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Betti Numbers of Locally Standard 2-Torus Manifolds*

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Abstract Let M^n be a smooth closed n-manifold with a locally standard $(\mathbb{Z}_2)^n$-action. This paper deals with the relationship among the mod 2 Betti numbers of M^n, the mod 2 Betti numbers and the h-vector of the orbit space of the action.

Keywords Locally standard 2-torus action, Manifold with corners, Betti number
2000 MR Subject Classification 57S17

1 Introduction

Locally standard $(\mathbb{Z}_2)^n$-actions on smooth closed n-manifolds belong to a class of particularly nicely behaving actions, introduced by Davis and Januszkiewicz [4]. Here a smooth connected closed n-manifold with a locally standard $(\mathbb{Z}_2)^n$-action is called a locally standard 2-torus manifold. Generally, the orbit space Q^n of a locally standard 2-torus manifold M^n is an n-dimensional nice manifold with corners. If Q^n is a simple convex polytope P^n, then M^n is called a small cover by Davis and Januszkiewicz. There are strong links of small covers with combinatorics of polytopes with the following two key points (see [4]):

(1) Each small cover $\pi : M^n \to P^n$ can be recovered from P^n with an associated characteristic function;

(2) The algebraic topology of a small cover $\pi : M^n \to P^n$, such as (equivariant) cohomology and mod 2 Betti numbers etc., can be explicitly expressed in terms of the combinatorics of P^n. For example, the mod 2 Betti numbers $\mathbf{b}(M^n) = (b_0(M^n), b_1(M^n), \cdots, b_n(M^n))$ of M^n agree with the h-vector $(h_0(P^n), h_1(P^n), \cdots, h_n(P^n))$ of P^n.

Given a locally standard 2-torus manifold $\pi : M^n \to Q^n$, generally Q^n is not a simple convex polytope, and it may be not contractible. We know from [6] that $\partial Q^n = \varnothing$ if and only if the action on M^n is free, so M^n can actually be regarded as a principal $(\mathbb{Z}_2)^n$-bundle over Q^n in this case. However, if $\partial Q^n \neq \varnothing$, then it admits a simplicial poset structure, so we can define an h-vector $\mathbf{h}(Q^n)$ on Q^n. In addition, like the case of small covers, Q^n also admits a characteristic function λ. As shown in [7], the pair (Q^n, λ) only provides information on the set of non-free orbits in M^n, so generally it is not enough to recover M^n (of course, it is enough if Q^n is contractible). Indeed, we also need another data (which provides information on the set of free orbits in M), i.e., a principal $(\mathbb{Z}_2)^n$-bundle ξ over Q^n determined uniquely by M^n

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up to isomorphism. Then it was shown in [7] that the orbit space Q^n with two data λ and ξ can reproduce M^n up to equivariant homeomorphism, denoted by $M(Q;\lambda,\xi)$. Generally the algebraic topology of a locally standard 2-torus manifold $\pi : M^n \to Q^n$ with $\partial Q^n \neq \varnothing$ is far from known except for the formula of the Euler characteristic of M^n in terms of Q^n (see [7]). The purpose of this paper is to consider the following question.

Question 1.1 Let $\pi : M^n \to Q^n$ be a locally standard 2-torus manifold. How do we read the mod 2 Betti numbers $\mathbf{b}(M^n)$ from Q^n?

We give an answer of Question 1.1 in the case where ∂Q^n is the boundary of a simple convex polytope. Our result is stated as follows.

Theorem 1.1 *Suppose that $\pi : M^n \to Q^n$ is a locally standard 2-torus manifold such that ∂Q^n is the boundary of a simple convex polytope P^n. Let $\xi = (E, p, Q)$ be the associated principal $(\mathbb{Z}_2)^n$-bundle over Q. Then for $n > 2$,*

$$b_i(M) = \begin{cases} 1, & \text{if } i = 0, n, \\ b_1(E) + h_1(Q) + 2^n - b_0(E), & \text{if } i = 1, \\ b_i(E) + h_i(Q), & \text{if } 1 < i < n, \end{cases}$$

and for $n = 2$,

$$b_i(M) = \begin{cases} 1, & \text{if } i = 0, 2, \\ 4b_1(Q) + h_1(Q), & \text{if } i = 1. \end{cases}$$

The arrangement of this paper is as follows. In Section 2, we introduce the concept of locally standard 2-torus action and manifold with corners, and show the reproducing process of locally standard 2-torus manifolds. In Section 3, when the boundary of the orbit space Q is the boundary of a simple convex polytope, we obtain the relationship among the Betti numbers of locally standard 2-torus manifold, the Betti numbers of the orbit space and the h-vector of the orbit space, which gives the proof of Theorem 1.1.

2 Locally Standard 2-Torus Manifolds

2.1 Manifold with corners

Let $\mathbb{R}^n_+ = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_i \geqslant 0, i = 1, \cdots, n\}$. For any $x = (x_1, \cdots, x_n) \in \mathbb{R}^n_+$, its codimension $c(x)$ is the number of x_i which are equal to 0.

Definition 2.1 (see [3]) *An n-manifold Q^n with corners is a Hausdorff space together with a maximal atlas of local charts onto open subsets of $\mathbb{R}^n_+$ so that the overlap maps are homeomorphisms (diffeomorphisms) which preserve codimension.*

Given a manifold Q^n with corners. For any $q \in Q^n$, let (U, φ) be a local chart of q. Then, the codimension $c(q)$ of q is defined as $c(\varphi(q))$. This is well defined because the overlap maps preserve codimension. An open face of Q^n of codimension l is a connected component of the inverse image $c^{-l}(l)$. A face is the closure of an open face. Specially, an open face of codimension one is called an open facet, and its closure is called a facet. For any $q \in Q^n$, let $\Sigma(q)$ be the set of facets which contain q. A manifold Q^n with corners is nice if $\mathrm{Card}(\Sigma(q)) = c(q)$ for any $q \in Q^n$.

Example 2.1 Any n-dimensional simple convex polytope is a nice manifold with corners.

Remark 2.1 It should be pointed out that the notion of a manifold with corners is also a natural generalization of the notions of ordinary manifolds with or without boundary. Actually, given a manifold Q^n with corners, it is easy to see that the codimension of each point in Q^n is zero if and only if the boundary of Q^n is empty. Furthermore, Q^n is a manifold with boundary if and only if there is at least one point q in Q^n such that $c(q) > 0$.

2.2 Locally standard 2-torus manifolds

The standard action of $(\mathbb{Z}_2)^n$ on $\mathbb{R}^n$ is defined by

$$((g_1, \cdots, g_n), (x_1, \cdots, x_n)) \longmapsto ((-1)^{g_1} x_1, \cdots, (-1)^{g_n} x_n).$$

Its orbit space is a positive cone $\mathbb{R}^n_+ = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_i \geqslant 0, i = 1, \cdots, n\}$. The action of $(\mathbb{Z}_2)^n$ on an n-dimensional smooth closed manifold M^n is locally standard if the action locally looks like the standard action of $(\mathbb{Z}_2)^n$ on $\mathbb{R}^n$. More precisely, for each point x in M^n, there is a $(\mathbb{Z}_2)^n$-invariant neighborhood V_x of x such that V_x is weakly equivariantly homeomorphic to a $(\mathbb{Z}_2)^n$-invariant open set W in the standard representation $\mathbb{R}^n$ (i.e., there is a homeomorphism $f : V_x \to W$ and an automorphism $\theta : (\mathbb{Z}_2)^n \to (\mathbb{Z}_2)^n$ such that $f(gv) = \theta(g)f(v)$ for $v \in V_x$).

An n-dimensional 2-torus manifold is a connected closed smooth manifold of dimension n with an effective smooth action of $(\mathbb{Z}_2)^n$. Generally, a 2-torus manifold may not be locally standard. We say that M is a locally standard 2-torus manifold if it is a 2-torus manifold and admits a locally standard action.

Example 2.2 The canonical $(\mathbb{Z}_2)^n$-action on RP^n, defined by

$$((g_1, \cdots, g_n), [x_0, x_1, \cdots, x_n]) = [x_0, (-1)^{g_1} x_1, \cdots, (-1)^{g_n} x_n],$$

is a locally standard 2-torus manifold.

Lemma 2.1 (see [3]) *Let $\pi : M \to Q$ be a locally standard 2-torus manifold. Then the orbit space Q is a nice manifold with corners, and for any $x \in M$, the rank of isotropy group G_x equals $c(\pi(x))$.*

Now let Q be a nice n-manifold with corners with $\partial Q \neq \varnothing$. Then it is easy to see that all faces of Q form a simplicial poset with ∂Q as the minimum element with respect to inverse inclusion. Following [4], let f_i denote the numbers of faces of codimension $i+1$ in Q. Define a polynomial $\Psi_Q(t)$ of degree n by

$$\Psi_Q(t) = (t-1)^n + \sum_{i=0}^{n-1} f_i (t-1)^{n-i-1}$$

and let h_i denote the coefficients of the term t^{n-i} in $\Psi_Q(t)$, where $0 \leqslant i \leqslant n$. Then $(h_0, h_1, \cdots, h_n)$ is called the h-vector of Q, denoted by $\mathbf{h}(Q)$.

A locally standard 2-torus manifold is called a small cover if its orbit space is a simple convex polytope. Davis and Januszkiewicz showed in [4] the following theorem.

Theorem 2.1 (Davis–Januszkiewicz) *Let $\pi : M \to P$ be a small cover. Then*

$$\mathbf{b}(M) = \mathbf{h}(P),$$

where $\mathbf{b}(M) = (b_0, b_1, \cdots, b_n)$ is the vector formed by all mod 2 Betti numbers of M.

2.3 Reconstruction of locally standard 2-torus manifolds

Suppose that $\pi : M \to Q$ is a locally standard 2-torus manifold.

If Q is a simple convex polytope P, then M is a small cover. In this case, Davis and Januszkiewicz showed in [4] that there exists a characteristic function λ from all facets of P to $(\mathbb{Z}_2)^n$, satisfying that whenever some facets $F_{i_1}, \cdots, F_{i_l}$ have a nonempty intersection, $\lambda(F_{i_1}), \cdots, \lambda(F_{i_l})$ are linearly independent. Furthermore, Davis and Januszkiewicz used λ to blow down the product bundle $(\mathbb{Z}_2)^n \times P$ to recover M.

In the setting of locally standard 2-torus manifolds, if the boundary of Q is nonempty, as shown in [7], an analogous reconstruction as above can still be carried out well, but two data for Q will be needed. One data is a characteristic function on Q, and the other data is a principal $(\mathbb{Z}_2)^n$-bundle over Q. Actually, if the boundary of Q is nonempty, since Q is nice, then ∂Q is the union of its all facets, and each facet of Q corresponds to a nonzero element $v \in (\mathbb{Z}_2)^n$ such that the inverse image of this facet is fixed by the rank-one subgroup determined by v. Thus there is a characteristic function

$$\lambda : \mathcal{F}(Q) \to (\mathbb{Z}_2)^n$$

satisfying the condition that whenever the intersection $\bigcap\limits_{j=1}^{l} F_{i_j}$ is nonempty, all elements of $\{\lambda(F_{i_j}) \mid j = 1, \cdots, l\}$ are linearly independent, where $\mathcal{F}(Q)$ denotes the set of all facets of Q. Note that for any k-face F of Q, since Q is nice, there exist $n-k$ facets $F_{i_1}, \cdots, F_{i_{n-k}}$ such that F is a component of $\bigcap\limits_{j=1}^{n-k} F_{i_j}$. Furthermore, $\lambda(F_{i_1}), \cdots, \lambda(F_{i_{n-k}})$ determine a rank $n-k$ subgroup of $(\mathbb{Z}_2)^n$, denoted by G_F. Besides, a principal $(\mathbb{Z}_2)^n$-bundle $\xi = (E, p, Q)$ over Q can be produced from $\pi : M \to Q$ in the following way: We take a small invariant open tubular neighborhood of $\pi^{-1}(F)$ in M for every $F \in \mathcal{F}(Q)$ and remove all such neighborhoods from M, where p is the projection. This gives a principal $(\mathbb{Z}_2)^n$-bundle over Q, which is unique up to isomorphism. Now we can reproduce a locally standard 2-torus manifold from these two data. First, define an equivalence relation $\sim$ on E as follows: For $u_1, u_2 \in E$,

$$u_1 \sim u_2 \Leftrightarrow p(u_1) = p(u_2) \text{ and } u_1 = u_2 g \text{ for some } g \in G_F,$$

where F is the face of Q containing $p(u_1) = p(u_2)$ in its relative interior and G_F is the subgroup of $(\mathbb{Z}_2)^n$ determined by F. Then the quotient space $E/\sim$, denoted by $M(Q, \xi, \lambda)$, naturally inherits the $(\mathbb{Z}_2)^n$-action from E. It was shown in [7] the following proposition.

Proposition 2.1 *Let $\pi : M \to Q$ with $\partial Q \neq \varnothing$ be an n-dimensional locally standard 2-torus manifold with ξ as the associated principal $(\mathbb{Z}_2)^n$-bundle and λ as the characteristic function. Then there is an equivariant homeomorphism from $M(Q, \xi, \lambda)$ to M which covers the identity on Q.*

Remark 2.2 Given a nice manifold with corners Q of dimension n, it is well-known that the isomorphism classes of all principal $(\mathbb{Z}_2)^n$-bundles over Q bijectively correspond to all elements of $H^1(Q; (\mathbb{Z}_2)^n)$. However, generally Q may not admit any characteristic function (see [4]). If Q admits a characteristic function, then in the above way we can construct all possible locally standard 2-torus manifolds over Q as

$$\{M(Q, \xi, \lambda) \mid \xi \in \mathcal{P}(Q), \lambda \in \Lambda(Q)\},$$

where $\mathcal{P}(Q)$ denotes the set of all principal $(\mathbb{Z}_2)^n$-bundles over Q and $\Lambda(Q)$ consists of all characteristic functions on Q. In particular, if Q is a simple convex polytope P, then the set

$$\{M(P, \lambda) \mid \lambda \in \Lambda(P)\}$$

consists of all possible small covers over P.

3 Betti Numbers of Locally Standard 2-Torus Manifold

Throughout the following, assume that Q is a connected nice n-manifold with corners such that ∂Q is the boundary of a simple convex polytope P, and Q (i.e., P) admits a characteristic function λ. Choose a principal $(\mathbb{Z}_2)^n$-bundle $\xi = (E, p, Q)$ over Q. Based upon the reconstruction of locally standard 2-torus manifolds, we can obtain a locally standard 2-torus manifold $\pi_Q : M(Q, \xi, \lambda) \to Q$ and a small cover $\pi_P : M(P, \lambda) \to P$. If $n = 1$, then Q will be a 1-simplex, so that $M(Q, \xi, \lambda)$ is exactly a circle with a refection. Thus, we assume that $n > 1$ in the following discussion.

Lemma 3.1 *Let $N = \pi_P^{-1}(\partial P)$ (so N is an $(n-1)$-dimensional CW complex). Then the mod 2 Betti numbers of N are*

$$\mathbf{b}(N) = (h_0(P), h_1(P), \cdots, h_{n-1}(P) + 2^n - 1).$$

Proof The simple convex polytope P has a natural CW complex structure such that every open face of P of dimension i is an i-cell. This induces the cell decomposition of quotient space $M(P, \lambda)$. Then we have the cellular chain complex with $\mathbb{Z}_2$ coefficients of $M(P, \lambda)$

$$0 \to C_n(M(P, \lambda)) \to C_{n-1}(M(P, \lambda)) \to \cdots \to C_0(M(P, \lambda) \to 0,$$

such that $\dim_{\mathbb{Z}_2}(C_i(M(P, \lambda))) = 2^i f_{n-i-1}(P)$, where $f_{n-i-1}(P)$ denotes the number of faces of dimension i in P. Obviously, N is the $(n-1)$-skeleton of $M(P, \lambda)$. So it has the following cellular chain complex with $\mathbb{Z}_2$ coefficients:

$$0 \to C_{n-1}(M(P, \lambda)) \to C_{n-2}(M(P, \lambda)) \to \cdots \to C_0(M(P, \lambda)) \to 0.$$

Then the required result follows from the facts that $\dim_{\mathbb{Z}_2}(C_n(M(P, \lambda))) = 2^n$ and $\mathbf{b}(M(P, \lambda)) = \mathbf{h}(P)$.

Lemma 3.2 *The total space E of $\xi = (E, p, Q)$ must still be a nice manifold with corners such that if $n > 2$, then its boundary ∂E is a disjoint union of 2^n copies of ∂Q; and if $n = 2$, then the ∂E is a disjoint union of 4 copies of ∂Q, or a disjoint union of 2 connected sums $\partial Q \sharp \partial Q$, or the connected sum of 4 copies of ∂Q.*

Proof It is obvious that E is a nice manifold with corners. Consider the inverse image $p^{-1}(\partial Q)$, it is easy to see that $p^{-1}(\partial Q) = \partial E$ and it is still a principal $(\mathbb{Z}_2)^n$-bundle. Since ∂Q is the boundary of P, we have that

$$H^1(\partial Q; (\mathbb{Z}_2)^n) = \begin{cases} 0, & \text{if } n > 2, \\ (\mathbb{Z}_2)^2, & \text{if } n = 2. \end{cases}$$

This implies that if $n > 2$, then $p^{-1}(\partial Q) \to \partial Q$ must be a trivial principal $(\mathbb{Z}_2)^n$-bundle. So $\partial E = p^{-1}(\partial Q) = (\mathbb{Z}_2)^n \times \partial Q$. If $n = 2$, since $H^1(\partial Q; (\mathbb{Z}_2)^2) = (\mathbb{Z}_2)^2$ is nontrivial, we have that $p^{-1}(\partial Q) \to \partial Q$ may not be trivial. Since $\partial Q = \partial P$ is the boundary of a polygon, an easy argument can induce the required result in this case.

Theorem 3.1 *For $n > 2$,*

$$b_i(M(Q, \xi, \lambda)) = \begin{cases} 1, & \textit{if } i = 0, n, \\ b_1(E) + h_1(Q) + 2^n - b_0(E), & \textit{if } i = 1, \\ b_i(E) + h_i(Q), & \textit{if } 1 < i < n, \end{cases}$$

and for $n = 2$,

$$b_i(M) = \begin{cases} 1, & \textit{if } i = 0, 2, \\ 4b_1(Q) + h_1(Q), & \textit{if } i = 1. \end{cases}$$

Proof We know from [1] that ∂Q has a collar in Q, i.e., there is an embedding

$$\varphi : [0, 1) \times \partial Q \to Q,$$

such that $\varphi(0, q) = q$ for any $q \in \partial Q$. Set $Q' = \varphi(\{\frac{1}{2}\} \times \partial Q)$, $Q_1 = \varphi([0, \frac{1}{2}] \times \partial Q)$ and $Q_2 = \overline{M/Q_1}$. Obviously, Q' is homeomorphic to S^{n-1}, $Q_1 \cap Q_2 = Q'$ and $Q_1 \cup Q_2 = Q$. It is easy to check that $(\pi_Q^{-1}(Q_1), \pi_Q^{-1}(Q_2))$ is an excisive couple of $M(Q, \xi, \lambda) = \pi_Q^{-1}(Q_1) \cup \pi_Q^{-1}(Q_2)$. Now let us look at the following Mayer-Vietoris sequence (with $\mathbb{Z}_2$ coefficients) of the couple $(\pi_Q^{-1}(Q_1), \pi_Q^{-1}(Q_2))$:

$$\begin{gathered} 0 \to H_n(\pi_Q^{-1}(Q')) \to H_n(\pi_Q^{-1}(Q_1)) \oplus H_n(\pi_Q^{-1}(Q_2)) \to H_n(M(Q, \xi, \lambda)) \to \\ \cdots \\ \to H_i(\pi_Q^{-1}(Q')) \to H_i(\pi_Q^{-1}(Q_1)) \oplus H_i(\pi_Q^{-1}(Q_2)) \to H_i(M(Q, \xi, \lambda)) \to \\ \cdots \\ \to H_0(\pi_Q^{-1}(Q')) \to H_0(\pi_Q^{-1}(Q_1)) \oplus H_0(\pi_Q^{-1}(Q_2)) \to H_0(M(Q, \xi, \lambda)) \to 0. \end{gathered}$$

Claim A If $n > 2$, then the mod 2 Betti numbers

$$b_i(\pi_Q^{-1}(Q')) = \begin{cases} 2^n, & \text{if } i = 0, n-1, \\ 0, & \text{otherwise.} \end{cases}$$

Since Q' is in the interior of Q, we see from the construction of $M(Q, \xi, \lambda)$ that $\pi_Q^{-1}(Q')$ is a principal $(\mathbb{Z}_2)^n$-bundle over Q'. Since Q' is homeomorphic to S^{n-1} and $n > 2$, similarly to the argument of Lemma 3.2, we have that $\pi_Q^{-1}(Q')$ is isomorphic to the product bundle $(\mathbb{Z}_2)^n \times Q'$, and then Claim A follows from this since Q' is homeomorphic to S^{n-1}.

Claim B If $n > 2$, then $\mathbf{b}(\pi_Q^{-1}(Q_1)) = \mathbf{b}(\pi_P^{-1}(\partial P))$.

Since ∂Q is the deformation retract of Q_1, this induces the deformation retraction of $\pi_Q^{-1}(Q_1)$ onto $\pi_Q^{-1}(\partial Q)$, so $\mathbf{b}(\pi_Q^{-1}(Q_1)) = \mathbf{b}(\pi_Q^{-1}(\partial Q))$. Then we have by Lemma 3.2 that $\mathbf{b}(\pi_Q^{-1}(Q_1)) = \mathbf{b}(\pi_Q^{-1}(\partial Q)) = \mathbf{b}(\pi_P^{-1}(\partial P))$ as desired. Furthermore, we obtain from Lemma 3.1 that

$$\mathbf{b}(\pi_Q^{-1}(Q_1)) = (h_0(P), h_1(P), \cdots, h_{n-2}(P), h_{n-1}(P) + 2^n - 1).$$

Claim C For the mod 2 Betti numbers of $\pi_Q^{-1}(Q_2)$, we have $\mathbf{b}(\pi_Q^{-1}(Q_2)) = \mathbf{b}(E)$.

This is because $\pi_Q^{-1}(Q_2)$ is the deformation retract of E.

Claim D $M(Q, \xi, \lambda)$ is connected.

For every point $x \in M$, there exists always a path connect x to $\pi_Q^{-1}(\partial Q)$. Also $\pi_Q^{-1}(\partial Q)$ is connected since $b_0(\pi_Q^{-1}(\partial Q)) = h_0(P) = 1$, so $M(Q, \xi, \lambda)$ is connected.

Together with Claims A–D, Lemma 3.2 and the Mayer-Vietoris sequence above, the required result in the case $n > 2$ follows immediately.

Finally, let us consider the case of $n = 2$. If $n = 2$ then we have from [7] that

$$\chi(M) = 4\chi(Q) - m,$$

where m is number of vertex of Q. It is easy to check $h_1(Q) = m - 2$, $\chi(M) = 2 - b_1(M)$ and $\chi(Q) = 1 - b_1(Q)$. Then we have

$$b_1(M) = 4b_1(Q) + h_1(Q)$$

as desired.

Corollary 3.1 *If E is trivial, then*

$$b_i(M) = 2^n(b_i(Q)) + h_i(Q), \quad i > 0.$$

Corollary 3.2 *If Q is a simple convex polytope, then*

$$b_i(M) = h_i(Q).$$

Remark 3.1 We see that Corollary 3.2 is just Theorem 2.1 of Davis-Januszkiewicz. So Theorem 3.1 is a generalization of Theorem 2.1.

Finally, we conclude this paper with the following remark on the construction of $M(Q, \xi, \lambda)$.

Remark 3.2 We see from Lemma 3.2 that if $n > 2$, ∂E is a disjoint union of 2^n copies of $\partial Q = \partial P$. Thus, we can obtain a closed manifold $\widehat{E}$ from E by gluing boundaries of 2^n copies $P_1, P_2, \cdots, P_{2^n}$ of P to the 2^n components of ∂E respectively. On the other hand, we know from the reconstruction of locally standard 2-torus manifolds that $M(P, \lambda)$ is actually obtained from the 2^n copies $P_1, P_2, \cdots, P_{2^n}$ of P along their boundaries via λ, while $M(Q, \xi, \lambda)$ is obtained from E by gluing the 2^n components (i.e., $\partial P_1, \cdots, \partial P_{2^n}$) of ∂E via λ. Thus we have that if $n > 2$, then $M(Q, \xi, \lambda) = M(P, \lambda)\sharp_{P_1, \cdots, P_{2^n}} \widehat{E}$, obtained by doing connected sums of 2^n times between $M(P, \lambda)$ and $\widehat{E}$ along the interiors of $P_1, \cdots, P_{2^n}$. Actually, $M(Q, \xi, \lambda)$ is exactly the equivariant connected sum of $M(P, \lambda)$ and $\widehat{E}$ along a free orbit, denoted by $M(Q, \xi, \lambda) = M(P, \lambda)\sharp^{\text{free}}_{(\mathbb{Z}_2)^n} \widehat{E}$.

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