Chin. Ann. Math.
35B(3), 2014, 469–482
DOI: 10.1007/s11401-014-0831-2

Chinese Annals of Mathematics, Series B

Two-Dimensional Parabolic Inverse Source Problem with Final Overdetermination in Reproducing Kernel Space*

Wenyan WANG[1] Masahiro YAMAMOTO[2] Bo HAN[3]

Abstract A new method of the reproducing kernel Hilbert space is applied to a two-dimensional parabolic inverse source problem with the final overdetermination. The exact and approximate solutions are both obtained in a reproducing kernel space. The approximate solution and its partial derivatives are proved to converge to the exact solution and its partial derivatives, respectively. A technique is proposed to improve some existing methods. Numerical results show that the method is of high precision, and confirm the robustness of our method for reconstructing source parameter.

Keywords Inverse source problem, Final overdetermination, Parabolic equation, Reproducing kernel
2000 MR Subject Classification 35K55, 47B32

1 Introduction

Inverse source identification problems are important in many branches of engineering sciences. For example, an accurate estimation of pollutant source is crucial to environmental safeguard in cities with high populations. Taking more and more important roles in the migration of groundwater, identification and control of pollution source and environmental protection (see [1]), the inverse source problems attracted much attention and have been studied by many authors (see [2–13]).

We consider the following inverse source problem of determining a pair of functions $w(x,t)$ and $p(x)$ satisfying

$$w_t(x,t) = \Delta w(x,t) + p(x)g(x,t), \quad (x,t) \in D \equiv \Omega \times [0,T], \tag{1.1}$$

$$w(x,0) = h(x), \quad x \in \Omega, \tag{1.2}$$

$$w(x,t) = 0, \quad x \in \partial\Omega,\ t \in [0,T], \tag{1.3}$$

and the overdetermination condition

$$w(x,T) = r(x), \quad x \in \Omega, \tag{1.4}$$

Manuscript received September 30, 2013.
[1]Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China; Department of Mathematics, Northeast Forestry University, Harbin 150040, China.
E-mail: wangwenyanwwy@gmail.com
[2]Graduate School of Mathematical Sciences, The University of Tokyo, Tokyo 153-8914, Japan.
E-mail: myama@ms.u-tokyo.ac.jp
[3]Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China.
E-mail: bohan@hit.edu.cn
*This work was supported by the National Natural Science Foundation of China (No. 91230119).

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $g(x,t)$ and $h(x)$ are known and sufficiently smooth functions, $r(x)$ is measurement data, and $g(x,T) \neq 0$. On the right-hand side of (1.1), $p(x)g(x,t)$ is interpreted as a heat source. Also, in the modeling of air pollution phenomena, $p(x)g(x,t)$ is considered as the source pollutant.

The existence and uniqueness of the solutions to this inverse problem are discussed in [11–13]. Various methods (see [14–20]) are developed for this inverse source problem and related inverse parabolic problems. In this paper, we use a reproducing kernel method to obtain the analytical solutions.

In recent years, there are broader interests in the use of reproducing kernels for the solutions to diverse inverse problems (see [21–22]). Those papers indicate that the reproducing kernel method (RKM) (see [23]) has many outstanding advantages. The most important advantages of RKM are as follows:

(i) The approximate solutions and their derivatives can be proved to converge uniformly to the exact solutions and their derivatives.

(ii) The structure of numerical programming is simple and the calculations are very fast.

In this paper, we represent an exact solution to problem (1.1)–(1.4) in a reproducing kernel space, and improve the existing methods as follows: First, we obtain reproducing kernel spaces by re-defining the inner products appropriately, and our reproducing kernel is simpler than [24]. Numerical calculations indicate that the method using our reproducing kernel can improve the precision and decrease the runtime, compared to the case where we use the reproducing kernel in [24]; second, this approach reduces problem (1.1)–(1.4) to a system of linear equations, and avoids the Gram-Schmidt orthogonalization process which is needed in [25]. Numerical calculations indicate that our method decreases the runtime, compared to the method using Gram-Schmidt orthogonalization process in [25].

Before applying our method to problem (1.1)–(1.4), we first transform (1.1) to an equation which is easy to solve by using the RKM. For simplicity, we take $\Omega = [0,1] \times [0,1] \subset \mathbb{R}^2$ and $T = 1$.

From (1.1) and (1.4), we obtain

$$p(x,y) = \frac{w_t(x,y,1) - \Delta r(x,y)}{g(x,y,1)}, \quad (x,y) \in \Omega. \tag{1.5}$$

Substituting (1.5) into (1.1) yields

$$\Delta w(x,y,t) - w_t(x,y,t) + \frac{g(x,y,t)}{g(x,y,1)} w_t(x,y,1) = \frac{g(x,y,t)}{g(x,y,1)} \Delta r(x,y).$$

Again we set

$$u(x,y,t) = w(x,y,t) - h(x,y). \tag{1.6}$$

Then we can further transform the original problem to

$$Lu(x,y,t) = f(x,y,t), \quad (x,y,t) \in D, \tag{1.7}$$

$$u(x,y,0) = 0, \quad (x,y) \in \Omega, \tag{1.8}$$

$$u(0, y, t) = u(1, y, t) = 0, \quad y, t \in [0, 1], \tag{1.9}$$

$$u(x, 0, t) = u(x, 1, t) = 0, \quad x, t \in [0, 1]. \tag{1.10}$$

Here we set

$$Lu = \Delta u - u_t + \frac{g(x,y,t)}{g(x,y,1)} u_t(x, y, 1),$$

$$f(x, y, t) = \frac{g(x,y,t)}{g(x,y,1)} \Delta r(x, y) - \Delta h(x, y).$$

In our method, the datum $r(x, y)$ with $\Delta r(x, y)$ is involved in (1.7). Thus we have to assume that $r(x, y) \in C^2(\Omega)$. This can be proved by the classical result for an initial-boundary value problem if we assume suitable smoothness assumptions on $g(x, y, t)$, $h(x, y)$ and compatibility conditions. In this paper, we omit the detailed proofs. In Section 4, we add noises in $L^\infty(\Omega)$ to $r(x, y)$ and test the robustness of our method with Tikhonov regularization for reconstructing $p(x, y)$.

We apply our numerical method to problem (1.7)–(1.10). Our method is composed of:

(i) Solving (1.7)–(1.10) which is an initial-boundary value problem for a non-classical heat equation.

(ii) Finding $p(x, y)$ by (1.5)–(1.6).

This paper is organized as follows: In Section 2, we construct reproducing kernel spaces according to (1.7)–(1.10). In Section 3, we give the exact and approximate solutions in the reproducing kernel space and show the convergence. In Section 4, numerical tests are done and we can conclude that our method is robust against errors. Finally a conclusion is given in Section 5.

2 Several Reproducing Kernel Spaces

In this section, referring to [23], we construct the reproducing kernel space $W_{(4,4,3)}(D)$ according to (1.7)–(1.10), which gives a simpler method than [24]. Henceforth "$\cdot'$" denotes the derivative in the variable under consideration.

First we define the reproducing kernel spaces $W_k[0, 1]$, $k = 4, 3, 2$.

We set $W_4[0, 1] = \{u \mid u, u', u''$ and $u^{(3)}$ are absolutely continuous real-valued functions in $[0, 1]$, $u^{(4)} \in L^2[0, 1], u(0) = 0, u(1) = 0\}$, and we define the inner product in $W_4[0, 1]$ by

$$\langle u, v\rangle_{W_4} = \sum_{i=1}^{2} u^{(i)}(0)v^{(i)}(0) + \int_0^1 u^{(4)}(x)v^{(4)}(x)\mathrm{d}x.$$

Next we set $W_3[0, 1] = \{u \mid u, u'$ and u'' are absolutely continuous real-valued functions in $[0, 1]$, $u^{(3)} \in L^2[0, 1], u(0) = 0\}$, and we define the inner product in $W_3[0, 1]$ by

$$\langle u, v\rangle_{W_3} = \sum_{i=1}^{2} u^{(i)}(0)v^{(i)}(0) + \int_0^1 u^{(3)}(x)v^{(3)}(x)\mathrm{d}x.$$

We set $W_2[0,1] = \{u \mid u$ and u' are absolutely continuous real-valued functions in $[0,1]$, $u'' \in L^2[0,1]\}$, and we define the inner product in $W_2[0,1]$ by

$$\langle u, v\rangle_{W_2} = \sum_{i=0}^{1} u^{(i)}(0)v^{(i)}(0) + \int_0^1 u''(x)v''(x)\mathrm{d}x.$$

We define the norms by $\|u\|_{W_k} = \sqrt{\langle u, u\rangle_{W_k}}$ for $k = 4, 3, 2$. We can prove that $W_4[0,1]$, $W_3[0,1]$, $W_2[0,1]$ are reproducing kernel spaces and the reproducing kernels $R1, R2, R3$ are defined respectively by (2.1)–(2.3) in [26].

Moreover we assume that $\{p_i(x)\}_{i=1}^{\infty}$ and $\{r_k(t)\}_{k=1}^{\infty}$ are respectively the orthonormal bases of the reproducing kernel spaces $W_4[0,1]$ and $W_3[0,1]$. Then we define $W_{(4,4,3)}(D)$ by

$$W_{(4,4,3)}(D) = \Big\{u \,\Big|\, u(x,y,t) = \sum_{i,j,k=1}^{\infty} c_{ijk}p_i(x)p_j(y)r_k(t),\ \sum_{i,j,k=1}^{\infty} |c_{ijk}|^2 < \infty,\ c_{ijk} \in \mathbb{R}\Big\}.$$

The inner product of $W_{(4,4,3)}(D)$ is defined dy

$$\langle u_1, u_2\rangle_{W_{(4,4,3)}} = \sum_{i,j,k=1}^{\infty} c_{ijk}d_{ijk},$$

where $u_1 = \sum\limits_{i,j,k=1}^{\infty} c_{ijk}p_i(x)p_j(y)r_k(t)$ and $u_2 = \sum\limits_{i,j,k=1}^{\infty} d_{ijk}p_i(x)p_j(y)r_k(t)$. The norm is denoted by

$$\|u\|^2_{W_{(4,4,3)}} = \langle u, u\rangle_{W_{(4,4,3)}}.$$

According to [23], we can obtain the following theorem.

Theorem 2.1 *$W_{(4,4,3)}(D)$ is a reproducing kernel space and its reproducing kernel is given by*

$$K(x,\xi,y,\zeta,t,\eta) = R1(x,\xi)R1(y,\zeta)R2(t,\eta),$$

where $R1(\cdot,\cdot)$ and $R2(\cdot,\cdot)$ are respectively the reproducing kernel of $W_4[0,1]$ and $W_3[0,1]$.

Similarly to $W_{(4,4,3)}(D)$, we can define the reproducing kernel space $W_{(2,2,2)}(D)$. It is easy to show that its reproducing kernel is

$$\overline{K}(x,\xi,y,\zeta,t,\eta) = R3(x,\xi)R3(y,\zeta)R3(t,\eta),$$

where $R3(\cdot,\cdot)$ is the reproducing kernel of $W_2[0,1]$.

3 The Solution of (1.7)–(1.10)

In this section, the exact solution of problem (1.7)–(1.10) is given in the reproducing kernel space $W_{(4,4,3)}(D)$.

We choose a countable dense subset $(x_i, y_i, t_i) \in D$, $i = 1, 2, 3, \cdots$. Put

$$\varphi_i(x,y,t) = \overline{K}(x, x_i, y, y_i, t, t_i), \quad \psi_i(x,y,t) = L^*\varphi_i(x,y,t),$$

where $\overline{K}$ is the reproducing kernel of $W_{(2,2,2)}(D)$ and L^* is the formal adjoint operator of L: $\langle L\varphi, \psi\rangle_{W_{(2,2,2)}} = \langle \varphi, L^*\psi\rangle_{W_{(4,3,3)}}$ for each $\varphi, \psi \in C_0^\infty(D)$. Here, noting that $r'_k \in C[0,1]$ by $r'_k \in W_2[0,1] \subset C[0,1]$ by the Sobolev embedding, we see that $L : W_{(4,4,3)}(D) \to W_{(2,2,2)}(D)$ is a bounded linear operator. Let $K(x,\xi,y,\zeta,t,\eta)$ be the reproducing kernel of $W_{(4,4,3)}(D)$.

We show the following Theorems 3.1–3.2.

Theorem 3.1 *Assume the uniqueness for the inverse problem* (1.1)–(1.4). *Then the system* $\{\psi_i\}_{i=1}^\infty$ *is complete in* $W_{(4,4,3)}(D)$, *and*

$$\begin{aligned}\psi_i(x,y,t) &= L_{(\xi,\zeta,\eta)}K(x,\xi,y,\zeta,t,\eta)|_{(\xi,\zeta,\eta)=(x_i,y_i,t_i)}\\ &= \Big(\frac{\partial^2 K}{\partial \xi^2} + \frac{\partial^2 K}{\partial \zeta^2}\Big)(x,x_i,y,y_i,t,t_i) - \partial_\eta K(x,x_i,y,y_i,t,t_i)\\ &\quad + \frac{g(x_i,y_i,t_i)}{g(x_i,y_i,1)}\partial_\eta K(x,x_i,y,y_i,t,1).\end{aligned}$$

We construct an orthonormal system $\{\overline{\psi}_i\}_{i=1}^\infty$ of $W_{(4,4,3)}(D)$ by the Gram-Schmidt orthogonalization process to $\{\psi_i\}_{i=1}^\infty$:

$$\overline{\psi}_i = \sum_{k=1}^{i} \beta_{ik}\psi_k \quad \text{with } \beta_{ii} > 0, \quad i = 1,2,\cdots.$$

Theorem 3.2 *A unique solution to* (1.7)–(1.10) *is expressed by*

$$u = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k,y_k,t_k)\overline{\psi}_i \tag{3.1}$$

in $W_{(4,4,3)}(D)$.

The proofs of the theorems are given in appendix and similar to those in [26].

Next we will find approximate solutions u_n in the form of

$$u_n = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} f(x_k,y_k,t_k)\overline{\psi}_i, \tag{3.2}$$

which is the n-term truncated Fourier series of the exact solution u in (1.7)–(1.10).

Theorem 3.3 *If u is the solution to* (1.7)–(1.10) *and $u_n = P_n u$, where P_n is the orthogonal projection from $W_{(4,4,3)}$ to* $\mathrm{Span}\{\overline{\psi}_i\}_{i=1}^n$, *then* $Lu_n(x_i,y_i,t_i) = f(x_i,y_i,t_i)$, $i = 1,2,\cdots,n$.

Since

$$\begin{aligned}Lu_n(x_i,y_i,t_i) &= \langle Lu_n, \varphi_i\rangle_{W_{(2,2,2)}} = \langle u_n, L^*\varphi_i\rangle_{W_{(4,4,3)}}\\ &= \langle P_n u, \psi_i\rangle_{W_{(4,4,3)}} = \langle u, P_n\psi_i\rangle_{W_{(4,4,3)}}\\ &= \langle u, \psi_i\rangle_{W_{(4,4,3)}} = \langle Lu, \varphi_i\rangle_{W_{(2,2,2)}}\\ &= Lu(x_i,y_i,t_i) = f(x_i,y_i,t_i), \quad i = 1,2,\cdots,n,\end{aligned}$$

the proof of Theorem 3.3 is seen.

On the other hand, we have

$$
\begin{aligned}
u_n(x,y,t) &= \sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik} f(x_k,y_k,t_k)\overline{\psi}_i(x,y,t) \\
&= \sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik} f(x_k,y_k,t_k)\sum_{l=1}^{i}\beta_{il}\psi_l(x,y,t) \\
&= \sum_{i=1}^{n} C_i\psi_i(x,y,t). \qquad (3.3)
\end{aligned}
$$

Here $C_i = \sum\limits_{l=i}^{n}\sum\limits_{k=1}^{l}\beta_{lk} f(x_k,y_k,t_k)\beta_{li}$, which is verified as follows.

From (3.2), we have

$$
u_n(x,y,t) = \sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik} f(x_k,y_k,t_k)\overline{\psi}_i(x,y,t) = \sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik} f(x_k,y_k,t_k)\sum_{l=1}^{i}\beta_{il}\psi_l(x,y,t).
$$

Let $\overline{C}_i = \sum\limits_{k=1}^{i}\beta_{ik} f(x_k,y_k,t_k)$. Then we obtain

$$
\begin{aligned}
u_n &= \sum_{i=1}^{n}\overline{C}_i\sum_{l=1}^{i}\beta_{il}\psi_l \\
&= \overline{C}_1\beta_{11}\psi_1 \\
&\quad + \overline{C}_2\beta_{21}\psi_1 + \overline{C}_2\beta_{22}\psi_2 \\
&\quad + \overline{C}_3\beta_{31}\psi_1 + \overline{C}_3\beta_{32}\psi_2 + \overline{C}_3\beta_{33}\psi_3 \\
&\quad + \cdots \\
&\quad + \overline{C}_n\beta_{n1}\psi_1 + \overline{C}_n\beta_{n2}\psi_2 + \overline{C}_n\beta_{n3}\psi_3 + \cdots + \overline{C}_n\beta_{nn}\psi_n \\
&= \sum_{i=1}^{n}\Big(\sum_{l=i}^{n}\overline{C}_l\beta_{li}\Big)\psi_i \\
&= \sum_{i=1}^{n}\Big(\sum_{l=i}^{n}\sum_{k=1}^{l}\beta_{lk} f(M_k)\beta_{li}\Big)\psi_i \\
&= \sum_{i=1}^{n} C_i\psi_i.
\end{aligned}
$$

Thus the proof is completed.

Then, from Theorem 3.3, we have

$$
Lu_n(x_j,y_j,t_j) = \sum_{i=1}^{n} C_i L\psi_i(x_j,y_j,t_j) = f(x_j,y_j,t_j), \quad j = 1,2,\cdots,n. \qquad (3.4)
$$

Thus, from (3.4), we can obtain $C_i, i = 1,2,\cdots,n$. Then we take them into (3.3) and obtain the approximate solution $u_n(x,y,t)$ to (1.7)–(1.10). Finally we may obtain the approximation of $(w(x,y,t),p(x,y))$ to the original inverse problem from (1.5)–(1.6).

Using (3.3) and (3.4) to solve (1.7)–(1.10), we can avoid the Gram-Schmidt orthogonalization process in [25] of $\{\psi_i\}_{i=1}^{\infty}$. Thus we can improve the precision and considerably decrease the runtime, compared to the method using the Gram-Schmidt orthogonalization process in [25]. Our method is efficiently applied for solving some model problems, and is of high precision.

We conclude this section with the convergence of the approximate solutions u_n. The exact solution and the n-term approximation solution to (1.7)–(1.10) are respectively denoted by u and u_n. Similarly to [27], we can obtain the following theorem.

Theorem 3.4 *Assume that* $u \in W_{(4,4,3)}(D)$. *Then*

(i) $\|u - u_n\|_{W_{(4,4,3)}} \to 0$, $n \to \infty$. *Moreover the sequence* $\|u - u_n\|_{W_{(4,4,3)}}$ *is monotonically decreasing in* n.

(ii)

$$\left\| \frac{\partial^{i+j+k} u}{\partial x^i \partial y^j \partial t^k} - \frac{\partial^{i+j+k} u_n}{\partial x^i \partial y^j \partial t^k} \right\|_{C(\overline{D})} \to 0, \quad n \to \infty$$

for $i, j = 0, 1, 2$, $k = 0, 1$, $i + j + k = 0, 1, 2$.

4 Numerical Examples

In this section, the numerical examples are studied to demonstrate that our method is effective and the accuracy of approximate solution is high. All computations are performed by Mathematica 5.0.

The domain D is divided into $m_1 \times m_2 \times m_3$ meshes with the step size $\frac{1}{m_3}$ in the t direction, and the step sizes $\frac{1}{m_1}$ and $\frac{1}{m_2}$ in the x and y directions, respectively, in which $m_1, m_2, m_3 \in \mathbb{N}$.

Example 4.1 Consider problem (1.1)–(1.4) with the following conditions:

$$\begin{cases} w(x,y,0) = \sin(x)\sin(1-x)\sin(y)\sin(1-y), & x,y \in [0,1], \\ w(0,y,t) = 0, \quad w(1,y,t) = 0, & y,t \in [0,1], \\ w(x,0,t) = 0, \quad w(x,1,t) = 0, & x,t \in [0,1], \\ g(x,y,t) = \exp(t), & x,y,t \in [0,1], \\ w(x,y,1) = \exp(1)\sin(x)\sin(1-x)\sin(y)\sin(1-y), & x,y \in [0,1]. \end{cases}$$

The exact solution is

$$w(x,y,t) = \exp(t)\sin(x)\sin(1-x)\sin(y)\sin(1-y)$$

and

$$\begin{aligned} p(x,y) = {} & 2\cos(x)\cos(1-x)\sin(y)\sin(1-y) \\ & - \frac{1}{2}(3\cos(1) - 7\cos(1-2y))\sin(x)\sin(1-x). \end{aligned}$$

With grid $m_1 \times m_2 \times m_3 = 5 \times 5 \times 5$, the absolute errors of $w(x,y,t)$ and $p(x,y)$ are respectively presented in Tables 1–2. Also the root mean square (RMS, for short) errors of $w(x,y,t)$ and $p(x,y)$, and CPU time are given in Table 3.

In the reproducing kernel method computations, the Gram-Schmidt orthogonal step is not included in the CPU time. We note that only the proof needs the Gram-Schmidt orthogonal step, but the numerical computations do not depend on such a step (see (3.3)–(3.4)).

Table 1 Absolute errors of $w(x,y,t)$ for Example 4.1

(x,y,t)	Absolute errors	(x,y,t)	Absolute errors	(x,y,t)	Absolute errors
$(\frac{1}{5},\frac{1}{5},\frac{1}{5})$	1.53203E-4	$(\frac{2}{5},\frac{2}{5},\frac{4}{5})$	2.28499E-4	$(\frac{3}{5},\frac{4}{5},\frac{1}{5})$	2.63810E-4
$(\frac{1}{5},\frac{1}{5},\frac{4}{5})$	4.45928E-4	$(\frac{2}{5},\frac{3}{5},\frac{1}{5})$	2.68327E-4	$(\frac{3}{5},\frac{4}{5},\frac{4}{5})$	5.15967E-4
$(\frac{1}{5},\frac{2}{5},\frac{1}{5})$	2.00921E-4	$(\frac{2}{5},\frac{3}{5},\frac{4}{5})$	2.68550E-4	$(\frac{4}{5},\frac{1}{5},\frac{1}{5})$	1.83683E-4
$(\frac{1}{5},\frac{2}{5},\frac{4}{5})$	4.41510E-4	$(\frac{2}{5},\frac{4}{5},\frac{1}{5})$	2.38590E-4	$(\frac{4}{5},\frac{1}{5},\frac{4}{5})$	4.96542E-4
$(\frac{1}{5},\frac{3}{5},\frac{1}{5})$	2.20419E-4	$(\frac{2}{5},\frac{4}{5},\frac{4}{5})$	4.55640E-4	$(\frac{4}{5},\frac{2}{5},\frac{1}{5})$	2.38786E-4
$(\frac{1}{5},\frac{3}{5},\frac{4}{5})$	4.88336E-4	$(\frac{3}{5},\frac{1}{5},\frac{1}{5})$	2.20609E-4	$(\frac{4}{5},\frac{2}{5},\frac{4}{5})$	4.56552E-4
$(\frac{1}{5},\frac{4}{5},\frac{1}{5})$	1.83490E-4	$(\frac{3}{5},\frac{1}{5},\frac{4}{5})$	4.89234E-4	$(\frac{4}{5},\frac{3}{5},\frac{1}{5})$	2.63909E-4
$(\frac{1}{5},\frac{4}{5},\frac{4}{5})$	4.95637E-4	$(\frac{3}{5},\frac{2}{5},\frac{1}{5})$	2.68471E-4	$(\frac{4}{5},\frac{3}{5},\frac{4}{5})$	5.16427E-4
$(\frac{2}{5},\frac{1}{5},\frac{1}{5})$	2.01015E-4	$(\frac{3}{5},\frac{2}{5},\frac{4}{5})$	2.69227E-4	$(\frac{4}{5},\frac{4}{5},\frac{1}{5})$	2.23939E-4
$(\frac{2}{5},\frac{1}{5},\frac{4}{5})$	4.41958E-4	$(\frac{3}{5},\frac{3}{5},\frac{1}{5})$	2.97001E-4	$(\frac{4}{5},\frac{4}{5},\frac{4}{5})$	5.65090E-4
$(\frac{2}{5},\frac{2}{5},\frac{1}{5})$	2.42961E-4	$(\frac{3}{5},\frac{3}{5},\frac{4}{5})$	3.17798E-4		

Table 2 Absolute errors of $p(x,y)$ for Example 4.1

(x,y)	Absolute errors	(x,y)	Absolute errors
$(\frac{1}{5},\frac{1}{5})$	1.06856E-3	$(\frac{3}{5},\frac{1}{5})$	2.09500E-3
$(\frac{1}{5},\frac{2}{5})$	1.71671E-3	$(\frac{3}{5},\frac{2}{5})$	3.37380E-3
$(\frac{1}{5},\frac{3}{5})$	2.09513E-3	$(\frac{3}{5},\frac{3}{5})$	4.08045E-3
$(\frac{1}{5},\frac{4}{5})$	1.79791E-3	$(\frac{3}{5},\frac{4}{5})$	3.46345E-3
$(\frac{2}{5},\frac{1}{5})$	1.71665E-3	$(\frac{4}{5},\frac{1}{5})$	1.79778E-3
$(\frac{2}{5},\frac{2}{5})$	2.75086E-3	$(\frac{4}{5},\frac{2}{5})$	2.92091E-3
$(\frac{2}{5},\frac{3}{5})$	3.37394E-3	$(\frac{4}{5},\frac{3}{5})$	3.46339E-3
$(\frac{2}{5},\frac{4}{5})$	2.92107E-3	$(\frac{4}{5},\frac{4}{5})$	2.84954E-3

Table 3 RMS errors of $w(x,y,t)$ and $p(x,y)$, and CPU time for Example 4.1

$m_1 \times m_2 \times m_3$	RMS errors of $w(x,y,t)$	RMS errors of $p(x,y)$	CPU time (s)
$2 \times 2 \times 2$	1.45741E-3	2.19094E-2	0.000
$4 \times 4 \times 4$	8.14157E-4	4.99276E-3	2.449
$5 \times 5 \times 5$	4.09812E-4	2.72072E-3	13.525

The above results show that as the step sizes decrease, the precision is improved.

In addition, in order to test the robustness of our method, uniform random noise creates noisy data by $r(x,y) + d \times \text{random}(x,y)$, where $\text{random}(x,y)$ describes a uniform random function with the range $[0,1]\times[0,1]$. The figures 1–4 show the results with grid $m_1 \times m_2 \times m_3 = 5 \times 5 \times 5$ and the noise factors $d = 0.1, 0.2, 0.3$. Our numerical method needs the second derivatives of r, but our available data with noise are not in $H^2(\Omega)$ in general and $L^\infty(\Omega)$-noises may cause instability. Therefore we apply the Tikhonov regularization for the stabilized reconstruction.

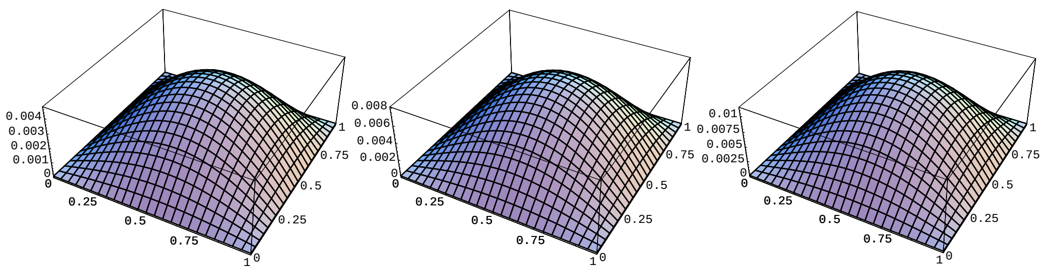

Figure 1 Error $|w(x,y,0.1) - w_{125}(x,y,0.1)|$ for Example 4.1: (left) $d = 0.1$, (middle) $d = 0.2$, (right) $d = 0.3$

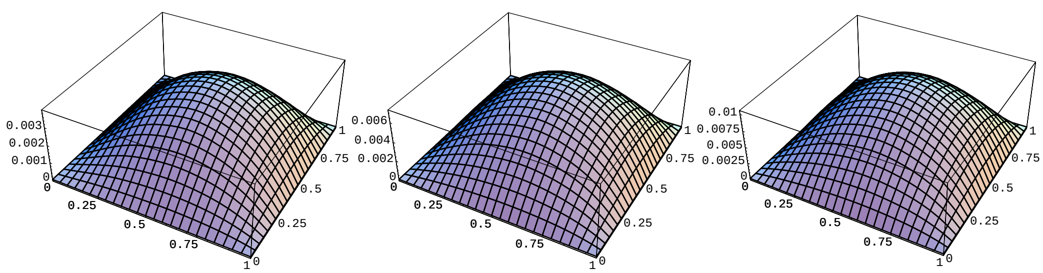

Figure 2 Error $|w(x,y,0.5) - w_{125}(x,y,0.5)|$ for Example 4.1: (left) $d = 0.1$, (middle) $d = 0.2$, (right) $d = 0.3$

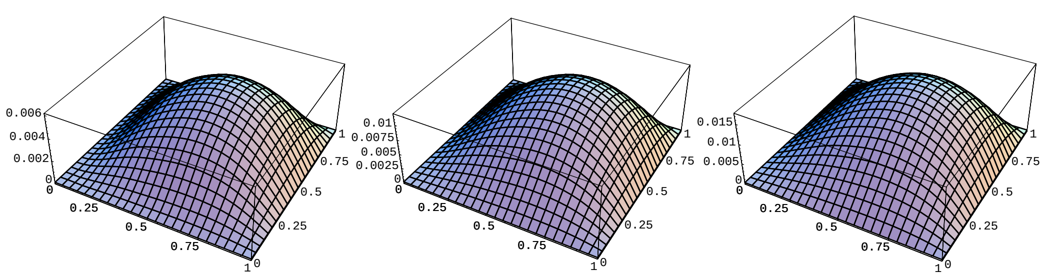

Figure 3 Error $|w(x,y,0.9) - w_{125}(x,y,0.9)|$ for Example 4.1: (left) $d = 0.1$, (middle) $d = 0.2$, (right) $d = 0.3$

From the figures, it can be seen that the results become worse as the noise factor d becomes larger, but even in the case of the largest noise level, i.e., $d = 0.3$, the absolute error $|p - p_{25}|$ is about 0.06, which can be considered as an acceptable numerical result. Thus, the figures confirm the robustness of our method for reconstructing $p(x,y)$.

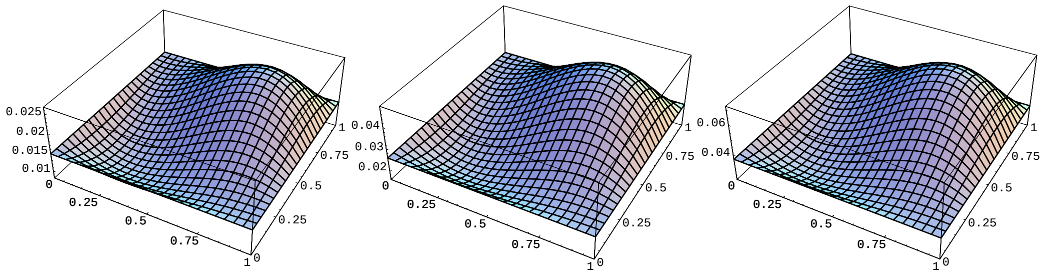

Figure 4 Error $|p - p_{25}|$ for Example 4.1: (left) $d = 0.1$, (middle) $d = 0.2$, (right) $d = 0.3$

Example 4.2 Consider problem (1.1)–(1.4) with the following conditions:

$$\begin{cases} w(x,y,0) = (1-x)y\sin(x)\sin(1-y), & x, y \in [0,1], \\ w(0,y,t) = 0, \quad w(1,y,t) = 0, & y, t \in [0,1], \\ w(x,0,t) = 0, \quad w(x,1,t) = 0, & x, t \in [0,1], \\ g(x,y,t) = \exp(t), & x, y, t \in [0,1], \\ w(x,y,1) = \exp(1)(1-x)y\sin(x)\sin(1-y), & x, y \in [0,1]. \end{cases}$$

The exact solution is $w(x,y,t) = \exp(t)(1-x)y\sin(x)\sin(1-y)$ and $p(x,y) = 2(1-x)\cos(1-y)\sin(x) + y(2\cos(x) + 3(1-x)\sin(x))\sin(1-y)$.

With grid $m_1 \times m_2 \times m_3 = 5 \times 5 \times 5$, the absolute errors of $w(x,y,t)$ and $p(x,y)$ are respectively presented in Tables 4–5. Also RMS errors of $w(x,y,t)$ and $p(x,y)$, and CPU time are given in Table 6.

Table 4 Absolute errors of $w(x,y,t)$ for Example 4.2

(x,y,t)	Absolute errors	(x,y,t)	Absolute errors	(x,y,t)	Absolute errors
$(\frac{1}{5},\frac{1}{5},\frac{1}{5})$	1.48500E-5	$(\frac{2}{5},\frac{2}{5},\frac{4}{5})$	7.40439E-5	$(\frac{3}{5},\frac{4}{5},\frac{1}{5})$	3.25325E-5
$(\frac{1}{5},\frac{1}{5},\frac{4}{5})$	5.20127E-5	$(\frac{2}{5},\frac{3}{5},\frac{1}{5})$	1.25608E-5	$(\frac{3}{5},\frac{4}{5},\frac{4}{5})$	5.29306E-5
$(\frac{1}{5},\frac{2}{5},\frac{1}{5})$	1.35525E-5	$(\frac{2}{5},\frac{3}{5},\frac{4}{5})$	6.65426E-5	$(\frac{4}{5},\frac{1}{5},\frac{1}{5})$	2.31230E-5
$(\frac{1}{5},\frac{2}{5},\frac{4}{5})$	1.81131E-5	$(\frac{2}{5},\frac{4}{5},\frac{1}{5})$	2.63316E-5	$(\frac{4}{5},\frac{1}{5},\frac{4}{5})$	6.25942E-5
$(\frac{1}{5},\frac{3}{5},\frac{1}{5})$	1.73686E-5	$(\frac{2}{5},\frac{4}{5},\frac{4}{5})$	4.04622E-5	$(\frac{4}{5},\frac{2}{5},\frac{1}{5})$	2.58179E-5
$(\frac{1}{5},\frac{3}{5},\frac{4}{5})$	2.59583E-5	$(\frac{3}{5},\frac{1}{5},\frac{1}{5})$	2.00571E-5	$(\frac{4}{5},\frac{2}{5},\frac{4}{5})$	2.78374E-5
$(\frac{1}{5},\frac{4}{5},\frac{1}{5})$	2.28123E-5	$(\frac{3}{5},\frac{1}{5},\frac{4}{5})$	3.08383E-5	$(\frac{4}{5},\frac{3}{5},\frac{1}{5})$	3.10066E-5
$(\frac{1}{5},\frac{4}{5},\frac{4}{5})$	7.05839E-5	$(\frac{3}{5},\frac{2}{5},\frac{1}{5})$	1.45909E-5	$(\frac{4}{5},\frac{3}{5},\frac{4}{5})$	3.90742E-5
$(\frac{2}{5},\frac{1}{5},\frac{1}{5})$	1.50457E-5	$(\frac{3}{5},\frac{2}{5},\frac{4}{5})$	6.24370E-5	$(\frac{4}{5},\frac{4}{5},\frac{1}{5})$	3.35229E-5
$(\frac{2}{5},\frac{1}{5},\frac{4}{5})$	2.10252E-5	$(\frac{3}{5},\frac{3}{5},\frac{1}{5})$	2.06593E-5	$(\frac{4}{5},\frac{4}{5},\frac{4}{5})$	8.62871E-5
$(\frac{2}{5},\frac{2}{5},\frac{1}{5})$	7.16726E-6	$(\frac{3}{5},\frac{3}{5},\frac{4}{5})$	5.31687E-5		

From the above results, we see that as the step sizes decrease, the precision improves.

In addition, the results with grid $m_1 \times m_2 \times m_3 = 5 \times 5 \times 5$ and the noise factors $d = 0.1, 0.2, 0.3$ are given in Figures 5–8.

From the figures, it can be seen that results become worse as noise factor d becomes larger, but even in the case where $d = 0.3$, the absolute error $|p - p_{25}|$ is about 0.02. Thus, the figures confirm the robustness of our method of reconstructing $p(x,y)$.

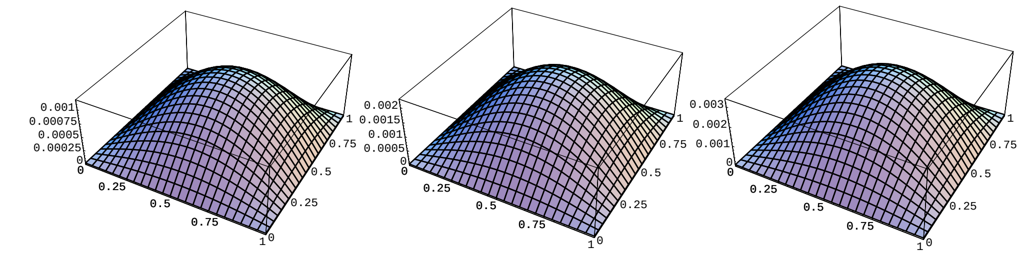

Figure 5 Error $|w(x,y,0.1) - w_{125}(x,y,0.1)|$ for Example 4.2: (left) $d = 0.1$, (middle) $d = 0.2$, (right) $d = 0.3$

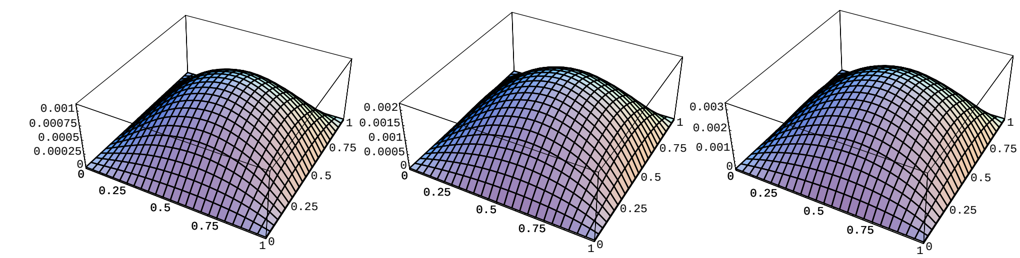

Figure 6 Error $|w(x,y,0.5) - w_{125}(x,y,0.5)|$ for Example 4.2: (left) $d = 0.1$, (middle) $d = 0.2$, (right) $d = 0.3$

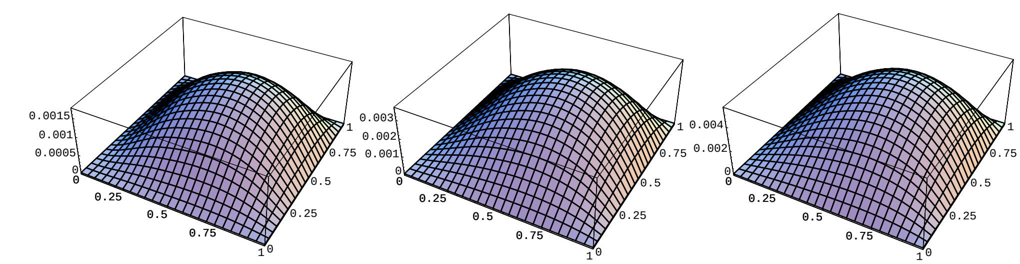

Figure 7 Error $|w(x,y,0.9) - w_{125}(x,y,0.9)|$ for Example 4.2: (left) $d = 0.1$, (middle) $d = 0.2$, (right) $d = 0.3$

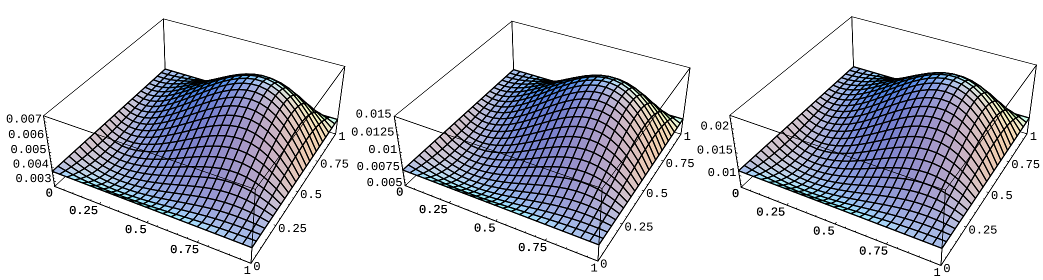

Figure 8 Error $|p - p_{25}|$ for Example 4.2: (left) $d = 0.1$, (middle) $d = 0.2$, (right) $d = 0.3$

Table 5 Absolute errors of $p(x, y)$ for Example 4.2

(x, y)	Absolute errors	(x, y)	Absolute errors
$(\frac{1}{5}, \frac{1}{5})$	1.85085E-4	$(\frac{3}{5}, \frac{1}{5})$	8.88742E-5
$(\frac{1}{5}, \frac{2}{5})$	2.73062E-4	$(\frac{3}{5}, \frac{2}{5})$	1.00934E-4
$(\frac{1}{5}, \frac{3}{5})$	1.94298E-4	$(\frac{3}{5}, \frac{3}{5})$	5.42988E-5
$(\frac{1}{5}, \frac{4}{5})$	2.28565E-5	$(\frac{3}{5}, \frac{4}{5})$	2.27713E-4
$(\frac{2}{5}, \frac{1}{5})$	2.19547E-4	$(\frac{4}{5}, \frac{1}{5})$	6.98635E-5
$(\frac{2}{5}, \frac{2}{5})$	3.15447E-4	$(\frac{4}{5}, \frac{2}{5})$	1.46284E-4
$(\frac{2}{5}, \frac{3}{5})$	1.81669E-4	$(\frac{4}{5}, \frac{3}{5})$	2.67697E-4
$(\frac{2}{5}, \frac{4}{5})$	5.49982E-5	$(\frac{4}{5}, \frac{4}{5})$	3.15736E-4

Table 6 RMS errors of $w(x, y, t)$ and $p(x, y)$, and CPU time for Example 4.2

$m_1 \times m_2 \times m_3$	RMS errors of $w(x, y, t)$	RMS errors of $p(x, y)$	CPU time (s)
$2 \times 2 \times 2$	1.09387E-4	2.52967E-2	0.000
$4 \times 4 \times 4$	2.10323E-4	1.37041E-3	3.197
$5 \times 5 \times 5$	4.23673E-5	1.93867E-4	17.175

5 Conclusions

In this article, our method has been successfully applied to a two-dimensional parabolic inverse source problem with the final overdetermination. Our method is based on the reproducing kernel Hilbert space, and improves some existing methods. The numerical results confirm that the accuracy of our method and the error of approximate solution are monotonically decreasing in the sense of $\|\cdot\|_{W_{(4,4,3)}}$. Moreover, our method is applicable to more general inverse source problem for parabolic equations, and we will discuss in a forthcoming paper.

6 The Proof of Theorems 3.1–3.2

6.1 The proof of Theorem 3.1

We have

$$\begin{aligned}\psi_i(x, y, t) &= (L^*\varphi_i)(x, y, t) = \langle (L^*\varphi_i)(\xi, \zeta, \eta), K(x, \xi, y, \zeta, t, \eta)\rangle \\ &= \langle \varphi_i(\xi, \zeta, \eta), L_{(\xi,\zeta,\eta)} K(x, \xi, y, \zeta, t, \eta)\rangle \\ &= L_{(\xi,\zeta,\eta)} K(x, \xi, y, \zeta, t, \eta)|_{(\xi,\zeta,\eta)=(x_i,y_i,t_i)} \\ &= \Big(\frac{\partial^2 K}{\partial \xi^2} + \frac{\partial^2 K}{\partial \zeta^2}\Big)(x, x_i, y, y_i, t, t_i) - \partial_\eta K(x, x_i, y, y_i, t, t_i) \\ &\quad + \frac{g(x_i, y_i, t_i)}{g(x_i, y_i, 1)} \partial_\eta K(x, x_i, y, y_i, t, 1).\end{aligned}$$

Clearly, $\psi_i \in W_{(4,4,3)}(D)$.

For each fixed $u \in W_{(4,4,3)}(D)$, let $\langle u, \psi_i\rangle_{W_{(4,4,3)}} = 0$, $i = 1, 2, \cdots$, which means that

$$\langle u, (L^*\varphi_i)\rangle_{W_{(4,4,3)}} = \langle Lu(\cdot), \varphi_i(\cdot)\rangle_{W_{(2,2,2)}} = (Lu)(x_i, y_i, t_i) = 0.$$

Note that $\{(x_i, y_i, t_i)\}_{i=1}^{\infty}$ is dense in D, and $(Lu)(x, y, t) = 0$. It follows that $u \equiv 0$ by the uniqueness assumption for the inverse problem. Thus the proof of Theorem 3.1 is completed.

6.2 The proof of Theorem 3.2

Applying Theorem 3.1, we easily see that $\{\overline{\psi}_i\}_{i=1}^{\infty}$ is a complete orthonormal system of $W_{(4,4,3)}(D)$.

Note that $\langle v, \varphi_i \rangle_{W_{(2,2,2)}} = v(x_i, y_i, t_i)$ for each $v \in W_{(2,2,2)}(D)$, and we have

$$\begin{aligned} u(x,y,t) &= \sum_{i=1}^{\infty} \langle u, \overline{\psi}_i \rangle_{W_{(4,4,3)}} \overline{\psi}_i(x,y,t) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle u, L^* \varphi_k \rangle_{W_{(4,4,3)}} \overline{\psi}_i(x,y,t) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle Lu, \varphi_k \rangle_{W_{(2,2,2)}} \overline{\psi}_i(x,y,t) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle f, \varphi_k \rangle_{W_{(2,2,2)}} \overline{\psi}_i(x,y,t) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(x_k, y_k, t_k) \overline{\psi}_i(x,y,t). \end{aligned}$$

The proof of the theorem is completed.

References

[1] Ebel, A. and Davitashvili, T., Air, water and soil quality modelling for risk and impact assessment, Springer, Dordrecht, 2007.

[2] Ling, L., Yamamoto, M., Hon, Y. C. and Takeuchi, T., Identification of source locations in two-dimensional heat equations, *Inverse Probl.*, **22**, 2006, 1289–1305.

[3] Cheng, W., Zhao, L. and Fu, C., Source term identification for an axisymmetric inverse heat conduction problem, *Comput. Math. Appl.*, **59**, 2010, 142–148.

[4] Badia, A. E. and Ha-Duong, T., On an inverse source problem for the heat equation, application to a pollution detection problem, *J. Inv. Ill-Posed Problems*, **10**, 2002, 585–599.

[5] Choulli, M. and Yamamoto, M., Conditional stability in determining a heat source, *J. Inv. Ill-Posed Problems*, **12**, 2004, 233–243.

[6] Dehghan, M. and Tatari, M., Determination of a control parameter in a one-dimensional parabolic equation using the method of radial basis functions, *Math. Comput. Model.*, **44**, 2006, 1160–1168.

[7] Sakamoto, K. and Yamamoto, M., Inverse heat source problem from time distributing overdetermination, *Appl. Anal.*, **88**, 2009, 735–748.

[8] Cannon, J. R. and Pérez-Esteva, S., Uniqueness and stability of 3D heat sources, *Inverse Probl.*, **7**, 1991, 57–62.

[9] Choulli, M. and Yamamoto, M., Generic well-posedness of an inverse parabolic problem—the Hölder-space approach, *Inverse Probl.*, **12**, 1996, 195–205.

[10] Fan, J. and Nakamura, G., Well-posedness of an inverse problem of Navier-Stokes equations with the final overdetermination, *J. Inv. Ill-Posed Problems*, **17**, 2009, 565–584.

[11] Choulli, M. and Yamamoto, M., Generic well-posedness of a linear inverse parabolic problem with diffusion parameters, *J. Inv. Ill-Posed Problems*, **7**, 1999, 241–254.

[12] Kamynin, V. L., On the unique solvability of an inverse problem for parabolic equations under a final overdetermination condition, *Math. Notes*, **73**, 2003, 202–211.

[13] Isakov, V., Inverse parabolic problems with the final overdetermination, *Commun. Pure Appl. Math.*, **44**, 1991, 185–209.

[14] Isakov, V., Inverse problems for partial differential equations, Applied Mathematical Sciences, Springer-Verlag, New York, 2006.

[15] Yi, Z. and Murio, D. A., Source term identification in 1-D IHCP, *Comput. Math. Appl.*, **47**, 2004, 1921–1933.

[16] Trong, D. D., Long, N. T. and Alain, P. N. D., Nonhomogeneous heat equation: identification and regularization for the inhomogeneous term, *J. Math. Anal. Appl.*, **312**, 2005, 93–104.

[17] Trong, D. D., Quan, P. H. and Alain, P. N. D., Determination of a two-dimensional heat source: uniqueness, regularization and error estimate, *J. Comput. Appl. Math.*, **191**, 2006, 50–67.

[18] Chen, Q. and Liu, J., Solving an inverse parabolic problem by optimization from final measurement data, *J. Comput. Appl. Math.*, **193**, 2006, 183–203.

[19] Choulli, M., An inverse problem for a semilinear parabolic equation, *Inverse Probl.*, **10**, 1994, 1123–1132.

[20] Yang, L., Yu, J. and Deng, Z., An inverse problem of identifying the coefficient of parabolic equation, *Appl. Math. Model.*, **32**, 2008, 1984–1995.

[21] Takeuchi, T. and Yamamoto, M., Tikhonov regularization by a reproducing kernel Hilbert space for the Cauchy problem for an elliptic equation, *SIAM J. Sci. Comput.*, **31**, 2008, 112–142.

[22] Hon, Y. C. and Takeuchi, T., Discretized Tikhonov regularization by reproducing kernel Hilbert space for backward heat conduction problem, *Adv. Comput. Math.*, **34**, 2011, 167–183.

[23] Cui, M. and Lin, Y., Nonlinear numerical analysis in the reproducing kernel space, Nova Science Publishers, Inc., New York, 2009.

[24] Cui, M. and Geng, F., A computational method for solving one-dimensional variable-coefficient Burgers equation, *Appl. Math. Comput.*, **188**, 2007, 1389–1401.

[25] Zhou, Y., Cui, M. and Lin, Y., Numerical algorithm for parabolic problems with non-classical conditions, *J. Comput. Appl. Math.*, **230**, 2009, 770–780.

[26] Wang, W., Han, B. and Yamamoto, M., Inverse heat problem of determining time-dependent source parameter in reproducing kernel space, *Nonlinear Anal. Real*, **14**, 2013, 875–887.

[27] Wang, W., Yamamoto, M. and Han, B., Numerical method in reproducing kernel space for an inverse source problem for the fractional diffusion equation, *Inverse Probl.*, **29**, 2013, 1–15.