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Poles of L-Functions on Quaternion Groups

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Abstract The author shows that the (partial) standard Langlands L-functions on quaternion groups have at most simple poles at certain positive integers.

Keywords Siegel Eisenstein series, L-Functions, Quaternion groups, Regularized Siegel-Weil formula
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1 Introduction

In the theory of automorphic forms, Eisenstein series and L-functions play critical roles. The analytic continuation of L-functions and information about the poles can be obtained by using Siegel Eisenstein series. To be able to get better results for location of poles, one needs to use the doubling method and the regularized Kudla-Rallis-Siegel-Weil formula. Poles of L-functions are important in the theory of theta correspondence. In this paper, we follow Kudla-Rallis' work on the orthogonal-symplectic case to have similar results for quaternion groups $O^*(4n)$ and $Sp^*(n,n)$. Here some Yamana's results on quaternion groups are crucial.

Let $\mathbf{k}$ be a totally real number field and let D be a quaternion division algebra over $\mathbf{k}$ with a main involution σ. Let ϵ be either 1 or -1. Let $V = D^m$ be a non-degenerate ϵ-Hermitian space equipped with a D-valued non-degenerate form $(\ ,\ )$ such that $(x,y)^\sigma = -\epsilon(y,x)$ and $(xa, yb) = a^\sigma(x,y)b$ for all $a, b \in D$ and $x, y \in V$. Assume that m is even. Let

$$H = \{h \in GL(m, D) : (hx, hy) = (x, y) \text{ for all } x, y \in V\}.$$

Let W' be a $2n$-dimensional vector space over D. Let W be a maximal totally isotropic subspace of W'. Let $S \in GL(2n, D)$ satisfy $S^* = \epsilon S$, where $S^* = {}^tS^\sigma$ is the conjugate transpose of S. S defines an ϵ-Hermitian form on W'. Let

$$G = G^n = \{g \in GL(2n, D) : g^*Sg = S\}.$$

Then G and H form a dual reductive pair in the sense of [5]. We can see these groups as quaternion orthogonal or symplectic groups. For example, G can be denoted as follows:

$$G \simeq \begin{cases} O^*(4n), & \text{if } \epsilon = -1 \quad (\text{case } 1), \\ Sp^*(n,n), & \text{if } \epsilon = +1 \quad (\text{case } 2). \end{cases}$$

Let P be the parabolic subgroup of G which stablizes a maximal isotropic subspace of W'. Such a parabolic subgroup is called a Siegel parabolic subgroup and it has the Levi decomposition $P = MN$, where

$$M = M_n = \Big\{ m(a) = \begin{pmatrix} a & 0 \\ 0 & a^{*-1} \end{pmatrix} : a \in GL(n, D) \Big\}$$

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is the standard Levi subgroup and

$$N = N_n = \left\{ n(b) = \begin{pmatrix} 1_n & b \\ 0 & 1_n \end{pmatrix} : b \in \mathrm{Mat}_n(D), b + \epsilon b^* = 0 \right\}$$

is the unipotent radical.

The modular function of P is given by $\delta_P(m) = |\det a|^{2\rho_{2n}}$ with

$$\rho_n = \begin{cases} \dfrac{n-1}{2}, & \text{if } \epsilon = -1, \\ \dfrac{n+1}{2}, & \text{if } \epsilon = +1, \end{cases}$$

where

$$m = \begin{pmatrix} a & 0 \\ 0 & a^{*-1} \end{pmatrix}$$

is in the standard Levi component of P.

Fix a non-trivial additive character ψ on $\mathbb{A}/\mathbf{k}$. Let $S(V^n(\mathbb{A}))$ be the space of Schwartz functions on $V^n(\mathbb{A})$. Let $\omega = \omega_\psi$ be the associated Weil representation of $G(\mathbb{A}) \times H(\mathbb{A})$ on $S(V^n(\mathbb{A}))$ (for the explicit definition see [9, 13]).

For $g \in G_{\mathbb{A}},\ \Phi \in S(V^n(\mathbb{A}))$ and $s \in \mathbf{C}$, we define the function

$$f_\Phi(g, s) = (\omega(g)\Phi)(0)|a(g)|^{s - s_0},$$

where $s_0 = m - \rho_{2n}$.

For a unitary idele-class character $\chi : \mathbb{A}^\times / \mathbf{k}^\times \to \mathbf{C}^\times$, let

$$I_n(s, \chi) = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\chi |\cdot|^s)$$

be the degenerate principal series representation of $G(\mathbb{A})$ consisting of functions f on $G(\mathbb{A})$ which are finite sums of monomials $\bigotimes_v f_v$, where at any archimedean v, f_v is K_v-finite and smooth, and at any non-archimedean v, f_v is locally constant and compactly supported, such that

$$f(nm(a)g, s) = \chi(\det a)|\det a|^{s + \rho_{2n}} f(g, s).$$

Note that the map $\Phi \mapsto f_\Phi(\cdot, s_0)$ defines a $G_{\mathbb{A}}$ intertwining map from $S(V^n(\mathbb{A}))$ to $I_n(s_0, \chi)$.

We fix a standard maximal compact subgroup $K = \prod_v K_v$ of $G(\mathbb{A})$. Then $I(s, \chi)$ is a representation of $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$, where $\mathfrak{g}_\infty$ is the Lie algebra of G_∞. A function

$$s \mapsto \Phi(s) \in I(s, \chi)$$

is called a standard section if the restriction of $\Phi(s)$ to K is independent of s.

For a standard section $\Phi(s) \in I_n(s, \chi)$ and $g \in G(\mathbb{A})$, we define an Eisenstein series

$$E(g, s, \Phi) = \sum_{\gamma \in P_n(\mathbf{k}) \backslash G(\mathbf{k})} \Phi(\gamma g, s),$$

which is absolutely convergent for $\mathrm{Re}(s) > \rho_{2n}$. Eisenstein series may have a pole when $\mathrm{Re}(s) \leq \rho_{2n}$. Now we define the normalized Siegel Eisenstein series by

$$E^*(g, s, \Phi) = b_n^S(s) E(g, s, \Phi),$$

where $b_n^S(s)$ is a certain product of Abelian L-series for a finite set of primes S which includes bad primes (see Lemma 2.1).

Suppose that π is a given irreducible cuspidal representation of $G(\mathbb{A})$ and χ was a given character of $\mathbb{A}^\times/\mathbf{k}^\times$. Let

$$L^S(s,\pi\otimes\chi)=\prod_{v\notin S}L^v(s,\pi_v\otimes\chi_v)$$

be the (partial) standard Langlands L-function associated to $\pi\otimes\chi$. An integral representation of this L-function was given in [9] by using a doubling method.

Define the set X_n^+ to be

$$\begin{aligned}
X_n^+&=\Big\{n-\frac{1}{2}-j\,\Big|\,j\in\mathbb{Z},0\le j\le n-1\Big\},\quad \text{if } \epsilon=-1,\\
X_n^+&=\Big\{n+\frac{1}{2}-j\,\Big|\,j\in\mathbb{Z},1\le j\le n\Big\},\quad \text{if } \epsilon=1,\ \chi\ne 1,\\
X_n^+&=\Big\{n+\frac{1}{2}-j\,\Big|\,j\in\mathbb{Z},0\le j\le n, j\ne 1\Big\},\quad \text{if } \epsilon=1,\ \chi=1.
\end{aligned}$$

Theorem 1.1 (Main Theorem) (1) *If $\chi^2\ne 1$, then $L^S(s,\pi\otimes\chi)$ is entire.*
(2) *If $\chi^2=1$, then $L^S(s,\pi\otimes\chi)$ has at most simple poles and these can only occur for*

$$s\in\Big\{t+\frac{1}{2}:t\in X_{2n}^+ \text{ and } t\le\rho_{4n}-n\Big\}=\Big\{1,2,\cdots,n+\frac{1+\epsilon}{2}\Big\},$$

where $s=2$ is not included in the set when $n=1$, $\epsilon=1$ and $\chi=1$.

The proof uses the information about the poles of normalized Siegel Eisenstein series, the regularized Kudla-Rallis-Siegel-Weil (KRSW for short) formula and the doubling method.

2 Poles of Normalized Siegel Eisenstein Series

For $\mathrm{Re}(s)>\rho_{2n}$ define an intertwining operator $M(s,\chi)=M_n(s,\chi)$,

$$M(s,\chi):\ I(\chi\,|\cdot|^s)\to I(\chi\,|\cdot|^{-s})$$

by

$$M(s,\chi)\Phi(g,s)=\int_{N_n(\mathbb{A})}\Phi(w_nng,s)\mathrm{d}n,$$

where w_n is the longest Weyl element

$$w_n=\begin{pmatrix}0_n, & 1_n,\\ -\epsilon 1_n, & 0_n\end{pmatrix}.$$

From the general theory of Eisenstein series (for example see [1, 8]), we have the following theorem.

Theorem 2.1 *The Eisenstein series $E(g,s,\Phi)$ is absolutely convergent in the half-plane $\mathrm{Re}\,(s)>\rho_{2n}$ and has a meromorphic continuation to the complex plane. Its meromorphic continuation satisfies the functional equation*

$$E(g,s,\Phi)=E(g,-s,M(s,\chi)\Phi).$$

Note that $M(s,\chi)$ has a meromorphic continuation as well.

For any non-archimedean place v of $\mathbf{k}$ at which χ_v is unramified, let $\Phi_v^0(s)$ be the spherical standard section of

$$I_{n,v}(s,\chi_v)=\mathrm{Ind}_{P(\mathbf{k}_v)}^{G(\mathbf{k}_v)}(\chi_v|\cdot|_v^s)$$

determined by $\Phi_v^0(s)(k) = 1$ for all $k \in K_v$. Since $I_n(s,\chi) = \bigotimes_v I_{n,v}(s,\chi_v)$, we can write

$$M(s,\chi) = \bigotimes_v M_v(s,\chi),$$

where, for $\Phi_v \in I_{n,v}(s,\chi_v)$;

$$M_v(s,\chi)\Phi(g,s) = \int_{N_n(\mathbf{k}_v)} \Phi_v(w_n n g, s)\mathrm{d}n.$$

The following calculation can be done by using a standard Gindikin-Karpelevich type argument (see [4]) or a Casselman type argument (see [2, Theorem 3.1, p. 397]).

Lemma 2.1 *If v is a non-archimedean place of $\mathbf{k}$ at which χ_v is unramified and D splits over $\mathbf{k}_v$, then*

$$M_v(s,\chi_v)\Phi_v^0(s) = \frac{a_{n,v}(s)}{b_{n,v}(s)}\Phi_v^0(-s),$$

$$a_{n,v}(s,\chi_v) = \begin{cases} \prod_{k=1}^{2n} \zeta_v(2s-4n+2k), & \text{(case1)}, \\ L_v(s+\rho_{2n}-4n,\chi_v)\prod_{k=1}^{2n} \zeta_v(2s-4n+2k), & \text{(case2)}, \end{cases}$$

$$b_{n,v}(s,\chi_v) = \begin{cases} \prod_{k=1}^{2n} \zeta_v(2s+4n-2k+1), & \text{(case1)}, \\ L_v(s+\rho_{2n},\chi_v)\prod_{k=1}^{2n} \zeta_v(2s+4n-2k+1), & \text{(case2)}. \end{cases}$$

Theorem 2.2 (see [13]) *Let $\Phi(s)$ be a holomorphic section of $I_n(s,\chi)$.*
(1) *If $\chi^2 \neq 1$, then $E^*(g,s,\Phi)$ is entire.*
(2) *Assume that $\chi^2 = 1$. Then the poles of $E^*(g,s,\Phi)$ in $\mathrm{Re}(s) \geq 0$ are at most simple and occur in the set X_n^+.*

Remark 2.1 Kudla and Rallis investigated the poles of the normalized Eisenstein series for orthogonal and symplectic groups in [6–7]. In [10] similar results are obtained for quaternion groups, where some possible poles on the left half-plane are not ruled out. By using a regularized Siegel-Weil formula for quaternion groups, Yamana were able to remove possible poles on the left-half plane in [13].

3 Regularized Kudla-Rallis-Siegel-Weil Formula

For $\Phi \in S(V(\mathbb{A})^n)$, $g \in G_{\mathbb{A}}$ and $h \in H_{\mathbb{A}}$, define a theta kernel:

$$\theta(g,h;\Phi) = \sum_{x \in V(\mathbf{k})^n} \omega(g,h)\Phi(x).$$

This is a smooth function on $G_{\mathbb{A}} \times H_{\mathbb{A}}$, left $G_{\mathbf{k}} \times H_{\mathbf{k}}$-invariant, and slowly increasing on $(G_{\mathbf{k}}\backslash G_{\mathbb{A}}) \times (H_{\mathbf{k}}\backslash H_{\mathbb{A}})$. By Weil's criterion (see [11, 13]), the integral

$$I_0(g;\Phi) = \int_{H(\mathbf{k})\backslash H(\mathbb{A})} \theta(g,h;\Phi)\mathrm{d}h$$

converges absolutely for all Φ if either $r = 0$ or $m - r > 2\rho_{2n}$, where r is the dimension of a maximal isotropic $\mathbf{k}$-subspace of V.

Assume that $r > 0$ and $m - r \leq 2\rho_{2n}$. The theta integral is not convergent anymore for all Φ. We need to have a regularization. From [7] and [13], one can regularize the theta integral by finding an element z_0 in the Bernstein center of H such that for all $\Phi \in S(V(\mathbb{A})^n)$

(1) $z_0\Phi(0) = \Phi(0)$,

(2) $F_{z_0\Phi} = 0$.

Here a local place v is fixed and F is defined to be a certain local $G_v \times H_v$-intertwining map from $S(V^n)$ to a local induced representation (see [13]).

From the construction, it can be seen that the kernel $\theta(g, h; z_0\Phi)$ is rapidly decreasing on $H(\mathbf{k})\backslash H(\mathbb{A})$ and the integral

$$I(g;\Phi) = \int_{H(\mathbf{k})\backslash H(\mathbb{A})} \theta(g, h; z_0\Phi)\mathrm{d}h$$

is independent of the choice of a local place v and z_0.

Now we can state a regularized SWKR-formula for quaternion groups.

Theorem 3.1 (see [13]) *Let Φ be a holomorphic section of $I(s,\chi)$, $s_0 = m - \rho_{2n}$ and $A_{-1}(g,\Phi) = \operatorname{Res}_{s_0} E^*(g, s, f_\Phi)$.*

(1) *Let $\epsilon = -1$. Then we have*

$$A_{-1}(g,\Phi) = I(g;\Phi).$$

(2) *Let $\epsilon = 1$.*

(a) *If $m \leq n$, then*

$$A_{-1}(g,\Phi) = \ell^{-1}\sum_{j=1}^{\ell} I_j(g;\Phi).$$

(b) *If $m \geq n+1$, then for every j*

$$A_{-1}(g,\Phi) = I_j(g;\Phi),$$

where $I_j(g;\Phi)$s denote to the theta integrals associated to the global equivalence classes locally isometric to V.

4 Poles of *L*-Functions

4.1 Zeta integrals of doubling method

Let $\pi = \bigotimes_v \pi_v$ be an irreducible automorphic cuspidal representation of $G(\mathbb{A})$. The Peterson pairing

$$\langle f_1, f_2\rangle = \int_{G_{\mathbf{k}}\backslash G_{\mathbb{A}}} f_1(g)\overline{f_2(g)}\mathrm{d}g$$

induces a pairing on π and we choose local pairings $\langle\, ,\, \rangle$ on π_v such that

$$\langle f_1, f_2\rangle = \prod_v \langle f_1, f_2\rangle_v,$$

where $f_i = \bigotimes_v f_{i,v}$ for $i = 1, 2$ are factorizable vectors. Here, local pairings are normalized so that $\langle f_v^0, f_v^0\rangle_v = 1$ for the spherical vector $f_v^0 \in \pi_v$.

The product $G \times G$ is embedded in G^{2n} as usual by

$$\iota_0 : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \to \begin{pmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & b' \\ c & 0 & d & 0 \\ 0 & c' & 0 & d' \end{pmatrix}.$$

We make a slight change in ι_0 to make it useful. Set

$$\iota(g_1, g_2) = \iota_0 \left(g_1, \begin{pmatrix} 1_n & 0_n \\ 0_n & -1_n \end{pmatrix} g_2 \begin{pmatrix} 1_n & 0_n \\ 0_n & -1_n \end{pmatrix} \right).$$

Let

$$\delta = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \in G^{2n}.$$

Choose $f_1, f_2 \in \pi$ such that $f_{i,v}$ is a spherical vector, that is K_v-invariant for all $v \notin S$. Choose a factorizable section

$$\Phi(s) = \prod_v \Phi_v(s) \in I_{2n}(s, \chi)$$

such that for all $v \notin S, \Phi_v(s)$ is the normalized spherical vector. Let $E(g, s, \Phi)$ be the Siegel Eisenstein series on $G^{2n}(\mathbb{A})$. Consider the zeta integral

$$Z(s, f_1, f_2, \Phi) = \int_{(G\times G)_{\mathbf{k}}\backslash(G\times G)_{\mathbb{A}}} f(g_1)\overline{f_2(g_2)}E(\iota(g_1, g_2), s, \Phi)\mathrm{d}g_1\mathrm{d}g_2.$$

From the doubling method of [3–4], we have that for $\mathrm{Re}(s) > \rho_{4n}$,

$$Z(s, f_1, f_2, \Phi) = \frac{1}{b_{2n}^S(s, \chi)} L^S\Big(s + \frac{1}{2}, \pi \otimes \chi\Big) \langle \pi_S(\Phi(s)) f_1, f_2\rangle,$$

where $\Phi_S(s) = \bigotimes\limits_{v\in S} \Phi_v(s)$ and

$$\langle \pi_S(\Phi(s)) f_1, f_2\rangle = \int_{G_S} \langle \pi(g) f_1, f_2\rangle \Phi_S(\delta \cdot \iota(g, 1), s)\mathrm{d}g.$$

Assume that f_1 and f_2 are factorizable vectors. Define local zeta integrals by

$$Z_v(s, f_{1,v}, f_{2,v}, \Phi_v) = \int_{G_v} \langle \pi_v(g) f_{1,v}, f_{2,v}\rangle_v \Phi_v(\delta\iota(g, 1), s)\mathrm{d}g.$$

To be able to have a result about poles of L-functions, we need to control local zeta integrals. As in the case of orthogonal-symplectic groups (see [7]), by choosing Φ whose support is sufficiently small, we have the following lemma for quaternion groups.

Lemma 4.1 (see [7, 12]) (1) *Assume that v is non-archimedean. For any $f_{1,v} \in \pi_v$, there exists a choice of $\Phi_v(s) \in I_{2n,v}(s, \chi)$ such that*

$$\pi(\Phi_v(s)) f_{1,v} = f_{1,v}$$

for all $s \in \mathbf{C}$. In particular,

$$Z_v(s, f_{1,v}, f_{2,v}, \Phi_v) = \langle \pi(\Phi_v(s)) f_{1,v}, f_{2,v}\rangle_v = \langle f_{1,v}, f_{2,v}\rangle_v$$

for such a choice.

(2) *Assume that v is archimedean. For any $s_0 \in \mathbf{C}$, there exist $f_{1,v}, f_{2,v}$ and Φ_v such that the local zeta integral is nonzero at s_0.*

Corollary 4.1 *Suppose that $f_1, f_2 \in \pi$ are K_v invariant for $v \in S$. Assume that for each $v \in S$, $f_{1,v}, f_{2,v}$ and Φ_v satisfy the conditions of Lemma* 4.1. *Then*

$$Z(s, f_1, f_2, \Phi) = \frac{1}{b_{2n}^S(s, \chi)} L^S\Big(s + \frac{1}{2}, \pi \otimes \chi\Big)\langle \pi(\Phi_\infty(s)) f_1, f_2\rangle.$$

From here we have

$$\begin{aligned} b_{2n}^S(s,\chi) Z(s, f_1, f_2, \Phi) &= \int_{(G\times G)_{\mathbf{k}}\backslash (G\times G)_{\mathbb{A}}} f(g_1)\overline{f_2(g_2)} E^*(\iota_0(g_1, g_2), s, \Phi) \mathrm{d}g_1 \mathrm{d}g_2 \\ &= L^S\Big(s + \frac{1}{2}, \pi \otimes \chi\Big)\langle \pi(\Phi_\infty(s)) f_1, f_2\rangle. \end{aligned} \tag{4.1}$$

Therefore, any pole of $L^S\big(s + \frac{1}{2}, \pi \otimes \chi\big)$ must be a pole of $E^*(g, s, \Phi)$ for a suitable choice of Φ and we have the following corollary.

Corollary 4.2 (1) *If $\chi^2 \neq 1$, then $L^S(s, \pi \otimes \chi)$ is entire.*
(2) *If $\chi^2 = 1$, then $L^S(s, \pi \otimes \chi)$ has at most simple poles and these can only occur for*

$$s \in \Big\{t + \frac{1}{2} : t \in X_{2n}^+\Big\}.$$

From now on assume that $\chi^2 = 1$. Let $s_0 = m - \rho_{4n}$ and assume that $s_0 \in X_{2n}^+$. Fix f_1 and f_2 and choose $\Phi(s)$ as in Corollary 4.1. From (4.1), we have

$$\begin{aligned} &\operatorname*{Res}_{s_0} L^S\Big(s + \frac{1}{2}, \pi \otimes \chi\Big)\langle \pi(\Phi_\infty(s)) f_1, f_2\rangle \\ &= \operatorname*{Res}_{s_0} b_{2n}^S Z(s, f_1, f_2, \Phi) \\ &= \int_{(G\times G)_{\mathbf{k}}\backslash (G\times G)_{\mathbb{A}}} f(g_1)\overline{f_2(g_2)} A_{-1}(\iota(g_1, g_2), \Phi) \mathrm{d}g_1 \mathrm{d}g_2. \end{aligned} \tag{4.2}$$

The regularized SWKR-formula and theta correspondence allow us to improve the result in Corollary 4.2. For $f \in \pi$ and $\Phi \in S(V(\mathbb{A})^n)$, the theta lifting of f is defined by

$$\theta(h; f, \Phi) = \int_{G_{\mathbf{k}}\backslash G_{\mathbb{A}}} f(g)\, \theta(g, h; \Phi) \mathrm{d}g.$$

Then $\Theta_V(\pi) = \{\theta(f, \Phi) \mid f \in \pi, \Phi \in S(V(\mathbb{A})^n)\}$ is an invariant subspace of the space of automorphic forms on $H(\mathbb{A})$. Following Proposition 7.2.4 of [7], we have the next proposition.

Proposition 4.1 *For $\Phi_1, \Phi_2 \in S(V_0(\mathbb{A})^n)$, let $\Phi = \Phi_1 \otimes \Phi_2 \in S(V_0(\mathbb{A})^{2n})$. Then we have*

$$\int_{(G\times G)_{\mathbf{k}}\backslash (G\times G)_{\mathbb{A}}} f(g_1)\overline{f_2(g_2)}\, I(\iota(g_1, g_2); \Phi) \mathrm{d}g_1 \mathrm{d}g_2 = \int_{H_{\mathbf{k}}\backslash H_{\mathbb{A}}} (\theta(h; f_1, \Phi_1)\overline{\theta(h; f_2, \Phi_2)}) z_0 \mathrm{d}h,$$

where c is a nonzero constant, z_0 is the element in the Bernstein center of H used in the regularization of theta integrals.

Let r be the dimension of a maximal totally isotropic subspace of V. We may write V as a sum $V = V_0 \oplus V_{r,r}$, where $V_{r,r}$ is a split ϵ-Hermitian space of dimension of $2r$. Let $m_0 = 2\rho_{4n} - m$. Then we have the following vanishing result in theta correspondence.

Lemma 4.2 (see [7]) $\Theta_{V_0}(\pi) = 0$ *if $m_0 = \dim V_0 < n$.*

Now we can improve the result in the previous corollary by removing about half of the possible poles.

Theorem 4.1 (Main Theorem) *If $\chi^2 = 1$, then $L^S(s, \pi \otimes \chi)$ has at most simple poles and these can only occur for $s \in \left\{t + \frac{1}{2} : t \in X_{2n}^+ \text{ and } t \leq \rho_{4n} - n\right\} = \left\{1, 2, \cdots, n + \frac{1+\epsilon}{2}\right\}$. Here $s = 2$ is not included in the set when $n = 1$, $\epsilon = 1$ and $\chi = 1$.*

Proof Suppose that there exists a pole at $s \in \left\{t + \frac{1}{2} : t \in X_{2n}^+\right\}$. Let $f = f_1 = f_2$ and $\Phi(s)$ as in Corollary 4.1. If $L^S\left(s + \frac{1}{2}, \pi \otimes \chi\right)$ has a pole (with a nonzero residue) at $s = s_0$, then from (4.2) we conclude that the integral of $A_{-1}(\iota(g_1, g_2), \Phi)$ against $f \otimes \overline{f}$ is nonzero. By using the regularized SWKR formula, we see that there exists at least one quadratic space V_0 of dimension m_0, a character $\chi_{V_0} = \chi$ and a function $\Phi = \Phi_1 \otimes \Phi_2 \in S(V_0(\mathbb{A})^{2n})$ such that the function $I(\iota(g_1, g_2); \Phi)$ has a nonzero integral against $f \otimes \overline{f}$. By Proposition 4.1, we see that $\theta(f, \Phi_1) \neq 0$. This means that $\Theta_{V_0}(\pi) \neq 0$. From the previous lemma, this is possible only if $m_0 \geq n$. Therefore, we have $2\rho_{4n} - m \geq n$ which implies $s_0 = m - \rho_{4n} \leq \rho_{4n} - n$.

Remark 4.1 In [13], Yamana gave a theorem in which the set of possible poles of $L^S(s, \pi \otimes \chi)$ includes negative numbers. In the main theorem, we improve this result by removing possible poles in the left-half plane.

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