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Monomial Base for Little q-Schur Algebra $u_k(2, r)$ at Even Roots of Unity

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Abstract Let $u_k(2, r)$ be a little q-Schur algebra over k, where k is a field containing an l'-th primitive root ε of 1 with $l' \geq 4$ even, the author constructs a certain monomial base for little q-Schur algebra $u_k(2, r)$.

Keywords Little q-Schur algebra, Infinitesimal q-Schur algebra, Quantum group
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1 Introduction

The infinitesimal Schur algebra $s(n, r)_h$ was introduced in [7] to study the polynomial representation of G_hT of a given degree r, where G_hT is the group scheme associated to the r-th Frobenius kernel G_h and a maximal torus T of the general linear group $\mathfrak{gl}_n$. The quantum version of the infinitesimal Schur algebra was studied by Cox in [2–3]. Semisimple infinitesimal Schur algebras were classfied in [6] and semisimple infinitesimal q-Schur algebras were classfied in [17]. The finite representation type of infinitesimal Schur algebra was classified in [8] and the tame representation type of infinitesimal Schur algebra was given in [4]. The finite representation type of infinitesimal q-Schur algebra was given in [16] and the tame representation type of infinitesimal q-Schur algebra was given in [18]. The structure of the endomorphism ring of tensor space as a module for the infinitesimal q-Schur algebra $s_q(2, r)_1$ was investigated in [13].

Little q-Schur algebra $u_k(n, r)$ was introduced in [11, 15] and the construction of various bases of monomial, BLM and PBW types for $u_k(n, r)$ was given. Here k is a field containing an l'-th primitive root ε of 1. There is a close relation between infinitesimal q-Schur algebra $s_q(n, r)_h$ and little q-Schur algebra $u_k(n, r)$. In fact, little q-Schur algebra can be considered as a subalgebra of infinitesimal q-Schur algebra (see [14]). The irreducible modules for little q-Schur algebras and semisimple little q-Schur algebras were classified in [12]. The basic algebra of the endomorphism ring of tensor space as a module for the little q-Schur algebra $u_k(2, r)$ was determined in [20].

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In [11, Theorem 8.2, Theorem 8.5], various bases for little q-Schur algebra $u_k(n,r)$ were obtained when $l' \geq 3$ odd. And the case when $l' \geq 4$ even was studied in [15]. In [19, Theorem 5.1], the construction of a $\mathscr{B}$-basis for little q-Schur algebra $u_{\mathscr{B}}(n,r)$ was given where $\mathscr{B} = \mathbb{Z}[\varepsilon, \varepsilon^{-1}]$ and ε is an l'-th primitive root of 1 with $l' \geq 3$ being odd. In this paper, we shall construct a certain monomial base for little q-Schur algebra $u_k(2,r)$ when $l' \geq 4$ even.

We organize this paper as follows. In Section 2 we recall the definition of little q-Schur algebras, given in [11, 15]. We shall construct a certain monomial base for little q-Schur algebra $u_k(2,r)$ in Section 3.

Throughout, let v be an indeterminate and let $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$. Let k be a field containing an l'-th primitive root ϵ of 1 with $l' \geq 3$. Let $l > 1$ be defined by

$$l = \begin{cases} l', & \text{if } l' \text{ is odd}, \\ \dfrac{l'}{2}, & \text{if } l' \text{ is even}. \end{cases}$$

Specializing v to ε, k will be viewed as a $\mathcal{Z}$-module.

2 The Little q-Schur Algebra

The quantized enveloping algebra of $\mathfrak{gl}_n$ is the algebra $\mathbf{U}(n) = \mathbf{U}(\mathfrak{gl}_n)$ over $\mathbb{Q}(v)$ generated by the elements

$$E_i,\ F_i,\ K_j^{\pm 1}\ (1 \leq i \leq n-1,\ 1 \leq j \leq n)$$

subject to the following relations:

(a) $K_iK_j = K_jK_i,\ K_iK_i^{-1} = 1$;

(b) $K_iE_j = v^{\epsilon(i,j)}E_jK_i$, where $\epsilon(i,i) = 1,\ \epsilon(i+1,i) = -1$, and $\epsilon(i,j) = 0$ otherwise;

(c) $K_iF_j = v^{-\epsilon(i,j)}F_jK_i$;

(d) $E_iE_j = E_jE_i,\ F_iF_j = F_jF_i$, if $|i-j| > 1$;

(e) $E_iF_j - F_jE_i = \delta_{ij}\frac{\widetilde{K}_i - \widetilde{K}_i^{-1}}{v - v^{-1}}$, where $\widetilde{K}_i := K_iK_{i+1}^{-1}$;

(f) $E_i^2E_j - (v+v^{-1})E_iE_jE_i + E_jE_i^2 = 0$, if $|i-j| = 1$;

(g) $F_i^2F_j - (v+v^{-1})F_iF_jF_i + F_jF_i^2 = 0$, if $|i-j| = 1$.

Following [21–22], let $U_{\mathcal{Z}}(n)$ be the $\mathcal{Z}$-subalgebra of $\mathbf{U}(n)$ generated by all $E_i^{(m)}$, $F_i^{(m)}$, $K_i^{\pm 1}$ and $\left[{K_i;0 \atop t} \right]$, where for $m, t \in \mathbb{N}$ and $c \in \mathbb{Z}$,

$$E_i^{(m)} = \frac{E_i^m}{[m]^!}, \quad F_i^{(m)} = \frac{F_i^m}{[m]^!}, \quad \left[{K_i;c \atop t} \right] = \prod_{s=1}^{t} \frac{K_iv^{c-s+1} - K_i^{-1}v^{-c+s-1}}{v^s - v^{-s}}$$

with $[m]^! = [1][2]\cdots[m]$ and $[i] = \frac{v^i - v^{-i}}{v - v^{-1}}$.

For any integers c, t with $t \geq 0$, let

$$\begin{bmatrix} c \\ t \end{bmatrix} = \prod_{s=1}^{t} \frac{v^{c-s+1} - v^{-c+s-1}}{v^s - v^{-s}} \in \mathcal{Z}.$$

If $c \geq t \geq 0$, we have $\left[\begin{smallmatrix} c \\ t \end{smallmatrix}\right] = \frac{[c]^!}{[t]^![c-t]^!}$, while if $t > c \geq 0$ we have $\left[\begin{smallmatrix} c \\ t \end{smallmatrix}\right] = 0$.

Let $U_k(n) = U_{\mathcal{Z}}(n) \otimes_{\mathcal{Z}} k$. We denote the images of E_i, F_i, K_j in $U_k(n)$ by the same letters. Let $\widetilde{u}_k(n)$ be the k-subalgebra of $U_k(n)$ generated by the elements E_i, F_i, $K_j^{\pm 1}$ for all i, j. Let $\widetilde{u}_k^0(n)$ be the k-subalgebra of $\widetilde{u}_k(n)$ generated by the elements $K_j^{\pm 1}$'s.

Let $\widetilde{\Xi}(n)$ be the set of all $n \times n$ matrices over $\mathbb{Z}$ with all off diagonal entries in $\mathbb{N}$, and let $\Xi(n) = M_n(\mathbb{N})$. Denote $\sigma : \Xi(n) \to \mathbb{N}$ be the map sending a matrix to the sum of its entries, and let $\Xi(n,r) = \sigma^{-1}(r)$, where $r \in \mathbb{N}$. For $1 \leq i, j \leq n$. let $E_{i,j} \in \Xi(n)$ be the matrix $(a_{k,l})$ with $a_{k,l} = \delta_{i,k}\delta_{j,l}$.

We denote

$$\Xi^{\pm}(n) = \{A = (a_{i,j}) \in \Xi(n) \mid a_{i,i} = 0, \forall i\},$$

$$\Xi^{0}(n) = \{A = (a_{i,j}) \in \Xi(n) \mid a_{i,j} = 0, \forall i \neq j\},$$

$$\Gamma(n) = \{A = (a_{i,j}) \in \widetilde{\Xi}(n) \mid a_{i,j} < l, \forall i \neq j \ \},$$

$$\Gamma^{\pm}(n) = \{A = (a_{i,j}) \mid A \in \widetilde{\Xi}(n),\ a_{i,j} < l, \forall i \neq j,\ a_{i,i} = 0 \text{ for all } i\}.$$

Let $\mathbf{U}_{\mathcal{Z}}(n,r)$ be the algebra over $\mathcal{Z}$ introduced in [1, §1.2]. It has a normalized $\mathcal{Z}$-basis $\{[A]\}_{A \in \Xi(n,r)}$ defined in [1]. From [9], the algebra $\mathbf{U}_{\mathcal{Z}}(n,r)$ is isomorphic to the q-Schur algebra $S_q(n,r)$. Let $\mathbf{U}(n,r) = \mathbf{U}_{\mathcal{Z}}(n,r) \otimes_{\mathcal{Z}} \mathbb{Q}(v)$, then we call both $\mathbf{U}_{\mathcal{Z}}(n,r)$ and $\mathbf{U}(n,r)$ as q-Schur algebras.

Given $r > 0$, $A \in \Xi^{\pm}(n)$ and $\mathbf{j} = (j_1, j_2, \cdots, j_n) \in \mathbb{Z}^n$, we define

$$A(\mathbf{j}, r) = \sum_{\substack{D \in \Xi^0(n) \\ \sigma(A+D) = r}} v^{\sum_i d_i j_i} [A + D] \in \mathbf{U}\ (n, r),$$

where $D = \mathrm{diag}(d_1, d_2, \cdots, d_n)$.

Theorem 2.1 (see [1]) *There exists an algebra epimorphism $\zeta_r : \mathbf{U}(n) \twoheadrightarrow \mathbf{U}(n,r)$ satisfying*

$$E_h \mapsto E_{h,h+1}(\mathbf{0}, r), \quad K_1^{j_1} K_2^{j_2} \cdots K_n^{j_n} \mapsto 0(\mathbf{j}, r), \quad F_h \mapsto E_{h+1,h}(\mathbf{0}, r).$$

Denote $\mathbf{e}_i = \zeta_r(E_i)$, $\mathbf{f}_i = \zeta_r(F_i)$, $\mathbf{k}_i = \zeta_r(K_i)$, $1 \leqslant i \leqslant n-1$, $1 \leqslant j \leqslant n$. For $\mathbf{t} = (t_1, t_2, \cdots, t_n) \in \mathbb{N}^n$, let $\mathbf{k}_{\mathbf{t}} = \prod\limits_{i=1}^{n} \left[\begin{smallmatrix} k_i; 0 \\ t_i \end{smallmatrix}\right]$.

Define a map $\bar{\ } : \mathbb{Z}^n \to \mathbb{Z}_{l'}^n$, $\overline{(j_1, j_2, \cdots, j_n)} = (\overline{j_1}, \overline{j_2}, \cdots, \overline{j_n})$. For $\mathbf{j}= (j_1, j_2, \cdots, j_n) \in \mathbb{N}^n$, let $\mathbf{k}^{\mathbf{j}} = \mathbf{k}_1^{j_1} \mathbf{k}_2^{j_2} \cdots \mathbf{k}_n^{j_n}$, $\sigma(\mathbf{j}) = j_1 + \cdots + j_n$, $\mathbb{N}_{l'} = \{0, 1, \cdots, l'-1\} \subseteq \mathbb{Z}$. Let $\Lambda(n,r) := \{\lambda \in \mathbb{N}^n \mid \sigma(\lambda) = r\}$.

Denote $\mathbf{U}_{\mathcal{Z}}^0(n,r)$ be the subalgebra generated by $\mathbf{k}_\lambda$ $(\lambda \in \Lambda(n,r))$ of $\mathbf{U}_{\mathcal{Z}}(n,r)$.

Theorem 2.2 (see [5, 10]) (1) *The set* $\{\mathbf{k}_\lambda \mid \lambda \in \Lambda(n,r)\}$ *is a complete set of orthogonal primitive idempotents (hence a basis) for* $\mathbf{U}_{\mathcal{Z}}(n,r)$. *In particular,* $1 = \sum_{\lambda\in\Lambda(n,r)} \mathbf{k}_\lambda$.

(2) *Let* $\lambda \in \Lambda(n,r)$. *Then* $\mathbf{k}_i\mathbf{k}_\lambda = v^{\lambda_i}\mathbf{k}_\lambda$ *for* $1 \le i \le n$.

Let $\mathbf{U}_k(n,r) = \mathbf{U}_{\mathcal{Z}}(n,r) \otimes_{\mathcal{Z}} k$. According to [9], we have $\zeta_r(\mathbf{U}_{\mathcal{Z}}(n)) = \mathbf{U}_{\mathcal{Z}}(n,r)$. Thus, ζ_r could naturally induce an epimorphism $\zeta_{r,k} := \zeta_r \otimes \mathrm{id} : \mathbf{U}_k(n) \twoheadrightarrow \mathbf{U}_k(n,r)$. For convenience, we denote $\mathbf{e}_i \otimes 1$, $\mathbf{f}_i \otimes 1$, $\mathbf{k}_j \otimes 1$ as $\mathbf{e}_i$, $\mathbf{f}_i$, $\mathbf{k}_j$, respectively. Then the algebra $u_k(n,r) = \zeta_{r,k}(\widetilde{u}_k(n))$ is called a little q-Schur algebra (see [11, 15]). Let $u_k^0(n,r) = \zeta_{r,k}(u_k^0(n))$.

Denote

$$\mathbf{p}_{\overline{\lambda}} = \begin{cases} \sum\limits_{\substack{\mu\in\Lambda(n,r)\\ \overline{\mu}=\overline{\lambda}}} \mathbf{k}_\mu, & \text{if } \overline{\lambda} \in \overline{\Lambda(n,r)},\\ 0, & \text{otherwise.}\end{cases}$$

For $A = (a_{i,j}) \in \Xi^{\pm}(n)$, we define

$$\mathbf{e}^{(A^+)} = \prod_{1\le i\le h<j\le n} \mathbf{e}_h^{(a_{i,j})}, \quad \mathbf{f}^{(A^-)} = \prod_{1\le j\le h<i\le n} \mathbf{f}_h^{(a_{i,j})}.$$

For $A \in \Gamma^{\pm}(n)$, $\overline{\lambda} \in \mathbb{Z}_{l'}^n$, we define

$$[\![A + \mathrm{diag}(\overline{\lambda}), r]\!] = \begin{cases} \sum\limits_{\substack{\mu\in\Lambda(n,r-\sigma(A))\\ \overline{\mu}=\overline{\lambda}}} [A + \mathrm{diag}(\mu)], & \text{if } \overline{\lambda} \in \overline{\Lambda(n, r-\sigma(A))},\\ 0, & \text{otherwise.}\end{cases}$$

For $A = (a_{s,t}) \in \widetilde{\Xi}(n)$ and $i<j$, let $\sigma_{i,j}(A) = \sum_{\substack{s\le i\\ t\ge j}} a_{s,t}$ and $\sigma_{j,i}(A) = \sum_{\substack{s\le i\\ t\ge j}} a_{t,s}$. Define $A' \preccurlyeq A$ if and only if $\sigma_{i,j}(A') \le \sigma_{i,j}(A)$ and $\sigma_{j,i}(A') \le \sigma_{j,i}(A)$ for all $1 \le i < j \le n$. Put $A' \prec A$ if $A' \preccurlyeq A$ and, for some pair (i,j) with $i<j$, either $\sigma_{i,j}(A') < \sigma_{i,j}(A)$ or $\sigma_{j,i}(A') < \sigma_{j,i}(A)$.

For $A = (a_{i,j}) \in \Xi(n)$, we define

$$\sigma_i(A) = a_{i,i} + \sum_{1\le j<i} (a_{i,j} + a_{j,i}).$$

Lemma 2.1 (see [11, Proposition 7.3], [15, Corollary 6.2]) *Each of the following sets forms a k-basis for the algebra* $u_k^0(n,r)$:

(1) $\mathcal{X}_{i_0} = \{\mathbf{k}^{\mathbf{j}} \mid \mathbf{j} \in \mathbb{N}_{l'}^n,\ j_{i_0} = 0,\ \sigma(\mathbf{j}) \le r\}$;

(2) $\mathcal{Y} = \{\mathbf{p}_{\overline{\lambda}} \mid \overline{\lambda} \in \overline{\Lambda(n,r)}\}$.

Lemma 2.2 (see [11, Theorem 8.2, Theorem 8.5], [15, Theorem 6.8, Theorem 6.9]) *Each of the following sets forms a k-basis for the little q-Schur algebra* $u_k(n,r)$:

(1) $\mathcal{M}_k = \{\mathbf{e}^{(A^+)}\mathbf{p}_{\overline{\lambda}}\mathbf{f}^{(A^-)} \mid A \in \Gamma^{\pm}(n),\ \lambda \in \Lambda(n,r),\ \lambda_i \ge \sigma_i(A),\ \forall i\}$;

(2) $\mathcal{L}_k = \{[\![A + \mathrm{diag}(\overline{\lambda}), r]\!] \mid A \in \Gamma^{\pm}(n),\ \sigma(A) \le r,\ \lambda \in \Lambda(n, r-\sigma(A))\}$;

(3) $\mathcal{N}_{i_0,k} = \{\mathbf{e}^{(A^+)}\mathbf{k}^{j}\mathbf{f}^{(A^-)} \mid A \in \Gamma^{\pm}(n),\ \mathbf{j} \in \mathbb{N}_{l'}^n,\ j_{i_0} = 0,\ \sigma(\mathbf{j}) + \sigma(A) \le r\}$.

Assume that l' is even. Let

$$\mathfrak{D}_{n,r} := \Big\{\mathbf{e}^{(A^+)}\prod_{i=1}^{n-1}\mathbf{k}_i^{\delta_i}\begin{bmatrix}\mathbf{k}_i;0\\ \lambda_i\end{bmatrix}\mathbf{f}^{(A^-)}\ \Big|\ A\in\Gamma^{\pm}(n),\ 0\le\lambda_i\le l-1,\ \delta_i=0,1,\ i=1,\cdots,n-1,$$
$$\sigma(A)+\sum_{i=1}^{n-1}(\lambda_i+\delta_i l)\le r\Big\}.$$

It is conjectured by Qiang Fu that the set $\mathfrak{D}_{n,r}$ forms a k-basis for $u_k(n,r)$. We will prove that this conjecture is true when $n=2$ and l' is even.

We have following results:

$$\begin{aligned}\mathbf{k}_i &= \sum_{\overline{\lambda}\in\overline{\Lambda(n,r)}}\varepsilon^{\lambda_i}\mathbf{p}_{\overline{\lambda}}, \quad 1\le i\le n;\\ \begin{bmatrix}\mathbf{k}_i;0\\ j\end{bmatrix} &= \sum_{\overline{\lambda}\in\overline{\Lambda(n,r)}}\begin{bmatrix}\lambda_i\\ j\end{bmatrix}\mathbf{p}_{\overline{\lambda}}, \quad 1\le i\le n;\\ \mathbf{p}_{\overline{\lambda}}\mathbf{p}_{\overline{\mu}} &= \delta_{\overline{\lambda},\overline{\mu}}\mathbf{p}_{\overline{\lambda}}.\end{aligned} \tag{2.1}$$

Lemma 2.3 *Assume that l' is even. There exists a bijection between the set*

$$\Big\{(A,\delta_1,\cdots,\delta_{n-1},\lambda_1,\cdots,\lambda_{n-1})\mid A\in\Gamma^{\pm}(n),\ 0\le\lambda_i\le l-1,\ \delta_i=0,1,\ i=1,2,\cdots,n-1,$$
$$\sigma(A)+\sum_{i=1}^{n-1}(\lambda_i+\delta_i l)\le r\Big\}$$

and the set

$$\{(A,\mathbf{j})\mid A\in\Gamma^{\pm}(n),\ \mathbf{j}\in\mathbb{N}_{l'}^n,\ j_n=0,\ \sigma(A)+\sigma(\mathbf{j})\le r\}.$$

Proof We establish a map between the two sets as

$$(A,\lambda_1,\cdots,\lambda_{n-1},\delta_1,\cdots,\delta_{n-1})\mapsto(A,\lambda_1+\delta_1 l,\cdots,\lambda_{n-1}+\delta_{n-1}l).$$

Then it is easy to know that the map is a bijection.

According to the lemma above, we can establish a bijection between the set $\mathfrak{D}_{n,r}$ and the set $\mathcal{N}_{i_0,k}$ in Lemma 2.2(3).

Similarly, there exists a bijection between the following two sets.

Lemma 2.4 *Assume that l' is even. There exists a bijection between the set*

$$\mathcal{T}_{n,r} := \Big\{(\lambda_1,\cdots,\lambda_{n-1},\delta_1,\cdots,\delta_{n-1})\mid 0\le\lambda_i\le l-1,\ \delta_i=0,1,\ i=1,\cdots,n-1,$$
$$\sum_{i=1}^{n-1}(\lambda_i+\delta_i l)\le r\Big\}$$

and the set

$$\mathcal{X}_{n,r} := \{\mathbf{j}=(j_1,\cdots,j_{n-1},0)\mid \mathbf{j}\in\mathbb{N}_{l'}^n,\ j_n=0,\ \sigma(\mathbf{j})\le r\}.$$

Then we can establish a bijection between the set $\mathfrak{B}_{n,r}$ and the set $\mathcal{X}_{i_0}$ in Lemma 2.1(1).

3 Monomial Base for $u_k(2,r)$

Denote

$$\begin{aligned}\mathcal{T}_{2,r} :=&\{(\lambda_1,\delta_1) \mid 0 \le \lambda_1 \le l-1, \delta_1 = 0,1,\ \lambda_1+\delta_1 l \le r\},\\ \Lambda(2,r) =&\{(0,r),(1,r-1),\cdots,(r-1,1),(r,0)\}.\end{aligned}$$

Proposition 3.1 *Assume that l' is even. Then the set*

$$\mathfrak{B}_{2,r} = \{\mathbf{k}_1^{\delta_1}\left[\begin{smallmatrix}\mathbf{k}_1;0\\ \lambda_1\end{smallmatrix}\right] \mid 0 \le \lambda_1 \le l-1,\ \lambda_1+\delta_1 l \le r,\ \delta_1 = 0,1\}$$

forms a k-basis for $u_k^0(2,r)$.

Proof We shall carry out the proof in three cases.

Case 1 $r < l$.

$$\overline{\Lambda(2,r)} = \{(\overline{0},\overline{r}),(\overline{1},\overline{r-1}),\cdots,(\overline{r-1},\overline{1}),(\overline{r},\overline{0})\}.$$

We fix an order in $\mathcal{Y} = \{\mathbf{p}_{\overline{\lambda}} \mid \overline{\lambda} \in \overline{\Lambda(2,r)}\}$:

$$\mathbf{p}_{(\overline{0},\overline{r})},\ \mathbf{p}_{(\overline{1},\overline{r-1})},\cdots,\ \mathbf{p}_{(\overline{r},\overline{0})}.$$

The elements (λ_1,δ_1) in $\mathcal{T}_{2,r}$ are

$$(0,0),\ (1,0),\ (2,0),\ \cdots,\ (r,0).$$

Hence, we will fix an order in $\mathfrak{B}_{2,r}$ as follows:

$$1,\begin{bmatrix}\mathbf{k}_1;0\\ 1\end{bmatrix},\begin{bmatrix}\mathbf{k}_1;0\\ 2\end{bmatrix},\cdots,\begin{bmatrix}\mathbf{k}_1;0\\ r\end{bmatrix}.$$

According to (2.1), each element $\left[\begin{smallmatrix}\mathbf{k}_1;0\\ \lambda_1\end{smallmatrix}\right]$ can be written as a k-linear combination of $\mathbf{p}_{\overline{\lambda}}$ ($\overline{\lambda} \in \overline{\Lambda(2,r)}$).

Applying the above order, we know that the matrix between vectors consisting of elements in $\mathfrak{B}_{2,r}$ and vectors consisting of elements in $\mathcal{Y}$ is a square matrix according to Lemma 2.4. We denote by this matrix B_r. By Lemma 2.1(2), $\mathcal{Y}$ is a k-basis of $u_k^0(2,r)$, it is sufficient to show that B_r is invertible.

B_r is the following matrix:

$$\begin{pmatrix}1 & 0 & 0 & \cdots & 0 & 0\\ 1 & 1 & 0 & \cdots & 0 & 0\\ 1 & \left[\begin{smallmatrix}2\\1\end{smallmatrix}\right] & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ 1 & \left[\begin{smallmatrix}r-1\\1\end{smallmatrix}\right] & \left[\begin{smallmatrix}r-1\\2\end{smallmatrix}\right] & \cdots & 1 & 0\\ 1 & \left[\begin{smallmatrix}r\\1\end{smallmatrix}\right] & \left[\begin{smallmatrix}r\\2\end{smallmatrix}\right] & \cdots & \left[\begin{smallmatrix}r\\r-1\end{smallmatrix}\right] & 1\end{pmatrix},$$

where the first column vector in B_r corresponds to the element 1 in $\mathfrak{B}_{2,r}$, the second column vector in B_r corresponds to the element $\genfrac{[}{]}{0pt}{}{\mathbf{k}_1;0}{1}$ in $\mathfrak{B}_{2,r}$, the third column vector corresponds to the element $\genfrac{[}{]}{0pt}{}{\mathbf{k}_1;0}{2}$ in $\mathfrak{B}_{2,r}$, $\cdots$, the $(r+1)$-th column vector in B_r corresponds to the element $\genfrac{[}{]}{0pt}{}{\mathbf{k}_1;0}{r}$ in $\mathfrak{B}_{2,r}$.

Apparently, B_r is a lower triangle matrix with diagonal elements 1 and hence, it is invertible.

Case 2 $l \le r \le l' - 1$.

Let $r = l + b$, $0 \le b \le l - 1$. We have

$$\overline{\Lambda(2,r)} = \{(\overline{0}, \overline{r}), (\overline{1}, \overline{r-1}), \cdots, (\overline{r-1}, \overline{1}), (\overline{r}, \overline{0})\}.$$

We also fix an order in $\mathcal{Y} = \{\mathbf{p}_{\overline{\lambda}} \mid \overline{\lambda} \in \overline{\Lambda(2,r)}\}$:

$$\mathbf{p}_{(\overline{0},\overline{r})},\ \mathbf{p}_{(\overline{1},\overline{r-1})}, \cdots,\ \mathbf{p}_{(\overline{r},\overline{0})}.$$

The elements (λ_1, δ_1) in $\mathcal{T}_{2,r}$ are

$$(0,0),\ (1,0),\ (2,0),\ \cdots,\ (l-1,0),\ (0,1),\ (1,1),\ (2,1),\ \cdots,\ (b,1).$$

Therefore, we shall fix an order in $\mathfrak{B}_{2,r}$ as follows:

$$1,\ \begin{bmatrix}\mathbf{k}_1;0\\1\end{bmatrix},\ \begin{bmatrix}\mathbf{k}_1;0\\2\end{bmatrix}, \cdots, \begin{bmatrix}\mathbf{k}_1;0\\l-1\end{bmatrix},\ \mathbf{k}_1,\ \mathbf{k}_1\begin{bmatrix}\mathbf{k}_1;0\\1\end{bmatrix},\ \mathbf{k}_1\begin{bmatrix}\mathbf{k}_1;0\\2\end{bmatrix}, \cdots, \mathbf{k}_1\begin{bmatrix}\mathbf{k}_1;0\\b\end{bmatrix}.$$

Similarly, each element $\mathbf{k}_1^{\delta_1}\genfrac{[}{]}{0pt}{}{\mathbf{k}_1;0}{\lambda_1}$ can be also written as a k-linear combination of $\mathbf{p}_{\overline{\lambda}}$ ($\overline{\lambda} \in \overline{\Lambda(2,r)}$) and we denote the matrix between vectors consisting of elements in $\mathfrak{B}_{2,r}$ and vectors consisting of elements in $\mathcal{Y}$ by B_r. Now we will show that B_r is invertible.

B_r is the following matrix:

$$\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & \cdots & 0\\
1 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \varepsilon & \varepsilon & 0 & \cdots & 0\\
1 & \genfrac{[}{]}{0pt}{}{2}{1} & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & \varepsilon^2 & \varepsilon^2\genfrac{[}{]}{0pt}{}{2}{1} & \varepsilon^2 & \cdots & 0\\
\vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots\\
1 & \genfrac{[}{]}{0pt}{}{b}{1} & \genfrac{[}{]}{0pt}{}{b}{2} & \cdots & 1 & 0 & \cdots & 0 & 0 & \varepsilon^b & \varepsilon^b\genfrac{[}{]}{0pt}{}{b}{1} & \varepsilon^b\genfrac{[}{]}{0pt}{}{b}{2} & \cdots & \varepsilon^b\\
1 & \genfrac{[}{]}{0pt}{}{b+1}{1} & \genfrac{[}{]}{0pt}{}{b+1}{2} & \cdots & \genfrac{[}{]}{0pt}{}{b+1}{b} & 1 & \cdots & 0 & 0 & \varepsilon^{b+1} & \varepsilon^{b+1}\genfrac{[}{]}{0pt}{}{b+1}{1} & \varepsilon^{b+1}\genfrac{[}{]}{0pt}{}{b+1}{2} & \cdots & \varepsilon^{b+1}\genfrac{[}{]}{0pt}{}{b+1}{b}\\
\vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots\\
1 & \genfrac{[}{]}{0pt}{}{l-2}{1} & \genfrac{[}{]}{0pt}{}{l-2}{2} & \cdots & \genfrac{[}{]}{0pt}{}{l-2}{b} & \genfrac{[}{]}{0pt}{}{l-2}{b+1} & \cdots & 1 & 0 & \varepsilon^{l-2} & \varepsilon^{l-2}\genfrac{[}{]}{0pt}{}{l-2}{1} & \varepsilon^{l-2}\genfrac{[}{]}{0pt}{}{l-2}{2} & \cdots & \varepsilon^{l-2}\genfrac{[}{]}{0pt}{}{l-2}{b}\\
1 & \genfrac{[}{]}{0pt}{}{l-1}{1} & \genfrac{[}{]}{0pt}{}{l-1}{2} & \cdots & \genfrac{[}{]}{0pt}{}{l-1}{b} & \genfrac{[}{]}{0pt}{}{l-1}{b+1} & \cdots & \genfrac{[}{]}{0pt}{}{l-1}{l-2} & 1 & \varepsilon^{l-1} & \varepsilon^{l-1}\genfrac{[}{]}{0pt}{}{l-1}{1} & \varepsilon^{l-1}\genfrac{[}{]}{0pt}{}{l-1}{2} & \cdots & \varepsilon^{l-1}\genfrac{[}{]}{0pt}{}{l-1}{b}\\
1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \varepsilon^l & 0 & 0 & \cdots & 0\\
1 & \genfrac{[}{]}{0pt}{}{l+1}{1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \varepsilon^{l+1} & \varepsilon^{l+1}\genfrac{[}{]}{0pt}{}{l+1}{1} & 0 & \cdots & 0\\
1 & \genfrac{[}{]}{0pt}{}{l+2}{1} & \genfrac{[}{]}{0pt}{}{l+2}{2} & \cdots & 0 & 0 & \cdots & 0 & 0 & \varepsilon^{l+2} & \varepsilon^{l+2}\genfrac{[}{]}{0pt}{}{l+2}{1} & \varepsilon^{l+2}\genfrac{[}{]}{0pt}{}{l+2}{2} & \cdots & 0\\
\vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots\\
1 & \genfrac{[}{]}{0pt}{}{l+b}{1} & \genfrac{[}{]}{0pt}{}{l+b}{2} & \cdots & \genfrac{[}{]}{0pt}{}{l+b}{b} & 0 & \cdots & 0 & 0 & \varepsilon^{l+b} & \varepsilon^{l+b}\genfrac{[}{]}{0pt}{}{l+b}{1} & \varepsilon^{l+b}\genfrac{[}{]}{0pt}{}{l+b}{2} & \cdots & \varepsilon^{l+b}\genfrac{[}{]}{0pt}{}{l+b}{b}
\end{pmatrix},$$

where the first column vector in B_r corresponds to the element 1 in $\mathfrak{B}_{2,r}$, the second column vector in B_r corresponds to the element $\genfrac{[}{]}{0pt}{}{\mathbf{k}_1;0}{1}$ in $\mathfrak{B}_{2,r}$, $\cdots$, the l-th column vector corresponds to the element $\left[{\mathbf{k}_1;0 \atop l-1}\right]$ in $\mathfrak{B}_{2,r}$, the $(l+1)$-th column vector corresponds to the element $\mathbf{k}_1$ in $\mathfrak{B}_{2,r}$, the $(l+2)$-th column corresponds to the element $\mathbf{k}_1\left[{\mathbf{k}_1;0 \atop 1}\right]$ in $\mathfrak{B}_{2,r}$, $\cdots$, the $(l+b+1)$-th column vector corresponds to the element $\mathbf{k}_1\left[{\mathbf{k}_1;0 \atop b}\right]$ in $\mathfrak{B}_{2,r}$.

Observe the determinant, there exists only one non-zero element 1 in l-th column which is in the l-th row. There exist two non-zero elements 1 in the $(l-1)$-th column which are in the $(l-1)$-th row and the l-th row. Counting the number of non-zero elements in each column in turn, we know that there are $l-b-1$ non-zero elements 1, $\left[{b+2 \atop b+1}\right]$, $\cdots$, $\left[{l-1 \atop b+1}\right]$ in the $(b+2)$-th column which are in the $(b+2)$-th, $(b+3)$-th, $\cdots$, l-th row, respectively.

Therefore, we can simplify the l-th, $(l-1)$-th, $\cdots$, $(b+2)$-th columns in turn. Then we have

$$|B_r| = \begin{vmatrix}
1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\
1 & 1 & 0 & \cdots & 0 & \varepsilon & \varepsilon & 0 & \cdots & 0 \\
1 & \left[{2 \atop 1}\right] & 1 & \cdots & 0 & \varepsilon^2 & \varepsilon^2\left[{2 \atop 1}\right] & \varepsilon^2 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\
1 & \left[{b-1 \atop 1}\right] & \left[{b-1 \atop 2}\right] & \cdots & 0 & \varepsilon^{b-1} & \varepsilon^{b-1}\left[{b-1 \atop 1}\right] & \varepsilon^{b-1}\left[{b-1 \atop 2}\right] & \cdots & 0 \\
1 & \left[{b \atop 1}\right] & \left[{b \atop 2}\right] & \cdots & 1 & \varepsilon^{b} & \varepsilon^{b}\left[{b \atop 1}\right] & \varepsilon^{b}\left[{b \atop 2}\right] & \cdots & \varepsilon^b \\
1 & 0 & 0 & \cdots & 0 & \varepsilon^{l} & 0 & 0 & \cdots & 0 \\
1 & \left[{l+1 \atop 1}\right] & 0 & \cdots & 0 & \varepsilon^{l+1} & \varepsilon^{l+1}\left[{l+1 \atop 1}\right] & 0 & \cdots & 0 \\
1 & \left[{l+2 \atop 1}\right] & \left[{l+2 \atop 2}\right] & \cdots & 0 & \varepsilon^{l+2} & \varepsilon^{l+2}\left[{l+2 \atop 1}\right] & \varepsilon^{l+2}\left[{l+2 \atop 2}\right] & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\
1 & \left[{l+b-1 \atop 1}\right] & \left[{l+b-1 \atop 2}\right] & \cdots & 0 & \varepsilon^{l+b-1} & \varepsilon^{l+b-1}\left[{l+b-1 \atop 1}\right] & \varepsilon^{l+b-1}\left[{l+b-1 \atop 2}\right] & \cdots & 0 \\
1 & \left[{l+b \atop 1}\right] & \left[{l+b \atop 2}\right] & \cdots & \left[{l+b \atop b}\right] & \varepsilon^{l+b} & \varepsilon^{l+b}\left[{l+b \atop 1}\right] & \varepsilon^{l+b}\left[{l+b \atop 2}\right] & \cdots & \varepsilon^{l+b}\left[{l+b \atop b}\right]
\end{vmatrix}.$$

Next we add -1 multiple of the first row vector to the $(b+2)$-th row vector and obtain

$$|B_r|$$

$$= \begin{vmatrix}
1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\
1 & 1 & 0 & \cdots & 0 & \varepsilon & \varepsilon & 0 & \cdots & 0 \\
1 & \left[{2 \atop 1}\right] & 1 & \cdots & 0 & \varepsilon^2 & \varepsilon^2\left[{2 \atop 1}\right] & \varepsilon^2 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\
1 & \left[{b-1 \atop 1}\right] & \left[{b-1 \atop 2}\right] & \cdots & 0 & \varepsilon^{b-1} & \varepsilon^{b-1}\left[{b-1 \atop 1}\right] & \varepsilon^{b-1}\left[{b-1 \atop 2}\right] & \cdots & 0 \\
1 & \left[{b \atop 1}\right] & \left[{b \atop 2}\right] & \cdots & 1 & \varepsilon^{b} & \varepsilon^{b}\left[{b \atop 1}\right] & \varepsilon^{b}\left[{b \atop 2}\right] & \cdots & \varepsilon^b \\
0 & 0 & 0 & \cdots & 0 & \varepsilon^{l}-1 & 0 & 0 & \cdots & 0 \\
1 & \left[{l+1 \atop 1}\right] & 0 & \cdots & 0 & \varepsilon^{l+1} & \varepsilon^{l+1}\left[{l+1 \atop 1}\right] & 0 & \cdots & 0 \\
1 & \left[{l+2 \atop 1}\right] & \left[{l+2 \atop 2}\right] & \cdots & 0 & \varepsilon^{l+2} & \varepsilon^{l+2}\left[{l+2 \atop 1}\right] & \varepsilon^{l+2}\left[{l+2 \atop 2}\right] & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\
1 & \left[{l+b-1 \atop 1}\right] & \left[{l+b-1 \atop 2}\right] & \cdots & 0 & \varepsilon^{l+b-1} & \varepsilon^{l+b-1}\left[{l+b-1 \atop 1}\right] & \varepsilon^{l+b-1}\left[{l+b-1 \atop 2}\right] & \cdots & 0 \\
1 & \left[{l+b \atop 1}\right] & \left[{l+b \atop 2}\right] & \cdots & \left[{l+b \atop b}\right] & \varepsilon^{l+b} & \varepsilon^{l+b}\left[{l+b \atop 1}\right] & \varepsilon^{l+b}\left[{l+b \atop 2}\right] & \cdots & \varepsilon^{l+b}\left[{l+b \atop b}\right]
\end{vmatrix}.$$

Note that the $(b+2)$-th row vector now has only one non-zero element $\varepsilon^l - 1$. So we simplify the $(b+2)$-th row in $|B_r|$ and we have

$$|B_r| = (\varepsilon^l - 1)\begin{vmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
1 & 1 & 0 & \cdots & 0 & \varepsilon & 0 & \cdots & 0 \\
1 & \left[{2 \atop 1}\right] & 1 & \cdots & 0 & \varepsilon^2\left[{2 \atop 1}\right] & \varepsilon^2 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
1 & \left[{b \atop 1}\right] & \left[{b \atop 2}\right] & \cdots & 1 & \varepsilon^b\left[{b \atop 1}\right] & \varepsilon^b\left[{b \atop 2}\right] & \cdots & \varepsilon^b \\
1 & \left[{l+1 \atop 1}\right] & 0 & \cdots & 0 & \varepsilon^{l+1}\left[{l+1 \atop 1}\right] & 0 & \cdots & 0 \\
1 & \left[{l+2 \atop 1}\right] & \left[{l+2 \atop 2}\right] & \cdots & 0 & \varepsilon^{l+2}\left[{l+2 \atop 1}\right] & \varepsilon^{l+2}\left[{l+2 \atop 2}\right] & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
1 & \left[{l+b \atop 1}\right] & \left[{l+b \atop 2}\right] & \cdots & \left[{l+b \atop b}\right] & \varepsilon^{l+b}\left[{l+b \atop 1}\right] & \varepsilon^{l+b}\left[{l+b \atop 2}\right] & \cdots & \varepsilon^{l+b}\left[{l+b \atop b}\right]
\end{vmatrix}.$$

It is a determinant of order $2b+1$.

Now, the first row vector has only one non-zero element 1. So we can simplify $|B_r|$ with respect to the first row and we have

$$|B_r| = (\varepsilon^l - 1)\begin{vmatrix}
1 & 0 & \cdots & 0 & 0 & \varepsilon & 0 & \cdots & 0 \\
\left[{2 \atop 1}\right] & 1 & \cdots & 0 & 0 & \varepsilon^2\left[{2 \atop 1}\right] & \varepsilon^2 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\
\left[{b-1 \atop 1}\right] & \left[{b-1 \atop 2}\right] & \cdots & 1 & 0 & \varepsilon^{b-1}\left[{b-1 \atop 1}\right] & \varepsilon^{b-1}\left[{b-1 \atop 2}\right] & \cdots & 0 \\
\left[{b \atop 1}\right] & \left[{b \atop 2}\right] & \cdots & \left[{b \atop b-1}\right] & 1 & \varepsilon^b\left[{b \atop 1}\right] & \varepsilon^b\left[{b \atop 2}\right] & \cdots & \varepsilon^b \\
\left[{l+1 \atop 1}\right] & 0 & \cdots & 0 & 0 & \varepsilon^{l+1}\left[{l+1 \atop 1}\right] & 0 & \cdots & 0 \\
\left[{l+2 \atop 1}\right] & \left[{l+2 \atop 2}\right] & \cdots & 0 & 0 & \varepsilon^{l+2}\left[{l+2 \atop 1}\right] & \varepsilon^{l+2}\left[{l+2 \atop 2}\right] & \cdots & 0 \\
\vdots & \vdots & & \vdots & 0 & \vdots & \vdots & & \vdots \\
\left[{l+b \atop 1}\right] & \left[{l+b \atop 2}\right] & \cdots & \left[{l+b \atop b-1}\right] & \left[{l+b \atop b}\right] & \varepsilon^{l+b}\left[{l+b \atop 1}\right] & \varepsilon^{l+b}\left[{l+b \atop 2}\right] & \cdots & \varepsilon^{l+b}\left[{l+b \atop b}\right]
\end{vmatrix}.$$

And it is a determinant of order $2b+1$.

Continuing in this way we finally get that

$$|B_r| = \prod_{j=1}^{b}(\varepsilon^{l+j} - \varepsilon^j)\left[{l+j \atop j}\right] \cdot \begin{vmatrix} 1 & 1 \\ 1 & \varepsilon^l \end{vmatrix} = \prod_{j=0}^{b}(\varepsilon^{l+j} - \varepsilon^j)\left[{l+j \atop j}\right].$$

In particular, in the case of $r = l'-1$, $B_{l'-1}$ is the following matrix:

$$\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
1 & 1 & 0 & \cdots & 0 & 0 & \varepsilon & \varepsilon & 0 & \cdots & 0 & 0 \\
1 & \left[{2 \atop 1}\right] & 1 & \cdots & 0 & 0 & \varepsilon^2 & \varepsilon^2\left[{2 \atop 1}\right] & \varepsilon^2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
1 & \left[{l-2 \atop 1}\right] & \left[{l-2 \atop 2}\right] & \cdots & 1 & 0 & \varepsilon^{l-2} & \varepsilon^{l-2}\left[{l-2 \atop 1}\right] & \varepsilon^{l-2}\left[{l-2 \atop 2}\right] & \cdots & \varepsilon^{l-2} & 0 \\
1 & \left[{l-1 \atop 1}\right] & \left[{l-1 \atop 2}\right] & \cdots & \left[{l-1 \atop l-2}\right] & 1 & \varepsilon^{l-1} & \varepsilon^{l-1}\left[{l-1 \atop 1}\right] & \varepsilon^{l-1}\left[{l-1 \atop 2}\right] & \cdots & \varepsilon^{l-1}\left[{l-1 \atop l-2}\right] & \varepsilon^{l-1} \\
1 & 0 & 0 & \cdots & 0 & 0 & \varepsilon^{l} & 0 & 0 & \cdots & 0 & 0 \\
1 & \left[{l+1 \atop 1}\right] & 0 & \cdots & 0 & 0 & \varepsilon^{l+1} & \varepsilon^{l+1}\left[{l+1 \atop 1}\right] & 0 & \cdots & 0 & 0 \\
1 & \left[{l+2 \atop 1}\right] & \left[{l+2 \atop 2}\right] & \cdots & 0 & 0 & \varepsilon^{l+2} & \varepsilon^{l+2}\left[{l+2 \atop 1}\right] & \varepsilon^{l+2}\left[{l+2 \atop 2}\right] & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
1 & \left[{l'-2 \atop 1}\right] & \left[{l'-2 \atop 2}\right] & \cdots & \left[{l'-2 \atop l-2}\right] & 0 & \varepsilon^{l'-2} & \varepsilon^{l'-2}\left[{l'-2 \atop 1}\right] & \varepsilon^{l'-2}\left[{l'-2 \atop 2}\right] & \cdots & \varepsilon^{l'-2}\left[{l'-2 \atop l-2}\right] & 0 \\
1 & \left[{l'-1 \atop 1}\right] & \left[{l'-1 \atop 2}\right] & \cdots & \left[{l'-1 \atop l-2}\right] & \left[{l'-1 \atop l-1}\right] & \varepsilon^{l'-1} & \varepsilon^{l'-1}\left[{l'-1 \atop 1}\right] & \varepsilon^{l'-1}\left[{l'-1 \atop 2}\right] & \cdots & \varepsilon^{l'-1}\left[{l'-1 \atop l-2}\right] & \varepsilon^{l'-1}\left[{l'-1 \atop l-1}\right]
\end{pmatrix}$$

and

$$|B_{l'-1}| = \prod_{j=1}^{l-1}(\varepsilon^{l+j} - \varepsilon^j)\left[{l+j \atop j}\right] \cdot \begin{vmatrix} 1 & 1 \\ 1 & \varepsilon^l \end{vmatrix} = \prod_{j=0}^{l-1}(\varepsilon^{l+j} - \varepsilon^j)\left[{l+j \atop j}\right].$$

Apparently, B_r is invertible.

Case 3 $r > l' - 1$.

Let $r = l' + b$, $b \geq 0$. We have that

$$\overline{\Lambda(2,r)} = \{(\overline{0}, \overline{r}), (\overline{1}, \overline{r-1}), \cdots, (\overline{l'-2}, \overline{b+2}), (\overline{l'-1}, \overline{b+1})\}.$$

And we choose an order in $\mathcal{Y} = \{ \mathbf{p}_{\overline{\lambda}} \mid \overline{\lambda} \in \overline{\Lambda(2,r)} \}$ as follows:

$$\mathbf{p}_{(\overline{0},\overline{r})},\ \mathbf{p}_{(\overline{1},\overline{r-1})}, \cdots,\ \mathbf{p}_{(\overline{l'-1},\overline{b+1})}.$$

The elements (λ_1, δ_1) in $\mathcal{T}_{2,r}$ are

$$(0,0),\ (1,0),\ (2,0),\ \cdots,\ (l-1,0),\ (0,1),\ (1,1),\ (2,1),\ \cdots,\ (l-1,1).$$

Similarly, we fix an order in $\mathfrak{B}_{2,r}$ as

$$1,\ \begin{bmatrix}\mathbf{k}_1;0\\1\end{bmatrix},\ \begin{bmatrix}\mathbf{k}_1;0\\2\end{bmatrix}, \cdots, \begin{bmatrix}\mathbf{k}_1;0\\l-1\end{bmatrix},\ \mathbf{k}_1,\ \mathbf{k}_1\begin{bmatrix}\mathbf{k}_1;0\\1\end{bmatrix},\ \mathbf{k}_1\begin{bmatrix}\mathbf{k}_1;0\\2\end{bmatrix}, \cdots, \mathbf{k}_1\begin{bmatrix}\mathbf{k}_1;0\\l-1\end{bmatrix},$$

and denote the matrix between vectors consisting of elements in $\mathfrak{B}_{2,r}$ and vectors consisting of elements in $\mathcal{Y}$ by B_r.

Since in the case $r \geq l' - 1$ we have $|\overline{\Lambda(2,r)}_{l'}| = |\overline{\Lambda(2,r+1)}_{l'}|$, it is easy to know that

$$B_{l'+j} = B_{l'+j+1},$$

where $j \geq -1$.

Therefore, we have $|B_r| \neq 0$ for any $r > 0$, that is, B_r is invertible. The assertion follows.

Now we give our main result in this paper which shows that the conjecture given above by Fu is true in the case $n = 2$.

Theorem 3.1 *Assume that l' is even. Then the set*

$$\mathfrak{D}_{2,r} = \Big\{ \mathbf{e}^{(A^+)}\mathbf{k}_1^{\delta_1}\begin{bmatrix}\mathbf{k}_1;0\\ \lambda_1\end{bmatrix}\mathbf{f}^{(A^-)} \mid A \in \Gamma^{\pm}(2),\ 0 \leq \lambda_1 \leq l-1,\ \delta_1 = 0,1,\ \sigma(A) + \lambda_1 + \delta_1 l \leq r \Big\}$$

forms a k-basis for $u_k(2,r)$.

Proof Obviously, $\sigma_1(A) = 0$ for any $A \in \Gamma^{\pm}(2)$. By Theorem 2.2, we have

$$\begin{aligned}
\mathbf{e}^{(A^+)}\mathbf{k}_1^{\delta_1}\begin{bmatrix}\mathbf{k}_1;0\\ \lambda_1\end{bmatrix}\mathbf{f}^{(A^-)} &= \mathbf{e}^{(A^+)}\mathbf{k}_1^{\delta_1}\begin{bmatrix}\mathbf{k}_1;0\\ \lambda_1\end{bmatrix} \cdot 1 \cdot \mathbf{f}^{(A^-)}\\
&= \mathbf{e}^{(A^+)}\mathbf{k}_1^{\delta_1}\begin{bmatrix}\mathbf{k}_1;0\\ \lambda_1\end{bmatrix}\Big(\sum_{\overline{\mu}\in\overline{\Lambda(2,r)}} \mathbf{k}_{\overline{\mu}}\Big)\mathbf{f}^{(A^-)}\\
&= \mathbf{e}^{(A^+)}\sum_{\overline{\mu}\in\overline{\Lambda(2,r)}} \varepsilon^{\mu_1\delta_1}\begin{bmatrix}\mu_1\\ \lambda_1\end{bmatrix}\mathbf{p}_{\overline{\mu}}\mathbf{f}^{(A^-)}\\
&= \sum_{\overline{\mu}\in\overline{\Lambda(2,r)}} \varepsilon^{\mu_1\delta_1}\begin{bmatrix}\mu_1\\ \lambda_1\end{bmatrix}\mathbf{e}^{(A^+)}\mathbf{p}_{\overline{\mu}}\mathbf{f}^{(A^-)}.
\end{aligned}$$

Then by [1], [19, Theorem 5.1(3)], [10, Theorem 5.5, Corollary 5.6] and Lemma 2.4, we obtain that

$$\begin{aligned}
\mathbf{e}^{(A^+)}\mathbf{k}_1^{\delta_1}\begin{bmatrix}\mathbf{k}_1;0\\ \lambda_1\end{bmatrix}\mathbf{f}^{(A^-)} &= \sum_{\overline{\mu}\in\overline{\Lambda(2,r)}} \varepsilon^{\mu_1\delta_1}\begin{bmatrix}\mu_1\\ \lambda_1\end{bmatrix}([\![A+\mathrm{diag}(\overline{\mu}-\overline{\sigma(A)}),r]\!]+g_{\overline{\mu}})\\
&= \sum_{\substack{\overline{\mu}\in\overline{\Lambda(2,r)}\\ \overline{\mu}-\overline{\sigma(A)}\in\overline{\Lambda(2,r-\sigma(A))}}} \varepsilon^{\mu_1\delta_1}\begin{bmatrix}\mu_1\\ \lambda_1\end{bmatrix}([\![A+\mathrm{diag}(\overline{\mu}-\overline{\sigma(A)}),r]\!]+g_{\overline{\mu}})\\
&= \sum_{\overline{\nu}\in\overline{\Lambda(2,r-\sigma(A))}} \varepsilon^{(\nu_1+\sigma_1(A))\delta_1}\begin{bmatrix}\nu_1+\sigma_1(A)\\ \lambda_1\end{bmatrix}([\![A+\mathrm{diag}(\overline{\nu}),r]\!]+g_{\overline{\nu}+\overline{\sigma(A)}})\\
&= \sum_{\overline{\nu}\in\overline{\Lambda(2,r-\sigma(A))}} \varepsilon^{\nu_1\delta_1}\begin{bmatrix}\nu_1\\ \lambda_1\end{bmatrix}([\![A+\mathrm{diag}(\overline{\nu}),r]\!]+g_{\overline{\nu}+\overline{\sigma(A)}})\\
&= \sum_{\overline{\nu}\in\overline{\Lambda(2,r-\sigma(A))}} \varepsilon^{\nu_1\delta_1}\begin{bmatrix}\nu_1\\ \lambda_1\end{bmatrix}([\![A+\mathrm{diag}(\overline{\nu}),r]\!])+h,
\end{aligned}$$

where $\nu=(\nu_1,\nu_2)$, $\nu_i:=\mu_i-\sigma_i(A)$ for $\overline{\mu}=\overline{\nu}$, $\overline{\mu}\in\overline{\Lambda(2,r)}$. In particular, $\nu_1=\mu_1-\sigma_1(A)=\mu_1$. For $A\in\Gamma^{\pm}(2)$,

$$\begin{aligned}
g_{\overline{\mu}} &= \sum_{B\prec A} f_B[\![B+\mathrm{diag}(\overline{\mu}-\overline{\sigma(B)}),r]\!],\\
h &= \sum_{\overline{\nu}\in\overline{\Lambda(2,r-\sigma(A))}} \varepsilon^{\nu_1\delta_1}[{}^{\nu_1}_{\lambda_1}]g_{\overline{\nu}+\overline{\sigma(A)}}
\end{aligned}$$

for some $f_B\in k$.

It is easy to know that the matrix $\left(\varepsilon^{\nu_1\delta_1}[{}^{\nu_1}_{\lambda_1}]\right)_{\lambda,\nu}$ is invertible. Thus, the elements in $\mathfrak{D}_{2,r}$ are k-linearly independent. And the assertion holds.

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