline{k}\|_{C([0,T])}). \tag{3.7}$$

This implies that $p\in C^1[0,T]$ and $k\in C[0,T]$.

Thus the mapping

$$S:\mathscr{Z}_{T,M}\to\mathbf{Y}_T,\quad(\overline{u},\overline{p},\overline{k})\mapsto(u,p,k)$$

given by (3.1)–(3.3) is well-defined.

Now we show that S maps $\mathscr{Z}_{T,M}$ into itself for sufficiently small $T>0$. More precisely, we have the following result.

Lemma 3.1 *For* $(\overline{u},\overline{p},\overline{k}),(\overline{U},\overline{P},\overline{K})\in\mathscr{Z}_{T,M}$, *define*

$$(u,p,k)=S(\overline{u},\overline{p},\overline{k}),\quad(U,P,K)=S(\overline{U},\overline{P},\overline{K}).$$

Then for properly small $\tau>0$, *we have*

$$\|(u,p,k)\|_{\mathbf{Y}_T}\le M \tag{3.8}$$

and

$$\|(u-U,p-P,k-K)\|_{\mathbf{Y}_T}\le\frac{1}{2}\|(\overline{u}-\overline{U},\overline{p}-\overline{P},\overline{k}-\overline{K})\|_{\mathbf{Y}_T} \tag{3.9}$$

for all $T\in(0,\tau]$.

Throughout the following proof, we use C to denote a constant which depends on Ω,α, the initial data a,b, the known functions f, ϕ_i and measurement data g_i, but independent of M and T.

Proof of Lemma 3.1 From Lemma 2.1 and (3.4), it follows that

$$\begin{aligned}\|u\|_{X_T}&\le C(1+T^{\alpha-1})\|a\|_{H^2(\Omega)}+C(1+T^{1-\frac{\alpha}{2}})\|b\|_{H^1(\Omega)}\\&\quad+C(T^{\alpha-1}+T)(\|\overline{k}*\overline{u}\|_{C([0,T];\mathcal{D}((-\mathscr{L})^{\frac{1}{\alpha}}))}+\|\overline{p}f\|_{C([0,T];\mathcal{D}((-\mathscr{L})^{\frac{1}{\alpha}}))})\\&\le C+C(T^{\alpha-1}+T^{1-\frac{\alpha}{2}})+C(T^{\alpha-1}+T)(TM^2+M).\end{aligned} \tag{3.10}$$

On the other hand, by (3.2)–(3.3), together with Lemma 2.4 and (3.6), we can show that $\|p\|_{C^1[0,T]}$ and $\|k\|_{C[0,T]}$ are bounded with

$$\begin{aligned}\|p\|_{C^1[0,T]}+\|k\|_{C[0,T]}&\le C(\|\mathscr{N}_1[u,\overline{l}\,]\|_{C^1[0,T]}+\|\mathscr{N}_2[u,\overline{l}\,]\|_{C^1[0,T]})\\&\le C(1+\|u\|_{X_T}+T^{\frac{1}{2}}\|\overline{l}\|_{C^1[0,T]})\\&\le C[1+\|u\|_{X_T}+T^{\frac{1}{2}}(1+T)M].\end{aligned} \tag{3.11}$$

Then, adding up (3.10)–(3.11) leads to

$$\|(u,p,k)\|_{\mathbf{Y}_T}\le C\omega_1(T)(1+M+M^2)+C, \tag{3.12}$$

where the function $\omega_1(T)$ is of the form

$$\omega_1(T) = T^{\alpha-1} + T^{1-\frac{\alpha}{2}} + T^{\alpha} + T + T^2 + T^{\frac{1}{2}} + T^{\frac{3}{2}}$$

and therefore satisfies $\lim_{T\to 0^+} \omega_1(T) = 0$. Now we take M, such that $M = 2C$ with the constant C in (3.12). Then there exists a sufficiently small $\tau > 0$, such that

$$\|(u,p,k)\|_{\mathbf{Y}_T} \le M \tag{3.13}$$

for all $T \in (0,\tau]$. That is, S maps $\mathcal{Z}_{T,M}$ into itself for each fixed $T \in (0,\tau]$.

Next we estimate the increment of operator S. To this end, we deduce the differences $(u-U, p-P, k-K)$ from (3.1)–(3.3) to yield

$$\begin{cases} \partial_t^\alpha (u-U) - \mathscr{L}[u-U] = -(\overline{k}-\overline{K}) * \overline{u} - \overline{K} * (\overline{u}-\overline{U}) + (\overline{p}-\overline{P})f, & (x,t) \in Q_T, \\ (u-U)(x,0) = (u-U)_t(x,0) = 0, & x \in \Omega, \\ (u-U)(x,t) = 0, & (x,t) \in \Sigma_T \end{cases} \tag{3.14}$$

and

$$p - P = \frac{1}{c_0}[-g_2(0)(\mathscr{N}_1[u,\overline{l}\,] - \mathscr{N}_1[U,\overline{L}\,]) + g_1(0)(\mathscr{N}_2[u,\overline{l}\,] - \mathscr{N}_2[U,\overline{L}\,])], \tag{3.15}$$

$$k - K = \frac{1}{c_0}[\mathscr{H}_1[f](D_t\mathscr{N}_2[u,\overline{l}\,] - D_t\mathscr{N}_2[U,\overline{L}]) - \mathscr{H}_2[f](D_t\mathscr{N}_1[u,\overline{l}\,] - D_t\mathscr{N}_1[U,\overline{L}\,])], \tag{3.16}$$

where $\overline{l}$ and $\overline{L}$ satisfy $\overline{l}(t) = \int_0^t \overline{k}(s)\mathrm{d}s$ and $L(t) = \int_0^t \overline{K}(s)\mathrm{d}s$, respectively.

Using Lemma 2.1 and (3.4), we obtain

$$\begin{aligned} &\|u-U\|_{X_T} \\ &\le C(T^{\alpha-1}+T)(\|(\overline{k}-\overline{K}) * \overline{u}\|_{C([0,T];\mathcal{D}((-\mathscr{L})^{\frac{1}{\alpha}}))} + \|\overline{K} * (\overline{u}-\overline{U})\|_{C([0,T];\mathcal{D}((-\mathscr{L})^{\frac{1}{\alpha}}))}) \\ &\quad + C(T^{\alpha-1}+T)\|(\overline{p}-\overline{P})f\|_{C([0,T];\mathcal{D}((-\mathscr{L})^{\frac{1}{\alpha}}))} \\ &\le C(T^{\alpha-1}+T)(TM\|\overline{k}-\overline{K}\|_{C[0,T]} + TM\|\overline{u}-\overline{U}\|_{C[0,T];H^2(\Omega))}) \\ &\quad + C(T^{\alpha-1}+T)\|\overline{p}-\overline{P}\|_{C[0,T]}\|f\|_{\mathcal{D}((-\mathscr{L})^{\frac{1}{\alpha}})} \\ &\le C\omega_2(T)(1+M)(\|\overline{u}-\overline{U}\|_{X_T} + \|\overline{p}-\overline{P}\|_{C[0,T]} + \|\overline{k}-\overline{K}\|_{C[0,T]}), \end{aligned} \tag{3.17}$$

where

$$\omega_2(T) = T^{\alpha-1} + T^{\alpha} + T + T^2$$

satisfies $\lim_{T\to 0^+} \omega_2(T) = 0$. Moreover, from (2.25) and $\phi_i \in H_0^2(\Omega)$, we can easily see that

$$\begin{aligned} D_t\mathscr{N}_i[u,\overline{l}\,] - D_t\mathscr{N}_i[U,\overline{L}\,] &= -\int_\Omega \phi_i(x)(\mathscr{L}[u_t] - \mathscr{L}[U_t])(x,t)\mathrm{d}x + [(\overline{l}-\overline{L}) * g_i']' \\ &= -\int_\Omega \mathscr{L}[\phi_i](x)(u-U)_t(x,t)\mathrm{d}x + (\overline{k}-\overline{K}) * g_i'. \end{aligned} \tag{3.18}$$

Therefore, it follows that

$$\begin{aligned} &\|\mathscr{N}_i[u,\overline{l}\,] - \mathscr{N}_i[U,\overline{L}\,]\|_{C^1[0,T]} \\ &\le \|\phi_i\|_{H^2(\Omega)}\|u-U\|_{C^1([0,T];L^2(\Omega))} + T^{\frac{1}{2}}\|g_i'\|_{L^2[0,T]}(\|\overline{l}-\overline{L}\|_{C[0,T]} + \|\overline{k}-\overline{K}\|_{C[0,T]}) \end{aligned}$$

$$\le C\|u-U\|_{C^1([0,T];L^2(\Omega))}+CT^{\frac{1}{2}}(T+1)\|\overline{k}-\overline{K}\|_{C[0,T]}. \tag{3.19}$$

Then, from (3.15)–(3.16), together with (3.19), we derive

$$\|p-P\|_{C^1[0,T]}+\|k-K\|_{C[0,T]}\le C\|u-U\|_{X_T}+C(T^{\frac{1}{2}}+T^{\frac{3}{2}})\|\overline{k}-\overline{K}\|_{C[0,T]}. \tag{3.20}$$

From (3.17) and (3.20), we deduce that

$$\|(u-U,p-P,k-K)\|_{\mathbf{Y}_T}\le C\omega_2(T)(1+M)\|(\overline{u}-\overline{U},\overline{p}-\overline{P},\overline{k}-\overline{K})\|_{\mathbf{Y}_T}. \tag{3.21}$$

Because of $\lim_{T\to 0^+}\omega_2(T)=0$, we can obtain (3.9), if we choose τ sufficiently small, such that $C\omega_2(T)(1+M)\le\frac{1}{2}$ for all $T\in(0,\tau]$. The proof is complete.

Now we are in position to prove Lemma 1.1.

Proof of Lemma 1.1 Lemma 3.1 shows that there exists a sufficiently small $\tau>0$, such that S is a contraction mapping from $\mathcal{Z}_{\tau,M}$ to $\mathcal{Z}_{\tau,M}$. Therefore the Banach fixed point theorem concludes that for sufficiently small τ, there exists a unique solution $(u,p,k)\in X_\tau\times C^1[0,\tau]\times C[0,\tau]$ to the problem constituted by (3.1)–(3.3). As a consequence, the problem constituted by (1.1) and (1.4) also admits a solution (u,p,k) in $[0,\tau]$ by Lemma 2.3. The proof is complete.

4 Proof of Lemma 1.2

In this section, we give the proof of the global in time uniqueness of solutions to our inverse problem, i.e., Lemma 1.2.

Proof of Lemma 1.2 By Lemma 2.3, we know that (2.20)–(2.22) is equivalent to our considered inverse problem. So, in the following we turn to prove the global uniqueness of (2.20)–(2.22).

Given any time T, we assume that (u_i,p_i,k_i) $(i=1,2)$ are two solutions to the problem (2.20)–(2.22) in $[0,T]$ with the regularity $(u_i,p_i,k_i)\in X_T\times C^1[0,T]\times C[0,T]$. This implies

$$\|(u_i,p_i,k_i)\|_{\mathbf{Y}_T}\le C^*,\quad i=1,2, \tag{4.1}$$

where C^* is depending on α,Ω,T, the initial data a and b, the known functions f,ϕ_i and the measurement data g_i.

Let

$$\widetilde{u}=u_1-u_2,\quad \widetilde{p}=p_1-p_2,\quad \widetilde{k}=k_1-k_2.$$

Then $(\widetilde{u},\widetilde{p},\widetilde{k})$ satisfies

$$\begin{cases}\partial_t^\alpha\widetilde{u}-\mathscr{L}[\widetilde{u}]=-\widetilde{k}*u_1-k_2*\widetilde{u}+\widetilde{p}f, & (x,t)\in Q_T,\\ \widetilde{u}(x,0)=\widetilde{u}_t(x,0)=0, & x\in\Omega,\\ \widetilde{u}(x,t)=0, & (x,t)\in\Sigma_T\end{cases} \tag{4.2}$$

and

$$\widetilde{p}=\frac{1}{c_0}[-g_2(0)(\mathscr{N}_1[u_1,l_1]-\mathscr{N}_1[u_2,l_2])+g_1(0)(\mathscr{N}_2[u_1,l_1]-\mathscr{N}_2[u_2,l_2])], \tag{4.3}$$

$$\widetilde{k}=\frac{1}{c_0}[\mathscr{H}_1[f](D_t\mathscr{N}_2[u_1,l_1]-D_t\mathscr{N}_2[u_2,l_2])-\mathscr{H}_2[f](D_t\mathscr{N}_1[u_1,l_1]-D_t\mathscr{N}_1[u_2,l_2])]. \tag{4.4}$$

Here, in a way similar to l, the functions l_i $(i=1,2)$ satisfy $l_i(t)=\int_0^t k_i(s)\mathrm{d}s$. We need to prove

$$\|(\widetilde{u},\widetilde{p},\widetilde{k})\|_{\mathbf{Y}_T}=0. \tag{4.5}$$

Define

$$T_0=\inf\{t\in(0,T]:\|(\widetilde{u},\widetilde{p},\widetilde{k})\|_{\mathbf{Y}_t}>0\}.$$

If (4.5) is not true, then it is obvious that T_0 is well-defined and satisfies $0<T_0<T$. Now we choose a monotone sequence $\{t_n\}\subset(0,T_0)$ satisfying $\lim\limits_{n\to\infty}t_n=T_0$. Since $T_0<T$, we can choose a sufficiently small $\varepsilon>0$, such that

$$t_n+\varepsilon\le T,\quad n=1,2,\cdots.$$

From the definition of T_0, we see that

$$\widetilde{u}=\widetilde{p}=\widetilde{k}=0\quad\text{in }[0,t_n]. \tag{4.6}$$

Applying Lemma 3.1 to (4.2) in $[0,t_n+\varepsilon]$ and using (3.4), (4.1) and (4.6), we obtain

$$\begin{aligned}\|\widetilde{u}\|_{X_{t_n+\varepsilon}}&\le C(T)(\|\widetilde{k}*u_1\|_{C([t_n,t_n+\varepsilon];\mathcal{D}((-\mathscr{L})^{\frac{1}{\alpha}}))}+\|k_1*\widetilde{u}\|_{C([t_n,t_n+\varepsilon];\mathcal{D}((-\mathscr{L})^{\frac{1}{\alpha}}))})\\&\quad+C(T)\|\widetilde{p}f\|_{C([t_n,t_n+\varepsilon];\mathcal{D}((-\mathscr{L})^{\frac{1}{\alpha}}))}\\&\le C(T)(\varepsilon C^*\|\widetilde{k}\|_{C[t_n,t_n+\varepsilon]}+\varepsilon C^*\|\widetilde{u}\|_{C([t_n,t_n+\varepsilon];H^2(\Omega))}+\|\widetilde{p}\|_{C[t_n,t_n+\varepsilon]}).\end{aligned} \tag{4.7}$$

Due to $\widetilde{p}(t_n)=0$, we have

$$\|\widetilde{p}\|_{C[t_n,t_n+\varepsilon]}=\max_{t_n\le t\le t_n+\varepsilon}\Big|\int_{t_n}^t\widetilde{p}'(s)\mathrm{d}s\Big|\le\varepsilon\|\widetilde{p}\|_{C^1[t_n,t_n+\varepsilon]}. \tag{4.8}$$

Substituting (4.8) into (4.7) yields

$$\|\widetilde{u}\|_{X_{t_n+\varepsilon}}\le C(T,C^*)\varepsilon\|(\widetilde{u},\widetilde{p},\widetilde{k})\|_{\mathbf{Y}_{t_n+\varepsilon}}. \tag{4.9}$$

Additionally, from a direct calculation, it follows for $i=1,2$ that

$$\mathscr{N}_i[u_1,l_1]-\mathscr{N}_i[u_2,l_2]=-\int_\Omega\mathscr{L}[\phi_i](x)\widetilde{u}(x,t)\mathrm{d}x+\int_0^t\widetilde{l}(t-s)g_i'(s)\mathrm{d}s \tag{4.10}$$

and

$$D_t\mathscr{N}_i[u_1,l_1]-D_t\mathscr{N}_i[u_2,l_2]=-\int_\Omega\mathscr{L}[\phi_i](x)\widetilde{u}_t(x,t)\mathrm{d}x+\int_0^t\widetilde{k}(t-s)g_i'(s)\mathrm{d}s, \tag{4.11}$$

by which and the Hölder's inequality, we have

$$\begin{aligned}&\|\mathscr{N}_i[u_1,l_1]-\mathscr{N}_i[u_2,l_2]\|_{C^1[t_n,t_n+\varepsilon]}\\&\le C\|\phi_i\|_{H^2(\Omega)}\|\widetilde{u}\|_{C^1([t_n,t_n+\varepsilon],L^2(\Omega))}+\varepsilon^{\frac{1}{2}}\|g_i'\|_{L^2(t_n,t_n+\varepsilon)}(\|\widetilde{l}\|_{C[t_n,t_n+\varepsilon]}+\|\widetilde{k}\|_{C[t_n,t_n+\varepsilon]})\\&\le C(T,\phi_i,g_i)(\|\widetilde{u}\|_{C^1([t_n,t_n+\varepsilon],L^2(\Omega))}+\varepsilon^{\frac{1}{2}}(\varepsilon+1)\|\widetilde{k}\|_{C[t_n,t_n+\varepsilon]}).\end{aligned} \tag{4.12}$$

Here we have used that

$$\|l\|_{C[t_n,t_n+\varepsilon]}=\max_{t_n\le t\le t_n+\varepsilon}\Big|\int_{t_n}^t\widetilde{k}(s)\mathrm{d}s\Big|\le\varepsilon\|\widetilde{k}\|_{C[t_n,t_n+\varepsilon]}.$$

Noticing that $\|\widetilde{p}\|_{C^1[0,t_n]} = \|\widetilde{k}\|_{C[0,t_n]} = 0$ and applying (4.3)–(4.4), (4.12), we have the following estimate for p and k:

$$\|\widetilde{p}\|_{C^1[0,t_n+\varepsilon]} + \|\widetilde{k}\|_{C[0,t_n+\varepsilon]} \leq C(T, \phi_i, g_i)(\|\widetilde{u}\|_{X_{t_n+\varepsilon}} + \varepsilon^{\frac{1}{2}}(\varepsilon+1)\|\widetilde{k}\|_{C[t_n,t_n+\varepsilon]}). \tag{4.13}$$

From (4.9) and (4.13), we deduce that

$$\|(\widetilde{u}, \widetilde{p}, \widetilde{k})\|_{\mathbf{Y}_{t_n+\varepsilon}} \leq C(T, \phi_i, g_i, C^*)\omega_3(\varepsilon)\|(\widetilde{u}, \widetilde{p}, \widetilde{k})\|_{\mathbf{Y}_{t_n+\varepsilon}} \tag{4.14}$$

with

$$\lim_{\varepsilon\to 0^+} \omega_3(\varepsilon) = \lim_{\varepsilon\to 0^+} (\varepsilon + \varepsilon^{\frac{1}{2}}(1+\varepsilon)) = 0.$$

So, for $\varepsilon > 0$ small enough, such that $C(T, \phi_i, g_i, C^*)\omega_3(\varepsilon) < 1$, we are led to

$$\|(\widetilde{u}, \widetilde{p}, \widetilde{k})\|_{\mathbf{Y}_{t_n+\varepsilon}} = 0, \quad n = 1, 2, \cdots .$$

By taking $n \to \infty$, we obtain

$$\widetilde{u} = \widetilde{p} = \widetilde{k} = 0 \quad \text{in } [0, T_0 + \varepsilon],$$

which contradicts with the definition of T_0. Therefore (4.5) is proved. Now we can conclude that

$$(u_1, p_1, k_1) = (u_2, p_2, k_2) \quad \text{in } [0, T]$$

for any time T. The proof for Lemma 1.2 is complete.

5 Proof of Theorem 1.1

Now we prove the global solvability Theorem 1.1 for our inverse problem. More precisely, for every given time $T > 0$, we will prove the existence of solutions to the problem constituted by (2.20)–(2.22), which is an equivalent form of our inverse problem.

In order to prove Theorem 1.1, we first show that local solution can be extended to a larger time interval in Subsection 5.1. Then, we give a preliminary estimate for the extension solution in Subsection 5.2. Finally, we prove Theorem 1.1 in Subsection 5.3.

5.1 Extension of the solution

Lemma 1.1 ensures that there exists a unique solution $(\widehat{u}, \widehat{p}, \widehat{k}) \in \mathbf{Y}_\tau$ to (2.20)–(2.22) for sufficiently small $\tau > 0$. In this subsection, we show that the unique solution $(\widehat{u}, \widehat{p}, \widehat{k})$ in $[0, \tau]$ can be extended to a larger time interval $[0, \tau']$, where $0 < \tau < \tau' \leq \min\{2\tau, T\}$. We state the result as follows.

Lemma 5.1 *Let $(\widehat{u}, \widehat{p}, \widehat{k}) \in \mathbf{Y}_\tau$ be the unique solution to* (2.20)–(2.22) *in $[0, \tau]$. Then there exists a $\tau' \in (\tau, \min\{2\tau, T\})$, such that $(\widehat{u}, \widehat{p}, \widehat{k})$ could be uniquely extended to a solution (u, p, k) in $[0, \tau']$, belonging to $\mathbf{Y}_{\tau'}$.*

Proof We consider

$$\begin{cases} \partial_t^\alpha u(x, \tau+t) - \mathscr{L}[u](x, \tau+t) + \displaystyle\int_0^{\tau+t} k(\tau+t-s)u(x,s)\mathrm{d}s \\ \quad = f(x)p(\tau+t), \quad (x,t) \in \Omega \times (0, \tau'-\tau), \\ u(x,\tau) = \widehat{u}(x,\tau), \quad u_t(x,\tau) = \widehat{u}_t(x,\tau), \quad x \in \Omega, \\ u(x, \tau+t) = 0, \quad (x,t) \in \partial\Omega \times [0, \tau'-\tau] \end{cases} \tag{5.1}$$

and

$$p(\tau+t)=\frac{1}{c_0}[-g_2(0)\mathscr{N}_1[u,l](\tau+t)+g_1(0)\mathscr{N}_2[u,l](\tau+t)],\quad t\in[0,\tau'-\tau], \tag{5.2}$$

$$k(\tau+t)=\frac{1}{c_0}[\mathscr{H}_1[f]D_t\mathscr{N}_2[u,l](\tau+t)-\mathscr{H}_2[f]D_t\mathscr{N}_1[u,l](\tau+t)],\quad t\in[0,\tau'-\tau]. \tag{5.3}$$

It suffices to show that (5.1)–(5.3) has a solution $(U(x,\tau+t),P(x,\tau+t),K(\tau+t))\in\mathbf{Y}_{\tau'-\tau}$. Then (u,p,k) defined by

$$(u(x,t),p(t),k(t))=\begin{cases}(\widehat{u}(x,t),\widehat{p}(t),\widehat{k}(t)), & \text{if } t\in[0,\tau],\\ (U(x,t),P(t),K(t)), & \text{if } t\in[\tau,\tau']\end{cases} \tag{5.4}$$

is an extension of $(\widehat{u},\widehat{p},\widehat{k})$ in $[0,\tau']$, which has the regularity $(u,p,k)\in\mathbf{Y}_{\tau'}$. Furthermore, from the global uniqueness result in Lemma 1.2, it follows that (u,p,k) given by (5.4) is the unique solution to the problem constituted by (2.20)–(2.22) in $t\in(0,\tau')$. This shows that the extension is unique.

Let

$$(u^\tau(x,t),p^\tau(t),k^\tau(t)):=(u(x,t+\tau),p(t+\tau),k(t+\tau)),\quad t\in[0,\tau'-\tau].$$

In the sequel, g_i^τ, l^τ and other functions with superscript τ are defined analogously. To prove the solvability of (5.1)–(5.3), we use the following splitting of convolution introduced by Colombo [11]:

$$\int_0^{\tau+t}k(\tau+t-s)u(x,s)\mathrm{d}s=k^\tau*\widehat{u}+\widehat{k}*u^\tau+\int_t^\tau\widehat{k}(\tau+t-s)\widehat{u}(x,s)\mathrm{d}s, \tag{5.5}$$

which was successfully used to prove the global uniqueness of an inverse problem concerning with the strongly damped wave equation with memory. In a way similar to (5.5), we have

$$\int_0^{\tau+t}l(\tau+t-s)g_i'(x,s)\mathrm{d}s=l^\tau*g_i'+\widehat{l}*(g_i')^\tau+\int_t^\tau\widehat{l}(\tau+t-s)g_i'(x,s)\mathrm{d}s. \tag{5.6}$$

Then we find that

$$\begin{aligned}\mathscr{N}_i[u,l](\tau+t)&=\partial_t^\alpha g_i^\tau-\mathscr{H}_i[\mathscr{L}[u^\tau]]+l^\tau*g_i'+\widehat{l}*(g_i^\tau)'+\int_t^\tau\widehat{l}(\tau+t-s)g_i'(s)\mathrm{d}s\\ &:=\widetilde{\mathscr{N}_i}[u^\tau,l^\tau]+\int_t^\tau\widehat{l}(\tau+t-s)g_i'(s)\mathrm{d}s,\quad t\in[0,\tau'-\tau],\ i=1,2.\end{aligned} \tag{5.7}$$

Based on (5.5) and (5.7), we rewrite the problem (5.1)–(5.3) as

$$\begin{cases}\partial_t^\alpha u^\tau(x,t)-\mathscr{L}[u^\tau](x,t)=-k^\tau(t)*\widehat{u}^\tau(x,t)-\widehat{k}(t)*u^\tau(x,t)\\ \quad -\displaystyle\int_t^\tau\widehat{k}(\tau+t-s)\widehat{u}(x,s)\mathrm{d}s+f(x)p^\tau,\quad (x,t)\in\Omega\times(0,\tau'-\tau),\\ u^\tau(x,0)=\widehat{u}(x,\tau),\quad u_t^\tau(x,0)=\widehat{u}_t(x,\tau),\quad x\in\Omega,\\ u^\tau(x,t)=0,\quad (x,t)\in\partial\Omega\times(0,\tau'-\tau)\end{cases} \tag{5.8}$$

and

$$p^\tau(t)=\frac{1}{c_0}[-g_2(0)\widetilde{\mathscr{N}_1}[u^\tau,l^\tau](t)+g_1(0)\widetilde{\mathscr{N}_2}[u^\tau,l^\tau](t)]+h_1(t),\quad t\in[0,\tau'-\tau], \tag{5.9}$$

$$k^\tau(t) = \frac{1}{c_0}[\mathscr{H}_1[f]D_t\widetilde{\mathscr{N}_2}[u^\tau, l^\tau](t) - \mathscr{H}_2[f]D_t\widetilde{\mathscr{N}_1}[u^\tau, l^\tau](t)] + h_2(t), \quad t \in [0, \tau' - \tau], \tag{5.10}$$

where h_1 and h_2 are known functions in terms of $\widehat{l}$ and g_i, defined by

$$h_1(t) = \frac{1}{c_0}\Big[- g_2(0)\int_t^\tau \widehat{l}(\tau + t - s)g_1'(s)\mathrm{d}s + g_1(0)\int_t^\tau \widehat{l}(\tau + t - s)g_2'(s)\mathrm{d}s\Big],$$
$$h_2(t) = \frac{1}{c_0}\Big[\mathscr{H}_1[f]\Big(\int_t^\tau \widehat{l}(\tau + t - s)g_2'(s)\mathrm{d}s\Big)_t - \mathscr{H}_2[f]\Big(\int_t^\tau \widehat{l}(\tau + t - s)g_1'(s)\mathrm{d}s\Big)_t\Big].$$

For given $(\widehat{u}, \widehat{p}, \widehat{k}) \in \mathbf{Y}_\tau$, we have $-\int_t^\tau \widehat{k}(\tau + t - s)\widehat{u}(x, s)\mathrm{d}s \in C([0, T]; \mathcal{D}((-\mathscr{L})^{\frac{1}{\alpha}}))$, $h_1 \in C^1[0, \tau' - \tau]$, $h_2 \in C[0, \tau' - \tau]$. Then repeating the arguments of the proof of Lemma 1.1, we could show that there exists a $\widehat{\tau} > 0$, such that (5.8)–(5.10) has a unique solution $(U(x, t + \tau), P(t+\tau), K(t+\tau))$ in $[0, \widehat{\tau}]$. Then (u, p, k) defined by (5.4) is the unique extension of $(\widehat{u}, \widehat{p}, \widehat{k})$ from $[0, \tau]$ to $[0, \tau']$, if we choose $\tau' = \min\{\tau + \widehat{\tau}, 2\tau, T\}$. This completes the proof of Lemma 5.1.

5.2 A preliminary estimate for the extended solution

We give here an a priori estimate on (u, p, k) defined by (5.4), which is the unique extension of $(\widehat{u}, \widehat{p}, \widehat{k})$ in $[0, \tau']$. It will be used to guarantee that the process of extension can be repeated.

Lemma 5.2 *Assume that the hypotheses* (H1)–(H5) *hold. Then for* $(u, p, k) \in \mathbf{Y}_{\tau'}$ *defined by* (5.4)*, we have the following estimate:*

$$\|(u, p, k)\|_{\mathbf{Y}_{\tau'}} \le C, \tag{5.11}$$

where C *depends on* $\alpha, \Omega, \tau, \tau'$*, the initial data* a *and* b*, the known functions* f, ϕ_i *and the measurement data* g_i*.*

In order to prove Lemma 5.2, we need the following two results.

Lemma 5.3 *Assume that the hypotheses* (H1)–(H5) *hold. Then for the solution* (u^τ, k^τ, p^τ) *to* (5.8)–(5.10)*, we have the following estimate:*

$$\|u^\tau\|_{X_t}^2 \le C\Big[1 + \|p^\tau\|_{C[0,t]}^2 + \int_0^t (\|u^\tau\|_{X_s}^2 + \|k^\tau\|_{C[0,s]}^2)\mathrm{d}s\Big] \tag{5.12}$$

for $t \in [0, \tau' - \tau]$*, where* C *depends on* $\alpha, \Omega, \tau, \tau'$*, the initial data* a *and* b*, the known functions* f, ϕ_i *and the measurement data* g_i*.*

Proof By Lemma 1.1, we see that there exists a positive constant C^*, depending on α, Ω, τ, the initial data a and b, the known functions f, ϕ_i and the measurement data g_i, such that

$$\|(\widehat{u}, \widehat{p}, \widehat{k})\|_{\mathbf{Y}_\tau} \le C^*. \tag{5.13}$$

Applying Lemma 2.1 to (5.8), we obtain for all $t \in [0, \tau' - \tau]$ that

$$\begin{aligned} &\|u^\tau(\cdot, t)\|_{H^2(\Omega)} + \|u_t^\tau(\cdot, t)\|_{L^2(\Omega)} \\ &\le C(1 + t^{\alpha-1})\|\widehat{u}(\cdot, \tau)\|_{H^2(\Omega)} + C(1 + t^{1-\frac{\alpha}{2}})\|\widehat{u}_t(\cdot, \tau)\|_{H^1(\Omega)} + C(t^{\alpha-1} + t) \\ &\quad\times \Big\|k^\tau * \widehat{u} - \widehat{k} * u^\tau - \int_t^\tau \widehat{k}(\tau + t - s)\widehat{u}(x, s)\mathrm{d}s + fp^\tau\Big\|_{C([0,t]; \mathcal{D}((-\mathscr{L})^{\frac{1}{\alpha}}))}. \end{aligned} \tag{5.14}$$

From (2.4) in Lemma 2.1 and (3.4), we deduce that

$$\begin{aligned}\|\widehat{u}_t(\cdot,\tau)\|_{H^1(\Omega)} &\le C\tau^{-1}\|a\|_{H^2(\Omega)} + C\|b\|_{H^1(\Omega)} + C\tau^{\alpha-1}\|\widehat{k}*\widehat{u}+f\widehat{p}\|_{C([0,\tau];\mathcal{D}((-\mathscr{L})^{\frac{1}{\alpha}}))} \\ &\le C\tau^{-1}\|a\|_{H^2(\Omega)} + C\|b\|_{H^1(\Omega)} \\ &\quad + C\tau^{\alpha-1}(\tau\|\widehat{k}\|_{C[0,\tau]}\|\widehat{u}\|_{C([0,\tau];H^2(\Omega))} + \|f\|_{\mathcal{D}((-\mathscr{L})^{\frac{1}{\alpha}})}\|\widehat{p}\|_{C[0,\tau]}) \\ &\le C(\tau,a,b,f,C^*). \end{aligned} \tag{5.15}$$

Note that $0\le t\le \tau'-\tau\le\tau$. Then by (4.4), we have

$$\Big\|\int_t^\tau \widehat{k}(\tau+t-s)\widehat{u}(x,s)\mathrm{d}s\Big\|_{C([0,t];\mathcal{D}((-\mathscr{L})^{\frac{1}{\alpha}}))} \le C(\tau)\|\widehat{k}\|_{C[0,\tau]}\|\widehat{u}\|_{C([0,\tau];H^2(\Omega))}. \tag{5.16}$$

Moreover, by the Hölder's inequality, we obtain for all $t\in[0,\tau'-\tau]$ that

$$\begin{aligned}\|k^\tau*\widehat{u}(\cdot,t)\|_{\mathcal{D}((-\mathscr{L})^{\frac{1}{\alpha}})} &\le C\|(k^\tau*\widehat{u})(\cdot,t)\|_{H^2(\Omega)} \\ &\le C\Big(\int_0^t |k^\tau(s)|^2\mathrm{d}s\int_0^t\|\widehat{u}(\cdot,t-s)\|^2_{H^2(\Omega)}\mathrm{d}s\Big)^{\frac{1}{2}} \\ &\le Ct^{\frac{1}{2}}\|\widehat{u}\|_{C([0,t];H^2(\Omega))}\Big(\int_0^t\|k^\tau\|^2_{C[0,s]}\mathrm{d}s\Big)^{\frac{1}{2}} \end{aligned} \tag{5.17}$$

and

$$\begin{aligned}\|\widehat{k}*u^\tau(\cdot,t)\|_{\mathcal{D}((-\mathscr{L})^{\frac{1}{\alpha}})} &\le Ct^{\frac{1}{2}}\|\widehat{k}\|_{C[0,t]}\Big(\int_0^t\|u^\tau(\cdot,s)\|^2_{H^2(\Omega)}\mathrm{d}s\Big)^{\frac{1}{2}} \\ &\le Ct^{\frac{1}{2}}\|\widehat{k}\|_{C[0,t]}\Big(\int_0^t\|u^\tau\|^2_{X_s}\mathrm{d}s\Big)^{\frac{1}{2}}. \end{aligned} \tag{5.18}$$

Finally, substituting (5.13), (5.15)–(5.18) into (5.14), and noticing that $t\le\tau$, we get (5.12). This completes the proof.

Lemma 5.4 *Assume that the hypotheses* (H1)–(H5) *hold. Then for the solution* (u^τ,k^τ,p^τ) *to* (5.8)–(5.10)*, we have the following estimate:*

$$\|p^\tau\|^2_{C^1[0,t]} + \|k^\tau\|^2_{C[0,t]} \le C\Big(1+\|u^\tau\|^2_{X_t} + \int_0^t\|k^\tau\|^2_{C[0,s]}\mathrm{d}s\Big) \tag{5.19}$$

for $t\in[0,\tau'-\tau]$*, where* C *depends on* α,Ω,τ,τ'*, the initial data* a *and* b*, the known functions* f,ϕ_i *and the measurement data* g_i*.*

Proof First by (5.9)–(5.10), we have

$$\|p^\tau\|_{C^1[0,t]} + \|k^\tau\|_{C[0,t]} \le C\sum_{i=1}^2(\|\widetilde{\mathscr{N}_i}[u^\tau,l^\tau]\|_{C^1[0,t]}) + \|h_1\|_{C^1[0,t]} + \|h_2\|_{C[0,t]}. \tag{5.20}$$

Next we estimate the two terms on the right-hand side of (5.20). We easily see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t l^\tau(t-s)g_i'(s)\mathrm{d}s = g_i'(0)l^\tau(t) + \int_0^t l^\tau(s)g_i''(t-s)\mathrm{d}s, \tag{5.21}$$

from which it follows that

$$\|l^\tau*g_i'\|_{C^1[0,t]} \le g_i'(0)\|l^\tau\|_{C[0,t]} + \|l^\tau\|_{C[0,t]}\|g_i''\|_{L^1(0,t)} + t^{\frac{1}{2}}\|l^\tau\|_{C[0,t]}\|g_i'\|_{L^2(0,t)}. \tag{5.22}$$

Furthermore, noticing that

$$\|g_i\|_{W^{2,1}(0,t)} \le C\|\partial_t^\alpha g_i\|_{C^1[0,t]}$$

(see Remark 1.1) and

$$\|l^\tau\|_{C[0,t]} = \Big\| \int_0^t k^\tau(s)\mathrm{d}s \Big\|_{C[0,t]} \le \int_0^t \|k^\tau(s)\|_{C[0,s]}\mathrm{d}s,$$

we have

$$\|l^\tau * g_i'\|_{C^1[0,t]} \le C(\tau,\tau',g_i)\int_0^t \|k^\tau(s)\|\mathrm{d}s. \tag{5.23}$$

Similarly, we have

$$\|\widehat{l} * (g_i^\tau)'\|_{C^1[0,t]} \le C(\tau,\tau',g_i^\tau)\int_0^t \|\widehat{k}(s)\|\mathrm{d}s \le C(\tau,\tau',g_i^\tau,C^*). \tag{5.24}$$

Then from the definitions of $\widetilde{\mathscr{N}_i}$ and (2.34), together with (5.22)–(5.23), we deduce that

$$\begin{aligned}
&\|\widetilde{\mathscr{N}_i}[u^\tau,l^\tau]\|_{C^1[0,t]}\\
&\le C\Big(\|\partial_t^\alpha g_i^\tau\|_{C^1[0,t]} + \Big\|\int_\Omega \mathscr{L}[\phi_i(x)]u^\tau(x,\cdot)\mathrm{d}x\Big\|_{C^1[0,t]} + \|l^\tau * g_i'\|_{C^1[0,t]} + \|\widehat{l} * (g_i^\tau)'\|_{C^1[0,t]}\Big)\\
&\le C(\tau,\tau',g_i,g_i^\tau,\phi_i,C^*)\Big(1+\|u^\tau\|_{C^1([0,t];L^2(\Omega))} + \int_0^t \|k^\tau\|_{C[0,s]}\mathrm{d}s\Big), \quad i=1,2. \qquad (5.25)
\end{aligned}$$

Moreover, a simple calculation gives

$$\|h_1\|_{C^1[0,t]} + \|h_2\|_{C[0,t]} \le C(\tau,\tau',C^*). \tag{5.26}$$

Finally, substituting the estimates (5.25)–(5.26) into (5.20) yields the desired estimate (5.19). The proof of Lemma 5.3 is complete.

Now we give the proof of Lemma 5.2.

Proof of Lemma 5.2 Noticing

$$\begin{aligned}
&(u(x,t),p(t),k(t)) = (\widehat{u}(x,t),\widehat{p}(t),k(t)), \quad t\in[0,\tau],\\
&(u(x,t),p(t),k(t)) = (u^\tau(x,t-\tau),p^\tau(x,t-\tau),k^\tau(t-\tau)), \quad t\in[\tau,\tau'],
\end{aligned}$$

it suffices to show that

$$\|(u^\tau,p^\tau,k^\tau)\|_{\mathbf{Y}_t} \le C, \quad \forall t\in[0,\tau'-\tau]. \tag{5.27}$$

Adding up (5.12) in Lemma 5.2 and (5.19) in Lemma 5.3 yields that, $\forall t\in[0,\tau'-\tau]$,

$$\|(u^\tau,p^\tau,k^\tau)\|_{\mathbf{Y}_t}^2 \le C + C\Big[\|p^\tau\|_{C[0,t]}^2 + \int_0^t (\|u^\tau\|_{X_s}^2 + \|k^\tau\|_{C[0,s]}^2)\mathrm{d}s\Big]. \tag{5.28}$$

Observe that, $\forall t\in(0,\tau'-\tau]$,

$$\|p^\tau\|_{C[0,t]} = \Big\|\int_0^t p_t^\tau(s)\mathrm{d}s + p^\tau(0)\Big\|_{C[0,t]} \le \int_0^t \|p^\tau\|_{C^1[0,s]}\mathrm{d}s + |\widehat{p}(\tau)|. \tag{5.29}$$

Inserting (5.29) into (5.28), we obtain

$$\|(u^\tau,p^\tau,k^\tau)\|_{\mathbf{Y}_t}^2 \le C + C\int_0^t \|(u^\tau,p^\tau,k^\tau)\|_{\mathbf{Y}_s}^2\mathrm{d}s. \tag{5.30}$$

Hence (5.27) can be obtained by applying the Gronwall's inequality to (5.30).

5.3 Proof of Theorem 1.1

Proof of Theorem 1.1 Set

$$\mathcal{T} := \{\tau \in (0,T] : \text{the problem constituted by (2.20)–(2.22) has at least a solution } (u,p,k) \in \mathbf{Y}_\tau \text{ in } [0,\tau]\}.$$

Obviously, we have $\mathcal{T} \neq \varnothing$ from Lemma 1.1. Define $T_1 := \sup(\mathcal{T})$. By a similar argument to the proof of Theorem 1.3 in [31], together with Lemmas 5.1–5.2, we could prove $T_1 = T$. This completes the proof of Theorem 1.1.

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