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Stability of Inverse Problems for Ultrahyperbolic Equations*

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(*Dedicated to the memory of Professor Arif Amirov*)

Abstract In this paper, the authors consider inverse problems of determining a coefficient or a source term in an ultrahyperbolic equation by some lateral boundary data. The authors prove Hölder estimates which are global and local and the key tool is Carleman estimate.

Keywords Ultrahyperbolic equation, Inverse problem, Stability, Carleman estimate
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1 Introduction

Let n, $m \in \mathbb{N}$ and $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$. We consider the ultrahyperbolic equation

$$\Delta_y u(x,y) - \Delta_x u(x,y) - p(x,y')u(x,y) = F(x,y) \tag{1.1}$$

in some bounded domain of $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$. We set

$$y = (y_1, y') \in \mathbb{R}^m, \quad y' = (y_2, \cdots, y_m) \in \mathbb{R}^{m-1}, \quad x = (x_1, \cdots, x_n) \in \mathbb{R}^n$$

and

$$\Delta_x = \sum_{i=1}^n \partial_{x_i}^2, \quad \Delta_y = \sum_{j=1}^m \partial_{y_j}^2, \quad \nabla_x = (\partial_{x_1}, \cdots, \partial_{x_n}), \quad \nabla_y = (\partial_{y_1}, \cdots, \partial_{y_m}),$$
$$\nabla_{x,y} = (\nabla_x, \nabla_y), \quad \partial_{x_i} = \frac{\partial}{\partial x_i}, \quad \partial_{y_j} = \frac{\partial}{\partial y_j}.$$

If $m = 1$, then (1.1) is a hyperbolic equation, where y_1 is the time variable. Generally, it is considered that one time dimension is fundamentally important in describing many dynamic evolutions of physical quantities in the classical and quantum fields. Multiple times have been considered rarely, because it is widely believed to violate the causality and lead to the instability yielding that the phenomena under consideration are not deterministic in a usual physical sense. However, certain developments in the theoretical physics such as the string theory require

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additional dimensions for the time, and for related literature, we refer to Bars [4], Craig and Weinstein [11], Sparling [29], and Tegmark [31]. In particular, the multiple dimensions are considered in the context of the twistor spaces (see [29]).

The quantum kinetic theory is one of the fields in which ultrahyperbolic type equations are arising. For example, let us consider the quantum kinetic equation:

$$\frac{\partial u}{\partial t}+\sum_{j=1}^{n} p_j\frac{\partial u}{\partial x_j}=\frac{\mathrm{i}}{(2\pi)^n h}\int_{\mathbb{R}^{2n}}\Big[\Phi\Big(x-\frac{h}{2}y,t\Big)-\Phi\Big(x+\frac{h}{2}y,t\Big)\Big] \\ \times\exp[\mathrm{i}y(p-\overline{p})]u(x,\overline{p},t)\mathrm{d}\overline{p}\mathrm{d}y+f$$

in a domain $\{(x,p,t);\, x=(x_1,x')\in\mathbb{R}^n,\ x_1>0,\ p\in\mathbb{R}^n,\ t\in\mathbb{R}\}$, where $u(x,p,t)$ is the quantum distribution function, h is Planck's constant, i is the imaginary unit, $\Phi(x,t)$ is the potential and $f(x,p,t)$ is the function characterizing the sources. Applying the Fourier transform with respect to p and the change of variables of the form

$$x-\frac{1}{2}hy=\xi,\quad x+\frac{1}{2}hy=\eta,$$

one can obtain the following ultrahyperbolic type equation:

$$\frac{\partial w}{\partial t}+\frac{\mathrm{i}}{2}h(\Delta_\eta-\Delta_\xi)w+\mathrm{i}[\Phi(\eta)-\Phi(\xi)]w=\widehat{f},$$

where $w(\xi,\eta,t)=\widehat{u}(x,y,t)$, $\widehat{u}$ and $\widehat{f}$ denote the Fourier transform of u and f respectively (see [2]). The ultrahyperbolic operator appears also as the stationary part of a generalized Schrödinger equation:

$$\mathrm{i}\partial_t u(x,y,t)=\Delta_x u(x,y,t)-\Delta_y u(x,y,t)$$

and for related nonlinear generalized Schrödinger equations, see [18–19, 30].

The solutions of some direct problems to ultrahyperbolic equations were investigated by Kostomarov [24–25] in the case of $n=3$, $m=2$ and $n=3$, $m=3$. As for the uniqueness and some mean value property of solutions to general ultrahyperbolic equations, see [10, 12, 14, 27], but there are very few results on the existence of the solution. In [11], the unique existence of solutions was proved for $\Delta_y u-\Delta_x u=0$ in $\mathbb{R}^{n+m}$ with suitably given initial data and also some non-uniqueness results were proved by some choice of hyperplanes, where the initial data are given. The proof in [11] assumed that all the coefficients in the ultrahyperbolic equation are constant in the whole domain because the key is the Fourier transform. There seems to be no result on the existence of the solution to a Cauchy problem of the ultrahyperbolic equation with non-analytic coefficients. In the case of analytic coefficients, by the Cauchy-Kovalevskaja theorem, we can establish the well-posedness of the initial value problem of determining the solution u to (1.1) satisfying $u(x,0,y')=a(x,y')$ and $\partial_{y_1}u(x,0,y')=b(x,y')$, where a and b are analytic.

In this article, we discuss inverse problems of determining a coefficient or a source term in an ultrahyperbolic equation. First we formulate an inverse source problem.

Let $D\subset\mathbb{R}^n$ be a bounded domain with smooth boundary ∂D and let

$$y=(y_1,y')\in\mathbb{R}^m,\quad y'=(y_2,\cdots,y_m)\in\mathbb{R}^{m-1}.$$

We arbitrarily fix

$$x_0 \notin \overline{D}.$$

Henceforth $(\,\cdot\,,\,\cdot\,)$ means the scalar product in $\mathbb{R}^n$ and $\mathbb{R}^m$. For $T, T_1 > 0$, we set

$$G(T,T_1) := \{y \in \mathbb{R}^m;\, |y_1| < T,\, |y'| < T_1\}, \quad G'(T,T_1) = G(T,T_1) \cap \{y_1 = 0\}.$$

In particular, we write

$$G = G(T,T), \quad G' = G'(T,T) = G \cap \{y_1 = 0\}.$$

Throughout this article, we identify $(0, y_2, \cdots, y_m) \in \mathbb{R}^m$ with $y' = (y_2, \cdots, y_m) \in \mathbb{R}^{m-1}$. Let $p(x, y')$ be given, $\nu(x) = (\nu_1(x), \cdots, \nu_n(x))$ denote the unit outward normal vector to ∂D, and $\partial_\nu u = (\nabla_x u, \nu)$. Moreover, let $T, T_1 > 0$ and $\Gamma \subset \partial D$ be given. We consider the following system:

$$Au = \Delta_y u(x,y) - \Delta_x u(x,y) - p(x,y')u(x,y) = f(x,y')R(x,y), \quad x \in D,\, y \in G(T,T_1), \tag{1.2}$$

$$u(x,0,y') = \partial_{y_1} u(x,0,y') = 0, \quad x \in D,\, y' \in G'(T,T_1), \tag{1.3}$$

$$u(x,y) = 0, \quad x \in \Gamma,\, y \in G(T,T_1). \tag{1.4}$$

We consider an inverse problem of determining $f(x, y')$ in (1.2) by extra data of the solution to (1.2)–(1.4).

Inverse Source Problem Let p, R be given suitably. Then determine $f(x, y')$, $x \in D$, $y' \in G'(T, T_1)$ by $\partial_\nu u|_{\Gamma \times G(T,T_1)}$. Here we do not assume the uniqueness of u, but its existence.

Next we discuss an inverse problem of determining a coefficient by overdetermining lateral boundary data. More precisely, we consider

$$\Delta_y v(x,y) - \Delta_x v(x,y) - p(x,y')v(x,y) = 0, \quad x \in D,\, y \in G(T,T_1), \tag{1.5}$$

$$v(x,0,y') = a(x,y'), \quad \partial_{y_1} v(x,0,y') = b(x,y'), \quad x \in D,\, y' \in G'(T,T_1), \tag{1.6}$$

$$v(x,y) = 0, \quad x \in \Gamma,\, y \in G(T,T_1). \tag{1.7}$$

Let $v = v(p)$ satisfy (1.5)–(1.7). We discuss the following problem.

Coefficient Inverse Problem Determine the coefficient $p(x, y')$, $(x, y') \in D \times G'(T, T_1)$ in (1.5) by extra data $\partial_\nu v(p)|_{\Gamma \times G}$.

Our main purpose is to establish the uniqueness and the stability for these inverse problems, assuming the existence of $v(p)$ and $v(q)$ within adequate classes.

The coefficient inverse problem is reduced to the inverse source problem as follows. Let $v(p)$ and $v(q)$ be two solutions of (1.5)–(1.7) with the coefficients p and q respectively. Here we do not assume the uniqueness of $v(p)$ and $v(q)$ but their existence.

The difference $u = v(p) - v(q)$ satisfies (1.2)–(1.4), where $f(x, y') = p(x, y') - q(x, y')$ and $R = v(q)(x, y)$. Therefore the determination of p, q is reduced to the inverse source problem.

Thus we first discuss the inverse source problem for (1.2)–(1.4). For the statements of the main results, we introduce the following notations. For $\delta > 0$, $x_0 \notin \overline{D}$ and $0 < \beta < 1$, we define the domains by

$$\Omega(\delta) = \{(x,y) \in D \times G(T,T_1);\, |x - x_0|^2 - \beta|y|^2 > \delta^2\},$$

$$\Omega'(\delta) = \Omega(\delta) \cap \{y_1 = 0\}.$$

We use the same notations $\Omega(\delta)$, $\Omega'(\delta)$ for $G = \{y \in \mathbb{R}^m;\ |y_1| < T,\ |y'| < T\}$ if there is no fear of confusion. Let $M > 0$ be arbitrarily fixed.

We are ready to state the following theorem.

Theorem 1.1 *We consider* (1.2)–(1.3) *in* $D \times G$. *Let*

$$f \in L^2(D \times G'), \quad p \in L^\infty(D \times G'), \quad \|p\|_{L^\infty(D\times G')} \le M,$$
$$R \in H^1(-T,T;L^\infty(D \times G')), \quad \|\partial_{y_1} R\|_{L^2(-T,T;L^\infty(D\times G'))} \le M.$$

We further assume that

$$\|f\|_{L^2(D\times G')} \le M, \quad \|\partial_{y_1} u\|_{H^2(D\times G)} \le M$$

and that there exists a constant $r_0 > 0$ *such that*

$$|R(x,0,y')| \ge r_0 > 0 \quad \text{in } D \times G'.$$

Finally we assume

$$\max_{x\in\overline{D}} |x - x_0| < \sqrt{\beta T^2 + \delta^2} \tag{1.8}$$

and that $\Gamma \subset \partial D$ *satisfies*

$$\Gamma \supset \partial D \cap \{|x - x_0| \ge \delta\}. \tag{1.9}$$

Then for any $\delta_1 > \delta$, *there exist constants* $C > 0$ *and* $\theta \in (0,1)$, *depending on* M, r_0 *such that*

$$\|f\|_{L_2(\Omega'(\delta_1))} \le C\|\partial_\nu \partial_{y_1} u\|^\theta_{L^2(\Gamma\times G)}. \tag{1.10}$$

Theorem 1.1 gives a local estimate. More precisely, given $\Gamma \subset \partial D$ and $T > 0$, we can find a subdomain $\Omega'(\delta_1)$, where the L^2-norm of f is estimated. For example, we choose $\widetilde{\delta}, \delta > 0$ with $\widetilde{\delta} > \delta$ and $0 < t_0 < T$ arbitrarily. We take β sufficiently small such that $\widetilde{\delta}^2 - \delta^2 > \beta t_0^2$. For this β, we choose $T > 0$ sufficiently large such that (1.8) holds. Then Theorem 1.1 asserts

$$\int_{|y'|<t_0} \int_{|x-x_0|>\widetilde{\delta}} |f(x,y')|^2 \mathrm{d}x\mathrm{d}y' \le C\Big(\int_{\Gamma\times G} |\partial_\nu \partial_{y_1} u|^2 \mathrm{d}S_y \mathrm{dS}_x\Big)^\theta. \tag{1.11}$$

In fact, we choose $\delta_1 > 0$ sufficiently close to δ such that $\widetilde{\delta} > \delta_1 > \delta$ and $\widetilde{\delta}^2 - \delta_1^2 > \beta t_0^2$. Since $|x-x_0|^2 - \beta|y'|^2 > \widetilde{\delta}^2 - \beta t_0^2 > \delta_1^2$ for $|x - x_0| > \widetilde{\delta}$ and $|y'| < t_0$, we see that $\Omega'(\delta_1) \supset \{|x - x_0| > \widetilde{\delta}\} \times \{|y'| < t_0\}$. Therefore (1.10) yields (1.11).

If we want to estimate f for larger t_0, then β has to be small and so we have to choose $T > 0$ very large, that is, we need to observe longer in y-direction as the right-hand side of (1.11) shows. In particular, for sufficiently large δ_1 we have $D \subset \{|x - x_0| > \delta_1\}$ and so the left-hand side of (1.11) estimates f over D provided that t_0 is small and $T > 0$ is very large. The above observation means that if we want to estimate f in a larger subdomain of D, then the size T of the "time" region has to be large. This fact corresponds to the finiteness of the propagation speed, which is a typical character for the case of $m = 1$.

In Theorem 1.1, it is not clear how large T and Γ are necessary for identifying f in a given subdomain or $D \times G'$. Next we derive the estimation of f in an arbitrarily given subdomain of $D \times G'$. For the statement, we recall

$$G(T, 2T) = \{y \in \mathbb{R}^m;\ |y_1| < T,\ |y'| < 2T\} \tag{1.12}$$

with $T > 0$. For $x_0 \notin \overline{D}$, we set

$$\partial D_+ = \{x \in \partial D; \, ((x - x_0), \nu) \geq 0\}. \tag{1.13}$$

We note that ∂D_+ is a proper subset of ∂D in general.

Theorem 1.2 *Let u satisfy (1.2)–(1.3) in $D \times G(T, 2T)$ and $u = 0$ on $\partial D \times G(T, 2T)$ and*

$$\|\partial_{y_1}^k u\|_{H^2(D\times G(T,2T))} \leq M, \quad k = 1, 2. \tag{1.14}$$

We further assume that $\beta > 0$ is sufficiently small and

$$T > \frac{1}{\sqrt{\beta}} \max_{x \in \overline{D}} |x - x_0|$$

and that

$$\|\partial_{y_1}^k R\|_{L^2(-T,T;L^\infty(D\times\{|y'|<2T\}))} \leq M, \quad k = 1, 2$$

and

$$|R(x, 0, y')| \neq 0, \quad x \in \overline{D}, \, |y'| \leq 2T.$$

Then for any small $\epsilon > 0$, there exist constants $C > 0$ and $\theta \in (0, 1)$ depending on ϵ, M, x_0, such that

$$\|f\|_{L^2(D\times\{|y'|<T-\epsilon\})} \leq C \sum_{k=1}^{2} \|\partial_\nu \partial_{y_1}^k u\|^\theta_{L^2(\partial D_+ \times G(T,2T))}. \tag{1.15}$$

Next we show stability results for the coefficient inverse problem. We state two results which correspond to Theorems 1.1–1.2, respectively.

Theorem 1.3 *We consider (1.5)–(1.7). Let*

$$\begin{aligned}
&p \in L^\infty(D \times G'), \quad \|p\|_{L^\infty(D\times G')}, \, \|q\|_{L^\infty(D\times G')} \leq M, \\
&v(p), v(q) \in H^1(-T, T; L^\infty(D \times G')), \\
&\|\partial_{y_1} v(p)\|_{H^2(D\times G)}, \|\partial_{y_1} v(q)\|_{H^2(D\times G)} \leq M, \\
&\|\partial_{y_1} v(p)\|_{L^2(-T,T;L^\infty(D\times G'))}, \|\partial_{y_1} v(q)\|_{L^2(-T,T;L^\infty(D\times G'))} \leq M, \\
&|a(x, y')| \geq r_0 \quad \text{on } \overline{D \times G'}
\end{aligned}$$

with some $r_0 > 0$. We assume that $0 < \beta < 1$, (1.8) and (1.9) hold. Then for any $\delta_1 > \delta$, there exist constants $C > 0$ and $\theta \in (0, 1)$, depending on M, r_0, such that

$$\|p - q\|_{L_2(\Omega'(\delta_1))} \leq C \|\partial_\nu \partial_{y_1} (v(p) - v(q))\|^\theta_{L^2(\Gamma\times G)}.$$

Theorem 1.4 *Let $v(p), v(q)$ satisfy (1.5)–(1.6) with p, q respectively and $v(p) = v(q)$ on $\partial D \times G(T, 2T)$. We assume*

$$\begin{aligned}
&\|\partial_{y_1} v(p)\|_{H^2(D\times G(T,2T))}, \|\partial_{y_1} v(q)\|_{H^2(D\times G(T,2T))} \leq M, \quad k = 1, 2, \\
&\|\partial_{y_1}^k v(p)\|_{L^2(-T,T;L^\infty(D\times\{|y'|<2T\}))}, \|\partial_{y_1}^k v(q)\|_{L^2(-T,T;L^\infty(D\times\{|y'|<2T\}))} \leq M, \\
&|a(x, y')| \geq r_0 \quad \text{on } \overline{D} \times \{|y'| \leq 2T\}
\end{aligned}$$

with some $r_0 > 0$. We further assume

$$T > \frac{1}{\sqrt{\beta}} \max_{x\in\overline{D}} |x - x_0|.$$

Then for any small $\epsilon > 0$, there exist constants $C > 0$ and $\theta \in (0,1)$ depending on ϵ, M, x_0, such that

$$\|p-q\|_{L^2(D\times\{|y'|<T-\epsilon\})} \le C\sum_{k=1}^{2} \|\partial_\nu \partial_{y_1}^k (v(p)-v(q))\|^{\theta}_{L^2(\partial D_+\times G(T,2T))}.$$

As is seen by the proof in Section 4, in Theorem 1.2, we can obtain the same stability in the case where $G(T,2T)$ is replaced by a smaller $G := \{y \in \mathbb{R}^m;\ |y_1| < T,\ |y'| < T\}$, if we can take also the norm of data on the other subboundary of G:

$$\|\partial_{y_1}^k u\|_{L^2(D\times(\partial G\setminus\{y_1=\pm T\}))}, \quad \|\partial_\nu \partial_{y_1}^k u\|_{L^2(D\times(\partial G\setminus\{y_1=\pm T\}))}, \tag{1.16}$$

where ν is the unit outward normal vector to $\partial G \setminus \{y_1 = \pm T\}$ and $\partial_\nu u = \nabla_y u \cdot \nu$. A similar remark holds for Theorem 1.4. Moreover, if we have an a priori Lipschitz estimate for the direct problem for (1.2)–(1.3) with $u = 0$ on $(\partial D \times G) \cup (D \times (\partial G \setminus \{y_1 = \pm T\}))$, then the same method as Imanuvilov and Yamamoto [16] can yield the Lipschitz stability, but we do not know such Lipschitz stability for the direct problem for $m \ge 2$. In the case of $m = 1$, that is, the inverse hyperbolic problem, we can replace (1.15) by the Lipschitz stability (see [15–16]) and we note that we need not fix $\epsilon > 0$. Moreover, for the uniqueness of f in $D \times \{|y'| < T\}$, we need a boundary datum $\partial_\nu u$ over $\partial D_+ \times \{(y_1, y');\ |y_1| < T,\ |y'| < 2T\}$, that is, we need a twice longer y'-region for the observation than the domain in y' where f is determined.

In Theorems 1.2 and 1.4, we can not take $\epsilon = 0$. However, since $\epsilon > 0$ is arbitrary, we can prove the uniqueness: For example, in Theorem 1.2, if $u(x,y) = 0$ for $x \in \partial D$, $|y_1| < T$ and $|y'| < 2T$ and $\partial_\nu u(x,y) = 0$ for $x \in \partial D_+$, $|y_1| < T$ and $|y'| < 2T$, then $f(x,y') = 0$ for $x \in D$ and $|y'| < T$.

The proofs of the theorems are based on the method by Bukhgeim and Klibanov [9]. In [9], the authors first applied a Carleman estimate which is an L^2-estimate with large parameters, and then established the uniqueness in determining a spatially varying coefficient by overdetermining lateral boundary data. After [9], there have been many works relying on that method with modified arguments. We refer to Amirov [1–2], Amirov and Yamamoto [3], Baudouin and Puel [5], Bellassoued [6], Bellassoued and Yamamoto [7–8], Imanuvilov and Yamamoto [15–16], Isakov [17], Khaĭdarov [20], Klibanov [21–22], Klibanov and Timonov [23], and Yamamoto [32]. Here we do not intend to give a complete list of the works and refer to the references therein. There are satisfactory amounts of works on classical equations in mathematical physics, but there are very few works for inverse problems of ultrahyperbolic equations. A key Carleman estimate was proved by Amirov [1–2] and Lavrent'ev, Romanov and Shishat·skiĭ [26], where they applied the Carleman estimates to the unique continuation and proved stability. See also Romanov [28] for a Carleman estimate for an ultrahyperbolic equation in a Riemannian manifold and an application to some unique continuation problems. In Chapter 4 of Amirov [2], the uniqueness for an inverse source problem of a different type was proved by using the Carleman estimate. To the best knowledge of the authors, there are no results on the conditional stability like Theorems 1.1–1.4.

This paper is composed of four sections and one appendix. In Section 2, we present two Carleman estimates. Sections 3–4 are devoted to the proofs of Theorems 1.1–1.2 respectively. The proof of the key Carleman estimate is given in Appendix.

2 Key Carleman Estimate

In this section, we show two Carleman estimates for an ultrahyperbolic equation. The former Carleman estimate is used for the proof of Theorem 1.1 and the latter for the proof of Theorem 1.2. As for the general theory of Carleman estimates for functions with compact supports, we refer to, for example, Hörmander [13] and Isakov [17], but we here give a direct proof because we need a Carleman estimate for functions not having compact supports and the proof of that Carleman estimate does not follow directly from [13, 17]. Another direct proof of a Carleman estimate for an ultrahyperbolic equation is found in [2, 26].

Here and henceforth let $\Gamma_x = \partial D \times G(T, T_1)$, $\Gamma_y = D \times \partial G(T, T_1)$, and let $\int_{\Gamma_x} \cdots \mathrm{dS}_x$ and $\int_{\Gamma_y} \cdots \mathrm{d}S_y$ be the boundary integrals on Γ_x and Γ_y, respectively.

We recall that for $x_0 \notin \overline{D}$, $y_0 \in \mathbb{R}^m$ and $\beta \in (0,1)$, we set

$$\varphi(x,y) = \mathrm{e}^{\gamma\psi(x,y)}, \quad \psi(x,y) = |x - x_0|^2 - \beta|y - y_0|^2,$$

where γ is a positive parameter. We consider the following equation:

$$\begin{aligned} Lu = \Delta_y u(x,y) - \Delta_x u(x,y) + \sum_{i=1}^{n} a_i(x,y)\partial_{x_i} u + \sum_{j=1}^{m} b_j(x,y)\partial_{y_j} u + a_0(x,y)u, \\ x \in D, \quad y \in G(T, T_1). \end{aligned} \tag{2.1}$$

Here we recall that

$$\Omega(\delta) = \{(x,y) \in D \times G(T,T_1);\ |x - x_0|^2 - \beta|y|^2 > \delta\}$$

and

$$\Omega'(\delta) = \Omega(\delta) \cap \{y_1 = 0\}, \quad G(T,T_1) = \{y \in \mathbb{R}^m;\ |y_1| < T,\ |y'| < T_1\}.$$

Let $\mu(x,y)$ be the outward unit normal vector to $\partial(D \times G(T,T_1))$ at (x,y) and let $\partial_\mu u = \nabla_{x,y} u \cdot \mu$. Henceforth we recall that $\partial D_+ \subset \partial D$ is defined by (1.13).

Theorem 2.1 *In* (2.1)*, let us assume that $a_i, b_j \in L^\infty(D \times G(T,T_1))$ for $0 \le i \le n$ and $1 \le j \le m$. Moreover, let $0 < \beta < 1$ be small and $\gamma > 0$ be sufficiently large, and let*

$$|x - x_0|^2 - \beta^2|y|^2 > \delta_0^2, \quad (x,y) \in D \times G(T,T_1) \tag{2.2}$$

with some $\delta_0 > 0$. Then there exist constants $C > 0$ and $s_0 > 0$ such that

$$\begin{aligned} &\int_{\Omega(\delta)} (s|\nabla_y u|^2 + s|\nabla_x u|^2 + s^3 u^2)\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}y \\ &\le C\int_{\Omega(\delta)} |Lu|^2 \mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}y + C\int_{\partial\Omega(\delta)} (s^3|u|^2 + s|\partial_\mu u|^2)\mathrm{e}^{2s\varphi}\mathrm{dS}_x\mathrm{dS}_y \end{aligned}$$

for all $u \in H^2(D \times G(T,T_1))$ and $s \ge s_0$.

Theorem 2.2 *Let $a_i, b_j \in L^\infty(D \times G(T,T_1))$, $0 \le i \le n$, $1 \le j \le m$ and (2.2) hold for $(x,y) \in D \times G(T,T_1)$. Then there exist constants $C > 0$ and $s_0 > 0$ such that*

$$\int_{D\times G(T,T_1)} (s|\nabla_y u|^2 + s|\nabla_x u|^2 + s^3u^2)\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}y$$
$$\le C\int_{D\times G(T,T_1)} |Lu|^2\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}y + C\int_{\partial D_+\times G(T,T_1)} s|\partial_\nu u|^2\mathrm{e}^{2s\varphi}\mathrm{dS}_x\mathrm{d}y$$

for all $s \ge s_0$ and all $u \in H^2(D \times G(T,T_1))$ satisfying

$$u = 0 \quad \text{on } \partial D \times G(T,T_1), \quad u = |\nabla_y u| = 0 \quad \text{on } D \times \partial G(T,T_1). \tag{2.3}$$

Theorem 2.1 gives a Carleman estimate which holds only in a level set $\Omega(\delta)$, while the Carleman estimate in Theorem 2.2 is global over the total domain $G \times (T,T_1)$. The proofs of Theorems 2.1–2.2 rely on an idea similar to Bellassoued and Yamamoto [8] and the proof is obtained only by integration by parts and is lengthy, so we give the proof in Appendix.

3 Proof of Theorem 1.1

The proofs of Theorems 1.3–1.4 are reduced to the proofs of Theorems 1.1–1.2, respectively, which this follows from setting $u = v(p) - v(q)$, $f = p - q$ and $R = v(q)$. Therefore it is sufficient to prove only Theorems 1.1–1.2. In this section, we will prove Theorem 1.1.

We set

$$\widetilde{r} = \max_{x\in\overline{D}} |x - x_0|, \quad G = G(T,T), \quad \psi(x,y) = |x - x_0|^2 - \beta|y|^2.$$

First by (1.8), we see that

$$\text{if } x \in D \text{ and } \psi(x,t) > \delta^2, \quad \text{then } |y| < T. \tag{3.1}$$

In fact, let $x \in D$ and $\psi(x,t) > \delta^2$. Then (1.8) yields

$$\delta^2 < \psi(x,y) \le \widetilde{r}^2 - \beta|y|^2 \le \beta T^2 + \delta^2 - \beta|y|^2.$$

Therefore, $\beta T^2 - \beta|y|^2 > 0$, that is, $|y| < T$. Thus (3.1) is verified.

Next, we characterize $\partial\Omega(\delta)$. We can easily have $\partial\Omega(\delta) = \bigcup_{j=1}^{3} \Gamma_j$, where

$$\Gamma_1 = (\partial D \times G) \cap \{(x,y) \in \overline{D \times G};\ \psi(x,y) > \delta^2\},$$
$$\Gamma_2 = (D \times \partial G) \cap \{(x,y) \in \overline{D \times G};\ \psi(x,y) > \delta^2\},$$
$$\Gamma_3 = (D \times G) \cap \{(x,y) \in \overline{D \times G};\ \psi(x,y) = \delta^2\}.$$

Then, similar to (3.1), we can prove that $\Gamma_2 = \emptyset$. In fact, (1.8) implies

$$\delta^2 = \psi(x,y) \le \widetilde{r}^2 - \beta|y|^2 \le \widetilde{r}^2 - \beta T^2 < \delta^2,$$

which is impossible.

Moreover, noting that $\Gamma_1 \subset (\partial D \times G) \cap \{(x,y) \in \overline{D \times G};\ |x - x_0| > \delta\}$, we see that

$$\Gamma_1 \subset \Gamma \times G.$$

Hence

$$\partial\Omega(\delta) \subset (\Gamma \times G) \cup ((D \times G) \cap \{(x,y) \in \overline{D \times G};\ \psi(x,y) = \delta^2\}). \tag{3.2}$$

Now we apply the Carleman estimate in Theorem 2.1, but no data are given on

$$(D \times G) \cap \{(x,y) \in \overline{D \times G};\ \psi(x,y) = \delta^2\}$$

and so we need a cut-off function.

Henceforth, $C > 0$ denotes a generic constant which is independent of s. We define a cut-off function $\chi \in C_0^\infty(\Omega(\delta))$ such that $0 \le \chi(x,y) \le 1$ and

$$\chi(x,y) = \begin{cases} 1, & (x,y) \in \Omega(\delta + 2\epsilon), \\ 0, & (x,y) \in \mathbb{R}^{n+m} \setminus \Omega(\delta + \epsilon). \end{cases} \tag{3.3}$$

Setting $z = (\partial_{y_1} u)\mathrm{e}^{s\varphi}\chi$, for $1 \le i \le n$, we have

$$\begin{aligned}
\partial_{x_i} z &= (\partial_{x_i}\partial_{y_1} u)\mathrm{e}^{s\varphi}\chi + s(\partial_{x_i}\varphi)z + (\partial_{y_1} u)\mathrm{e}^{s\varphi}\partial_{x_i}\chi, \\
\partial_{x_i}^2 z &= (\partial_{x_i}^2\partial_{y_1} u)\mathrm{e}^{s\varphi}\chi + s(\partial_{x_i}\varphi)(\partial_{x_i} z - s(\partial_{x_i}\varphi)z) \\
&\quad + s(\partial_{x_i}^2\varphi)z + s(\partial_{x_i}\varphi)(\partial_{x_i} z) + 2(\partial_{x_i}\partial_{y_1} u)\mathrm{e}^{s\varphi}\partial_{x_i}\chi + (\partial_{y_1} u)\mathrm{e}^{s\varphi}\partial_{x_i}^2\chi,
\end{aligned}$$

that is,

$$\begin{aligned}
\Delta_x z &= (\Delta_x \partial_{y_1} u)\mathrm{e}^{s\varphi}\chi + 2s(\nabla_x\varphi, \nabla_x z) + z(s\Delta_x\varphi - s^2|\nabla_x\varphi|^2) \\
&\quad + 2(\nabla_x(\partial_{y_1} u), \nabla_x\chi)\mathrm{e}^{s\varphi} + (\partial_{y_1} u)\mathrm{e}^{s\varphi}\Delta_x\chi.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\Delta_y z &= (\Delta_y \partial_{y_1} u)\mathrm{e}^{s\varphi}\chi + 2s(\nabla_y\varphi, \nabla_y z) + z(s\Delta_y\varphi - s^2|\nabla_y\varphi|^2) \\
&\quad + 2(\nabla_y\partial_{y_1} u, \nabla_y\chi)\mathrm{e}^{s\varphi} + (\partial_{y_1} u)\mathrm{e}^{s\varphi}(\Delta_y\chi).
\end{aligned}$$

From (1.2), we obtain

$$\begin{aligned}
Az &= f(\partial_{y_1} R)\mathrm{e}^{s\varphi}\chi + s\{2(\nabla_y\varphi, \nabla_y z) - 2(\nabla_x\varphi, \nabla_x z) + (\Delta_y\varphi - \Delta_x\varphi)z\} \\
&\quad - s^2(|\nabla_y\varphi|^2 - |\nabla_x\varphi|^2)z + 2\mathrm{e}^{s\varphi}\{(\nabla_y\partial_{y_1} u, \nabla_y\chi) - (\nabla_x\partial_{y_1} u, \nabla_x\chi)\} \\
&\quad + (\partial_{y_1} u)\mathrm{e}^{s\varphi}(\Delta_y\chi - \Delta_x\chi).
\end{aligned} \tag{3.4}$$

In particular, setting $w = \chi(\partial_{y_1} u)$ and $s = 0$ in (3.4) we have

$$Aw = f(\partial_{y_1} R)\chi + 2(\nabla_y\partial_{y_1} u, \nabla_y\chi) - 2(\nabla_x\partial_{y_1} u, \nabla_x\chi) + (\partial_{y_1} u)(\Delta_y\chi - \Delta_x\chi). \tag{3.5}$$

By (3.2)–(3.3), we see that

$$w = |\nabla_y w| = |\nabla_x w| = 0 \quad \text{ on } (D \times G) \cap \{(x,y) \in \overline{D \times G};\ \psi(x,y) = \delta^2\}.$$

By (1.4) we have $|\nabla_y u| = 0$ and $\nabla_x\partial_{y_1} u = (\partial_\nu\partial_{y_1} u)\nu$ on $\Gamma \times G$. Moreover, by $0 < \beta < 1$ we note that

$$|x - x_0|^2 - \beta^2|y|^2 > |x - x_0|^2 - \beta|y|^2 > \delta^2 > 0.$$

Thus the assumptions in Theorem 2.1 are satisfied in $\Omega(\delta)$. We apply the Carleman estimate given by Theorem 2.1, and we obtain

$$\begin{aligned}
&\int_{\Omega(\delta)} (s|\nabla_y w|^2 + s|\nabla_x w|^2 + s^3|w|^2)\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}y \\
\le{}& C\int_{\Omega(\delta)} |f(\partial_{y_1}R)\chi|^2\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}y \\
&+ C\int_{\Omega(\delta)} |2(\nabla_y\partial_{y_1}u, \nabla_y\chi) - (\nabla_x\partial_{y_1}u, \nabla_x\chi) \\
&+ (\partial_{y_1}u)(\Delta_y\chi - \Delta_x\chi)|^2\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}y \\
&+ Cs^3\int_{\Gamma\times G} |\partial_\nu\partial_{y_1}u|^2\mathrm{e}^{2s\varphi}\mathrm{dS}_x\mathrm{d}y. \qquad (3.6)
\end{aligned}$$

Here we also used

$$\partial_\nu w = (\partial_\nu\partial_{y_1}u)\chi + (\partial_\nu\chi)\partial_{y_1}u.$$

Since $z = w\mathrm{e}^{s\varphi}$, we have

$$s^3z^2 = s^3w^2\mathrm{e}^{2s\varphi} \qquad (3.7)$$

and

$$\begin{aligned}
s(|\nabla_y z|^2 + |\nabla_x z|^2) &= s(|\nabla_y(w\mathrm{e}^{s\varphi})|^2 + |\nabla_x(w\mathrm{e}^{s\varphi})|^2) \\
&= s(|(\nabla_y w)\mathrm{e}^{s\varphi} + ws(\nabla_y\varphi)\mathrm{e}^{s\varphi}|^2 + |(\nabla_x w)\mathrm{e}^{s\varphi} + ws(\nabla_x\varphi)\mathrm{e}^{s\varphi}|^2) \\
&\le C(s(|\nabla_y w|^2 + |\nabla_x w|^2) + s^3w^2)\mathrm{e}^{2s\varphi}. \qquad (3.8)
\end{aligned}$$

We set

$$\mu_2 = \mathrm{e}^{\gamma(\delta+2\epsilon)^2}, \quad \mu_3 = \mathrm{e}^{\gamma(\delta+3\epsilon)^2}.$$

On the other hand, by (3.3), the supports of the functions $\nabla_x\chi, \nabla_y\chi, \Delta_x\chi, \Delta_y\chi$ are the subsets of $\Omega(\delta+\epsilon)\setminus\Omega(\delta+2\epsilon)$, so that

$$\begin{aligned}
&\int_{\Omega(\delta)} |2(\nabla_y\partial_{y_1}u, \nabla_y\chi) - (\nabla_x\partial_{y_1}u, \nabla_x\chi) + (\partial_{y_1}u)(\Delta_y\chi - \Delta_x\chi)|^2\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}y \\
\le{}& C\mathrm{e}^{2s\mu_2}\int_{\Omega(\delta+\epsilon)\setminus\Omega(\delta+2\epsilon)} (|\partial_{y_1}u|^2 + |\nabla_y\partial_{y_1}u|^2 + |\nabla_x\partial_{y_1}u|^2)\mathrm{d}x\mathrm{d}y \\
\le{}& C\mathrm{e}^{2s\mu_2}\|u\|^2_{H^2(\Omega(\delta))} \\
\le{}& C\mathrm{e}^{2s\mu_2}M^2. \qquad (3.9)
\end{aligned}$$

In terms of (3.7)–(3.9), we rewrite (3.6) as

$$\begin{aligned}
&\int_{\Omega(\delta)} (s|\nabla_y z|^2 + s|\nabla_x z|^2 + s^3z^2)\mathrm{d}x\mathrm{d}y \\
\le{}& C\int_{\Omega(\delta)} |f(\partial_{y_1}R)\chi|^2\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}y + C\mathrm{e}^{2s\mu_2}M^2 + C\mathrm{e}^{Cs}\int_{\Gamma\times G} |\partial_\nu\partial_{y_1}u|^2\mathrm{dS}_x\mathrm{d}y. \qquad (3.10)
\end{aligned}$$

We set

$$\Omega_\delta^- := \{(x, y) \in \Omega(\delta);\, y_1 < 0\}.$$

We multiply (3.4) by $\partial_{y_1} z$ and integrate it over Ω_δ^- to have

$$\begin{aligned}\int_{\Omega_\delta^-} Az(\partial_{y_1} z)\mathrm{d}x\mathrm{d}y = &\int_{\Omega_\delta^-} f(\partial_{y_1} R)\mathrm{e}^{s\varphi}\chi(\partial_{y_1} z)\mathrm{d}x\mathrm{d}y \\ &+ \int_{\Omega_\delta^-} \{s((2\nabla_y\varphi, \nabla_y z) - (2\nabla_x\varphi, \nabla_x z))\partial_{y_1} z \\ &+ (\Delta_y\varphi - \Delta_x\varphi)z\partial_{y_1} z - s^2(|\nabla_y\varphi|^2 - |\nabla_x\varphi|^2)z\partial_{y_1} z\}\mathrm{d}x\mathrm{d}y \\ &+ \int_{\Omega_\delta^-} 2\mathrm{e}^{s\varphi}((\nabla_y\partial_{y_1} u, \nabla_y\chi) - (\nabla_x\partial_{y_1} u, \nabla_x\chi))\partial_{y_1} z\mathrm{d}x\mathrm{d}y \\ &+ \int_{\Omega_\delta^-} (\partial_{y_1} u)\mathrm{e}^{s\varphi}(\Delta_y\chi - \Delta_x\chi)\partial_{y_1} z\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{3.11}$$

We denote the left-hand side of (3.11) by I_1 and the right-hand side by I_2. By (3.2), we note that

$$\partial\Omega_\delta^- \subset (\Gamma \times G) \cup \{(x, y) \in \overline{D \times G};\ \psi(x, y) = \delta^2\} \cup (D \times G').$$

Moreover, by (1.3)–(1.4) and (3.3), we have

$$\partial_{y_j} u = \partial_{y_j}\partial_{y_k} u = 0 \quad \text{on } (\Gamma \times G) \cup \{(x, y) \in \overline{D \times G};\ \psi(x, y) = \delta^2\},\ 1 \le k, j \le m$$

and

$$\partial_{y_1} u = \partial_{y_1}\partial_{y_k} u = \partial_{x_j}\partial_{y_1} u = 0 \quad \text{on } D \times G',\ 2 \le k \le m,\ 1 \le j \le n.$$

Then, we have

$$z = |\nabla_x z| = |\nabla_{y'} z| = 0 \quad \text{on } \partial\Omega_\delta^-. \tag{3.12}$$

Therefore, using the integration by parts, we obtain

$$\begin{aligned}\mathrm{I}_1 &= \int_{\Omega_\delta^-} Az(\partial_{y_1} z)\mathrm{d}x\mathrm{d}y \\ &= \int_{\Omega_\delta^-} \frac{1}{2}\Big(\frac{\partial|\partial_{y_1} z|^2}{\partial y_1} + \frac{\partial|\nabla_x z|^2}{\partial y_1} - \frac{\partial|\nabla_{y'} z|^2}{\partial y_1} - p\frac{\partial|z|^2}{\partial y_1}\Big)\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Consequently, we have

$$\mathrm{I}_1 = \frac{1}{2}\int_{\partial\Omega_\delta^-} |\partial_{y_1} z|^2\nu_{y_1}\mathrm{dS}_x\mathrm{dS}_y + \frac{1}{2}\int_{\partial\Omega_\delta^-} (|\nabla_x z|^2 - |\nabla_{y'} z|^2 - p|z|^2)\nu_{y_1}\mathrm{dS}_x\mathrm{dS}_y.$$

Here ν_{y_1} is the y_1-component of the unit outward normal vector ν to $\partial\Omega_\delta^-$. We see that $\nu_{y_1} = 0$ on $\Gamma \times G$. Moreover, $\nu_{y_1} = 0$ on $\partial\Omega_\delta^- \cap \{y_1 = 0\}$. Therefore, (3.12) yields

$$\mathrm{I}_1 = \frac{1}{2}\int_{\Omega'(\delta)} |\partial_{y_1} z(x, 0, y')|^2\mathrm{d}x\mathrm{d}y'. \tag{3.13}$$

From (1.2), we have

$$\begin{aligned}\partial_{y_1} z(x, 0, y') &= ((\partial_{y_1}^2 u)\mathrm{e}^{s\varphi}\chi)(x, 0, y') + (\partial_{y_1} u)(x, 0, y')\partial_{y_1}(\chi\mathrm{e}^{s\varphi})(x, 0, y') \\ &= (\Delta_x u - \Delta_{y'} u + pu + fR)(x, 0, y')(\mathrm{e}^{s\varphi}\chi)(x, 0, y') \\ &= f(x, y')R(x, 0, y')\mathrm{e}^{s\varphi(x,0,y')}\chi(x, 0, y').\end{aligned}$$

Thus by (3.3) and the condition $|R(x,0,y')| \geq r_0 > 0$, we see that

$$\begin{aligned} \mathrm{I}_1 &= \frac{1}{2}\int_{\Omega'(\delta)} |f(x,y')R(x,0,y')\mathrm{e}^{s\varphi(x,0,y')}\chi(x,0,y')|^2 \mathrm{d}x\mathrm{d}y' \\ &\geq \frac{r_0^2}{2}\int_{\Omega'(\delta+2\epsilon)} f^2(x,y')\mathrm{e}^{2s\varphi(x,0,y')}\mathrm{d}x\mathrm{d}y'. \end{aligned} \tag{3.14}$$

Next we estimate I_2. Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\Big|\int_{\Omega_\delta^-} 2\mathrm{e}^{s\varphi}((\nabla_y\partial_{y_1}u,\nabla_y\chi)-(\nabla_x\partial_{y_1}u,\nabla_x\chi))\partial_{y_1}z\mathrm{d}x\mathrm{d}y\Big| \\ \leq &\int_{\Omega_\delta^-}|\partial_{y_1}z|^2\mathrm{d}x\mathrm{d}y + \int_{\Omega_\delta^-}\mathrm{e}^{2s\varphi}|(\nabla_y\partial_{y_1}u,\nabla_y\chi)-(\nabla_x\partial_{y_1}u,\nabla_x\chi)|^2\mathrm{d}x\mathrm{d}y. \end{aligned}$$

Therefore we absorb the terms including $\int_{\Omega_\delta^-}|\partial_{y_1}z|^2\mathrm{d}x\mathrm{d}y$ into $s\int_{\Omega_\delta^-}|\nabla_y z|^2\mathrm{d}x\mathrm{d}y$, and we obtain

$$\begin{aligned} \mathrm{I}_2 \leq\ & C\int_{\Omega_\delta^-} f^2|\partial_{y_1}R|^2\mathrm{e}^{2s\varphi}\chi^2\mathrm{d}x\mathrm{d}y + C\int_{\Omega_\delta^-}(s|\nabla_y z|^2+s|\nabla_x z|^2+s^3|z|^2)\mathrm{d}x\mathrm{d}y \\ &+\int_{\Omega_\delta^-}\mathrm{e}^{2s\varphi}|(\nabla_y\partial_{y_1}u,\nabla_y\chi)-(\nabla_x\partial_{y_1}u,\nabla_x\chi)|^2\mathrm{d}x\mathrm{d}y \\ &+C\int_{\Omega_\delta^-}|\partial_{y_1}u|^2\mathrm{e}^{2s\varphi}(\Delta_y\chi-\Delta_x\chi)^2\mathrm{d}x\mathrm{d}y. \end{aligned}$$

Now by noting that $\Omega_\delta^- \subset \Omega(\delta)$, (3.3) and (3.10), we have

$$\begin{aligned} \mathrm{I}_2 \leq\ & C\int_{\Omega(\delta)} f^2|\partial_{y_1}R|^2\chi^2\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}y + C\mathrm{e}^{2s\mu_2}M^2 + C\mathrm{e}^{Cs}\int_{\Gamma\times G}|\partial_\nu\partial_{y_1}u|^2\mathrm{dS}_x\mathrm{d}y \\ &+C\int_{\Omega(\delta+\varepsilon)\setminus\Omega(\delta+2\epsilon)}(|\partial_{y_1}u|^2+|\nabla_x\partial_{y_1}u|^2+|\nabla_y\partial_{y_1}u|^2)\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}y. \end{aligned}$$

By (3.3) and the a priori boundedness on u, we have

$$\int_{\Omega(\delta+\epsilon)\setminus\Omega(\delta+2\epsilon)}(|\partial_{y_1}u|^2+|\nabla_x\partial_{y_1}u|^2+|\nabla_y\partial_{y_1}u|^2)\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}y \leq C\mathrm{e}^{2s\mu_2}M^2$$

and so

$$\mathrm{I}_2 \leq C\int_{\Omega(\delta)} f^2|\partial_{y_1}R|^2\chi^2\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}y + C\mathrm{e}^{2s\mu_2}M^2 + C\mathrm{e}^{Cs}\int_{\Gamma\times G}|\partial_\nu\partial_{y_1}u|^2\mathrm{dS}_x\mathrm{d}y. \tag{3.15}$$

Now we will consider the first term on the right-hand side of (3.15). Since $R \in H^1(-T,T; L^\infty\ (D\times G'))$ and $R(x,0,y') \geq r_0 > 0$ on $\overline{D\times G'}$, we can define a function $g_0 \in L^2(-T,T)$ by

$$g_0(y_1) = \sup_{(x,y')\in D\times G'}\frac{|\partial_{y_1}R(x,y)|}{|R(x,0,y')|}.$$

Then we can write

$$
\begin{aligned}
&\int_{\Omega(\delta)} f^2|\partial_{y_1}R|^2\chi^2 \mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}y \\
&= \int_{\Omega(\delta)\setminus\Omega(\delta+2\varepsilon)} f^2|\partial_{y_1}R|^2\chi^2\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}y + \int_{\Omega(\delta+2\varepsilon)} f^2|\partial_{y_1}R|^2\chi^2\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}y \\
&\le \mathrm{e}^{2s\mu_2}\int_{\Omega(\delta)\setminus\Omega(\delta+2\varepsilon)} f^2|\partial_{y_1}R|^2\mathrm{d}x\mathrm{d}y + \int_{\Omega(\delta+2\varepsilon)} f^2|\partial_{y_1}R|^2\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}y \\
&\le C\mathrm{e}^{2s\mu_2}M^2 + C\int_{\Omega(\delta+2\varepsilon)} f^2|R(x,0,y')|^2|g_0(y_1)|^2\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}y \\
&\le C\mathrm{e}^{2s\mu_2}M^2 + C\int_{\Omega(\delta+2\varepsilon)} f^2|g_0(y_1)|^2\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

On the other hand, we have

$$\Omega(\delta+2\epsilon)\subset\Omega'(\delta+2\epsilon)\times(-T,T).$$

In fact, let $(x,y_1,y')\in\Omega(\delta+2\epsilon)$. Then

$$|x-x_0|^2-\beta|y'|^2-\beta|y_1|^2>(\delta+2\epsilon)^2.$$

Hence (1.8) implies $\widetilde{r}^2<\beta T^2+\delta^2$ and so

$$\beta|y_1|^2<|x-x_0|^2-\beta|y'|^2-(\delta+2\epsilon)^2\le\widetilde{r}^2-(\delta+2\epsilon)^2<\beta T^2,$$

that is, $|y_1|<T$. Since

$$|x-x_0|^2-\beta|y'|^2\ge|x-x_0|^2-\beta|y|^2>(\delta+2\epsilon)^2,$$

we see that $(x,0,y')\in\Omega'(\delta+2\epsilon)$.

Consequently we obtain

$$\int_{\Omega(\delta)} f^2|\partial_{y_1}R|^2\chi^2\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}y\le C\mathrm{e}^{2s\mu_2}M^2+C\int_{\Omega'(\delta+2\epsilon)} f^2\mathrm{e}^{2s\varphi(x,0,y')}G_0(x,y')\mathrm{d}x\mathrm{d}y',$$

where

$$G_0(x,y')=\int_{-T}^{T}|g_0(y_1)|^2\mathrm{e}^{2s\varphi(x,0,y')(\mathrm{e}^{-\gamma\beta y_1^2}-1)}\mathrm{d}y_1.$$

Moreover, $|g_0|^2\in L^1(-T,T)$ and the Lebesgue theorem imply

$$\sup_{\substack{x\in D\\|y'|<T}} G_0(x,y')=o(1)\quad \text{as } s\to\infty.$$

Henceforth, we set

$$d^2=\int_{\Gamma\times G}|\partial_\nu\partial_{y_1}u|^2\mathrm{d}\mathrm{S}_x\mathrm{d}y.$$

Then (3.15) yields

$$\mathrm{I}_2\le o(1)\int_{\Omega'(\delta+2\epsilon)} f^2\mathrm{e}^{2s\varphi(x,0,y')}\mathrm{d}x\mathrm{d}y'+C\mathrm{e}^{2s\mu_2}M^2+C\mathrm{e}^{Cs}d^2. \tag{3.16}$$

From (3.14) and (3.16), we obtain

$$o(1)\int_{\Omega'(\delta+2\epsilon)} f^2 \mathrm{e}^{2s\varphi(x,0,y')}\mathrm{d}x\mathrm{d}y' + C\mathrm{e}^{2s\mu_2}M^2 + C\mathrm{e}^{Cs}d^2 \ge \int_{\Omega'(\delta+2\epsilon)} f^2(x,y')\mathrm{e}^{2s\varphi(x,0,y')}\mathrm{d}x\mathrm{d}y'.$$

Finally

$$(1-o(1))\int_{\Omega'(\delta+2\epsilon)} |f(x,y')|^2\mathrm{e}^{2s\varphi(x,0,y')}\mathrm{d}x\mathrm{d}y' \le C\mathrm{e}^{2s\mu_2}M^2 + C\mathrm{e}^{Cs}d^2 \tag{3.17}$$

for all large $s \ge s_0$, where s_0 is some constant. Reducing the integral on the left-hand side to $\Omega'(\delta+3\epsilon)$, we have

$$\mathrm{e}^{2s\mu_3}\int_{\Omega'(\delta+3\epsilon)} f^2\mathrm{d}x\mathrm{d}y' \le C\mathrm{e}^{2s\mu_2}M^2 + C\mathrm{e}^{Cs}d^2,$$

that is,

$$\int_{\Omega'(\delta+3\epsilon)} f^2\mathrm{d}x\mathrm{d}y' \le C\mathrm{e}^{-2s\mu}M^2 + C\mathrm{e}^{Cs}d^2 \tag{3.18}$$

for all $s \ge s_0$, where $\mu = \mu_3 - \mu_2$. Replacing C by $C\mathrm{e}^{Cs_0}$, we see that (3.18) holds for all $s \ge 0$. First, let $M \ge d$. Choosing $s \ge 0$ such that

$$M^2\mathrm{e}^{-2s\mu} = \mathrm{e}^{Cs}d^2, \quad \text{that is,} \quad s = \frac{2}{C+2\mu}\log\frac{M}{d} \ge 0,$$

we obtain

$$\int_{\Omega'(\delta+3\epsilon)} |f|^2\mathrm{d}x\mathrm{d}y' \le 2M^{\frac{2C}{C+2\mu}}d^{\frac{4\mu}{C+2\mu}}.$$

Second, let $M < d$. Then setting $s = 0$ in (3.18), we have

$$\int_{\Omega'(\delta+3\epsilon)} |f|^2\mathrm{d}x\mathrm{d}y' \le 2Cd^2.$$

Therefore

$$\int_{\Omega'(\delta+3\epsilon)} |f|^2\mathrm{d}x\mathrm{d}y' \le C(d^{2\theta} + d^2).$$

By the a priori boundedness $\|\partial_{y_1}u\|_{H^2(D\times G)} \le M$ and the trace theorem, we have $d \le CM$ and so we can have $d^{2\theta} + d^2 \le Cd^{2\theta}$. Since $\epsilon > 0$ is arbitrarily small, the proof of Theorem 1.1 is completed.

4 Proof of Theorem 1.2

The proof relies on Theorem 2.2 and is similar to that in [16].

Since u itself does not satisfy (2.3), we have to introduce a cut-off function. Moreover, we have to apply a Carleman estimate by shifting the domain along the y'-direction. Thus we need to introduce several notations. We set

$$\widetilde{r} = \max_{x\in\overline{D}}|x-x_0|, \quad r = \min_{x\in\overline{D}}|x-x_0|.$$

By $x_0 \notin \overline{D}$, we see that $r > 0$. We choose $\rho > 1$ sufficiently large so that

$$\frac{\widetilde{r}}{r} < \rho. \tag{4.1}$$

By (4.1) and the assumption on T, we have

$$\frac{\beta T^2}{\rho^2} < r^2 < \widetilde{r}^2 < \beta T^2. \tag{4.2}$$

Furthermore, if necessary, we choose smaller β such that

$$r^2 > \beta^2 T^2. \tag{4.3}$$

We arbitrarily choose $y_0' = (y_2^0, \cdots, y_m^0) \in \mathbb{R}^{m-1}$ satisfying

$$|y_0'| \le T - \frac{T}{\rho} - \epsilon. \tag{4.4}$$

We set

$$G(y_0') = \{y \in \mathbb{R}^m;\, |y_1| < T,\, |y' - y_0'| < T\}, \quad G'(y_0') = G(y_0') \cap \{y_1 = 0\}$$

and we recall

$$G(T, 2T) = \{y \in \mathbb{R}^m;\, |y_1| < T,\, |y'| < 2T\}.$$

Moreover let

$$\psi(x,y) = |x - x_0|^2 - \beta|y_1|^2 - \beta|y' - y_0'|^2, \quad \varphi(x,t) = \mathrm{e}^{\gamma\psi(x,y)}.$$

Then (4.2) yields

$$\psi(x, \pm T, y') \le |x - x_0|^2 - \beta T^2 \le \widetilde{r}^2 - \beta T^2 < 0, \quad \text{if } x \in D \text{ and } |y' - y_0'| \le T, \tag{4.5}$$

$$\psi(x, y_1, y') \le \widetilde{r}^2 - \beta T^2 < 0, \quad \text{if } x \in D \text{ and } |y_1| \le T \tag{4.6}$$

and

$$\psi(x, 0, y') \ge r^2 - \beta\frac{T^2}{\rho^2} > 0, \quad \text{if } x \in D \text{ and } |y' - y_0'| \le \frac{T}{\rho}. \tag{4.7}$$

Therefore, for small $\epsilon > 0$ there exists $\delta > 0$ such that

$$\psi(x, y_1, y') < -\epsilon, \quad x \in D, \tag{4.8}$$

if $T - 2\delta \le |y_1| \le T$ or $T - 2\delta \le |y' - y_0'| \le T$ and

$$\psi(x, y_1, y') > \epsilon, \quad x \in D,\ |y_1| < \delta,\ |y' - y_0'| \le \frac{T}{\rho}. \tag{4.9}$$

In order to apply Theorem 2.2, we introduce a cut-off function $\chi(y)$ and define $\chi_0 \in C_0^\infty(\mathbb{R})$ such that $0 \le \chi_0 \le 1$ and

$$\chi_0(\xi) = \begin{cases} 0, & T - \delta \le |\xi| \le T, \\ 1, & |\xi| \le T - 2\delta. \end{cases}$$

Setting $\chi(y_1, y) = \chi_0(y_1)\chi_0(|y' - y_0'|)$, we see that $\chi \in C_0^\infty(\mathbb{R}^m)$, $0 \le \chi \le 1$ and

$$\chi(y_1, y) = \begin{cases} 0, & T - \delta \le |y_1| \le T \text{ or } T - \delta \le |y' - y_0'| \le T, \\ 1, & |y_1| \le T - 2\delta \text{ and } |y' - y_0'| \le T - 2\delta. \end{cases} \tag{4.10}$$

By choosing $\delta > 0$ smaller if necessary, we assume

$$\frac{T}{\rho} < T - 2\delta. \tag{4.11}$$

We set

$$w_k = (\partial_{y_1}^k u)\chi, \quad k = 1, 2.$$

Then

$$Aw_k = \chi f(x, y')\partial_{y_1}^k R(x, y) + 2(\nabla_y \chi, \nabla_y \partial_{y_1}^k u) + (\Delta_y \chi)\partial_{y_1}^k u, \\ x \in D,\ y \in G(y_0'),\ k = 1, 2 \tag{4.12}$$

and

$$\begin{aligned} &w_k = |\nabla_y w_k| = 0 \quad \text{on } D \times \partial G(y_0'), \\ &w_k = 0 \quad \text{on } \partial D \times G(y_0'). \end{aligned} \tag{4.13}$$

From (4.4), we note that

$$G(y_0') \subset G(T, 2T). \tag{4.14}$$

By (4.12)–(4.13), we can apply the Carleman estimate (see Theorem 2.2) to w_1, w_2:

$$\begin{aligned} &\int_{D\times G(y_0')} \sum_{k=1}^{2} (s|\nabla_{x,y} w_k|^2 + s^3 w_k^2)\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}y \\ &\le C\int_{D\times G(y_0')} \sum_{k=1}^{2} \chi^2 f^2 |\partial_{y_1}^k R(x, y)|^2 \mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}y \\ &\quad + C\int_{D\times G(y_0')} \sum_{k=1}^{2} |2(\nabla_y \chi, \nabla_y \partial_{y_1}^k u) + (\Delta_y \chi)\partial_{y_1}^k u|^2 \mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}y \\ &\quad + C\int_{\partial D_+ \times G(y_0')} \sum_{k=1}^{2} s|\partial_\nu w_k|^2 \mathrm{e}^{2s\varphi}\mathrm{d}\mathrm{S}_x\mathrm{d}y \\ &:= \mathrm{S}_1 + \mathrm{S}_2 + \mathrm{S}_3. \end{aligned} \tag{4.15}$$

Here and henceforth, $C > 0$ denotes a generic constant which is independent of $s > 0$. From the assumption on R, we have

$$\mathrm{S}_1 \le C\int_{D\times G(y_0')} \chi^2 f^2 \mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}y.$$

By (4.10), we see that $|y_1| \le T - 2\delta$ or $T - \delta \le |y_1| \le T$ implies $\partial_{y_1}\chi = \partial_{y_1}^2 \chi = 0$, and $|y' - y_0'| \le T - 2\delta$ or $T - \delta \le |y' - y_0'| \le T$ implies $\partial_{y_k}\chi = \partial_{y_k}^2 \chi = 0$ for $2 \le k \le m$. Therefore, if $|y_1| \in [0, T - 2\delta] \cup [T - \delta, T]$ and $|y' - y_0'| \in [0, T - 2\delta] \cup [T - \delta, T]$, then $|\nabla_y \chi| = \Delta_y \chi = 0$. Hence

$$\begin{aligned} \mathrm{S}_2 = C\Big(&\int_{\{T-2\delta<|y_1|<T-\delta\}\cap(D\times G(y_0'))} + \int_{\{T-2\delta<|y'-y_0'|<T-\delta\}\cap(D\times G(y_0'))}\Big) \sum_{k=1}^{2} |2(\nabla_y \chi, \nabla_y \partial_{y_1}^k u) \\ &+ (\Delta_y \chi)\partial_{y_1}^k u|^2 \mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}y. \end{aligned}$$

By (4.8), we have $\psi(x, y) < -\epsilon$ in the regions of the above integrals. Hence

$$\mathrm{S}_2 \le C\int_{D\times G(y_0')} M^2 \exp(2s\mathrm{e}^{-\gamma\epsilon})\mathrm{d}x\mathrm{d}y \le CM^2 \mathrm{e}^{2s\kappa_1}. \tag{4.16}$$

Here and henceforth we set

$$\kappa_1 = \mathrm{e}^{-\gamma\epsilon}, \quad \kappa_2 = \mathrm{e}^{\gamma\epsilon}.$$

Finally, we obtain

$$\begin{aligned} \mathrm{S}_3 &\le C\mathrm{e}^{Cs}\int_{\partial D_+\times G(y_0')}\sum_{k=1}^{2}|\partial_\nu\partial_{y_1}^k u|^2 \mathrm{dS}_x\mathrm{d}y \\ &\le C\mathrm{e}^{Cs}\int_{\partial D_+\times G(T,2T)}\sum_{k=1}^{2}|\partial_\nu\partial_{y_1}^k u|^2 \mathrm{dS}_x\mathrm{d}y := C\mathrm{e}^{Cs}d^2. \end{aligned} \tag{4.17}$$

Here we used (4.14).

Consequently (4.15) yields

$$\begin{aligned} &\int_{D\times G(y_0')}\sum_{k=1}^{2}(s|\nabla_x w_k|^2 + s|\nabla_y w_k|^2 + s^3 w_k^2)\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}y \\ \le{}& C\int_{D\times G(y_0')}\chi^2 f^2\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}y + CM^2\mathrm{e}^{2s\kappa_1} + C\mathrm{e}^{Cs}d^2. \end{aligned} \tag{4.18}$$

Next, since $\chi(-T, y') = 0$ for $y' \in G'(y_0')$ by (4.10), the Cauchy-Schwarz inequality yields

$$\begin{aligned} &\int_{D\times G'(y_0')}\chi^2(0,y')(\partial_{y_1}^2 u(x,0,y'))^2\mathrm{e}^{2s\varphi(x,0,y')}\mathrm{d}x\mathrm{d}y' \\ ={}& \int_{-T}^{0}\partial_{y_1}\Big(\int_{D\times G'(y_0')}\chi^2(y_1,y')(\partial_{y_1}^2 u(x,y_1,y'))^2\mathrm{e}^{2s\varphi(x,y_1,y')}\mathrm{d}x\mathrm{d}y'\Big)\mathrm{d}y_1 \\ ={}& \int_{-T}^{0}\int_{D\times G'(y_0')}(2(\partial_{y_1}\chi)\chi(\partial_{y_1}^2 u(x,y_1,y'))^2 + 2\chi^2(\partial_{y_1}^2 u)(\partial_{y_1}^3 u) \\ &+ \chi^2(\partial_{y_1}^2 u)^2 2s(\partial_{y_1}\varphi))\mathrm{e}^{2s\varphi(x,y_1,y')}\mathrm{d}x\mathrm{d}y \\ \le{}& \int_{-T}^{T}\int_{D\times G'(y_0')}|\partial_{y_1}\chi|^2|\partial_{y_1}^2 u|^2\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}y \\ &+ C\int_{-T}^{T}\int_{D\times G'(y_0')}(|\chi\partial_{y_1}^2 u|^2 + |\chi\partial_{y_1}^3 u|^2 + s|\chi\partial_{y_1}^2 u|^2)\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}y \\ \le{}& CM^2\mathrm{e}^{2s\kappa_1} + C\int_{D\times G(y_0')}(|w_2|^2 + |\partial_{y_1}w_2|^2 + s|w_2|^2)\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}y. \end{aligned}$$

For the last inequality, we used (4.16) and

$$\chi\partial_{y_1}^3 u = \partial_{y_1}(\chi\partial_{y_1}^2 u) - (\partial_{y_1}\chi)\partial_{y_1}^2 u = \partial_{y_1}w_2 - (\partial_{y_1}\chi)\partial_{y_1}^2 u.$$

Hence

$$\begin{aligned} &\int_{D\times G'(y_0')}|\chi(0,y')|^2|\partial_{y_1}^2 u(x,0,y')|^2\mathrm{e}^{2s\varphi(x,0,y')}\mathrm{d}x\mathrm{d}y' \\ \le{}& CM^2\mathrm{e}^{2s\kappa_1} + C\int_{D\times G(y_0')}(s|w_2|^2 + |\partial_{y_1}w_2|^2)\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}y. \end{aligned}$$

Applying (4.18), we have

$$\begin{aligned}&\int_{D\times G'(y_0')}|\chi_0(|y'-y_0'|)|^2|\partial_{y_1}^2u(x,0,y')|^2\mathrm{e}^{2s\varphi(x,0,y')}\mathrm{d}x\mathrm{d}y'\\ &\le \frac{C}{s}\int_{D\times G(y_0')}\chi_0^2(y_1)\chi_0^2(|y'-y_0'|)|f|^2\mathrm{e}^{2s\varphi(x,y_1,y')}\mathrm{d}x\mathrm{d}y+CM^2\mathrm{e}^{2s\kappa_1}+C\mathrm{e}^{Cs}d^2\\ &\le \frac{C}{s}\int_{D\times G'(y_0')}\chi_0^2(|y'-y_0'|)|f|^2\mathrm{e}^{2s\varphi(x,0,y')}\mathrm{d}x\mathrm{d}y'+CM^2\mathrm{e}^{2s\kappa_1}+C\mathrm{e}^{Cs}d^2. \end{aligned}\tag{4.19}$$

Here we used $|\chi_0(y_1)|\le 1$ and $\mathrm{e}^{2s\varphi(x,y_1,y')}\le \mathrm{e}^{2s\varphi(x,0,y')}$ for $x\in D$ and $y\in G(y_0')$.

On the other hand, substituting $y_1=0$ in (1.2) and applying $u(x,0,y')=0$ and $R(x,0,y')\ne 0$ for $x\in\overline{D}$ and $|y'|\le 2T$, we have

$$f(x,y')=\frac{\partial_{y_1}^2u(x,0,y')}{R(x,0,y')},\quad x\in\overline{D},\ \ |y'|\le 2T. \tag{4.20}$$

Noting by (4.4) that if $|y'-y_0'|<T$, then $|y'|<2T$, we apply (4.20) in (4.19), so that

$$\begin{aligned}&\int_{D\times G'(y_0')}\chi_0^2(|y'-y_0'|)|f|^2\mathrm{e}^{2s\varphi(x,0,y')}\mathrm{d}x\mathrm{d}y'\\ &\le \frac{C}{s}\int_{D\times G'(y_0')}\chi_0^2(|y'-y_0'|)|f|^2\mathrm{e}^{2s\varphi(x,0,y')}\mathrm{d}x\mathrm{d}y'+CM^2\mathrm{e}^{2s\kappa_1}+C\mathrm{e}^{Cs}d^2\end{aligned}$$

for all large $s>0$. Absorbing the first term on the right-hand side into the left-hand side by choosing $s>0$ large, we obtain

$$\int_{D\times G'(y_0')}\chi_0^2(|y'-y_0'|)|f|^2\mathrm{e}^{2s\varphi(x,0,y')}\mathrm{d}x\mathrm{d}y'\le CM^2\mathrm{e}^{2s\kappa_1}+C\mathrm{e}^{Cs}d^2$$

for all large $s>0$.

Replacing the integration domain on the left-hand side by $D\times\left\{y';\,|y'-y_0'|<\frac{T}{\rho}\right\}\subset D\times G'(y_0')$ and using (4.9)–(4.11), we see that $\psi(x,0,y')>\epsilon$, $\chi_0(|y'-y_0'|)=1$ there and

$$\mathrm{e}^{2s\kappa_2}\int_{D\times\{y';\,|y'-y_0'|<\frac{T}{\rho}\}}|f|^2\mathrm{d}x\mathrm{d}y'\le CM^2\mathrm{e}^{2s\kappa_1}+C\mathrm{e}^{Cs}d^2$$

for all $s\ge s_0$, where s_0 is some constant. By the definition, we have $\kappa_2>\kappa_1$ and set $\kappa=\kappa_2-\kappa_1>0$. Then the last inequality implies

$$\int_{D\times\{y';\,|y'-y_0'|<\frac{T}{\rho}\}}|f|^2\mathrm{d}x\mathrm{d}y'\le CM^2\mathrm{e}^{-2s\kappa}+C\mathrm{e}^{Cs}d^2 \tag{4.21}$$

for all $s\ge s_0$. By the same argument as in the proof of Theorem 1.1 after (3.18), we can choose $\theta\in(0,1)$ such that

$$\int_{D\times\left\{y';\,|y'-y_0'|<\frac{T}{\rho}\right\}}|f|^2\mathrm{d}x\mathrm{d}y'\le C(d^{2\theta}+d^2)$$

for all $y_0'\in\mathbb{R}^{m-1}$ satisfying $|y_0'|<T-\frac{T}{\rho}-\epsilon$. By $\|\partial_{y_1}u\|_{H^2(D\times G)}\le M$, the trace theorem yields $d\le CM$, which implies $d\le Cd^\theta$. Varying y_0' and noting

$$\bigcup\left\{y'\in\mathbb{R}^{m-1};\,|y'-y_0'|\le\frac{T}{\rho},\,|y_0'|\le T-\frac{T}{\rho}-\epsilon\right\}=\{y'\in\mathbb{R}^{m-1};\,|y'|<T-\epsilon\},$$

we obtain

$$\int_{D\times\{y';\,|y'|<T-\epsilon\}} |f(x,y')|^2 \mathrm{d}x\mathrm{d}y' \le Cd^{2\theta}.$$

Thus the proof of Theorem 1.2 is completed.

5 Appendix

Thanks to the large parameter s, it is sufficient to prove Theorems 2.1–2.2 in the case of $a_i = b_j = 0$ in (2.1). Let us set

$$L_0 u := \Delta_y u(x,y) - \Delta_x u(x,y) = F. \tag{5.1}$$

We prove only Theorem 2.2 and the proof of Theorem 2.1 is obtained by replacing the domain $D \times G(T,T_1)$ by $\Omega(\delta)$. Henceforth, we write $z_{y_k} = \partial_{y_k} z$, $z_{x_j} = \partial_{x_j} z$, $z_{x_j y_k} = \partial_{x_j}\partial_{y_k} z$ and use ν to denote the unit outward normal vector to a hypersurface under consideration and we set $\partial_\nu z = (\nabla_x z, \nu)$ or $\partial_\nu z = (\nabla_y z, \nu)$. Moreover, we set

$$\Omega = D \times G(T,T_1), \quad \Gamma_x = \partial D \times G(T,T_1), \quad \Gamma_y = D \times \partial G(T,T_1) \tag{5.2}$$

and

$$z(x,y) = \mathrm{e}^{s\varphi(x,y)} u(x,y), \quad Pz(x,y) = \mathrm{e}^{s\varphi} L_0(z\mathrm{e}^{-s\varphi}). \tag{5.3}$$

By (5.3), we calculate

$$Pz = P^+ z + P^- z, \tag{5.4}$$

where

$$P^+ z = \Delta_y z - \Delta_x z + s^2(|\nabla_y \varphi|^2 - |\nabla_x \varphi|^2) z, \tag{5.5}$$

$$P^- z = -2s((\nabla_y \varphi, \nabla_y z) - (\nabla_x \varphi, \nabla_x z)) - s(\Delta_y \varphi - \Delta_x \varphi) z. \tag{5.6}$$

The first term on the right-hand side of the Carleman estimate is $\|Pz\|^2_{L^2(\Omega)}$ and it suffices to make a lower estimation of $\|Pz\|^2_{L^2(\Omega)}$. Since

$$\begin{aligned}\|Pz\|^2_{L^2(\Omega)} &= \|P^+ z\|^2_{L^2(\Omega)} + \|P^- z\|^2_{L^2(\Omega)} + 2(P^+ z, P^- z)_{L^2(\Omega)} \\ &\ge 2(P^+ z, P^- z)_{L^2(\Omega)},\end{aligned}$$

we will estimate $(P^+ z, P^- z)_{L^2(\Omega)}$ as follows. Using (5.5)–(5.6), we obtain

$$(P^+ z, P^- z)_{L^2(\Omega)} = \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4 + \mathrm{I}_5 + \mathrm{I}_6,$$

where

$$\begin{aligned}
\mathrm{I}_1 &= -2s\int_\Omega \Delta_y z((\nabla_y\varphi,\nabla_y z)-(\nabla_x\varphi,\nabla_x z))\mathrm{d}x\mathrm{d}y,\\
\mathrm{I}_2 &= -s\int_\Omega \Delta_y z(\Delta_y\varphi-\Delta_x\varphi)z\mathrm{d}x\mathrm{d}y,\\
\mathrm{I}_3 &= 2s\int_\Omega \Delta_x z((\nabla_y\varphi,\nabla_y z)-(\nabla_x\varphi,\nabla_x z))\mathrm{d}x\mathrm{d}y,\\
\mathrm{I}_4 &= s\int_\Omega \Delta_x z(\Delta_y\varphi-\Delta_x\varphi)z\mathrm{d}x\mathrm{d}y,\\
\mathrm{I}_5 &= -2s^3\int_\Omega (|\nabla_y\varphi|^2-|\nabla_x\varphi|^2)z((\nabla_y\varphi,\nabla_y z)-(\nabla_x\varphi,\nabla_x z))\mathrm{d}x\mathrm{d}y,\\
\mathrm{I}_6 &= -s^3\int_\Omega (|\nabla_y\varphi|^2-|\nabla_x\varphi|^2)(\Delta_y\varphi-\Delta_x\varphi)|z|^2\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Now, we will estimate the terms I_k, $1\le k\le 6$, using the integration by parts and the boundary condition of z. Then we have

$$\begin{aligned}
\mathrm{I}_1 &= -2s\int_\Omega \Delta_y z((\nabla_y\varphi,\nabla_y z)-(\nabla_x\varphi,\nabla_x z))\mathrm{d}x\mathrm{d}y\\
&= 2s\int_\Omega (\nabla_y z,\nabla_y(\nabla_y\varphi,\nabla_y z))\mathrm{d}x\mathrm{d}y - 2s\int_{\Gamma_y}(\partial_\nu z)(\nabla_y\varphi,\nabla_y z)\mathrm{d}S_y\mathrm{d}x\\
&\quad - 2s\int_\Omega (\nabla_y z,\nabla_y(\nabla_x\varphi,\nabla_x z))\mathrm{d}x\mathrm{d}y + 2s\int_{\Gamma_y}(\partial_\nu z)(\nabla_x\varphi,\nabla_x z)\mathrm{d}S_y\mathrm{d}x\\
&= 2s\sum_{k,j=1}^m\int_\Omega z_{y_k}(\varphi_{y_j}z_{y_j})_{y_k}\mathrm{d}x\mathrm{d}y - 2s\int_{\Gamma_y}(\partial_\nu z)(\nabla_y\varphi,\nabla_y z)\mathrm{d}S_y\mathrm{d}x\\
&\quad - 2s\sum_{k=1}^m\sum_{j=1}^n\int_\Omega z_{y_k}(\varphi_{x_j}z_{x_j})_{y_k}\mathrm{d}x\mathrm{d}y + 2s\int_{\Gamma_y}(\partial_\nu z)(\nabla_x\varphi,\nabla_x z)\mathrm{d}S_y\mathrm{d}x\\
&= 2s\sum_{k,j=1}^m\int_\Omega z_{y_k}z_{y_j}\varphi_{y_jy_k}\mathrm{d}x\mathrm{d}y - s\int_\Omega |\nabla_y z|^2\Delta_y\varphi\mathrm{d}x\mathrm{d}y\\
&\quad - 2s\int_{\Gamma_y}(\partial_\nu z)(\nabla_y\varphi,\nabla_y z)\mathrm{d}S_y\mathrm{d}x + s\int_{\Gamma_y}(\partial_\nu\varphi)|\nabla_y z|^2\mathrm{d}S_y\mathrm{d}x\\
&\quad - 2s\sum_{k=1}^m\sum_{j=1}^n\int_\Omega z_{y_k}z_{x_j}\varphi_{x_jy_k}\mathrm{d}x\mathrm{d}y + s\int_\Omega |\nabla_y z|^2\Delta_x\varphi\mathrm{d}x\mathrm{d}y\\
&\quad + 2s\int_{\Gamma_y}(\partial_\nu z)(\nabla_x\varphi,\nabla_x z)\mathrm{d}S_y\mathrm{d}x - s\int_{\Gamma_x}(\partial_\nu\varphi)|\nabla_y z|^2\mathrm{d}\mathrm{S}_x\mathrm{d}y,\\
\mathrm{I}_2 &= -s\int_\Omega (\Delta_y z)(\Delta_y\varphi-\Delta_x\varphi)z\mathrm{d}x\mathrm{d}y\\
&= s\int_\Omega |\nabla_y z|^2(\Delta_y\varphi-\Delta_x\varphi)\mathrm{d}x\mathrm{d}y + \frac{s}{2}\int_\Omega(\nabla_y(|z|^2),\nabla_y(\Delta_y\varphi-\Delta_x\varphi))\mathrm{d}x\mathrm{d}y\\
&\quad - s\int_{\Gamma_y}(\partial_\nu z)(\Delta_y\varphi-\Delta_x\varphi)z\mathrm{d}S_y\mathrm{d}x\\
&= s\int_\Omega |\nabla_y z|^2(\Delta_y\varphi-\Delta_x\varphi)\mathrm{d}x\mathrm{d}y - \frac{s}{2}\int_\Omega |z|^2\Delta_y(\Delta_y\varphi-\Delta_x\varphi)\mathrm{d}x\mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&\quad - s\int_{\Gamma_y}(\partial_\nu z)(\Delta_y\varphi-\Delta_x\varphi)z\mathrm{d}S_y\mathrm{d}x+\frac{s}{2}\int_{\Gamma_y}\partial_\nu(\Delta_y\varphi-\Delta_x\varphi)|z|^2\mathrm{d}S_y\mathrm{d}x,\\
\mathrm{I}_3 &= 2s\int_\Omega \Delta_x z(\nabla_y\varphi,\nabla_y z)\mathrm{d}x\mathrm{d}y-2s\int_\Omega(\Delta_x z)(\nabla_x\varphi,\nabla_x z)\mathrm{d}x\mathrm{d}y\\
&= -2s\sum_{k=1}^n\sum_{j=1}^m\int_\Omega z_{x_k}(\varphi_{y_j}z_{y_j})_{x_k}\mathrm{d}x\mathrm{d}y+2s\sum_{k,j=1}^n\int_\Omega z_{x_k}(\varphi_{x_j}z_{x_j})_{x_k}\mathrm{d}x\mathrm{d}y\\
&\quad + 2s\int_{\Gamma_x}(\partial_\nu z)(\nabla_y\varphi,\nabla_y z)\mathrm{dS}_x\mathrm{d}y-2s\int_{\Gamma_x}(\partial_\nu z)(\nabla_x\varphi,\nabla_x z)\mathrm{dS}_x\mathrm{d}y\\
&= -2s\sum_{k=1}^n\sum_{j=1}^m\int_\Omega z_{x_k}z_{y_j}\varphi_{y_jx_k}\mathrm{d}x\mathrm{d}y+s\int_\Omega|\nabla_x z|^2\Delta_y\varphi\mathrm{d}x\mathrm{d}y\\
&\quad - s\int_{\Gamma_y}(\partial_\nu\varphi)|\nabla_x z|^2\mathrm{d}S_y\mathrm{d}x-s\int_\Omega|\nabla_x z|^2\Delta_x\varphi\mathrm{d}x\mathrm{d}y\\
&\quad + s\int_{\Gamma_x}(\partial_\nu\varphi)|\nabla_x z|^2\mathrm{dS}_x\mathrm{d}y+2s\sum_{k,j=1}^n\int_\Omega z_{x_k}z_{x_j}\varphi_{x_jx_k}\mathrm{d}x\mathrm{d}y\\
&\quad + 2s\int_{\Gamma_x}(\partial_\nu z)(\nabla_y\varphi,\nabla_y z)\mathrm{dS}_x\mathrm{d}y-2s\int_{\Gamma_x}(\partial_\nu z)(\nabla_x\varphi,\nabla_x z)\mathrm{dS}_x\mathrm{d}y,\\
\mathrm{I}_4 &= s\int_\Omega(\Delta_x z)(\Delta_y\varphi-\Delta_x\varphi)z\mathrm{d}x\mathrm{d}y\\
&= -s\int_\Omega(\nabla_x z,\nabla_x((\Delta_y\varphi-\Delta_x\varphi)z))\mathrm{d}x\mathrm{d}y+s\int_{\Gamma_x}(\partial_\nu z)(\Delta_y\varphi-\Delta_x\varphi)z\mathrm{dS}_x\mathrm{d}y\\
&= -s\int_\Omega|\nabla_x z|^2(\Delta_y\varphi-\Delta_x\varphi)\mathrm{d}x\mathrm{d}y+\frac{s}{2}\int_\Omega|z|^2\Delta_x(\Delta_y\varphi-\Delta_x\varphi)\mathrm{d}x\mathrm{d}y\\
&\quad + s\int_{\Gamma_x}(\partial_\nu z)(\Delta_y\varphi-\Delta_x\varphi)z\mathrm{dS}_x\mathrm{d}y-\frac{s}{2}\int_{\Gamma_x}\partial_\nu(\Delta_y\varphi-\Delta_x\varphi)|z|^2\mathrm{dS}_x\mathrm{d}y,\\
\mathrm{I}_5 &= -2s^3\int_\Omega(|\nabla_y\varphi|^2-|\nabla_x\varphi|^2)z((\nabla_y\varphi,\nabla_y z)-(\nabla_x\varphi,\nabla_x z))\mathrm{d}x\mathrm{d}y\\
&= -s^3\int_\Omega((\nabla_y\varphi,\nabla_y(|z|^2))-(\nabla_x\varphi,\nabla_x(|z|^2)))(|\nabla_y\varphi|^2-|\nabla_x\varphi|^2)\mathrm{d}x\mathrm{d}y\\
&= -s^3\int_\Omega(\nabla_y\varphi,\nabla_y(|z|^2(|\nabla_y\varphi|^2-|\nabla_x\varphi|^2)))\mathrm{d}x\mathrm{d}y\\
&\quad + s^3\int_\Omega(\nabla_y\varphi,|z|^2\nabla_y(|\nabla_y\varphi|^2-|\nabla_x\varphi|^2))\mathrm{d}x\mathrm{d}y\\
&\quad + s^3\int_\Omega(\nabla_x\varphi,\nabla_x(|z|^2(|\nabla_y\varphi|^2-|\nabla_x\varphi|^2)))\mathrm{d}x\mathrm{d}y\\
&\quad - s^3\int_\Omega(\nabla_x\varphi,|z|^2\nabla_x(|\nabla_y\varphi|^2-|\nabla_x\varphi|^2))\mathrm{d}x\mathrm{d}y\\
&= s^3\int_\Omega|z|^2(|\nabla_y\varphi|^2-|\nabla_x\varphi|^2)(\Delta_y\varphi-\Delta_x\varphi)\mathrm{d}x\mathrm{d}y\\
&\quad + s^3\int_\Omega|z|^2(\nabla_y\varphi,\nabla_y(|\nabla_y\varphi|^2-|\nabla_x\varphi|^2))\mathrm{d}x\mathrm{d}y\\
&\quad - s^3\int_\Omega|z|^2(\nabla_x\varphi,\nabla_x(|\nabla_y\varphi|^2-|\nabla_x\varphi|^2))\mathrm{d}x\mathrm{d}y
\end{aligned}
$$

$$- s^3 \int_{\Gamma_y} (\partial_\nu \varphi)|z|^2(|\nabla_y \varphi|^2 - |\nabla_x \varphi|^2)\mathrm{d}S_y \mathrm{d}x$$
$$+ s^3 \int_{\Gamma_x} (\partial_\nu \varphi)|z|^2(|\nabla_y \varphi|^2 - |\nabla_x \varphi|^2)\mathrm{dS}_x \mathrm{d}y$$

and

$$\mathrm{I}_6 = -s^3 \int_\Omega (|\nabla_y \varphi|^2 - |\nabla_x \varphi|^2)(\Delta_y \varphi - \Delta_x \varphi)|z|^2 \mathrm{d}x\mathrm{d}y.$$

Therefore, we can rewrite

$$(P^+ z, P^- z)_{L^2(\Omega)} = \mathrm{J}_1 + \mathrm{J}_2 + \mathrm{J}_3 + \mathrm{J}_4 + \mathrm{J}_5 + \mathrm{B}_0, \tag{5.7}$$

where

$$\mathrm{J}_1 = 2s \sum_{k,j=1}^{m} \int_\Omega z_{y_k} z_{y_j} \varphi_{y_j y_k} \mathrm{d}x\mathrm{d}y - 4s \sum_{k=1}^{m}\sum_{j=1}^{n} \int_\Omega z_{y_k} z_{x_j} \varphi_{x_j y_k} \mathrm{d}x\mathrm{d}y,$$
$$\mathrm{J}_2 = 2s \sum_{k,j=1}^{n} \int_\Omega z_{x_k} z_{x_j} \varphi_{x_j x_k} \mathrm{d}x\mathrm{d}y,$$
$$\mathrm{J}_3 = -\frac{s}{2} \int_\Omega |z|^2 \Delta_y(\Delta_y \varphi - \Delta_x \varphi)\mathrm{d}x\mathrm{d}y,$$
$$\mathrm{J}_4 = \frac{s}{2} \int_\Omega |z|^2 \Delta_x(\Delta_y \varphi - \Delta_x \varphi)\mathrm{d}x\mathrm{d}y,$$
$$\mathrm{J}_5 = s^3 \int_\Omega |z|^2 (\nabla_y \varphi, \nabla_y(|\nabla_y \varphi|^2 - |\nabla_x \varphi|^2))\mathrm{d}x\mathrm{d}y$$
$$- s^3 \int_\Omega |z|^2 (\nabla_x \varphi, \nabla_x(|\nabla_y \varphi|^2 - |\nabla_x \varphi|^2))\mathrm{d}x\mathrm{d}y$$

and

$$\mathrm{B}_0 = 2s \int_{\Gamma_y} (\partial_\nu z)((\nabla_x \varphi, \nabla_x z) - (\nabla_y \varphi, \nabla_y z))\mathrm{d}S_y \mathrm{d}x$$
$$+ s \int_{\Gamma_y} (\partial_\nu \varphi)(|\nabla_y z|^2 - |\nabla_x z|^2)\mathrm{d}S_y \mathrm{d}x - s^3 \int_{\Gamma_y} (\partial_\nu \varphi)|z|^2(|\nabla_y \varphi|^2 - |\nabla_x \varphi|^2)\mathrm{d}S_y \mathrm{d}x$$
$$- s \int_{\Gamma_y} (\partial_\nu z)(\Delta_y \varphi - \Delta_x \varphi) z \mathrm{d}S_y \mathrm{d}x + \frac{s}{2} \int_{\Gamma_y} \partial_\nu(\Delta_y \varphi - \Delta_x \varphi)|z|^2 \mathrm{d}S_y \mathrm{d}x$$
$$+ 2s \int_{\Gamma_x} (\partial_\nu z)((\nabla_y \varphi, \nabla_y z) - (\nabla_x \varphi, \nabla_x z))\mathrm{dS}_x \mathrm{d}y$$
$$- s \int_{\Gamma_x} (\partial_\nu \varphi)(|\nabla_y z|^2 - |\nabla_x z|^2)\mathrm{dS}_x \mathrm{d}y + s^3 \int_{\Gamma_x} (\partial_\nu \varphi)|z|^2(|\nabla_y \varphi|^2 - |\nabla_x \varphi|^2)\mathrm{dS}_x \mathrm{d}y$$
$$+ s \int_{\Gamma_x} (\partial_\nu z)(\Delta_y \varphi - \Delta_x \varphi) z \mathrm{dS}_x \mathrm{d}y - \frac{s}{2} \int_{\Gamma_x} \partial_\nu(\Delta_y \varphi - \Delta_x \varphi)|z|^2 \mathrm{dS}_x \mathrm{d}y.$$

Next we calculate J_k, $1 \le k \le 5$ by substituting the concrete form of φ. Setting

$$d_1(\psi) = \Delta_y \psi - \Delta_x \psi, \quad d_2(\psi) = |\nabla_y \psi|^2 - |\nabla_x \psi|^2,$$

we have

$$
\begin{aligned}
&\varphi_{x_i x_i} = \gamma\varphi(2 + \gamma\psi_{x_i}^2), && \varphi_{x_i y_j} = \gamma^2\varphi\psi_{x_i}\psi_{x_j},\\
&\varphi_{x_i x_j} = \gamma\varphi(\psi_{x_i x_j} + \gamma\psi_{x_i}\psi_{x_j}), && \varphi_{y_i y_j} = \gamma\varphi(\psi_{y_i y_j} + \gamma\psi_{y_i}\psi_{y_j}),\\
&\nabla_x\varphi = \gamma\varphi\nabla_x\psi, && \nabla_y\varphi = \gamma\varphi\nabla_y\psi,\\
&\Delta_x\varphi = \gamma\varphi(\Delta_x\psi + \gamma|\nabla_x\psi|^2), && \Delta_y\varphi = \gamma\varphi(\Delta_y\psi + \gamma|\nabla_y\psi|^2),\\
&\Delta_y\varphi - \Delta_x\varphi = \gamma\varphi d_1(\psi) + \gamma^2\varphi d_2(\psi).
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
\mathrm{J}_1 &= 2s\sum_{k,j=1}^{m}\int_\Omega z_{y_k}z_{y_j}\varphi_{y_j y_k}\mathrm{d}x\mathrm{d}y - 4s\sum_{k=1}^{m}\sum_{j=1}^{n}\int_\Omega z_{y_k}z_{x_j}\varphi_{x_j y_k}\mathrm{d}x\mathrm{d}y\\
&= \sum_{k,j=1}^{m}\int_\Omega 2s\gamma\varphi(\psi_{y_j y_k} + \gamma\psi_{y_k}\psi_{y_j})z_{y_k}z_{y_j}\mathrm{d}x\mathrm{d}y\\
&\quad - \sum_{k=1}^{m}\sum_{j=1}^{n}\int_\Omega 4s\gamma^2\varphi\psi_{y_k}\psi_{x_j}z_{y_k}z_{x_j}\mathrm{d}x\mathrm{d}y\\
&= \sum_{k,j=1}^{m}\int_\Omega 2s\gamma\varphi\psi_{y_j y_k}z_{y_k}z_{y_j}\mathrm{d}x\mathrm{d}y + \int_\Omega 2s\gamma^2\varphi\Big(\sum_{j=1}^{m}\psi_{y_j}z_{y_j}\Big)^2\mathrm{d}x\mathrm{d}y\\
&\quad - \sum_{k=1}^{m}\sum_{j=1}^{n}\int_\Omega 4s\gamma^2\varphi\psi_{y_k}\psi_{x_j}z_{y_k}z_{x_j}\mathrm{d}x\mathrm{d}y
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{J}_2 &= 2s\sum_{k,j=1}^{n}\int_\Omega z_{x_k}z_{x_j}\varphi_{x_j x_k}\mathrm{d}x\mathrm{d}y\\
&= \sum_{k,j=1}^{n}\int_\Omega 2s\gamma\varphi z_{x_k}z_{x_j}(\psi_{x_j x_k} + \gamma\psi_{x_k}\psi_{x_j})\mathrm{d}x\mathrm{d}y\\
&= \sum_{k,j=1}^{n}\int_\Omega 2s\gamma\varphi z_{x_k}z_{x_j}\psi_{x_j x_k}\mathrm{d}x\mathrm{d}y + \int_\Omega 2s\gamma^2\varphi\Big(\sum_{k=1}^{n}z_{x_k}\psi_{x_k}\Big)^2\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

We can directly verify

$$
\begin{aligned}
\Delta_y(\varphi d_2(\psi)) &= \gamma\varphi(\Delta_y\psi)d_2(\psi) + \gamma^2\varphi|\nabla_y\psi|^2 d_2(\psi)\\
&\quad + 2\gamma\varphi(\nabla_y\psi, \nabla_y(d_2(\psi))) + \varphi\Delta_y(d_2(\psi)).
\end{aligned}
$$

In fact,

$$
\begin{aligned}
\Delta_y(\varphi d_2(\psi)) &= \sum_{j=1}^{m}(\varphi_{y_j y_j}d_2(\psi) + 2\varphi_{y_j}(d_2(\psi))_{y_j} + \varphi(d_2(\psi))_{y_j y_j})\\
&= \sum_{j=1}^{m}\gamma\varphi\psi_{y_j y_j}d_2(\psi) + \sum_{j=1}^{m}\gamma^2\varphi\psi_{y_j}^2 d_2(\psi)\\
&\quad + \sum_{j=1}^{m}(2\gamma\varphi\psi_{y_j}(d_2(\psi))_{y_j} + \varphi(d_2(\psi))_{y_j y_j})
\end{aligned}
$$

$$= \gamma\varphi(\Delta_y\psi)\ d_2(\psi) + \gamma^2\varphi|\nabla_y\psi|^2 d_2(\psi) + 2\gamma\varphi(\nabla_y\psi, \nabla_y(d_2(\psi))) + \varphi\Delta_y(d_2(\psi)).$$

Consequently, we see that

$$\begin{aligned}
\mathrm{J}_3 &= -\frac{s}{2}\int_\Omega |z|^2\Delta_y(\Delta_y\varphi - \Delta_x\varphi)\mathrm{d}x\mathrm{d}y \\
&= -\frac{s}{2}\int_\Omega |z|^2\Delta_y(\gamma\varphi d_1(\psi) + \gamma^2\varphi d_2(\psi))\mathrm{d}x\mathrm{d}y \\
&= -\int_\Omega \frac{s}{2}\gamma^2\varphi|z|^2(d_1(\psi)\Delta_y\psi + \Delta_y(d_2(\psi)))\mathrm{d}x\mathrm{d}y \\
&\quad - \int_\Omega \frac{s}{2}\gamma^3\varphi|z|^2(d_1(\psi)|\nabla_y\psi|^2 + (\Delta_y\psi)d_2(\psi) + 2(\nabla_y\psi, \nabla_y(d_2(\psi))))\mathrm{d}x\mathrm{d}y \\
&\quad - \int_\Omega \frac{s}{2}\gamma^4\varphi|z|^2|\nabla_y\psi|^2 d_2(\psi)\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Since

$$\begin{aligned}
\Delta_x(\varphi d_2(\psi)) &= \gamma\varphi(\Delta_x\psi)d_2(\psi) + \gamma^2\varphi|\nabla_x\psi|^2 d_2(\psi) \\
&\quad + 2\gamma\varphi(\nabla_x\psi, \nabla_x(d_2(\psi))) + \varphi\Delta_x(d_2(\psi)),
\end{aligned}$$

we have

$$\begin{aligned}
\mathrm{J}_4 &= \frac{s}{2}\int_\Omega |z|^2\Delta_x(\Delta_y\varphi - \Delta_x\varphi)\mathrm{d}x\mathrm{d}y \\
&= \int_\Omega \frac{s}{2}\gamma d_1(\psi)|z|^2\Delta_x\varphi\mathrm{d}x\mathrm{d}y + \int_\Omega \frac{s}{2}\gamma^2|z|^2\Delta_x(\varphi d_2(\psi))\mathrm{d}x\mathrm{d}y \\
&= \int_\Omega \frac{s}{2}\gamma^2\varphi|z|^2(d_1(\psi)\Delta_x\psi + \Delta_x(d_2(\psi)))\mathrm{d}x\mathrm{d}y \\
&\quad + \int_\Omega \frac{s}{2}\gamma^3\varphi|z|^2((\Delta_x\psi)d_2(\psi) + 2(\nabla_x\psi, \nabla_x(d_2(\psi))) + d_1(\psi)|\nabla_x\psi|^2)\mathrm{d}x\mathrm{d}y \\
&\quad + \int_\Omega \frac{s}{2}\gamma^4\varphi|z|^2|\nabla_x\psi|^2 d_2(\psi)\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

We can directly verify

$$\begin{aligned}
(\nabla_y\varphi, \nabla_y(|\nabla_y\varphi|^2 - |\nabla_x\varphi|^2)) &= (\nabla_y\varphi, \nabla_y(\gamma^2\varphi^2 d_2(\psi))) \\
&= \gamma^2(\nabla_y\varphi, d_2(\psi)\nabla_y(\varphi^2)) + \gamma^2(\nabla_y\varphi, \varphi^2\nabla_y(d_2(\psi))) \\
&= 2\gamma^4\varphi^3 d_2(\psi)(\nabla_y\psi, \nabla_y\psi) + \gamma^3\varphi^3(\nabla_y\psi, \nabla_y(d_2(\psi)))
\end{aligned}$$

and

$$(\nabla_x\varphi, \nabla_x(|\nabla_y\varphi|^2 - |\nabla_x\varphi|^2)) = 2\gamma^4\varphi^3 d_2(\psi)(\nabla_x\psi, \nabla_x\psi) + \gamma^3\varphi^3(\nabla_x\psi, \nabla_x(d_2(\psi))).$$

Therefore, we conclude that

$$\begin{aligned}
\mathrm{J}_5 &= s^3\int_\Omega |z|^2(\nabla_y\varphi, \nabla_y(|\nabla_y\varphi|^2 - |\nabla_x\varphi|^2))\mathrm{d}x\mathrm{d}y \\
&\quad - s^3\int_\Omega |z|^2(\nabla_x\varphi, \nabla_x(|\nabla_y\varphi|^2 - |\nabla_x\varphi|^2))\mathrm{d}x\mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&= s^3\int_\Omega |z|^2\{2\gamma^4\varphi^3 d_2(\psi)(\nabla_y\psi,\nabla_y\psi)+\gamma^3\varphi^3(\nabla_y\psi,\nabla_y(d_2(\psi)))\}\mathrm{d}x\mathrm{d}y\\
&\quad - s^3\int_\Omega |z|^2\{2\gamma^4\varphi^3 d_2(\psi)(\nabla_x\psi,\nabla_x\psi)+\gamma^3\varphi^3(\nabla_x\psi,\nabla_x(d_2(\psi)))\}\mathrm{d}x\mathrm{d}y\\
&= \int_\Omega s^3\gamma^3\varphi^3|z|^2\{(\nabla_y\psi,\nabla_y(d_2(\psi)))-(\nabla_x\psi,\nabla_x(d_2(\psi)))\}\mathrm{d}x\mathrm{d}y\\
&\quad + \int_\Omega 2s^3\gamma^4\varphi^3|z|^2(d_2(\psi))^2\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

Finally, we obtain the boundary term as follows:

$$
\begin{aligned}
\mathrm{B}_0 &= \int_{\Gamma_y} 2s\gamma\varphi(\partial_\nu z)((\nabla_x\psi,\nabla_x z)-(\nabla_y\psi,\nabla_y z))\mathrm{dS}_y\mathrm{d}x\\
&\quad + \int_{\Gamma_y} s\gamma\varphi(\partial_\nu\psi)(|\nabla_y z|^2-|\nabla_x z|^2)\mathrm{dS}_y\mathrm{d}x - \int_{\Gamma_y} s^3\gamma^3\varphi^3(\partial_\nu\psi)|z|^2 d_2(\psi)\mathrm{dS}_y\mathrm{d}x\\
&\quad - \int_{\Gamma_y} s(\gamma\varphi d_1(\psi)+\gamma^2\varphi d_2(\psi))z(\partial_\nu z)\mathrm{dS}_y\mathrm{d}x\\
&\quad + \int_{\Gamma_y} \frac{s}{2}((\gamma^2\varphi d_1(\psi)+\gamma^3\varphi d_2(\psi))\partial_\nu\psi+\gamma^2\varphi\partial_\nu(d_2(\psi)))|z|^2\mathrm{dS}_y\mathrm{d}x\\
&\quad + \int_{\Gamma_x} 2s\gamma\varphi(\partial_\nu z)((\nabla_y\psi,\nabla_y z)-(\nabla_x\psi,\nabla_x z))\mathrm{dS}_x\mathrm{d}y\\
&\quad - \int_{\Gamma_x} s\gamma\varphi(\partial_\nu\psi)(|\nabla_y z|^2-|\nabla_x z|^2)\mathrm{dS}_x\mathrm{d}y + \int_{\Gamma_x} s^3\gamma^3\varphi^3(\partial_\nu\psi)|z|^2 d_2(\psi)\mathrm{dS}_x\mathrm{d}y\\
&\quad + \int_{\Gamma_x} s(\gamma\varphi d_1(\psi)+\gamma^2\varphi d_2(\psi))z(\partial_\nu z)\mathrm{dS}_x\mathrm{d}y\\
&\quad - \int_{\Gamma_x} \frac{s}{2}((\gamma^2\varphi d_1(\psi)+\gamma^3\varphi d_2(\psi))(\partial_\nu\psi)+\gamma^2\varphi\partial_\nu(d_2(\psi)))|z|^2\mathrm{dS}_x\mathrm{d}y. \qquad (5.8)
\end{aligned}
$$

Then from (5.7), we have

$$
\begin{aligned}
(P^+z,P^-z)_{L^2(\Omega)} &= \sum_{k,j=1}^m\int_\Omega 2s\gamma\varphi\psi_{y_jy_k}z_{y_k}z_{y_j}\mathrm{d}x\mathrm{d}y + \int_\Omega 2s\gamma^2\varphi\Big(\sum_{j=1}^m\psi_{y_j}z_{y_j}\Big)^2\mathrm{d}x\mathrm{d}y\\
&\quad - \sum_{k=1}^m\sum_{j=1}^n\int_\Omega 4s\gamma^2\varphi\psi_{y_k}\psi_{x_j}z_{y_k}z_{x_j}\mathrm{d}x\mathrm{d}y\\
&\quad + \sum_{k,j=1}^n\int_\Omega 2s\gamma\varphi z_{x_k}z_{x_j}\psi_{x_jx_k}\mathrm{d}x\mathrm{d}y + \int_\Omega 2s\gamma^2\varphi\Big(\sum_{k=1}^n z_{x_k}\psi_{x_k}\Big)^2\mathrm{d}x\mathrm{d}y\\
&\quad - \int_\Omega \frac{s}{2}\gamma^2\varphi|z|^2 d_3(\psi)\mathrm{d}x\mathrm{d}y - \int_\Omega s\gamma^3\varphi|z|^2 d_5(\psi)\mathrm{d}x\mathrm{d}y\\
&\quad - \int_\Omega \frac{s}{2}\gamma^4\varphi|z|^2(d_2(\psi))^2\mathrm{d}x\mathrm{d}y + \int_\Omega s^3\gamma^3\varphi^3|z|^2 d_4(\psi)\mathrm{d}x\mathrm{d}y\\
&\quad + \int_\Omega 2s^3\gamma^4\varphi^3|z|^2(d_2(\psi))^2\mathrm{d}x\mathrm{d}y + \mathrm{B}_0, \qquad (5.9)
\end{aligned}
$$

where

$$
d_3 := d_3(\psi) = (d_1(\psi))^2 + \Delta_y(d_2(\psi)) - \Delta_x(d_2(\psi)),
$$

$$d_4 := d_4(\psi) = (\nabla_y \psi, \nabla_y (d_2(\psi))) - (\nabla_x \psi, \nabla_x (d_2(\psi))),$$
$$d_5 := d_5(\psi) = d_1(\psi) d_2(\psi) + (\nabla_y \psi, \nabla_y (d_2(\psi))) - (\nabla_x \psi, \nabla_x (d_2(\psi))).$$

We have $\varphi \ge 1$ on $\overline{\Omega}$ and so $\varphi^k \le C\varphi^4$, $k = 1, 2, 3$ on $\overline{\Omega}$. Moreover the second, third and fifth terms on the right-hand side of (5.9) are summed up into

$$\int_\Omega 2s\gamma^2 \varphi \Big(\sum_{j=1}^m \psi_{y_j} z_{y_j} - \sum_{k=1}^n \psi_{x_k} z_{x_k} \Big)^2 \mathrm{d}x\mathrm{d}y \ge 0.$$

Hence

$$(P^+ z, P^- z)_{L^2(\Omega)} \ge -\int_\Omega 4\beta s\gamma\varphi |\nabla_y z|^2 \mathrm{d}x\mathrm{d}y + \int_\Omega 4s\gamma\varphi |\nabla_x z|^2 \mathrm{d}x\mathrm{d}y$$
$$+ \int_\Omega 2s^3\gamma^4\varphi^3 d_2^2 |z|^2 \mathrm{d}x\mathrm{d}y + \int_\Omega o(s^3\gamma^4\varphi^3)|z|^2 \mathrm{d}x\mathrm{d}y + \mathrm{B}_0.$$

By the assumption (2.2), we have

$$d_2^2 = 16(|x - x_0|^2 - \beta^2 |y - y_0|^2)^2 \ge 16\delta_0^2.$$

Therefore, we can write

$$(P^+ z, P^- z)_{L^2(\Omega)} \ge -\int_\Omega 4\beta s\gamma\varphi |\nabla_y z|^2 \mathrm{d}x\mathrm{d}y + \int_\Omega 4s\gamma\varphi |\nabla_x z|^2 \mathrm{d}x\mathrm{d}y$$
$$+ 32\delta_0^2 \int_\Omega s^3\gamma^4\varphi^3 |z|^2 \mathrm{d}x\mathrm{d}y + \int_\Omega o(s^3\gamma^4\varphi^3)|z|^2 \mathrm{d}x\mathrm{d}y + \mathrm{B}_0. \tag{5.10}$$

In (5.10), the signs of the terms of $|\nabla_x z|^2$ and $|\nabla_y z|^2$ are different. Thus we need to perform another estimation for

$$\int_\Omega (P^+ z + P^- z)\varphi z \mathrm{d}x\mathrm{d}y.$$

Multiplying the equation $Pz = F\mathrm{e}^{s\varphi}$ by φz and applying the integration by parts, we have

$$\int_\Omega (P^+ z + P^- z)\varphi z \mathrm{d}x\mathrm{d}y$$
$$= \int_\Omega (\Delta_y z)\varphi z \mathrm{d}x\mathrm{d}y - \int_\Omega (\Delta_x z)\varphi z \mathrm{d}x\mathrm{d}y + \int_\Omega s^2(|\nabla_y \varphi|^2 - |\nabla_x \varphi|^2)\varphi |z|^2 \mathrm{d}x\mathrm{d}y$$
$$- 2s \int_\Omega ((\nabla_y \varphi, \nabla_y z) - (\nabla_x \varphi, \nabla_x z))\varphi z \mathrm{d}x\mathrm{d}y - \int_\Omega s(\Delta_y \varphi - \Delta_x \varphi)|z|^2 \mathrm{d}x\mathrm{d}y$$
$$:= \mathrm{K}_1 + \mathrm{K}_2 + \mathrm{K}_3 + \mathrm{K}_4 + \mathrm{K}_5.$$

Now we estimate the terms K_j, $1 \le j \le 5$ as follows:

$$\mathrm{K}_1 = \int_\Omega (\Delta_y z)(\varphi z) \mathrm{d}x\mathrm{d}y$$
$$= -\int_\Omega (\nabla_y z, \nabla_y (\varphi z)) \mathrm{d}x\mathrm{d}y + \int_{\Gamma_y} (\partial_\nu z)(\varphi z) \mathrm{d}S_y \mathrm{d}x$$
$$= -\int_\Omega |\nabla_y z|^2 \varphi \mathrm{d}x\mathrm{d}y - \frac{1}{2}\int_\Omega (\nabla_y(|z|^2), \nabla_y \varphi) \mathrm{d}x\mathrm{d}y + \int_{\Gamma_y} (\partial_\nu z)(\varphi z) \mathrm{d}S_y \mathrm{d}x$$

$$
\begin{aligned}
&= -\int_\Omega \varphi|\nabla_y z|^2 \mathrm{d}x\mathrm{d}y + \frac{1}{2}\int_\Omega \gamma\varphi|z|^2\Delta_y\psi \mathrm{d}x\mathrm{d}y \\
&\quad + \frac{1}{2}\int_\Omega \gamma^2\varphi|z|^2|\nabla_y\psi|^2\mathrm{d}x\mathrm{d}y + \int_{\Gamma_y}(\partial_\nu z)(\varphi z)\mathrm{d}S_y\mathrm{d}x - \frac{1}{2}\int_{\Gamma_y}\gamma\varphi(\partial_\nu\psi)|z|^2\mathrm{d}S_y\mathrm{d}x,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{K}_2 &= -\int_\Omega(\Delta_x z)(\varphi z)\mathrm{d}x\mathrm{d}y \\
&= \int_\Omega \varphi|\nabla_x z|^2\mathrm{d}x\mathrm{d}y - \frac{1}{2}\int_\Omega |z|^2\Delta_x\varphi\mathrm{d}x\mathrm{d}y \\
&\quad - \int_{\Gamma_x}(\partial_\nu z)(\varphi z)\mathrm{dS}_x\mathrm{d}y + \frac{1}{2}\int_{\Gamma_x}(\partial_\nu\varphi)|z|^2\mathrm{dS}_x\mathrm{d}y \\
&= \int_\Omega \varphi|\nabla_x z|^2\mathrm{d}x\mathrm{d}y - \frac{1}{2}\int_\Omega \gamma\varphi|z|^2\Delta_x\psi\mathrm{d}x\mathrm{d}y \\
&\quad - \frac{1}{2}\int_\Omega \gamma^2\varphi|z|^2|\nabla_x\psi|^2\mathrm{d}x\mathrm{d}y - \int_{\Gamma_x}(\partial_\nu z)(\varphi z)\mathrm{dS}_x\mathrm{d}y + \frac{1}{2}\int_{\Gamma_x}\gamma\varphi(\partial_\nu\psi)|z|^2\mathrm{dS}_x\mathrm{d}y,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{K}_3 &= \int_\Omega s^2(|\nabla_y\varphi|^2 - |\nabla_x\varphi|^2)|z|^2\varphi\mathrm{d}x\mathrm{d}y \\
&= \int_\Omega s^2\gamma^2\varphi^3(|\nabla_y\psi|^2 - |\nabla_x\psi|^2)|z|^2\mathrm{d}x\mathrm{d}y \\
&= \int_\Omega s^2\gamma^2\varphi^3 d_2(\psi)|z|^2\mathrm{d}x\mathrm{d}y,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{K}_4 &= -\int_\Omega 2s((\nabla_y\varphi,\nabla_y z) - (\nabla_x\varphi,\nabla_x z))\varphi z\mathrm{d}x\mathrm{d}y \\
&= -\int_\Omega s(\nabla_y\varphi,\nabla_y(|z|^2\varphi))\mathrm{d}x\mathrm{d}y + \int_\Omega s(\nabla_y\varphi,|z|^2\nabla_y\varphi)\mathrm{d}x\mathrm{d}y \\
&\quad + \int_\Omega s(\nabla_x\varphi,\nabla_x(|z|^2\varphi))\mathrm{d}x\mathrm{d}y - \int_\Omega s(\nabla_x\varphi,|z|^2\nabla_x\varphi)\mathrm{d}x\mathrm{d}y \\
&= \int_\Omega s\varphi|z|^2\Delta_y\varphi\mathrm{d}x\mathrm{d}y + \int_\Omega s|z|^2|\nabla_y\varphi|^2\mathrm{d}x\mathrm{d}y - \int_\Omega s\varphi|z|^2\Delta_x\varphi\mathrm{d}x\mathrm{d}y \\
&\quad - \int_\Omega s|z|^2|\nabla_x\varphi|^2\mathrm{d}x\mathrm{d}y - s\int_{\Gamma_y}(\partial_\nu\varphi)|z|^2\varphi\mathrm{d}S_y\mathrm{d}x + s\int_{\Gamma_x}(\partial_\nu\varphi)|z|^2\varphi\mathrm{dS}_x\mathrm{d}y \\
&= \int_\Omega s\gamma\varphi^2|z|^2 d_1(\psi)\mathrm{d}x\mathrm{d}y + 2\int_\Omega s\gamma^2\varphi^2|z|^2 d_2(\psi)\mathrm{d}x\mathrm{d}y \\
&\quad - s\int_{\Gamma_y}\gamma\varphi^2(\partial_\nu\psi)|z|^2\mathrm{d}S_y\mathrm{d}x + s\int_{\Gamma_x}\gamma\varphi^2(\partial_\nu\psi)|z|^2\mathrm{dS}_x\mathrm{d}y,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{K}_5 &= -\int_\Omega s(\Delta_y\varphi - \Delta_x\varphi)|z|^2\varphi\mathrm{d}x\mathrm{d}y \\
&= -\int_\Omega s\gamma\varphi^2(\Delta_y\psi + \gamma|\nabla_y\psi|^2)|z|^2\mathrm{d}x\mathrm{d}y + \int_\Omega s\gamma\varphi^2(\Delta_x\psi + \gamma|\nabla_x\psi|^2)|z|^2\mathrm{d}x\mathrm{d}y \\
&= -\int_\Omega s\gamma\varphi^2|z|^2 d_1(\psi)\mathrm{d}x\mathrm{d}y - \int_\Omega s\gamma^2\varphi^2|z|^2 d_2(\psi)\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

Therefore we see that

$$
\begin{aligned}
&\int_\Omega (P^+z + P^-z)\varphi z\mathrm{d}x\mathrm{d}y \\
&= -\int_\Omega \varphi|\nabla_y z|^2\mathrm{d}x\mathrm{d}y + \frac{1}{2}\int_\Omega \gamma\varphi|z|^2 d_1(\psi)\mathrm{d}x\mathrm{d}y
\end{aligned}
$$

$$+\frac{1}{2}\int_\Omega \gamma^2\varphi|z|^2 d_2(\psi)\mathrm{d}x\mathrm{d}y + \int_\Omega \varphi|\nabla_x z|^2\mathrm{d}x\mathrm{d}y$$
$$+\int_\Omega s^2\gamma^2\varphi^3 d_2(\psi)|z|^2\mathrm{d}x\mathrm{d}y + \int_\Omega s\gamma^2\varphi^2|z|^2 d_2(\psi)\mathrm{d}x\mathrm{d}y + \mathrm{B}_1. \tag{5.11}$$

Here

$$\begin{aligned}\mathrm{B}_1 = &\int_{\Gamma_y}(\partial_\nu z)(\varphi z)\mathrm{dS}_y\mathrm{d}x - \frac{1}{2}\int_{\Gamma_y}\gamma\varphi(1+2s\varphi)(\partial_\nu\psi)|z|^2\mathrm{d}S_y\mathrm{d}x\\ &-\int_{\Gamma_x}(\partial_\nu z)(\varphi z)\mathrm{dS}_x\mathrm{d}y + \frac{1}{2}\int_{\Gamma_x}\gamma\varphi(1+2s\varphi)(\partial_\nu\psi)|z|^2\mathrm{dS}_x\mathrm{d}y\\ =\,& 0,\end{aligned}$$

because $z=0$ on $\Gamma_x\cup\Gamma_y$ by (2.3). Now we calculate B_0 given by (5.8), while (2.3) implies $\nabla_y z = 0$ and $\nabla_x z = (\partial_\nu z)\nu$ on Γ_x and all the integrations on Γ_y vanish. Hence

$$\begin{aligned}\mathrm{B}_0 &= -\int_{\Gamma_x}2s\gamma\varphi(\partial_\nu z)(\nabla_x\varphi,\nabla_x z)\mathrm{dS}_x\mathrm{d}y + \int_{\Gamma_x}s\gamma\varphi(\partial_\nu z)|\nabla_x z|^2\mathrm{dS}_x\mathrm{d}y\\ &= -4\int_{\Gamma_x}s\gamma\varphi|\partial_\nu z|^2((x-x_0),\nu)\mathrm{dS}_x\mathrm{d}y + 2\int_{\Gamma_x}s\gamma\varphi((x-x_0),\nu)|\partial_\nu z|^2\mathrm{dS}_x\mathrm{d}y\\ &= -2\int_{\Gamma_x}s\gamma\varphi|\partial_\nu z|^2((x-x_0),\nu)\mathrm{dS}_x\mathrm{d}y\\ &\ge -2\int_{\Gamma_x\cap\{((x-x_0),\nu)\ge 0\}}s\gamma\varphi|\partial_\nu z|^2((x-x_0),\nu)\mathrm{dS}_x\mathrm{d}y.\end{aligned} \tag{5.12}$$

So multiplying (5.11) by $-s\gamma(4\beta+\mu)$, where we choose $\mu>0$ later, we have

$$\begin{aligned}&-\int_\Omega(4\beta+\mu)s\gamma\varphi(Pz)z\mathrm{d}x\mathrm{d}y\\ =&\int_\Omega(4\beta+\mu)s\gamma\varphi|\nabla_y z|^2\mathrm{d}x\mathrm{d}y - \int_\Omega(4\beta+\mu)s\gamma\varphi|\nabla_x z|^2\mathrm{d}x\mathrm{d}y\\ &+\int_\Omega o(s^3\gamma^4\varphi^3)|z|^2\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{5.13}$$

We add (5.10) and (5.13) to have

$$\begin{aligned}&(P^+z,P^-z)_{L^2(\Omega)} - (4\beta+\mu)\int_\Omega(Pz)s\gamma\varphi z\mathrm{d}x\mathrm{d}y\\ \ge\,&\mu\int_\Omega s\gamma\varphi|\nabla_y z|^2\mathrm{d}x\mathrm{d}y + (4-4\beta-\mu)\int_\Omega s\gamma\varphi|\nabla_x z|^2\mathrm{d}x\mathrm{d}y\\ &+32\delta_0^2\int_\Omega s^3\gamma^4\varphi^3|z|^2\mathrm{d}x\mathrm{d}y + \int_\Omega o(s^3\gamma^4\varphi^3)|z|^2\mathrm{d}x\mathrm{d}y + \mathrm{B}_0.\end{aligned}$$

On the other hand, since

$$\begin{aligned}(P^+z,P^-z)_{L^2(\Omega)} &= \frac{1}{2}(\|P^+z+P^-z\|^2_{L^2(\Omega)} - \|P^+z\|^2_{L^2(\Omega)} - \|P^-z\|^2_{L^2(\Omega)})\\ &\le \frac{1}{2}\|Pz\|^2_{L^2(\Omega)},\end{aligned}$$

by the Cauchy-Schwarz inequality, we see

$$[\text{left-hand side}] \le \frac{1}{2}\|Pz\|^2_{L^2(\Omega)} + (4\beta+\mu)\Big\{\frac{1}{2}\int_\Omega|Pz|^2\mathrm{d}x\mathrm{d}y + \frac{1}{2}\int_\Omega s^2\gamma^2\varphi^2|z|^2\mathrm{d}x\mathrm{d}y\Big\}.$$

By $0 < \beta < 1$, we can choose $\mu > 0$ sufficiently small, so that

$$4 - 4\beta - \mu > 0.$$

Absorbing the term of $|z|^2$ with $o(s^3\gamma^4\varphi^2)$, we complete the proof of Theorem 2.2.

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