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The ∂-Stabilization of a Heegaard Splitting with Distance at Least 6 is Unstabilized*

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Abstract Let M be a compact orientable 3-manifold with ∂M connected. If $V \cup_S W$ is a Heegaard splitting of M with distance at least 6, then the ∂-stabilization of $V \cup_S W$ along ∂M is unstabilized. Hence M has at least two unstabilized Heegaard splittings with different genera. The basic tool is a result on disk complex given by Masur and Schleimer.

Keywords Heegaard splitting, Distance, Stabilization
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1 Introduction

Let M be a compact orientable 3-manifold. If there exists a closed surface X which cuts M into two compression body V and W so that $X = \partial_+ V = \partial_+ W$, then we say M has a Heegaard splitting, denoted by $M = V \cup_X W$. In this case, X is called a Heegaard surface, and $g(X)$ is called the genus of the Heegaard splitting. The Heegaard splitting $M = V \cup_X W$ is said to be stabilized if there exist essential disks B in V and D in W such that B intersects D in just one point; otherwise, it is said to be unstabilized. $M = V \cup_X W$ is said to be reducible if there exists an essential simple closed curve on X which bounds disks in both V and W; otherwise, it is said to be irreducible. $M = V \cup_X W$ is said to be weakly reducible if there exist an essential disk D in V and B in W such that $D \cap B = \varnothing$; otherwise, it is said to be strongly irreducible. $M = V \cup_X W$ is said to be ∂-reducible if there exists an essential disk of M which intersects X in an essential simple closed curve; otherwise, it is said to be ∂-irreducible. The distance of two essential simple closed curves α and β on X, denoted by $d(\alpha, \beta)$, is defined to be the smallest integer $n \geq 0$ so that there exists a sequence of essential simple closed curves $\alpha_0 = \alpha, \cdots, \alpha_n = \beta$ on X such that α_{i-1} is disjoint from α_i for $1 \leq i \leq n$. The distance of the Heegaard surface X, denoted by $d(X)$, is defined to be $\min\{d(\alpha, \beta)\}$, where α bounds a disk in V and β bounds a disk in W (see [2, 4]).

Let M be a compact orientable 3-manifold, and F be a component of ∂M. Let $M = V \cup_X W$ be a Heegaard splitting. Then $V \cup_X W$ induces another Heegaard splitting of M called the

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∂-stabilization of $V \cup_X W$ as follow.

Without loss of generality, we may assume that $F \subset \partial_- W$. Now there exists an essential disk B which divides W into $F \times I$ and $W - F \times (0, 1)$. Assume that $F = F \times \{0\}$. Then $F \times \{1\} - \operatorname{int} B$ is a sub-surface of X. Let p be a point on F, and $N(p)$ be a regular neighborhood of p on F such that $N(p) \times \{1\}$ is disjoint from B. Now let $V^* = V \cup N(p) \times I \cup F \times [0, 1]$, and W^* be the closure of $M - V^*$. Then V^* and W^* are two compression bodies such that $\partial_+ V^* = \partial_+ W^* = X^*$. Hence $V^* \cup_{X^*} W^*$ is also a Heegaard splitting of M, called the ∂-stabilization of $V \cup_X W$. In this case, $g(X^*) = g(X) + g(F)$ (see [8]).

Now a natural question is the following question.

Question 1.1 Let $M = V \cup_X W$ be an unstabilized Heegaard splitting, and $M = V^* \cup_{X^*} W^*$ be the ∂-stabilization of $V \cup_X W$. Is $M = V^* \cup_{X^*} W^*$ unstabilized?

Remark 1.1 If $M = V^* \cup_{X^*} W^*$ is unstabilized, then M has two unstabilized Heegaard splittings with different Heegaard genera. Moreover, this implies a way to find Haken closed 3-manifolds which have unstabilized Heegaard splittings with different Heegaard genera: Let M be a Haken closed 3-manifold, and F be a closed incompressible surface which cuts M into two 3-manifolds M_1 and M_2 with ∂M_i connected. Now let $M_i = V_i \cup_{X_i} W_i$ be a Heegaard splitting, and $M_i = V_i^* \cup_{X_i^*} W_i^*$ be the ∂-stabilization of $M_i = V_i \cup_{X_i} W_i$. Now if one of $M_1 = V_1^* \cup_{X_1^*} W_1^*$ and $M_2 = V_2^* \cup_{X_2^*} W_2^*$, say $M_1 = V_1^* \cup_{X_1^*} W_1^*$, is unstabilized, then M has two natural Heegaard splittings, one of which is the amalgamation of $M_1 = V_1 \cup_{X_1} W_1$ and $M_2 = V_2 \cup_{X_2} W_2$, and the other is the amalgamation of $M_1 = V_1^* \cup_{X_1^*} W_1^*$ and $M_2 = V_2 \cup_{X_2} W_2$. Thus we can only consider if the two amalgamations are unstabilized. Bachman [1] announced a result on this topic.

Scharlemann and Tomova [9] proved that if $M = V \cup_X W$ is a Heegaard splitting, then, for any Heegaard splitting $M = V^* \cup_{X^*} W^*$, either $g(X^*) \geq 2d(X)$ or $M = V^* \cup_{X^*} W^*$ is obtained by doing ∂-stabilizations and stabilizations from $M = V \cup_X W$. Scharlemann-Tomova theorem implies that if $M = V \cup_X W$ has high distance while ∂M has at least two components, then the ∂-stabilization of $M = V \cup_X W$ along a minimal genus component of ∂M is unstabilized. We know little on Question 1.1 when ∂M is connected except that M is an I-bundle of a genus g closed surface F_g. In this case, the ∂-stabilization of the trivial Heegaard splitting of $F_g \times I$ is unstabilized. The main result of this paper is the following theorem.

Theorem 1.1 *Let M be a compact orientable 3-manifold with ∂M connected. Then the ∂-stabilization of a Heegaard splitting of M with distance at least 6 is unstabilized. Furthermore, M admits two unstabilized Heegaard splittings with different genera.*

2 Some Known Results on Arc and Curve Complexes

In this section, we assume that S is a compact orientable surface of genus g with at least one boundary component. A simple closed curve in S is said to be essential if it does not bound a disk in S and not parallel to ∂S. A properly embedded arc in S is said to be essential if it is not parallel to ∂S.

Suppose that $g \geq 2$. Harvey [3] defined the curve complex $\mathcal{C}(S)$ as follows: The vertices of $\mathcal{C}(S)$ are the isotopy classes of essential simple closed curves on S, and $k+1$ distinct vertices

$x_0, x_1, \cdots, x_k$ determine a k-simplex of $\mathcal{C}(S)$ if and only if they are represented by pairwise disjoint simple closed curves. For two vertices x and y of $\mathcal{C}(S)$, the distance of x and y, denoted by $d_{\mathcal{C}(S)}(x,y)$, is defined to be the smallest integer $n \geq 0$ so there exists a sequence of vertices $x_0 = x, \cdots, x_n = y$ such that x_{i-1} and x_i are represented by two disjoint simple closed curves on S for each $1 \leq i \leq n$. For two sets of vertices in $\mathcal{C}(S)$, $d(X,Y)$ is defined to be $\min\big\{d_{\mathcal{C}(S)}(x,y) \mid x \in X,\ y \in Y\big\}$. For a Heegaard splitting $V \cup_X W$ with genus at least 2, if we denote by A the isotopy class of essential simple closed curves on X which bounds a disk in V, and B the isotopy class of essential simple closed curves on X which bounds a disk in W, then $d(X) = d(A,B)$. Now let S be a once-punctured torus or a torus. In this case, Masur and Minsky [5–6] define $\mathcal{C}(S)$ as follows: The vertices of $\mathcal{C}(S)$ are the isotopy classes of essential simple closed curves or essential arcs on S, and $k+1$ distinct vertices $x_0, x_1, \cdots, x_k$ determine a k-simplex of $\mathcal{C}(S)$ if and only if x_{i-1} and x_i are represented by two simple closed curves c_{i-1} and c_i on S such that c_{i-1} intersects c_i in just one point for each $1 \leq i \leq k$.

Masur and Minsky define the arc and curve complex $\mathcal{AC}(S)$ as follows: The vertices of $\mathcal{C}(S)$ are the isotopy classes of essential simple closed curves and essential arcs on S. Then $\mathcal{AC}(S)$ and $d_{\mathcal{AC}(S)}(x,y)$ can be defined in the same way with $\mathcal{C}(S)$.

In the following argument, we assume that X is a closed surface of genus at least two and S is a once-punctured subsurface of X with $g(S) \geq 1$. S is said to be essential and proper if ∂S does not bound a disk on X. Define the maps κ_S: $\mathcal{C}(X) \to \mathcal{AC}(S) \cup \{\varnothing\}$ and σ_S: $\mathcal{AC}(S) \to \mathcal{C}(S)$ as follows:

Let $\alpha \in \mathcal{C}(X)$, and α_c be a simple closed curve in the isotopy class α. α_c is tight to ∂S if the geometry intersection number of α_c and ∂S is minimal among all the simple closed curves in α. Now for $\alpha \in \mathcal{C}(X)$, and $\alpha_c \in \alpha$ which is tight to S, let $\kappa_S(\alpha) = \alpha \cap S$. For any $\alpha \in \mathcal{C}(X)$, $\alpha' \in \sigma_S(\alpha)$ if and only if α' is a boundary component of a regular neighborhood of $\alpha \cup \partial S$ and essential. Specially, let $\sigma_S(\varnothing) = \varnothing$. Now let $\pi_S = \sigma_S \circ \kappa_S$. We say $\alpha \in \mathcal{C}(X)$ cuts S if $\pi_S(\alpha) \neq \varnothing$. If α,$\beta \in \mathcal{C}(X)$ both cut S, we write $d_{\mathcal{C}(S)}(\alpha, \beta) = \mathrm{diam}_{\mathcal{C}(S)}(\pi_S(\alpha), \pi_S(\beta))$.

Lemma 2.1 *Let S be an essential subsurface of X. Suppose α,$\beta \in \mathcal{C}(X)$ are disjoint in X and both cut S. Then $d_{\mathcal{C}(S)}(\alpha, \beta) = \mathrm{diam}_{\mathcal{C}(S)}(\pi_S(\alpha), \pi_S(\beta)) \leq 2$.*

Proof The lemma is immediately from Lemma 2.2 in [6].

Suppose V is a genus at least 2 handlebody with $\partial V = X$. Define disk complex $\mathcal{D}(V)$ to be the collection of essential disk $D \subset V$, up to isotopy. Place an edge between any two vertices D_1,$D_2 \in \mathcal{D}(V)$ if D_1 and D_2 can be isotopic to being disjoint in V. Let S be a once-punctured essential subsurface of X. S is called a hole for $\mathcal{D}(V)$ if, for any $D \in \mathcal{D}(V)$, ∂D cuts S.

A role tool of this paper is the following.

Lemma 2.2 *Suppose S is a hole for $\mathcal{D}(V)$, $S \subset \partial V$. Then for any essential disk D cuts S, there exists an essential disk D' with the following properties:*

(1) *∂S and $\partial D'$ are tight.*

(2) *If S is incompressible, then D' is not boundary compressible into S and $d_{\mathcal{AC}(S)}(D, D') \leq 3$.*

(3) *If S is compressible, then $\partial D' \subset S$ and $d_{\mathcal{AC}(S)}(D, D') \leq 3$.*

Proof See the proof of Lemma 11.7 in [7].

3 The Proof of Theorem 1.1

Theorem 3.1 *Let M be a compact orientable 3-manifold with ∂M connected. Then the ∂-stabilization of a Heegaard splitting of M with distance at least 6 is unstabilized. Furthermore, M admits two unstabilized Heegaard splittings with different genera.*

Proof Let $M = V \cup_X W$ be a Heegaard splitting with distance at least 6. Recalling the definition of the ∂-stabilization of $V \cup_X W$:

In this case, we may assume that $F = \partial M = \partial_- W$. As defined in Section 1, V^* and W^* are two compression bodies such that $\partial_+ V^* = \partial_+ W^* = X^*$. And $V^* \cup_{X^*} W^*$ is also a Heegaard splitting of M, called the ∂-stabilization of $V \cup_X W$. Since $\partial M = F$ is connected, W^* is a handlebody of genus $g(X^*) = g(X) + g(F)$. See Figure 1.

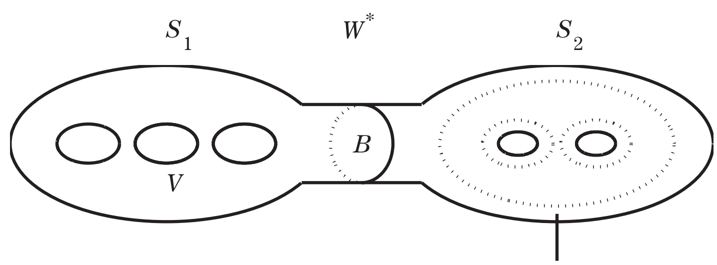

Figure 1 ∂-stabilization

By the definition, ∂B cuts X^* into a subsurface of X, say S_1, and a subsurface of $F \times \{1\}$, say S_2. See Figure 1.

Claim 3.1 S_2 is incompressible in W^*.

Proof Since $d(X) \geq 6$, by definitions in Section 1, $V \cup_X W$ is strongly irreducible and ∂-irreducible. Hence M is irreducible and ∂-irreducible (see [2]). This means that F is incompressible in M. If S_2 is compressible in W^*, then ∂B bounds a disk in W^*, say B'; otherwise, F is compressible in M. Now $B \cup B'$ is a sphere in W such that X and F lie in the two sides of $B \cup B'$. This means that the compression body W is reducible, a contradiction.

Suppose $V^* \cup_{X^*} W^*$ is stabilized. Since $g(X^*) \geq 2$, $M = V^* \cup_{S^*} W^*$ is a reducible Heegaard splitting. Hence there exists a sphere P which intersects X^* in an essential simple closed curve, say C. Thus C cuts P into an essential disk D_1 in V^* and an essential disk E_1 in W^*. We may assume that $|C \cap \partial B|$ is minimal among all reducing sphere of $V^* \cup_{X^*} W^*$. By Claim 3.1, if $C \cap \partial B = \varnothing$, then $C \subset S_1$. In this case, by the proof of Claim 3.1, C is not parallel to ∂B. This means that C is essential on X. This means that $V \cup_X W$ is reducible, a contradiction. Hence we have $|C \cap \partial B| > 0$.

Claim 3.2 (1) S_1 is compressible in W^*.

(2) S_1 is a hole for $\mathcal{D}(W^*)$.

Proof (1) By the definition of $V^* \cup_{S^*} W^*$, $N(p) \times I$ is disjoint from $W - F \times (0,1)$. Hence S_1 is compressible in W^*.

(2) Let D be an essential disk in W^*. By Claim 3.1, S_2 is incompressible in W^*. Hence either $\partial D \subset S$ or D can be isotoped so that each component of $\partial D \cap S_1$ and $\partial D \cap S_2$ is essential in S_1 or S_2. By the definition, S_1 is a hole for $\mathcal{D}(W^*)$.

Note that B cuts V^* into V and $F \times I$. Now consider the two essential disks D_1 in V^* and D_2 in W^*. By the minimality of $C \cap \partial B$, each component of $\partial D \cap S_1$ and $\partial D \cap S_2$ is essential in S_1 or S_2. We may assume that each component of $D_1 \cap B$ is an arc on both D_1 and B. Let a be an outermost component of $D_1 \cap B$ relative to D_1. This means that a, together with an arc on ∂D_1, bounds a disk D_2 such that $\operatorname{int} D_2$ is disjoint from B, and $D_2 \subset V$ since F is incompressible in M. Thus a, together with an arc on ∂B, does also bound a disk D_3 in V. Furthermore, ∂D_3 is essential in X. Since E_1 is an essential disk of W^*, by Lemma 2.2, there exists an essential disk E_2 in W^* such that $d_{\mathcal{AC}(S_1)}(\partial E_2, \partial E_1) \leq 3$, and $\partial E_2 \subset S_1$.

By Lemma 2.1, and since ∂E_2 are contained in S_1, $d_{\mathcal{C}(S_1)}(\partial E_2, \partial E_1) \leq 5$. Note that $\partial D_1 = \partial E_1 = C$, $d_{\mathcal{C}(S_1)}(\partial D_1, \partial E_2) \leq 5$. Then $d_{\mathcal{C}(S_1)}(\partial D_3, \partial E_2) \leq 5$. Since both ∂D_3 and ∂E_2 are essential curves in S_1, and S_1 is obtained by removing a disk B from X, we have that any vertex in the path of $\mathcal{C}(S_1)$ connecting ∂D_3 and ∂E_2 is essential in X. So $d_{\mathcal{C}(X)}(\partial D_3, \partial E_2) \leq 5$. This means that $d(X) \leq 5$, a contradiction. Now $V \cup_X W$ and $V^* \cup_{X^*} W^*$ are two unstabilized Heegaard splittings with genera $g(X)$ and $g(X) + g(\partial M)$.

Remark 3.1 In fact, Lemma 2.2 is also true when V is a compression body. By the proof of Theorem 1.1, it is also true when ∂M is not connected. We omit the argument.

Now an interesting question is to determine the sharp lower bound of $d(X)$, say b, so that the ∂-stabilization of $V \cup_X W$ is unstabilized. Let M be a compact orientable 3-manifold with ∂M connected, and $V \cup_X W$ be a Heegaard splitting of M. We may assume that $\partial M = \partial_- W$. $V \cup_X W$ is said to be primitive if there exist an essential disk D in V and a spine annulus A in W such that D intersects A in just one point. If $V \cup_X W$ is primitive, then $d(X) \leq 2$, and the ∂-stabilization of $V \cup_X W$ is stabilized. Furthermore, there exists primitive Heegaard splittings with distance 2. For example, Morimoto [8] constructed a non-trivial knot whose complement admits a genus two primitive Heegaard splitting $V \cup_X W$. Hence $d(X) = 2$. In this case, $b \geq 3$. So we have the following conjecture.

Conjecture 3.1 Let M be a compact orientable 3-manifold with ∂M connected. Then the ∂-stabilization of a Heegaard splitting of M with distance at least 3 is unstabilized.

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